Groups Geom. Dyn. 16 (2022), 779–842
DOI 10.4171/GGD/653

Character varieties of a transitioning Coxeter 4-orbifold

Stefano Riolo and Andrea Seppi

Abstract. In 2010, Kerckhoff and Storm discovered a path of hyperbolic 4-polytopes eventually collapsing to an ideal right-angled cuboctahedron. This is expressed by a deformation of the inclusion of a discrete reflection group (a right-angled Coxeter group) in the isometry group of hyperbolic 4-space. More recently, we have shown that the path of polytopes can be extended to Anti-de Sitter geometry so as to have geometric transition on a naturally associated 4-orbifold, via a transitional half-pipe structure. In this paper, we study the hyperbolic, Anti-de Sitter, and half-pipe character varieties of Kerckhoff and Storm's right-angled Coxeter group near each of the found holonomy representations, including a description of the singularity that appears at the collapse. An essential tool is the study of some rigidity properties of right-angled cusp groups in dimension four.

1. Introduction

In the seventies, Thurston [29] introduced the notion *degeneration* of (G, X)-structures, later widely studied and used in dimension three [4, 7, 12, 13, 16–19, 21–24, 28]. Typical instances of this phenomenon are paths of hyperbolic cone structures on a 3-manifold eventually collapsing to some lower-dimensional orbifold, whose geometric structure is said to *regenerate* to 3-dimensional hyperbolic structures.

In his thesis [8], Danciger showed that when the limit is 2-dimensional and hyperbolic, it often regenerates to Anti-de Sitter (AdS) structures as well, so as to have *geometric transition* from hyperbolic to AdS structures (see also [1, 9–11, 30]). To that purpose, he introduced half-pipe (HP) geometry, which is a *limit geometry* [6] of both hyperbolic and AdS geometries inside projective geometry and encodes the behaviour of such a collapse "at the first order". One can indeed suitably "rescale" the structures inside the "ambient" projective geometry along the direction of collapse, so as to get at the limit a 3-dimensional "transitional" HP structure.

Concerning dimension four, Kerckhoff and Storm [15] described a path $t \mapsto \mathcal{P}_t$, $t \in (0, 1]$, of hyperbolic 4-polytopes which collapse as $t \to 0$ to a 3-dimensional ideal right-angled cuboctahedron. This induces a path of incomplete hyperbolic structures on a naturally associated 4-orbifold $\mathcal{O}$. The orbifold fundamental group of $\mathcal{O}$ is a rank-22 right-angled Coxeter group Γ_{22}, which embeds in $\mathrm{Isom}(\mathbb{H}^4)$ as a discrete reflection group

2020 Mathematics Subject Classification. Primary 57M50; Secondary 14M35, 52B11, 57K32.
Keywords. Character varieties, 4-dimensional geometric structures, geometric transition, Coxeter orbifolds, cusp rigidity.

when $t = 1$. In [26] (see also [27]), we found a similar path of AdS 4-polytopes such that the two paths, suitably rescaled, can be joined so as to give geometric transition on the orbifold $\mathcal{O}$. In particular, there is a transitional HP orbifold structure on $\mathcal{O}$ joining the two paths.

Keckhoff and Storm's deformation has been studied and used in [20] to show, among other things, the first examples of collapse of 4-dimensional hyperbolic cone structures to 3-dimensional ones. Similarly, thanks to the found AdS deformation and HP transitional structure, in [26] the authors provided the first examples of geometric transition from hyperbolic to AdS cone structures in dimension four.

The goal of this paper is to describe the hyperbolic, AdS, and HP character varieties of the right-angled Coxeter group Γ_{22}, including a study of the behaviour at the collapse. The results are summarised in Theorem 1.1 below.

1.1. The three character varieties of Γ_{22}

Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or the group G_{HP^4} of transformations of HP geometry, and let $G^+ < G$ be the subgroup of orientation-preserving transformations.

Recall that $\mathrm{Hom}(\Gamma_{22}, G)$ is naturally a real algebraic affine set [31]. We call *character variety* of Γ_{22} the (topological) quotient

$$X(\Gamma_{22}, G) = \mathrm{Hom}(\Gamma_{22}, G)/G^+$$

by the action of G^+ by conjugation. When G is reductive, that is in the hyperbolic and AdS settings, it is also possible to define the GIT quotient, which has a structure of real semialgebraic set by general results [25]. We will come back to this point of view at the end of the subsection.

The holonomy representations of the geometric structures on the orbifold $\mathcal{O}$ constructed in [15, 26] provide a smooth path $t \mapsto [\rho_t^G]$ in $X(\Gamma_{22}, G)$. This path was originally defined in [15] when $G = \mathrm{Isom}(\mathbb{H}^4)$ only for $t \in (0, 1]$ and is easily continued analytically also for non-positive times. The AdS path, introduced in [26], is only defined for $t \in (-1, 1)$ and diverges as $|t| \to 1^-$, while for $G = G_{\mathsf{HP}^4}$ there is a "trivial" path of non-equivalent HP representations (defined for $t \in \mathbb{R}$, and diverging as $|t| \to +\infty$) differing from one another by "stretching" in the ambient real projective space (see below in Section 1.2).

The representations obtained at $t = 0$ correspond geometrically to a "collapse" and play a special role in two ways. First, they correspond to a "symmetry" in the character varieties, since the representations ρ_t^G and ρ_{-t}^G are conjugated in G but not in G^+. Second, interpreting $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, and G_{HP^4} as subgroups of $\mathrm{PGL}(5, \mathbb{R})$, the three representations ρ_0^G coincide. They correspond to a representation (we omit the superscript G here)

$$\rho_0 \colon \Gamma_{22} \to \mathrm{Stab}(\mathbb{H}^3) < G,$$

for a fixed copy of $\mathbb{H}^3$ in $\mathbb{H}^4$, $\mathbb{A}\mathrm{d}\mathbb{S}^4$ or HP^4, respectively. Projecting the image of ρ_0 in

Figure 1. On the left, a topological picture of $X(\Gamma_{22}, G)$ near the collapse, which corresponds to the point $[\rho_0]$. The vertical component $\mathcal{V}$ is $\{[\rho_t^G]\}_t$. The horizontal component $\mathcal{H}$ is 12-dimensional and corresponds to the deformations of the complete hyperbolic structure of the ideal right-angled cuboctahedron. On the right, the corresponding neighbourhood in $\mathrm{Hom}(\Gamma_{22}, G)/G$, i.e., in the further quotient of $X(\Gamma_{22}, G)$ by $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$.

$\mathrm{Stab}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3) \times \mathbb{Z}/2\mathbb{Z}$ to $\mathrm{Isom}(\mathbb{H}^3)$ gives the reflection group of an ideal right-angled cuboctahedron (see Figure 5).

Our main result is resumed as follows. (A precise explanation of the terminology is given below the statement; see Figure 1 for a schematic picture.)

Theorem 1.1. *Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or G_{HP^4}. A neighbourhood $\mathcal{U}$ of $[\rho_0]$ in $X(\Gamma_{22}, G)$ consists of two smooth, transverse, components $\mathcal{V}$ and $\mathcal{H}$ satisfying $\mathcal{V} \cap \mathcal{H} = \{[\rho_0]\}$:*

- *the curve $\mathcal{V}$ of the conjugacy classes of all the holonomy representations ρ_t^G;*
- *a 12-dimensional ball $\mathcal{H}$, identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{U}$ fixing $\mathcal{H}$ point-wise and sending $[\rho_t^G]$ to $[\rho_{-t}^G]$.

Let us include some comments to elucidate the content of Theorem 1.1. First, our proofs actually show that the representation ρ_0 has a neighbourhood in $\mathrm{Hom}(\Gamma_{22}, G)$ that is homeomorphic to $(\mathcal{H} \cup \mathcal{V}) \times G^+$, in such a way that the action of G^+ corresponds to obvious left multiplication by G^+ on the second factor (see Remark 4.5).

Let $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$, and $\widetilde{\mathcal{H}}$ be the preimages in $\mathrm{Hom}(\Gamma_{22}, G)$ of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{H}$, respectively. By "smoothness" of the "components" $\mathcal{V}$ and $\mathcal{H}$ of $\mathcal{U}$ we actually refer to $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{H}}$, and $\widetilde{\mathcal{U}}$, respectively. In particular, $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ are smooth manifolds (of dimension 11 and 22, respectively). The smoothness of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$, together with the local product structure in a neighbourhood of ρ_0, induces a smooth structure on the components $\mathcal{H}$ and $\mathcal{V}$ in the quotient.

The "transversality" of $\mathcal{V}$ and $\mathcal{H}$ is defined as follows: $\widetilde{\mathcal{V}} \cap \widetilde{\mathcal{H}}$ is the G-orbit of ρ_0, and the quotient to $\mathcal{V}$ and $\mathcal{H}$, respectively. Moreover, the Zariski tangent spaces of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ intersect transversely in the Zariski tangent space of $\mathrm{Hom}(\Gamma_{22}, G)$ at ρ_0 (and hence at any other point of its orbit). In particular, every infinitesimal deformation tangent to both $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ is tangent to the G^+-orbit of ρ_0. (See Section 1.5 below and Remark 7.9 for more details.)

Our analysis will also show that, when G is $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the character variety $X(\Gamma_{22}, G)$ is homeomorphic to the GIT quotient $\mathrm{Hom}(\Gamma_{22}, G) /\!/ G^+$ near each $[\rho_t]$ (see Remark 4.6). In other words, $X(\Gamma_{22}, G)$ is Hausdorff near $[\rho_t]$. Moreover, the natural smooth structure of each component is coherent with the real semialgebraic structure of the GIT quotient (see also Remark 7.9).

In order to further discuss our results, we first need to describe the three paths of geometric representations of Theorem 1.1.

1.2. The three deformations

For $t = 1$, the hyperbolic polytope $\mathcal{P}_1 \subset \mathbb{H}^4$ is obtained in [15] from the ideal right-angled 24-cell by removing two opposite bounding hyperplanes. So $\mathcal{P}_1$ has two "Fuchsian ends", and in particular its volume is infinite. The reflection group

$$\Gamma_{22} < \mathrm{Isom}(\mathbb{H}^4)$$

associated to $\mathcal{P}_1$ is thus a right-angled Coxeter group obtained by removing from the reflection group Γ_{24} of the ideal right-angled 24-cell two generators (reflections at two opposite facets).

As a sort of "reflective hyperbolic Dehn filling", Kerckhoff and Storm show that the inclusion $\Gamma_{22} \to \mathrm{Isom}(\mathbb{H}^4)$ is not locally rigid. This is done by moving the bounding hyperplanes of $\mathcal{P}_1$ in such a way that the orthogonality conditions given by the relations of Γ_{22} are maintained, and thus obtaining a path $\rho_t^{\mathbb{H}^4}$ of geometric representations of Γ_{22}.

As t decreases from 1, the combinatorics of $\mathcal{P}_t$ changes a few times, until the volume of $\mathcal{P}_t$ becomes finite. Most of the dihedral angles of $\mathcal{P}_t$ are constantly right, while the varying ones are all equal and tend to π as $t \to 0$, when $\mathcal{P}_t$ collapses to the cuboctahedron. As an abstract group, Γ_{22} can be identified to the orbifold fundamental group of an orbifold $\mathcal{O}$ supported on the complement in $\mathcal{P}_t$ of the ridges with non-constant dihedral angle.

Kerckhoff and Storm show moreover that the space of conjugacy classes of representations $\Gamma_{22} \to \mathrm{Isom}(\mathbb{H}^4)$ deforming the inclusion is a smooth curve outside of the collapse. In other words, for $t \neq 0$ the only non-trivial deformation (up to conjugacy) is given by the found holonomies $\rho_t^{\mathbb{H}^4}$.

In [26], we produced a path of AdS 4-polytopes with the same combinatorics of the hyperbolic polytope $\mathcal{P}_t$ for $t \in (0, \varepsilon)$, such that the same orthogonality conditions between the bounding hyperplanes are satisfied, and again collapsing to an ideal right-angled cuboctahedron in a spacelike hyperplane $\mathbb{H}^3$ of $\mathbb{A}\mathrm{d}\mathbb{S}^4$. Some bounding hyperplanes are spacelike, and some others are timelike. We have, in particular, a path of AdS orbifold structures on $\mathcal{O}$, with holonomy representation $\rho_t^{\mathbb{A}\mathrm{d}\mathbb{S}^4} \colon \Gamma_{22} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ given by sending each generator to the corresponding AdS reflection.

We moreover find in [26] a one-parameter family of transitional HP structures on $\mathcal{O}$, with holonomy $\rho_t^{\mathsf{HP}^4}$. To interpret distinct elements in this family, recall that in half-pipe space there is a preferred direction under which the HP metric is degenerate. An HP structure is never rigid, because one can always conjugate with a transformation which "stretches" the degenerate direction and obtain a new structure equivalent to the initial one *as a real projective structure*, but inequivalent *as a half-pipe structure*. We discover here (this is part of the content of Theorem 1.1) that such stretchings are the only possible deformations, so that the found HP structures are essentially unique.

Finally, we remark that the geometric transition described in [26] induces a continuous deformation connecting in the $\mathrm{PGL}(5,\mathbb{R})$-character variety "half" of the path in

$$\mathcal{V} \subset X\big(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4)\big)$$

(which is exactly the path of hyperbolic representations exhibited by Kerckhoff and Storm, for $t \in (0, 1]$) and "half" of the analogous path in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$, going through a single HP representation $\rho_{t_0}^{\mathsf{HP}^4}$ with $t_0 \neq 0$ (this value of t_0 can be chosen arbitrarily, up to reparameterising the entire deformation).

1.3. About the result

Theorem 1.1 contains several novelties. First, while the smoothness of the $\mathrm{Isom}(\mathbb{H}^4)$-character variety for $t > 0$ was proved in [15], the smoothness on the AdS and HP sides is completely new. Second, the study of the character variety at the "collapsed" point $[\rho_0]$ is a new result in all three settings. Some motivations follow.

First of all, we found worthwhile analysing the behaviour of the deformation space of the AdS orbifold $\mathcal{O}$—equivalently, the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-character variety of Γ_{22} near $[\rho_t]$—and comparing it with the hyperbolic counterpart. In fact, the literature seems to miss a study of deformations of AdS polytopes in this spirit. With respect to hyperbolic geometry, one may expect more flexibility in AdS geometry, but we find that the behaviour on the AdS side is the same as the hyperbolic counterpart. (In regard, see however [3, Question 9.3] and the related discussion.)

Regarding the collapse, in [15, Section 14] Kerckhoff and Storm mention that the family of hyperbolic polytopes $\mathcal{P}_t \subset \mathbb{H}^4$, $t > 0$, is expected to have interesting geometric limits by rescaling in the direction transverse to the collapse. On the other hand, they assert that "providing the details of this geometric construction would require more space than perhaps is merited here".

The work [26] provides a complete description of such a geometric limit in HP geometry, which, after the work of Danciger, seems the best suited in order to analyse this kind of collapse. One can in fact consider the limiting HP structure as an object which encodes the collapse *at the first order*, essentially keeping track of the derivatives of all the associated geometric quantities. Thus, if [26] describes the collapse *at level of geometric structures*, Theorem 1.1 (in the hyperbolic and AdS setting) gives a precise description of the collapse *at level of the character variety*.

Finally, our study of the HP character variety shows that the only deformations of the found HP orbifold structure are obtained by "stretching" in the degenerate direction. The presence of many commutation relations forces the rigidity of the HP structures.

Together with the hyperbolic and AdS picture, this shows that "nearby" there is no collapsing path of hyperbolic or AdS orbifold structures other than the ones we found (up to reparameterisation). This should be compared with some 3-dimensional examples found by Danciger [9, Section 6], where the transitional HP structure deforms non-trivially to nearby HP structures that regenerate to non-equivalent AdS structures, despite not regenerating to hyperbolic structures.

All in all, Theorem 1.1 exhibits a strong lack of flexibility around this example. Its proof, explained in the next section, suggests that this could be more generally due to dimension issues, confirming the usual feeling that "the rigidity increases with the dimension".

An overview of the ideas behind the proof of Theorem 1.1 follows. We start in Section 1.4 with the geometric tools, which lead to the topological description of the neighbourhood $\mathcal{U}$ of $[\rho_0]$. In Section 1.5, we then outline the algebraic aspects, which lead to the description of the Zariski tangent space and the transversality statement.

1.4. Cusp rigidity in dimension four

The holonomy representations ρ_t have the property that each generator in the standard presentation of Γ_{22} is sent by ρ_t to a (hyperbolic or AdS) reflection, and this property is preserved by small deformations. As in [15], we thus reduce to studying the configurations of hyperplanes of reflections satisfying certain orthogonality conditions. Once this set-up is established, there are two main facts to prove: the smoothness of the character variety outside the collapse and the description of the collapse itself.

For the first fact, the proof on the AdS side follows the general lines of the proof given in [15] for the hyperbolic case. However, different arguments are required for one point of fundamental importance concerning a property of rigidity of cusp representations in dimension four.

In fact, in [15] a preliminary lemma is proved, which can be summarised by saying that in dimension four "cusp groups stay cusp groups". More precisely, if we consider the orbifold fundamental group of a Euclidean cube Γ_{cube}, this property states that any representation of Γ_{cube} into $\text{Isom}(\mathbb{H}^4)$ sending the six standard generators to reflections in six distinct hyperplanes with the property of being asymptotic to a common point at infinity (a "cusp group") can only be deformed by preserving this property. Note that the analogue fact is false in dimension three, where the situation is more flexible.

We do prove the analogous property for AdS geometry in dimension four (Proposition 3.15), where a cusp group is defined analogously. There are however remarkable differences due to the different nature of hyperbolic and AdS geometries, for instance a cusp group in $\mathbb{A}\mathrm{d}\mathbb{S}^4$ will be generated by 4 reflections in timelike hyperplanes and 2 reflections in spacelike hyperplanes. The proof of this rigidity property in AdS uses therefore ad hoc arguments and is somehow more surprising than its hyperbolic counterpart, as in general a little more flexibility might be expected for AdS geometry.

Once this fundamental property is established, the proof of the smoothness of the curve is based on a careful analysis of the structure of the group Γ_{22} and the possible deformations of the polytope $\mathcal{P}_t$, relying on the application of the above rigidity property to each peripheral subgroup (there is a cusp group $\Gamma_{\mathrm{cube}} < G$ associated to each ideal vertex of $\mathcal{P}_t$). The methods are rather elementary, although some intricate computation is necessary, and the general strategy is similar to that of the hyperbolic analogue provided in [15].

Let us now explain our arguments to analyse the collapse in both the $\mathbb{H}^4$ and $\mathbb{A}\mathrm{d}\mathbb{S}^4$ character variety. The proof is essentially the same for both cases, so let us focus on the hyperbolic case (that is, $G = \mathrm{Isom}(\mathbb{H}^4)$) in this introduction for definiteness.

It is not difficult to describe the two components $\mathcal{V}$ and $\mathcal{H}$ of the neighbourhood $\mathcal{U}$ from a geometric point of view: the "vertical" curve $\mathcal{V}$ consists of the conjugacy classes of the holonomy representations ρ_t^G of $\mathcal{O}$, $t > 0$, plus the natural extension of the path for $t < 0$ given by $r \circ \rho_{-t}^G \circ r$. Here r is the reflection in the totally geodesic copy of $\mathbb{H}^3$ to which the polytope collapses as $t = 0$. On the other hand, the "horizontal" 12-dimensional component $\mathcal{H}$ consists of representations which fix setwise this copy of $\mathbb{H}^3$, and deform the reflection group of the ideal right-angled cuboctahedron in $\mathbb{H}^3$.

One then has to show that there exists a neighbourhood of $[\rho_0]$ such that every point in this neighbourhood belongs to one of these two components—namely, there are no other conjugacy classes of representations nearby $[\rho_0]$. To prove this, we refine the study of the rigidity properties of the cusps. We introduce a notion of *collapsed cusp group*: a representation of Γ_{cube} defined similarly to cusp groups, but allowing that two generators are sent to reflections in the same hyperplane. The restriction of ρ_0 to each peripheral subgroup is in fact a collapsed cusp group. Then we prove a more general version of the aforementioned rigidity property "cusp groups stay cusp groups", by showing that "collapsed cusp groups either stay collapsed or deform to cusp groups". More precisely, representations nearby a collapsed cusp group either keep the property that two opposite generators are sent to the same reflection or become cusp groups in the usual sense.

By an analysis of the character variety in the spirit of Kerckhoff and Storm, we show that the "vertical" curve $\mathcal{V}$ is smooth also at $t = 0$ if we impose that the asymptoticity conditions are preserved. Applying the more general property of rigidity which includes the "collapsed" case is then the fundamental step to conclude the proof.

Concerning the proof for the HP case, it follows a similar line, but many steps are dramatically simpler. The key point is again a rigidity property for 4-dimensional (collapsed) cusp groups, which is shown rather easily by using the isomorphism between G_{HP^4} and the group of isometries of Minkowski space $\mathbb{R}^{1,3}$, which is a semidirect product $\mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$. The proof then parallels the steps for the hyperbolic and AdS case, except that the smoothness of the vertical component $\mathcal{V}$ is granted by the fact that—thanks to this semidirect product structure of G_{HP^4}—$\mathcal{V}$ identifies with the first cohomology vector space $H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. The proof that this vector space is 1-dimensional (see (2) below) requires a certain amount of technicality and relies on a precise study of the group-theoretical structure of Γ_{22}.

1.5. The Zariski tangent space and the first cohomology group

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation. The Zariski tangent space to $\mathrm{Hom}(\Gamma_{22}, G)$ at ρ is naturally identified to the space $Z^1_{\mathrm{Ad}\,\rho}(\Gamma_{22}, \mathfrak{g})$ of cocycles with coefficients twisted by $\mathrm{Ad} \circ \rho$. Roughly speaking, the cohomology group $H^1_{\mathrm{Ad}\,\rho}(\Gamma_{22}, \mathfrak{g})$ plays the same role for the quotient $X(\Gamma, G)$ at $[\rho]$. Indeed the coboundaries $B^1_{\mathrm{Ad}\,\rho}(\Gamma_{22}, \mathfrak{g})$ are precisely the infinitesimal deformations tangent to the orbit. See [14,31] for more details.

Since ρ_0 preserves a totally geodesic copy of $\mathbb{H}^3$, we have a natural decomposition:

$$H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) \cong H^1_{\mathrm{Ad}\,\rho_0}\big(\Gamma_{22}, \mathfrak{o}(1,3)\big) \oplus H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3}). \tag{1}$$

We show that the vector space $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$ is 13-dimensional. In the decomposition (1), the first factor is 12-dimensional and "tangent" to $\mathcal{H}$, while the second factor is 1-dimensional and "tangent" to $\mathcal{V}$; moreover integrable vectors are precisely those lying in one of these two factors. This statement is made more precise by looking at the representation variety $\mathrm{Hom}(\Gamma_{22}, G)$, where the tangent spaces of the smooth varieties $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ are generated by the preimages in $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$ of the two factors in the decomposition (1). Hence the intersection $\widetilde{\mathcal{V}} \cap \widetilde{\mathcal{H}}$ is transverse and consists precisely of the orbit of ρ_0.

As mentioned above, we prove (in Proposition 6.5) that the second factor in the decomposition (1) has dimension one, namely

$$H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \cong \mathbb{R}. \tag{2}$$

The non-trivial elements in this vector space are obtained geometrically from the HP holonomy representations that we constructed in [26], and are easily shown to be "first-order deformations" of paths in $\mathcal{V}$. On the other hand, the first factor in the decomposition (1) is 12-dimensional by general reasons, namely by an orbifold version of hyperbolic Dehn filling (note that the cuboctahedron has exactly 12 vertices). Its elements are again integrable and tangent to deformations in $\mathcal{H}$. This cohomological computation is the main algebraic step in the proof of Theorem 1.1.

As another noteworthy comment on the consequences of (2), recall Danciger's result [9, Theorem 1.2]: the existence of geometric transition on a compact HP 3-manifold $\mathcal{X}$, with singular locus a knot Σ, is proved under the sole condition that

$$H^1_{\mathrm{Ad}\,\rho_0}\big(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathfrak{so}(1,2)\big) \cong \mathbb{R}. \tag{3}$$

Now, for any representation $\rho\colon \Gamma \to \mathrm{O}(1,2)$ with Γ finitely generated there is a natural identification

$$H^1_{\mathrm{Ad}\,\rho}\big(\Gamma, \mathfrak{so}(1,2)\big) \cong H^1_{\rho}(\Gamma, \mathbb{R}^{1,2}).$$

In presence of a collapse of hyperbolic or AdS structures of dimension n, the holonomy representations of a rescaled HP structure naturally provide elements of the cohomology group $H^1_{\rho_0}(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathbb{R}^{1,n-1})$. In particular, the correct generalisation of Danciger's condition (3) to any dimension n would be

$$H^1_{\rho_0}\big(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathbb{R}^{1,n-1}\big) \cong \mathbb{R}, \tag{4}$$

in agreement with what we found for Γ_{22} (the orbifold fundamental group of $\mathcal{O}$)—compare with (2). In conclusion, this suggests that a higher-dimensional regeneration result in the spirit of Danciger, although far from reach at the present time, might be reasonable.

Organisation of the paper

In Section 2, we establish the set-up for the proof of Theorem 1.1 in the hyperbolic and AdS cases. In Section 3, which may be of independent interest, we study deformations of some right-angled Coxeter groups represented as "cusp groups" in $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ for $n = 3, 4$. In Section 4, we study the $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ character varieties of Γ_{22}, concluding the first part of the proof of Theorem 1.1 for the hyperbolic and AdS case. Section 5 is the analogue of Sections 2 and 3 for HP geometry. In Section 6, we provide the explicit computation of (2) and use it for the first part of the proof of Theorem 1.1 in the HP case. In Section 7, we conclude the proof of Theorem 1.1 in all the three cases, relying on algebraic computations of the first cohomology group.

2. Reflections in hyperbolic and AdS geometry

In this section, we establish the set-up for the study of hyperbolic and AdS character varieties of right-angled Coxeter groups.

2.1. Hyperbolic and AdS geometry

We begin with the necessary definitions and notation. Let $q_{\pm 1}$ be the quadratic form on $\mathbb{R}^{n+1}$ of signature $(-, +, \dots, +, \pm)$ defined by

$$q_{\pm 1}(x) = -x_0^2 + x_1^2 + \dots + x_{n-1}^2 \pm x_n^2,$$

and let $b_{\pm 1}$ be the associated bilinear form.

The n-dimensional *hyperbolic space* $\mathbb{H}^n$ is defined via the "hyperboloid model" as

$$\mathbb{H}^n = \big\{x \in \mathbb{R}^{n+1} \mid q_1(x) = -1,\ x_0 > 0\big\}.$$

The restriction of q_1 endows $\mathbb{H}^n$ with a complete Riemannian metric of constant negative sectional curvature, whose isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is identified to an index-two subgroup of $\mathrm{O}(1, n)$, namely the subgroup of those linear isometries of q_1 which preserve $\mathbb{H}^n$. Despite that this subgroup is defined by an inequality, if n is even, then $\mathrm{Isom}(\mathbb{H}^n)$ is naturally isomorphic to the algebraic Lie group $\mathrm{SO}(1, n)$ (via $A \mapsto A$ if A preserves $\mathbb{H}^n \subset \mathbb{R}^{n+1}$, and $A \mapsto -A$ otherwise). The *boundary at infinity* of $\mathbb{H}^n$ is the projectivisation of the cone of null directions for the quadratic form q_1:

$$\partial\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid q_1(x) = 0\}/\mathbb{R}^*.$$

Hence both $\mathbb{H}^n$ and $\partial\mathbb{H}^n$ can be seen as subsets of $\mathbb{R}\mathrm{P}^n$. The topology of $\mathbb{R}\mathrm{P}^n$ induces a natural topology on $\mathbb{H}^n \cup \partial\mathbb{H}^n$, which makes it homeomorphic to the closed n-ball D^n. Finally, given a subset $A \subset \mathbb{H}^n$ that is closed as a subspace of $\mathbb{H}^n$, its *ideal closure* $\overline{A}$ is the closure of A in $\mathbb{R}\mathrm{P}^n$. In particular, we have $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial\mathbb{H}^n$.

The n-dimensional *AdS space* is defined as

$$\mathbb{A}\mathrm{d}\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid q_{-1}(x) = -1\},$$

and the restriction of q_{-1} endows $\mathbb{A}\mathrm{d}\mathbb{S}^n$ with a Lorentzian metric of constant negative sectional curvature. Observe that $\mathbb{A}\mathrm{d}\mathbb{S}^n$ is homeomorphic to $S^1 \times \mathbb{R}^{n-1}$. Its isometry group $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ is identified to the algebraic Lie groups $\mathrm{O}(q_{-1}) \cong \mathrm{O}(2, n-1)$. The *boundary at infinity* of $\mathbb{A}\mathrm{d}\mathbb{S}^n$ is defined as the image of the null directions for the quadratic form q_{-1}, now in the projective sphere $\widetilde{\mathbb{R}\mathrm{P}}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}_{>0}$:

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid q_{-1}(x) = 0\}/\mathbb{R}_{>0}.$$

Interpreting $\mathbb{A}\mathrm{d}\mathbb{S}^n$ and $\partial\mathbb{A}\mathrm{d}\mathbb{S}^n$ as subsets of $\widetilde{\mathbb{R}\mathrm{P}}^n$ gives $\mathbb{A}\mathrm{d}\mathbb{S}^n \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^n$ a natural topology, which makes it homeomorphic to $S^1 \times D^{n-1}$. Again, the *ideal closure* of a subset $A \subset \mathbb{A}\mathrm{d}\mathbb{S}^n$ that is closed in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ is its closure $\overline{A}$ in $\widetilde{\mathbb{R}\mathrm{P}}^n$, and we have $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^n} = \mathbb{A}\mathrm{d}\mathbb{S}^n \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^n$.

2.2. Hyperplanes and reflections

A *hyperplane* of $\mathbb{H}^n$ (resp. $\mathbb{A}\mathrm{d}\mathbb{S}^n$) is the intersection, when non-empty, of a linear hyperplane of $\mathbb{R}^{n+1}$ with $\mathbb{H}^n$ (resp. $\mathbb{A}\mathrm{d}\mathbb{S}^n$). Notice that, unlike the hyperbolic case, the intersection of a linear hyperplane with $\mathbb{A}\mathrm{d}\mathbb{S}^n$ is always non-empty, and sometimes disconnected (see the discussion preceding Lemma 2.2). In both cases, hyperplanes are totally geodesic.

A *reflection* in hyperbolic geometry, resp. in AdS geometry, is a non-trivial involution $r \in \mathrm{Isom}(\mathbb{H}^n)$, resp. $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, that fixes point-wise a hyperplane.

Let us denote by $\perp_1$ the orthogonal complement with respect to the bilinear form b_1, and let $X \in \mathbb{R}^{n+1}$ be a vector. The linear hyperplane $X^{\perp_1}$ of $\mathbb{R}^{n+1}$ intersects $\mathbb{H}^n$ if and only if $q_1(X) > 0$, i.e., if X is *spacelike* for q_1. Hence for every q_1-spacelike vector X there is a hyperplane

$$H_X = X^{\perp_1} \cap \mathbb{H}^n.$$

It is clearly harmless to assume that $q_1(X) = 1$, so that the vector X is uniquely determined up to changing the sign.

Given a hyperplane H_X in $\mathbb{H}^n$, there is a unique reflection r_X fixing the given hyperplane. Indeed, the reflection r_X is the linear transformation in $\mathrm{O}(q_1)$ which fixes $X^{\perp_1}$ and acts on the subspace generated by X as minus the identity. Two spacelike unit vectors X and Y give the same reflection if and only if $X = \pm Y$. Finally, it is a simple exercise to show that two reflections r_X and r_Y commute if and only if either $X = \pm Y$ or X and Y are orthogonal for the bilinear form b_1.

We summarise the above considerations in the following statement:

Lemma 2.1. *There is a two-sheeted covering map*

$$\left\{X \in \mathbb{R}^{n+1} : q_1(X) = +1\right\} \to \left\{r \in \mathrm{Isom}(\mathbb{H}^n) : r \text{ is a reflection}\right\},$$

which maps a spacelike unit vector X to the unique reflection r_X fixing H_X point-wise. Moreover, two distinct reflections r_X and r_Y commute if and only if $b_1(X, Y) = 0$.

The subset of vectors in $\mathbb{R}^{n+1}$ such that $q_1(X) = +1$ is usually called *de Sitter space*.

Let us now move to AdS geometry. A hyperplane is called *spacelike*, *timelike* or *lightlike* if the induced bilinear form, obtained as the restriction of the Lorentzian metric of $\mathbb{A}\mathrm{d}\mathbb{S}^n$, is positive definite, indefinite or degenerate, respectively. Spacelike hyperplanes are disconnected, and each of the two connected components is an isometrically embedded copy of $\mathbb{H}^{n-1}$. Timelike hyperplanes are isometrically embedded copies of $\mathbb{A}\mathrm{d}\mathbb{S}^{n-1}$.

Let us denote by $\perp_{-1}$ the orthogonality relation with respect to b_{-1}. We have the following lemma.

Lemma 2.2. *Given a vector $X \in \mathbb{R}^{n+1}$, the intersection $X^{\perp_{-1}} \cap \mathbb{A}\mathrm{d}\mathbb{S}^n$ is non-empty, and is*

- *a spacelike hyperplane if $q_{-1}(X) < 0$,*
- *a timelike hyperplane if $q_{-1}(X) > 0$,*
- *a lightlike hyperplane if $q_{-1}(X) = 0$.*

The hyperplane of fixed points of an AdS reflection is either spacelike or timelike. Similarly to the hyperbolic case, given a vector X such that $q_{-1}(X) = \pm 1$, the unique reflection fixing

$$H_X = X^{\perp_{-1}} \cap \mathbb{A}\mathrm{d}\mathbb{S}^n$$

is induced by the linear transformation in $\mathrm{O}(q_{-1})$ acting on $X^{\perp_{-1}}$ as the identity and on $\mathrm{Span}(X)$ (which is in direct sum with $X^{\perp_{-1}}$ since $q_{-1}(X) \neq 0$) as minus the identity.

In conclusion, we have another summarising statement:

Lemma 2.3. *There is a two-sheeted covering map*

$$\left\{X \in \mathbb{R}^{n+1} : q_{-1}(X) = \pm 1\right\} \to \left\{r \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n) : r \text{ is a reflection}\right\},$$

which maps a spacelike or timelike unit vector X to the unique reflection r_X fixing H_X point-wise. Moreover, two distinct reflections r_X and r_Y commute if and only if $b_{-1}(X, Y) = 0$.

The space $\{X \in \mathbb{R}^{n+1} : q_{-1}(X) = \pm 1\}$ has two connected components, as well as the space of reflections. The component defined by $q_{-1}(X) = -1$ is a copy of AdS space itself.

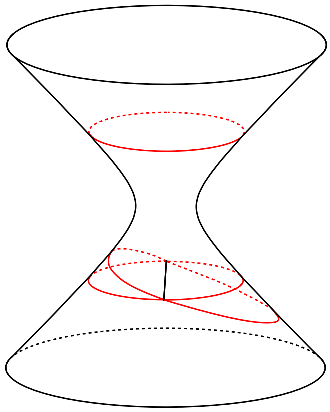

Figure 2. Two spacelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ can intersect in a totally geodesic spacelike hyperplane, at a point at infinity, or be disjoint. The picture ($n = 3$) is in an affine chart for the real projective sphere, where AdS space is the interior of a one-sheeted hyperboloid. Each disk drawn in the picture represents a connected component of a spacelike hyperplane, and has an isometric copy on the "opposite" affine chart of $\widetilde{\mathbb{R}\mathrm{P}}^n$, obtained by applying minus the identity to $\mathbb{A}\mathrm{d}\mathbb{S}^n$.

2.3. Relative position of hyperplanes

It will be useful to discuss the relative position of hyperplanes. For hyperbolic geometry, this is easily summarised:

Lemma 2.4. *Let H_X and H_Y be two distinct hyperplanes in $\mathbb{H}^n$, for $q_1(X) = q_1(Y) = 1$. Then the following hold:*

- *H_X and H_Y intersect in $\mathbb{H}^n$ if and only if $|b_1(X,Y)| < 1$;*
- *$\overline{H}_X$ and $\overline{H}_Y$ intersect in $\partial\mathbb{H}^n$ if and only if $|b_1(X,Y)| = 1$;*
- *$\overline{H}_X$ and $\overline{H}_Y$ are disjoint in $\overline{\mathbb{H}}^n$ if and only if $|b_1(X,Y)| > 1$.*

In the second item of Lemma 2.4, $\overline{H}_X$ and $\overline{H}_Y$ intersect in exactly one point at infinity $p \in \partial\mathbb{H}^n$. In this case, we say that H_X and H_Y are *asymptotic (at p)*. By little abuse, sometimes we also say that a hyperplane H is *asymptotic* to a point at infinity p if $p \in \overline{H}$. Note the difference between the two notions: if the first item holds, then H_X and H_Y are not asymptotic, despite being asymptotic to p for any point at infinity $p \in \overline{H}_X \cap \overline{H}_Y \neq \emptyset$.

For AdS hyperplanes, it is necessary to distinguish several cases. Here we will only consider the cases of interest for the proofs of our main results. For spacelike hyperplanes, we have (see Figure 2) the following lemma.

Lemma 2.5. *Let H_X and H_Y be two distinct spacelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$, for $q_{-1}(X) = q_{-1}(Y) = -1$. Then,*

- *H_X and H_Y intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ if and only if $|b_{-1}(X,Y)| > 1$;*
- *$\overline{H}_X$ and $\overline{H}_Y$ intersect in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^n$ if and only if $|b_{-1}(X,Y)| = 1$;*
- *$\overline{H}_X$ and $\overline{H}_Y$ are disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^n}$ if and only if $|b_{-1}(X,Y)| < 1$.*

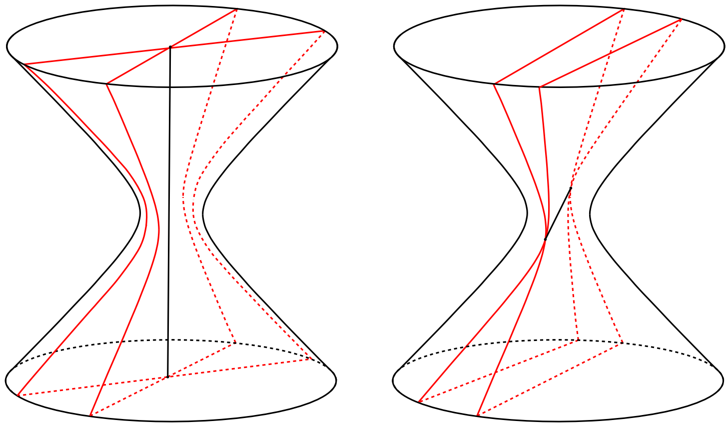

Figure 3. Two timelike planes in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ intersecting in a timelike (left) or spacelike (right) line. One should keep in mind that the hyperplanes enter into the "opposite" affine chart, where one has a completely analogous picture.

Similar terminology for asymptoticity is adopted when the second item of Lemma 2.5 occurs, the only difference with the hyperbolic case being that now $\bar{H}_X$ and $\bar{H}_Y$ intersect in exactly two antipodal points at infinity $\pm p \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^n$.

For two AdS timelike hyperplanes the situation is different, as explained in the following lemma. See also Figure 3.

Lemma 2.6. *Let H_X and H_Y be two distinct timelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$, for $q_{-1}(X) = q_{-1}(Y) = 1$. Then H_X and H_Y intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ and the intersection $H_X \cap H_Y$ is*

- *spacelike if and only if $|b_{-1}(X,Y)| > 1$;*
- *lightlike if and only if $|b_{-1}(X,Y)| = 1$;*
- *timelike if and only if $|b_{-1}(X,Y)| < 1$.*

In the first case of Lemmas 2.5 and 2.6, the intersection $H_X \cap H_Y$ consists of two totally geodesic copies of $\mathbb{H}^{n-2}$; in the third case of Lemma 2.6 it is a totally geodesic copy of $\mathbb{A}\mathrm{d}\mathbb{S}^{n-2}$.

3. Right-angled cusp groups in hyperbolic and AdS geometry

In this section, which may be of independent interest, we study deformations of some right-angled Coxeter groups represented as "cusp groups" in $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, for $n = 3, 4$. Let us begin in the next subsection with a general set up.

3.1. Coxeter groups and representation varieties

Given a finitely presented group Γ and an algebraic Lie group G, we denote by $\mathrm{Hom}(\Gamma, G)$ the space of representations $\rho\colon \Gamma \to G$. Since $\mathrm{Hom}(\Gamma, G)$ is naturally an affine algebraic set (see also Section 7.1), it is called *representation variety*.

In the remainder of the paper, we will restrict the attention to the case where Γ is a right-angled Coxeter group, which we now define.

Definition 3.1 (RACG). Given a finite set S and a subset R of (unordered) pairs of distinct elements of S, the associated *right-angled Coxeter group* has the presentation

$$\langle S \mid s^2 = 1\ \forall s \in S,\ s_1 s_2 = s_1 s_2\ \forall (s_1, s_2) \in R \rangle.$$

For instance, the group generated by the reflections in the sides of a right-angled Euclidean or hyperbolic polytope is a right-angled Coxeter group.

We will only be interested in representations of a right-angled Coxeter group Γ which send every generator to a reflection. Let us introduce more formally this space.

Definition 3.2 (The set $\mathrm{Hom}_{\mathrm{refl}}$). Let G be $\mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ and Γ a right-angled Coxeter group as above. We define $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ as the subset of $\mathrm{Hom}(\Gamma, G)$ of representations ρ such that:

- for every $s \in S$, the isometry $\rho(s)$ is a reflection, and
- for every $(s_1, s_2) \in R$, the reflections $\rho(s_1)$ and $\rho(s_2)$ are distinct.

Reflections constitute a connected component in the space of order-two isometries in $\mathrm{Isom}(\mathbb{H}^n)$, while in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ they constitute two connected components, given by reflections in spacelike and timelike hyperplanes. Moreover, by Lemmas 2.1 and 2.3, two distinct reflections commute if and only if their fixed hyperplanes are orthogonal. Hence we immediately get the following lemma.

Lemma 3.3. *The subset* $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ *is clopen in* $\mathrm{Hom}(\Gamma, G)$.

To simplify the computations, we will follow [15] and adopt a local model for the representation variety which is well adapted to our setting. Roughly speaking, we only consider the deformations of the hyperplanes fixed by the reflection associated to each generator. This will reduce significantly the complexity of the problem, since (in dimension n) for each generator we have a vector of $n + 1$ entries (giving the hyperplane of reflection) in place of an $(n + 1) \times (n + 1)$ matrix (giving the reflection itself).

More precisely, a local parametrisation of the set $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ is given by the following lemma.

Lemma 3.4. *The set* $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ *is finitely covered by a disjoint union of subsets of* $\mathbb{R}^{(n+1)|S|}$ *defined by the vanishing of* $|S| + |R|$ *quadratic conditions.*

Proof. Let us first give the proof for $G = \mathrm{Isom}(\mathbb{H}^n)$. Let us identify $\mathbb{R}^{(n+1)|S|}$ to the vector space of functions $f : S \to \mathbb{R}^{n+1}$. For every representation $\rho : \Gamma \to G$ in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$, we can choose a function f such that $q_1(f(s)) = 1$ and $\rho(s) = r_{f(s)}$ for every generator s. In fact there are $2^{|S|}$ possible choices of such an f, differing by changing sign to the image of each generator, and they all satisfy the following conditions:

(1) the vector $f(\boldsymbol{s})$ is unitary, meaning that $q_1(f(\boldsymbol{s})) = 1$, hence giving $|S|$ quadratic conditions;

(2) for each of the commutation relations $\boldsymbol{s}_i \boldsymbol{s}_j = \boldsymbol{s}_j \boldsymbol{s}_i$ in Γ, by Lemma 2.1 the corresponding vectors $f(\boldsymbol{s}_i)$ and $f(\boldsymbol{s}_j)$ are orthogonal with respect to b_1.

Conversely, every function f satisfying these conditions induces the representation ρ in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ defined by $\rho(\boldsymbol{s}) = r_{f(\boldsymbol{s})}$. Define a function

$$g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$$

by

$$g(f) = \Big(\big(q_1\big(f(\boldsymbol{s})\big) - 1\big)_{\boldsymbol{s}\in S}, \big(b_1\big(f(\boldsymbol{s}_i), f(\boldsymbol{s}_j)\big)\big)_{(\boldsymbol{s}_i,\boldsymbol{s}_j)\in R}\Big).$$

We have shown that $g^{-1}(0)$ is a $2^{|S|}$-sheeted covering of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, \mathrm{Isom}(\mathbb{H}^n))$, with deck transformations given by the group $(\mathbb{Z}/2\mathbb{Z})^{|S|}$.

The proof for the AdS case is analogous, except that we have to distinguish several cases, depending on whether $\rho(\boldsymbol{s})$ is a reflection in a spacelike or timelike hyperplane. In the former case, we must impose $q_{-1}(f(\boldsymbol{s})) = -1$, and in the latter $q_{-1}(f(\boldsymbol{s})) = 1$ (see Lemma 2.2). The orthogonality conditions are the same, but now using the bilinear form b_{-1}, by Lemma 2.3. In conclusion, we have that $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n))$ is finitely covered by a disjoint union of $|S|$ subsets each defined by the vanishing of a quadratic function $g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$. ■

Remark 3.5. The covering map of Lemma 3.4 is equivariant with respect to the following two actions of $\mathrm{O}(q_{\pm 1})$:

- the action on the space of functions $f\colon S \to \mathbb{R}^{n+1}$ by post-composition $(A \cdot f)(\boldsymbol{s}) = A(f(\boldsymbol{s}))$. This action preserves the zero locus of the defining functions $g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$ introduced in the proof of Lemma 3.4;
- the action on $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ given by the G-action by conjugation. This action preserves the clopen subset $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ of $\mathrm{Hom}(\Gamma, G)$.

3.2. Flexibility in dimension three

Let Γ_{rect} denote the right-angled Coxeter group generated by the reflections along the sides of a Euclidean rectangle. The standard presentation of Γ_{rect} has four generators (one for each side of the rectangle), and relations such that each generator has order two and reflections in adjacent sides commute.

Definition 3.6 (Cusp group in dimension three). The image of a representation of Γ_{rect} into $\mathrm{Isom}(\mathbb{H}^3)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ is called a *cusp group* if the four generators are sent to reflections in four distinct planes asymptotic to a common point at infinity.

We will also consider other similar representations of Γ_{rect}, which occur in correspondence to a collapse, when two non-commuting generators are sent to the same reflection. Let us begin with the hyperbolic case:

Definition 3.7 (Collapsed cusp group for $\mathbb{H}^3$). The image of a representation of Γ_{rect} into $\mathrm{Isom}(\mathbb{H}^3)$ is called a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes asymptotic to a common point at infinity.

Let ρ' be a representation near a given $\rho \in \mathrm{Hom}_{\mathrm{refl}}(\Gamma_{\mathrm{rect}}, \mathrm{Isom}(\mathbb{H}^3))$ and $\boldsymbol{s}$ a generator of Γ_{rect}. In virtue of Lemma 3.3 and the discussion below, we refer to the fixed-point set of $\rho'(\boldsymbol{s})$ as a *plane* of ρ'.

In [15, Lemma 5.1], the following property of cusp groups is proved:

Proposition 3.8. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{H}^3)$ be a representation whose image is a cusp group. For all nearby representations whose image is not a cusp group, a pair of opposite planes intersect in $\mathbb{H}^3$, while the other pair of opposite planes have disjoint ideal closures in $\overline{\mathbb{H}}^3$.*

In fact, a simple adaptation of the proof shows the following proposition.

Proposition 3.9. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{H}^3)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds.*

(1) *If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.*

(2) *The image of ρ' is a cusp group.*

(3) *A pair of opposite planes intersect in $\mathbb{H}^3$, while the other pair of opposite planes have disjoint ideal closures in $\overline{\mathbb{H}}^3$.*

The first case may hold only if ρ is a collapsed cusp group. Under this hypothesis, Proposition 3.9 can be rephrased by saying that a deformation of a collapsed cusp group either preserves the property that two planes corresponding to non-adjacent sides of the rectangle coincide (which is the case when the collapsed cusp group remains a collapsed cusp group, for instance) or falls in the class of representations described in Proposition 3.8, namely, deformations of non-collapsed cusp groups. If ρ is a cusp group, then the content of Proposition 3.9 is the same as Proposition 3.8.

We now move to the AdS version of Propositions 3.8 and 3.9, for which we will give a complete proof. Proofs for the hyperbolic case are easier and can be repeated by mimicking the AdS case.

Note that for an AdS cusp group the four planes necessarily satisfy the orthogonality conditions as in a rectangle, and therefore two of them are spacelike and two timelike. We will show the following proposition, which is the AdS version of Proposition 3.8.

Proposition 3.10. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ be a representation whose image is a cusp group. For all nearby representations whose image is not a cusp group, exactly one of the following possibilities holds:*

(1) *the ideal closures of the two spacelike planes are disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}}^3$, whereas the two timelike planes intersect in a timelike geodesic of $\mathbb{A}\mathrm{d}\mathbb{S}^3$;*

(2) *the two spacelike planes intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, whereas the two timelike planes intersect in two spacelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.*

Proposition 3.10 follows from the more general Proposition 3.12 below, which also includes the collapsed case. We will consider only the degeneration of cusp groups to a collapsed cusp group when the two planes which coincide are spacelike, as in the following definition:

Definition 3.11 (Collapsed cusp group for $\mathbb{A}\mathrm{d}\mathbb{S}^3$). The image of a representation of Γ_{rect} into $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ is called a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes, two timelike and one spacelike, asymptotic to a common point at infinity.

Proposition 3.12. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) *if $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$ is a reflection in a spacelike plane, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$;*

(2) *the image of ρ' is a cusp group;*

(3) *the ideal closures of the two spacelike planes are disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}}^3$, whereas the two timelike planes intersect in a timelike geodesic of $\mathbb{A}\mathrm{d}\mathbb{S}^3$;*

(4) *the two spacelike planes intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, whereas the two timelike planes intersect in two spacelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.*

Proof. By Lemmas 3.3 and 3.4, it is sufficient to analyse a neighbourhood of a lift $f\colon S \to \mathbb{R}^4$ of ρ, where S is the standard generating set of Γ_{rect}. Let us denote $\boldsymbol{s}_1, \boldsymbol{s}_2$ the generators which are sent by ρ to a reflection in a spacelike plane, and $\boldsymbol{t}_1, \boldsymbol{t}_2$ those sent to a reflection in a timelike plane. The same will occur for representations nearby ρ.

Let us fix a nearby representation ρ' and a lift $f'\colon S \to \mathbb{R}^4$. Let us denote $X_i = f'(\boldsymbol{s}_i)$ and $Y_i = f'(\boldsymbol{t}_i)$. Recall that X_i is orthogonal to Y_j for $i, j = 1, 2$.

Suppose that $X_1 \neq \pm X_2$, for otherwise we are in the case of item (1). Up to the action of $\mathrm{O}(q_{-1})$ (see Remark 3.5) and up to changing signs, we can assume once and forever that

$$X_1 = (1, 0, 0, 0) \quad \text{and} \quad Y_1 = (0, 1, 0, 0).$$

Suppose first that the hyperplanes H_{Y_2}, H_{X_1}, and H_{Y_1} are asymptotic to the same point at infinity p. (As a side remark, observe that in this case they are also asymptotic to the antipodal point $-p$.) We can assume that $p = (0, 0, 1, 1) \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. Together with the orthogonality of Y_2 with X_1, this implies (up to changing the sign if necessary) that

$$Y_2 = (0, 1, a, -a)$$

for some parameter $a \neq 0$. Applying the orthogonality of X_2 with Y_1 and Y_2, we now find (always up to a sign) that

$$X_2 = (1, 0, b, -b)$$

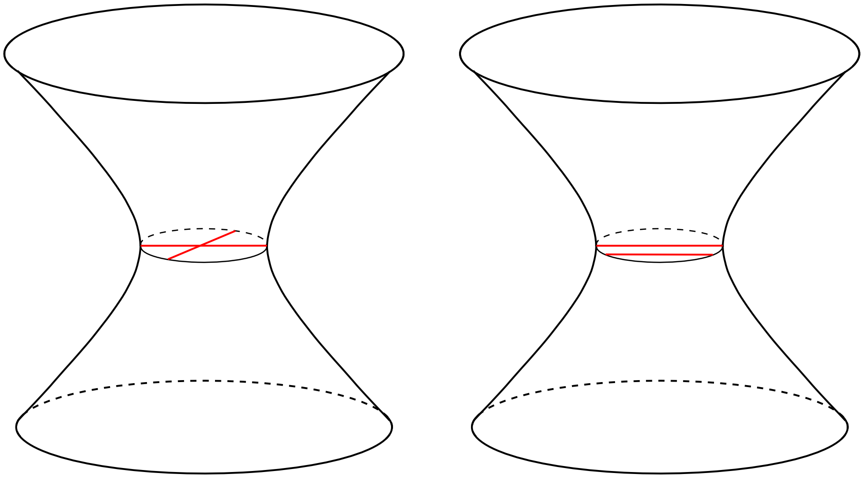

Figure 4. The two configurations for the geodesics ℓ_1 and ℓ_2 in the "horizontal" spacelike plane H_{X_1}, as in the proof of Proposition 3.12. The two timelike planes H_{Y_i} containing ℓ_i and orthogonal to H_{X_1} are as in Figure 3, left and right figure, respectively.

for some b, which implies that H_{X_2} also is asymptotic to the point $p = (0, 0, 1, 1)$. Thus we still have a cusp group and we are in the case of item (2).

Suppose instead that $\overline{H}_{Y_2} \cap \overline{H}_{X_1} \cap \overline{H}_{Y_1} = \emptyset$. Consider a connected component $H^0_{X_1}$ of H_{X_1}, which is a copy of $\mathbb{H}^2$. We have two geodesics in $H^0_{X_1}$: $\ell_1 = H_{Y_1} \cap H^0_{X_1}$ and $\ell_2 = H_{Y_2} \cap H^0_{X_1}$. There are two possibilities: either ℓ_1 and ℓ_2 intersect in $H^0_{X_1}$, or they are ultraparallel. See Figure 4 (and Figure 3).

If ℓ_1 and ℓ_2 intersect in $H^0_{X_1}$, we can assume that $\ell_1 \cap \ell_2 = \{(0, 0, 0, 1)\}$. Equivalently,

$$Y_2 = (0, \cos\theta, \sin\theta, 0),$$

where θ is the angle between the two geodesics in $H^0_{X_1}$. In this case, the two timelike planes H_{Y_1} and H_{Y_2} have timelike intersection by Lemma 2.6 (the intersection is indeed the timelike geodesic $(\cos(s), 0, 0, \sin(s))$). Imposing the orthogonality of H_{X_2} with H_{Y_1} and H_{Y_2}, we find (up to a sign) that

$$X_2 = (\cos\varphi, 0, 0, \sin\varphi),$$

which means that H_{X_1} and H_{X_2} are disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ by Lemma 2.5. (The parameter φ is indeed the timelike distance between H_{X_1} and H_{X_2}, which is achieved on the timelike geodesic we have just introduced.) So, in this case item (3) of the statement holds.

If ℓ_1 and ℓ_2 are ultraparallel, we can assume that

$$Y_2 = (0, \cosh\theta, 0, \sinh\theta),$$

where θ is now the distance between the two aforementioned geodesics in $H^0_{X_1}$. In this case, H_{Y_1} and H_{Y_2} have spacelike intersection (which is the geodesic $(\cosh(s), 0, \sinh(s), 0)$, see Lemma 2.6). Imposing again the orthogonality of H_{X_2} with H_{Y_1} and H_{Y_2}, and changing sign if necessary, we find

$$X_2 = (\cosh\varphi, 0, \sinh\varphi, 0),$$

namely, H_{X_1} and H_{X_2} intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ by Lemma 2.5 (the parameter φ now being their angle of intersection). Thus, item (4) of the statement holds. This concludes the proof. ■

Remark 3.13. In the proof of Proposition 3.12, we have used only that ρ can be continuously deformed to ρ'. Hence the conclusions of Propositions 3.12 and 3.9 actually hold on the entire connected component of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{\mathrm{rect}}, G)$ containing ρ.

3.3. Rigidity in dimension four

Let us now move to dimension four.

Let Γ_{cube} be the group generated by the reflections in the faces of a Euclidean cube. The group Γ_{cube} has six generators, one for each face, and 12 commutation relations, one for each edge of the cube, involving the two faces adjacent to that edge. Of course, there is also a square-type relation for each generator. There is no relation between the generators corresponding to opposite faces.

Definition 3.14 (Cusp group in dimension four). The image of a representation of Γ_{cube} into $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or $\mathrm{Isom}(\mathbb{H}^4)$ is called a *cusp group* if the six generators are sent to reflections in six distinct hyperplanes asymptotic to a common point at infinity.

In the AdS case, among these six hyperplanes, two opposite hyperplanes are necessarily spacelike, while the other four are timelike.

The following proposition is the fundamental property that can be roughly rephrased as "cusp groups stay cusp groups". Its hyperbolic counterpart is proved in [15, Lemma 5.3].

Proposition 3.15. *Let $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ be a representation whose image is a cusp group. Then all nearby representations are cusp groups.*

Similarly to dimension three, we will obtain Proposition 3.15 as a special case of a more general statement including the collapsed case. Let us first give the definition of a collapsed cusp group, where two non-commuting generators can be sent to the same reflection (along a spacelike hyperplane in the AdS case):

Definition 3.16 (Collapsed cusp group in dimension four). The image of a representation of Γ_{cube} into $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ is called a *collapsed cusp group* if the six generators are sent to reflections along five distinct hyperplanes asymptotic to a common point at infinity. In the AdS case, we require that the unique reflection associated to two generators is along a spacelike hyperplane.

Let us now formulate and prove the more general version of Proposition 3.15.

Proposition 3.17. *Let $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) *if s_1 and s_2 are generators such that $\rho(s_1) = \rho(s_2)$ is a reflection in a spacelike hyperplane, then $\rho'(s_1) = \rho'(s_2)$;*

(2) *the image of ρ' is a cusp group.*

Proof. Similarly to the 3-dimensional case treated in the previous section, any representation ρ' nearby ρ lies in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{\mathrm{cube}}, G)$, hence it sends the six standard generators of Γ_{cube} to reflections. Moreover, the hyperplanes of ρ' have the same type (spacelike or timelike) as for ρ.

Let us pick a lift $f'\colon S \to \mathbb{R}^5$ of ρ', for S the standard generating set of Γ_{cube}. Denote by s_1, s_2 the two generators corresponding to opposite faces of the cube which are sent to reflections in spacelike hyperplanes, and $X_i = f'(s_i)$ (so that $q_{-1}(X_i) = -1$). Similarly, we define Y_i and Z_i for $i = 1, 2$, on which q_{-1} takes value 1. Hence each of these six vectors is orthogonal to four of the others: more precisely, A_i is orthogonal to all the others except A_j, for $A \in \{X, Y, Z\}$ and $i, j = 1, 2$.

Now, let us assume that the hyperplanes H_{X_1} and H_{X_2} do not coincide, that is $X_1 \neq \pm X_2$. We shall show that the image of ρ' is still a cusp group.

Let us start by considering the intersection with a connected component $H^0_{X_1}$ of H_{X_1}, which is a copy of $\mathbb{H}^3$. Here we see the (2-dimensional) planes $H_{Y_1} \cap H^0_{X_1}$, $H_{Y_2} \cap H^0_{X_1}$, $H_{Z_1} \cap H^0_{X_1}$, and $H_{Z_2} \cap H^0_{X_1}$, whose associated reflections give a representation $\Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{H}^3)$ which is nearby a (rectangular) cusp group. As in the proof of Proposition 3.10, it is easy to see that if this representation of Γ_{rect} is a cusp group in $H^0_{X_1}$, then necessarily also H_{X_2} is asymptotic to a common point at infinity with H_{Y_1}, H_{Y_2}, H_{Z_1}, H_{Z_2}. Therefore, the image of Γ_{cube} is still a cusp group, since we are assuming that $H_{X_2} \neq H_{X_1}$.

Hence let us assume that the representation of Γ_{rect} is not a cusp group, and we will derive a contradiction. By Proposition 3.8 (up to relabelling) we may assume that $H_{Y_1} \cap H^0_{X_1}$ and $H_{Y_2} \cap H^0_{X_1}$ intersect in $H^0_{X_1}$, while $H_{Z_1} \cap H^0_{X_1}$ and $H_{Z_2} \cap H^0_{X_1}$ are disjoint in $H^0_{X_1}$. This implies that $H_{Y_1} \cap H_{Y_2}$ is a timelike plane (i.e., a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^2$), while $H_{Z_1} \cap H_{Z_2}$ is spacelike (i.e., a copy of $\mathbb{H}^2$). To see this, one can in fact assume that, up to the signs,

$$X_1 = (1,0,0,0,0), \quad Y_1 = (0,1,0,0,0), \quad Y_2 = (0,\cos\theta,\sin\theta,0,0),$$

and apply Lemma 2.6—and similarly for Z_1 and Z_2.

Now, let us consider the intersection with H_{Y_1}, which is a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. We have thus a representation of Γ_{rect} acting on this copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ as a cusp group or collapsed cusp group, with generators which are reflections in $H_{Z_1} \cap H_{Y_1}$, $H_{Z_2} \cap H_{Y_1}$, $H_{X_1} \cap H_{Y_1}$, and $H_{X_2} \cap H_{Y_1}$. Since $H_{Z_1} \cap H_{Z_2}$ is spacelike, then $H_{Z_1} \cap H_{Z_2} \cap H_{Y_1}$ is also spacelike, and therefore we are in the situation of Proposition 3.12 item (4), recalling that $H_{X_1} \neq H_{X_2}$ by our assumption. This implies that $H_{X_1} \cap H_{Y_1}$ and $H_{X_2} \cap H_{Y_1}$ intersect in $H_{Y_1} \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$.

On the other hand, considering the intersection with H_{Z_1}, which is again a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, since $H_{Y_1} \cap H_{Y_2}$ is timelike, we find that $H_{Y_1} \cap H_{Y_2} \cap H_{Z_1}$ is a timelike geodesic. By Proposition 3.12 item (3), $H_{X_1} \cap H_{Z_1}$ and $H_{X_2} \cap H_{Z_1}$ do not intersect in H_{Z_1}, which in turn implies (since H_{X_1} and H_{X_2} are both orthogonal to H_{Z_1}) that H_{X_1} and H_{X_2} are disjoint in $\mathbb{A}\mathrm{d}\mathbb{S}^4$. This contradicts the conclusion of the previous paragraph. ∎

Remark 3.18. In case (1) of Proposition 3.17, i.e., when $\rho(s_1) = \rho(s_2)$, the following possibility is not excluded: for some deformation ρ' of ρ, the remaining four generators $s_3, \ldots, s_6$ (which are sent by ρ to a rectangular cusp group in a copy of $\mathbb{H}^3$) are *not* sent by ρ' to a cusp group.

The analogous property for $\mathbb{H}^4$, which is a generalisation of [15, Lemma 5.3], can be proved along the same lines. We state it here:

Proposition 3.19. *Let $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{H}^4)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds.*

(1) *If s_1 and s_2 are generators such that $\rho(s_1) = \rho(s_2)$, then $\rho'(s_1) = \rho'(s_2)$.*

(2) *The image of ρ' is a cusp group.*

4. The hyperbolic and AdS character varieties of Γ_{22}

In this section, we study the $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ character varieties of the group Γ_{22} near the conjugacy classes of the holonomy representations ρ_t found in [15, 26]. We prove here the "topological part" of Theorem 1.1 (Theorem 4.16) in the hyperbolic and AdS case.

4.1. The group Γ_{22}

As in [15], we define

$$\Gamma_{22} < \mathrm{Isom}(\mathbb{H}^4)$$

as the group generated by the hyperbolic reflections along the hyperplanes determined by the 22 vectors in Table 1. These hyperplanes bound a right-angled polytope in $\mathbb{H}^4$ of infinite volume, which is obtained by "removing two opposite walls" from the ideal right-angled 24-cell.

All the dihedral angles between two intersecting hyperplanes are right. Therefore, Γ_{22} is a right-angled Coxeter group. We will consider Γ_{22} as an abstract group, that is the right-angled Coxeter groups on 22 generators

$$\mathbf{0^+}, \ldots, \mathbf{7^+}, \mathbf{0^-}, \ldots, \mathbf{7^-}, \boldsymbol{A}, \ldots, \boldsymbol{F}$$

satisfying the following relations:

- $s^2 = 1$ for each generator s,

$$
\begin{aligned}
\mathbf{0^+} &= (\sqrt{2}, +1, +1, +1, +1), & \mathbf{0^-} &= (\sqrt{2}, +1, +1, +1, -1),\\
\mathbf{1^+} &= (\sqrt{2}, +1, -1, +1, -1), & \mathbf{1^-} &= (\sqrt{2}, +1, -1, +1, +1),\\
\mathbf{2^+} &= (\sqrt{2}, +1, -1, -1, +1), & \mathbf{2^-} &= (\sqrt{2}, +1, -1, -1, -1),\\
\mathbf{3^+} &= (\sqrt{2}, +1, +1, -1, -1), & \mathbf{3^-} &= (\sqrt{2}, +1, +1, -1, +1),\\
\mathbf{4^+} &= (\sqrt{2}, -1, +1, -1, +1), & \mathbf{4^-} &= (\sqrt{2}, -1, +1, -1, -1),\\
\mathbf{5^+} &= (\sqrt{2}, -1, +1, +1, -1), & \mathbf{5^-} &= (\sqrt{2}, -1, +1, +1, +1),\\
\mathbf{6^+} &= (\sqrt{2}, -1, -1, +1, +1), & \mathbf{6^-} &= (\sqrt{2}, -1, -1, +1, -1),\\
\mathbf{7^+} &= (\sqrt{2}, -1, -1, -1, -1), & \mathbf{7^-} &= (\sqrt{2}, -1, -1, -1, +1),\\
\boldsymbol{A} &= (1, +\sqrt{2}, 0, 0, 0), & \boldsymbol{B} &= (1, 0, +\sqrt{2}, 0, 0),\\
\boldsymbol{C} &= (1, 0, 0, +\sqrt{2}, 0), & \boldsymbol{D} &= (1, 0, 0, -\sqrt{2}, 0),\\
\boldsymbol{E} &= (1, 0, -\sqrt{2}, 0, 0), & \boldsymbol{F} &= (1, -\sqrt{2}, 0, 0, 0).
\end{aligned}
$$

Table 1. The 22 unit vectors defining the bounding hyperplanes of a right-angled polytope in $\mathbb{H}^4$. The reflections in these hyperplanes generate the Coxeter group Γ_{22}. Adding the vectors $(1, 0, 0, 0, \pm\sqrt{2})$ to this list, one obtains the ideal right-angled 24-cell.

- $s_1 s_2 = s_2 s_1$ for each pair s_1, s_2 of generators such that the corresponding vectors in Table 1 are orthogonal with respect to the bilinear form b_1.

The generators are partitioned into three *types*: *positive* $\mathbf{0^+}, \dots, \mathbf{7^+}$, *negative* $\mathbf{0^-}, \dots, \mathbf{7^-}$, and *letters* $\boldsymbol{A}, \dots, \boldsymbol{F}$. The type is inherited from the standard 3-colouring of the facets of the 24-cell (see [15] for more details).

The reader can check from Table 1 that there are no commutation condition between two generators of the same type, that every $\boldsymbol{i}^+$ commutes with four vectors of type $\boldsymbol{j}^-$ (including $\boldsymbol{i}^-$), and every $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$ commutes with $\boldsymbol{i}^-$ and $\boldsymbol{i}^+$ for four choices of $\boldsymbol{i} \in \{\mathbf{0}, \dots, \mathbf{7}\}$. Hence there are $8 \cdot 4 + 6 \cdot 8 = 80$ commutation relations. Altogether, there are $102 = 22 + 80$ relations.

We would like to stress once more that throughout the following (with a few exceptions which will be remarked opportunely) we will use the symbols $\boldsymbol{i}^+ \in \{\mathbf{0^+}, \dots, \mathbf{7^+}\}$, $\boldsymbol{i}^- \in \{\mathbf{0^-}, \dots, \mathbf{7^-}\}$, and $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$ to denote the 22 abstract generators of Γ_{22} (rather than vectors in $\mathbb{R}^5$).

4.2. A curve of geometric representations

Let us now introduce the representations of our interest, which appear in the statement of Theorem 1.1. Unlike the introduction, we will omit the superscript G hereafter, and the ambient geometry we consider will be clear from the context.

$f_t(\mathbf{0^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, +t, +t, +t, +1)$,	$f_t(\mathbf{0^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, +1, +1, +1, -t)$,
$f_t(\mathbf{1^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, +t, -t, +t, -1)$,	$f_t(\mathbf{1^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, +1, -1, +1, +t)$,
$f_t(\mathbf{2^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, +t, -t, -t, +1)$,	$f_t(\mathbf{2^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, +1, -1, -1, -t)$,
$f_t(\mathbf{3^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, +t, +t, -t, -1)$,	$f_t(\mathbf{3^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, +1, +1, -1, +t)$,
$f_t(\mathbf{4^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, -t, +t, -t, +1)$,	$f_t(\mathbf{4^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, -1, +1, -1, -t)$,
$f_t(\mathbf{5^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, -t, +t, +t, -1)$,	$f_t(\mathbf{5^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, -1, +1, +1, +t)$,
$f_t(\mathbf{6^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, -t, -t, +t, +1)$,	$f_t(\mathbf{6^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, -1, -1, +1, -t)$,
$f_t(\mathbf{7^+}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}\,t, -t, -t, -t, -1)$,	$f_t(\mathbf{7^-}) = \frac{1}{\sqrt{1+t^2}}(\sqrt{2}, -1, -1, -1, +t)$,
$f_t(\boldsymbol{A}) = (1, +\sqrt{2}, 0, 0, 0)$,	$f_t(\boldsymbol{B}) = (1, 0, +\sqrt{2}, 0, 0)$,
$f_t(\boldsymbol{C}) = (1, 0, 0, +\sqrt{2}, 0)$,	$f_t(\boldsymbol{D}) = (1, 0, 0, -\sqrt{2}, 0)$,
$f_t(\boldsymbol{E}) = (1, 0, -\sqrt{2}, 0, 0)$,	$f_t(\boldsymbol{F}) = (1, -\sqrt{2}, 0, 0, 0)$.

Table 2. The list of vectors X, satisfying $q_1(X) = 1$, in Definition 4.1. The representation ρ_t maps each generator $\boldsymbol{s}$ to the hyperbolic reflection in the orthogonal complement of $f_t(\boldsymbol{s})$.

Definition 4.1 (The two paths ρ_t). For $t \in (-1, 1)$, we define ρ_t to be the representation of Γ_{22} in $\mathrm{Isom}(\mathbb{H}^4)$ (resp. $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$) sending each generator $\boldsymbol{s}$ of Γ_{22} to the hyperbolic (resp. AdS) reflection $r_{f_t(\boldsymbol{s})}$ associated to the corresponding vector $f_t(\boldsymbol{s})$ of Table 2 (resp. Table 3).

Some comments are necessary to explain Definition 4.1 and the tables involved.

(1) It can be checked that all the orthogonality relations (with respect to the bilinear form b_1) between vectors in Table 1 are maintained for the vectors in Table 2 with respect to b_1, and in Table 3 with respect to b_{-1}. This shows that Definition 4.1 is well-posed, meaning that ρ_t are representations of Γ_{22} by Lemma 2.3.

(2) By construction, the representations ρ_t are in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$, for $G = \mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ (see Definition 3.2). Tables 2 and 3 exhibit continuous lifts $f_t\colon S \to \mathbb{R}^{110}$ as in Lemma 3.4, taking values in a subset of $\mathbb{R}^{110}$ defined by the vanishing of 102 quadratic conditions, for S the standard generating set of Γ_{22}.

(3) The vectors of Table 2 coincide with those of Table 1 for $t = 1$. Hence, in the hyperbolic case, the path of representations ρ_t is a deformation of the reflection group of the aforementioned right-angled polytope with 22 facets. For $t \in (0, 1)$, this coincides with the path of representations exhibited in [15]. For $t \in (-1, 0)$, the representation ρ_t is obtained by conjugating ρ_{-t} by the reflection r in the "horizontal" hyperplane $x_4 = 0$

$$r\colon (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, x_1, x_2, x_3, -x_4). \tag{5}$$

(This is seen immediately using Remark 3.5.)

$$
\begin{array}{ll}
f_t(\mathbf{0^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, +t, +t, +t, +1), & f_t(\mathbf{0^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, +1, +1, +1, +t), \\
f_t(\mathbf{1^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, +t, -t, +t, -1), & f_t(\mathbf{1^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, +1, -1, +1, -t), \\
f_t(\mathbf{2^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, +t, -t, -t, +1), & f_t(\mathbf{2^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, +1, -1, -1, +t), \\
f_t(\mathbf{3^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, +t, +t, -t, -1), & f_t(\mathbf{3^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, +1, +1, -1, -t), \\
f_t(\mathbf{4^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, -t, +t, -t, +1), & f_t(\mathbf{4^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, -1, +1, -1, +t), \\
f_t(\mathbf{5^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, -t, +t, +t, -1), & f_t(\mathbf{5^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, -1, +1, +1, -t), \\
f_t(\mathbf{6^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, -t, -t, +t, +1), & f_t(\mathbf{6^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, -1, -1, +1, +t), \\
f_t(\mathbf{7^+}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}\,t, -t, -t, -t, -1), & f_t(\mathbf{7^-}) = \frac{1}{\sqrt{1-t^2}}(\sqrt{2}, -1, -1, -1, -t), \\
f_t(\boldsymbol{A}) = (1, +\sqrt{2}, 0, 0, 0), & f_t(\boldsymbol{B}) = (1, 0, +\sqrt{2}, 0, 0), \\
f_t(\boldsymbol{C}) = (1, 0, 0, +\sqrt{2}, 0), & f_t(\boldsymbol{D}) = (1, 0, 0, -\sqrt{2}, 0), \\
f_t(\boldsymbol{E}) = (1, 0, -\sqrt{2}, 0, 0), & f_t(\boldsymbol{F}) = (1, -\sqrt{2}, 0, 0, 0).
\end{array}
$$

Table 3. The list of vectors for the definition of ρ_t, in the AdS case. The quadratic form q_{-1} takes value -1 on the vectors $f_t(\boldsymbol{i}^+)$, and $+1$ on the vectors $f_t(\boldsymbol{i}^-)$ and $f_t(\boldsymbol{X})$.

(4) On the AdS side, the path ρ_t has been exhibited in [26] for $t \in (-1, 0)$. Again, the path is extended here for positive times by conjugation by r.

(5) Both these paths occur as the holonomy representations of a deformation of hyperbolic and AdS cone-orbifold structures. The purpose of our previous work [26] was to describe the geometric transition from hyperbolic ($t > 0$) to AdS ($t < 0$) structures. Since here we are interested in the $\mathrm{Isom}(\mathbb{H}^4)$- and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-character varieties on their own, we found it more useful to treat the two paths ρ_t separately, and extend each of them by conjugation with the orientation-reversing transformation r also for negative (resp. positive) times.

4.3. The collapsed representation and the cuboctahedron

For $t = 0$, the hyperbolic and AdS representations ρ_0 take value in the stabiliser of the hyperplane given by $\{x_4 = 0\}$. Unlike the case $t \neq 0$, these representations are not holonomies of hyperbolic/AdS orbifold structures, but correspond to what we call the *collapse* of the respective geometric structures.

Let us consider $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ as subgroups of $\mathrm{GL}(5, \mathbb{R})$. Then the representations ρ_0 agree for the hyperbolic and AdS case. Indeed, defining

$$\mathbb{H}^3 := \mathbb{H}^4 \cap \{x_4 = 0\} = \mathbb{A}\mathrm{d}\mathbb{S}^4 \cap \{x_4 = 0,\ x_0 > 0\} \subset \mathbb{R}^5,$$

its stabiliser

$$G_0 = \mathrm{Stab}_{\mathrm{Isom}(\mathbb{H}^4)}(\mathbb{H}^3) = \mathrm{Stab}_{\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)}(\mathbb{H}^3) < \mathrm{GL}(5, \mathbb{R})$$

$v_{\mathbf{0}} = (\sqrt{2}, +1, +1, +1)$,	
$v_{\mathbf{1}} = (\sqrt{2}, +1, -1, +1)$,	$v_{\boldsymbol{A}} = (1, +\sqrt{2}, 0, 0)$,
$v_{\mathbf{2}} = (\sqrt{2}, +1, -1, -1)$,	$v_{\boldsymbol{B}} = (1, 0, +\sqrt{2}, 0)$,
$v_{\mathbf{3}} = (\sqrt{2}, +1, +1, -1)$,	$v_{\boldsymbol{C}} = (1, 0, 0, +\sqrt{2})$,
$v_{\mathbf{4}} = (\sqrt{2}, -1, +1, -1)$,	$v_{\boldsymbol{D}} = (1, 0, 0, -\sqrt{2})$,
$v_{\mathbf{5}} = (\sqrt{2}, -1, +1, +1)$,	$v_{\boldsymbol{E}} = (1, 0, -\sqrt{2}, 0)$,
$v_{\mathbf{6}} = (\sqrt{2}, -1, -1, +1)$,	$v_{\boldsymbol{F}} = (1, -\sqrt{2}, 0, 0)$.
$v_{\mathbf{7}} = (\sqrt{2}, -1, -1, -1)$,	

Table 4. The vectors $v_{\boldsymbol{i}}, v_{\boldsymbol{X}} \in \mathbb{R}^{1,3}$ defining the bounding planes of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. These vectors are involved also in Definition 6.3, introducting the cocycles τ_λ in the vector space $Z^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$.

consists of matrices in the block form

$$\left(\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & \pm 1 \end{array}\right).$$

The stabiliser G_0 is isomorphic to $\operatorname{Isom}(\mathbb{H}^3) \times \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$-factor is generated by the reflection r of equation (5), which acts by switching the two sides of $\{x_4 = 0\}$. Under this isomorphism, the representation ρ_0 reads as

$$\begin{aligned} \rho_0(\boldsymbol{i}^+) &= r && \text{for each } \boldsymbol{i} \in \{\mathbf{0}, \dots, \mathbf{7}\}, \\ \rho_0(\boldsymbol{i}^-) &= r_{v_{\boldsymbol{i}}} && \text{for each } \boldsymbol{i} \in \{\mathbf{0}, \dots, \mathbf{7}\}, \\ \rho_0(\boldsymbol{X}) &= r_{v_{\boldsymbol{X}}} && \text{for each } \boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}, \end{aligned} \tag{6}$$

where the vectors $v_{\boldsymbol{i}}, v_{\boldsymbol{X}} \in \mathbb{R}^{1,3}$, collected in Table 4, define the bounding planes $H_{v_{\boldsymbol{i}}}$ and $H_{v_{\boldsymbol{X}}}$ of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. The triangular faces of this cuboctahedron are of type $\boldsymbol{i}$, while the quadrilateral faces are of type $\boldsymbol{X}$ (see Figure 5).

4.4. The conjugacy action

We are ready to start the study of the hyperbolic and AdS character varieties of Γ_{22} near the representation ρ_t introduced in Definition 4.1.

We begin by analysing the action of G by conjugation on $\operatorname{Hom}(\Gamma_{22}, G)$. For $t \neq 0$, nearby ρ_t the action of G is “good”, namely is free and proper, as we will see in the following two lemmas.

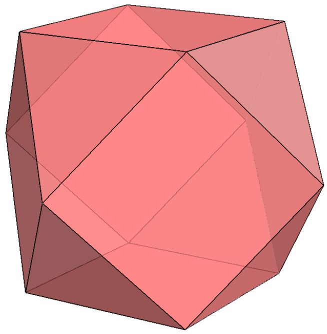

Figure 5. A cuboctahedron is the convex envelop of the midpoints of the edges of a regular cube (or octahedron). It is realised in $\mathbb{H}^3$ as an ideal right-angled polytope.

Lemma 4.2. *For $t \neq 0$, the stabiliser of ρ_t in G is trivial. The stabiliser of ρ_0 in G is the order-two subgroup generated by the reflection r in the hyperplane $\{x_4 = 0\}$.*

Proof. We give the proof for the hyperbolic and AdS case at the same time, since they are completely analogous. By Remark 3.5, any element in the stabiliser of ρ_t is induced by a matrix $A \in \mathrm{O}(q_{\pm 1})$ which maps every vector $f_t(\boldsymbol{s})$ in Table 2 or Table 3 either to itself or to its opposite. Since the six vectors $f_t(\boldsymbol{A}), \ldots, f_t(\boldsymbol{F})$ do not depend on t and generate the hyperplane $\{x_4 = 0\}$, the matrix A must preserve the hyperplane $\{x_4 = 0\}$.

Moreover, let $\mathcal{P}_t$ be the polytope bounded by the 22 hyperplanes orthogonal to the vectors of Tables 2 or 3. It was proved in [20, Proposition 3.19] and [26, Proposition 7.21] that the intersection of $\mathcal{P}_t$ with the hyperplane defined by the equation $x_4 = 0$ is constant and is an ideal right-angled cuboctahedron in $\mathbb{H}^3$ (see Section 4.3). Since the action of A on $\mathbb{H}^3$ necessarily preserves each face of the cuboctahedron, it follows that A must act on the linear hyperplane $\{x_4 = 0\}$ as $\pm\mathrm{id}$.

This shows that the only non-trivial candidates for A are $\pm r$, where r is the reflection of equation (5). For $t = 0$, the reflection r preserves all the hyperplanes orthogonal to the vectors of Tables 2 or 3, hence the associated element in G generates the stabiliser of ρ_0. When $t \neq 0$, the reflection r does not preserve any of the hyperplanes of the form $H_{f_t(\boldsymbol{i}+)}$ or $H_{f_t(\boldsymbol{i}-)}$, hence the stabiliser of ρ_t is trivial in this case. ∎

The next lemma will be useful to show that the action of G^+ by conjugation is proper, in a suitable region of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$.

Lemma 4.3. *Suppose that η_n is a sequence in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ converging to some ρ_t, and h_n is a sequence in G such that $h_n \cdot \eta_n$ converges. Then h_n has a subsequence that converges in G.*

Proof. Suppose that $\eta_n \to \rho_t$ and h_n is a sequence in G such that

$$h_n \cdot \eta_n \to \eta_\infty.$$

Since $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ is clopen in the representation variety, the limit point η_∞ is in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$. Passing to the finite cover $g^{-1}(0)$ of Lemma 3.4, and up to taking subsequences, we can then assume to have a sequence f_n in $g^{-1}(0)$ (projecting to η_n) such that $f_n \to f_\infty$ and $h_n \cdot f_n \to \hat{f}_\infty$. Here we are thinking of $f_n, f_\infty, \hat{f}_\infty$ as functions from the standard generators of Γ_{22} to $\mathbb{R}^5$, and (by a small abuse of notation) h_n is a sequence in $\mathrm{O}(q_{\pm 1})$ acting by the obvious action on $\mathbb{R}^5$ (see Remark 3.5).

We have to show that h_n converges in $\mathrm{O}(q_{\pm 1})$ up to subsequences. Recall that f_∞ is a lift in $g^{-1}(0)$ of ρ_t, and therefore (up to changes of sign) the vectors $f_\infty(\boldsymbol{s})$ are given by Table 2 or Table 3 for some value of t. Take five generators $\boldsymbol{s}_1, \dots, \boldsymbol{s}_5$ of Γ_{22} such that $f_\infty(\boldsymbol{s}_1), \dots, f_\infty(\boldsymbol{s}_5)$ are linearly independent, for instance $\boldsymbol{0}^-$, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$. Since linear independence is an open condition, $\{f_n(\boldsymbol{s}_1), \dots, f_n(\boldsymbol{s}_5)\}$ forms a basis of $\mathbb{R}^5$ for large n.

The linear isometry $h_n \in \mathrm{O}(q_{\pm 1})$, considered as a 5-by-5 matrix, is therefore determined by the condition that h_n sends the basis $\{f_n(\boldsymbol{s}_1), \dots, f_n(\boldsymbol{s}_5)\}$ to $\{h_n \cdot f_n(\boldsymbol{s}_1), \dots, h_n \cdot f_n(\boldsymbol{s}_5)\}$. More concretely, we can write h_n (as a matrix) as $(h_{n,1})^{-1} \circ h_{n,2}$, where $h_{n,1}$ is the matrix sending the standard basis to the basis $\{f_n(\boldsymbol{s}_1), \dots, f_n(\boldsymbol{s}_5)\}$, and $h_{n,2}$ is the matrix sending the standard basis to the basis $\{h_n \cdot f_n(\boldsymbol{s}_1), \dots, h_n \cdot f_n(\boldsymbol{s}_5)\}$. Since f_n and $h_n \cdot f_n$ are converging sequences, we have that $h_{n,1} \to h_{\infty,1}$ and $h_{n,2} \to h_{\infty,2}$, and moreover $h_{\infty,1}$ is invertible since $f_\infty(\boldsymbol{s}_1), \dots, f_\infty(\boldsymbol{s}_5)$ is a basis.

Therefore, h_n converges to a 5-by-5 matrix $h_\infty = (h_{\infty,1})^{-1} \circ h_{\infty,2}$, which is still in $\mathrm{O}(q_{\pm 1})$ since $\mathrm{O}(q_{\pm 1})$ is closed in the space of 5-by-5 matrices. This concludes the proof. ∎

The two lemmas have some consequences on the character variety, which we now define as follows:

Definition 4.4 (Character variety). Let G be $\mathrm{Isom}(\mathbb{H}^n)$, G_{HP^n} or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, and let G^+ denote its subgroup of orientation-preserving transformations. Given a finitely generated group Γ, we call *character variety* of Γ in G the quotient

$$X(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/G^+$$

by the action of G^+ by conjugation.

Remark 4.5. It follows from Lemmas 4.2 and 4.3 that the G^+-action is locally a product in a neighbourhood of ρ_t. More precisely, there is a neighbourhood of the G^+-orbit of ρ_t in $\mathrm{Hom}(\Gamma_{22}, G)$ homeomorphic to $\mathcal{U} \times G^+$, where $\mathcal{U}$ is a neighbourhood of $[\rho_t]$ in $X(\Gamma_{22}, G)$. Moreover, the action of G^+ corresponds, under this homeomorphism, to left multiplication on the second factor.

Remark 4.6. Let us suppose that G is $\mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$; in other words G is reductive. Thanks to some well-known results from GIT (see for instance the concise exposition in [5, Section 2] and the references therein), the GIT quotient $\mathrm{Hom}(\Gamma_{22}, G)/\!/G^+$ can be identified with the "Hausdorff quotient" of the representation variety by conjugation: namely, the quotient by G^+ of the subset of $\mathrm{Hom}(\Gamma, G)$ consisting of points with closed G^+-orbits.

In the portion of the character variety of our interest, no non-Hausdorff pathological situation arises. More precisely, the GIT quotient $\mathrm{Hom}(\Gamma_{22}, G) /\!\!/ G^+$ coincides with the ordinary topological quotient in a neighbourhood of each $[\rho_t]$. (This holds similarly for $\mathrm{Hom}(\Gamma_{22}, G) /\!\!/ G$.) For that, it follows from Lemma 4.3 that

- the G^+-action is proper on $G^+ \cdot \{\rho_t \mid t \in (-1, 1)\}$;
- for each t, the G^+-orbit of ρ_t is closed. (This follows by applying Lemma 4.3 to the constant sequence $\eta_n \equiv \rho_t$).

Actually the latter is true in a neighbourhood of $\{\rho_t \mid t \in (-1, 1)\}$, since in the proof of Lemma 4.3 we only used that, for five generators $s_1, \dots, s_5$ of Γ_{22}, the corresponding vectors in $\mathbb{R}^5$ are linearly independent, and this is still true in an open neighbourhood.

In fact, our argument shows a little more, namely that if ρ is in such a neighbourhood, then $[\rho]$ is separated from any other point in $X(\Gamma_{22}, G)$. This is because, if $[\rho]$ were not separated from $[\rho']$, we would have a sequence $\rho_n \to \rho$ and a sequence h_n such that $h_n \rho_n h_n^{-1}$ converges to ρ'. But Lemma 4.3 shows that $h_n \to h_\infty$ up to subsequences, hence by continuity h_∞ conjugates ρ and ρ', namely $[\rho] = [\rho']$.

4.5. A smoothness result

In the hyperbolic case, the smoothness of the $\mathrm{Isom}(\mathbb{H}^4)$-character variety near the points $[\rho_t]$ with $t \neq 0$ has been essentially proved in [15, Theorem 12.3]:

Proposition 4.7. *For $t \in (0, 1)$, the set $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4))$ is a smooth 11-dimensional manifold near ρ_t.*

(Recalling that for negative times ρ_t is a conjugate of ρ_{-t}, the result holds for $t \in (-1, 0)$ as well.) Our main purpose is to extend and generalise the analysis for $t = 0$, and do similarly for the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-representation variety.

Let us first briefly sketch the lines of the proof of Proposition 4.7 given in [15]. By Lemma 3.4 (recall $g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$ from the proof), it suffices to show that $g^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^{110}$ near any preimage of ρ_{t_0}, for all $t_0 \in (0, 1)$.

Let

$$f_t\colon \{\text{standard generators of } \Gamma_{22}\} \to \mathbb{R}^5 \tag{7}$$

be as in Table 2, so giving an embedding of $(0, 1)$ into $g^{-1}(0) \subset \mathbb{R}^{110}$ going through a preimage of ρ_{t_0}. The proof in [15] essentially consists in showing that the kernel of $g\colon \mathbb{R}^{110} \to \mathbb{R}^{102}$ is 11-dimensional for $t \in (0, 1)$. Since there is a 10-dimensional smooth orbit given by the action of $\mathrm{Isom}^+(\mathbb{H}^4)$, the proof boils down to showing that the tangent space to the orbit has a 1-dimensional complement, which is indeed given by the tangent space to the 1-dimensional submanifold $\{f_t \mid t \in (0, 1)\}$.

Since the action of $\mathrm{Isom}^+(\mathbb{H}^4)$ is smooth, it then follows that the $\mathrm{Isom}^+(\mathbb{H}^4)$-orbit of the curve $\{\rho_t \mid t \in (0, 1)\}$ is a smooth 11-dimensional manifold, on which the $\mathrm{Isom}^+(\mathbb{H}^4)$-action by conjugation is free and proper by Lemmas 4.2 and 4.3. Hence it follows from Proposition 4.7 that $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4))$ is a 1-dimensional smooth manifold near $[\rho_t]$, for $t \in (0, 1)$.

In the next sections, we will prove the analogous of Proposition 4.7 for the AdS case. However, we are interested also in the study of the character variety near “the collapse”, that is the point $[\rho_0]$. Hence we will prove a more detailed statement.

Let G be as usual $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

Definition 4.8 (The set Hom_0). We define $\mathrm{Hom}_0(\Gamma_{22}, G)$ as the subset of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ of representations ρ such that the following holds. Let s_1, s_2 be any pair of generators of Γ_{22} such that the hyperplanes fixed by $\rho_t(s_1)$ and $\rho_t(s_2)$ are either asymptotic or equal for some $t \neq 0$. Then, so are the hyperplanes fixed by $\rho(s_1)$ and $\rho(s_2)$.

Recall from Lemmas 2.4 and 2.5 that two hyperplanes are asymptotic or equal if and only if, using the bilinear form b_1 for $\mathbb{H}^4$ and b_{-1} for $\mathbb{A}\mathrm{d}\mathbb{S}^4$, the product of their orthogonal unit vectors is 1 in absolute value. It is thus easy to check from Tables 2 and 3 that this condition is preserved by the deformation f_t for all t both in the hyperbolic and AdS case, and thus the definition is well posed (i.e., it does not depend on the choice of $t \neq 0$).

In the setting of Lemma 3.4, $\mathrm{Hom}_0(\Gamma_{22}, G)$ corresponds to a subset of $g^{-1}(0) \subset \mathbb{R}^{110}$ defined by the vanishing of 36 more quadratic conditions. Indeed, for each of the 12 ideal vertices of the polytope $\mathcal{P}_t$ bounded by the hyperplanes of Tables 2 and 3, we have three asymptoticity conditions (see [26, Proposition 7.13]). Hence $\mathrm{Hom}_0(\Gamma_{22}, G)$ is locally homeomorphic to the zero locus of a function $g_0\colon \mathbb{R}^{110} \to \mathbb{R}^{138}$ extending g. More precisely, the following lemma holds:

Lemma 4.9. *The set* $\mathrm{Hom}_0(\Gamma_{22}, G)$ *is finitely covered by a disjoint union of subsets of* $\mathbb{R}^{110}$ *defined by the vanishing of* 138 *quadratic conditions.*

Remark 4.10. For simplicity of exposition, from now on we will work in the AdS setting, i.e., in the case $G = \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. All what follows can be easily adapted to the hyperbolic case. We will therefore omit the proofs and only highlight the points where differences with respect to the AdS case occur.

The essential property we will prove is that near each of the representations ρ_t the variety $\mathrm{Hom}_0(\Gamma_{22}, G)$ is smooth. Hence the goal of the next two sections is to prove the following:

Proposition 4.11. *For* $t \in (-1, 1)$, *the set* $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ *is a smooth* 11-*dimensional manifold near* ρ_t.

The proof of Proposition 4.11 will be given at the end of Section 4.7. From the results on cusp rigidity established in Section 3.3, we obtain the smoothness of

$$\mathrm{Hom}\left(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)\right) \quad \text{for } t \neq 0$$

as a direct corollary:

Corollary 4.12. *For* $t \in (-1, 1) \smallsetminus \{0\}$, *the set* $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ *is a smooth* 11-*dimensional manifold near* ρ_t.

Proof. It is not difficult to check that, when $t \neq 0$, for every pair of generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ of Γ_{22} such that the associated hyperplanes $H_{f_t(\boldsymbol{s}_1)}$ and $H_{f_t(\boldsymbol{s}_2)}$ are asymptotic, there are four other generators $\boldsymbol{s}_3, \dots, \boldsymbol{s}_6$ such that the reflections $r_{\boldsymbol{s}_1}, \dots, r_{\boldsymbol{s}_6}$ generate a cusp group in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. By Lemma 3.15, the asymptoticity conditions are preserved since cusp groups stay cusp groups under small deformations. Hence a neighbourhood of ρ_t in $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ is actually contained in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. The proof now follows from Proposition 4.11. ∎

The next sections will be devoted to the proof of Proposition 4.11. We will adapt some of the ideas of [15, Sections 5, 11, and 12] used in the proof of Proposition 4.7 in the hyperbolic case. An analogous argument shows that the statement of Proposition 4.11 holds also for the $\mathbb{H}^4$-character variety, which for $t = 0$ is new with respect to the results of [15].

4.6. Infinitesimal deformations of the letter generators

Recall Lemma 4.9. Throughout this and the following sections, we denote by

$$g_0 : \mathbb{R}^{110} \to \mathbb{R}^{138}$$

the quadratic function defining the clopen subset of $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ that contains the lifts of the representations ρ_t. A continuous lift of the path $t \mapsto \rho_t$ is defined by f_t in Table 3. To prove Proposition 4.11 in the AdS case, it then suffices to show that for all $t \in (-1, 1)$ the set $g_0^{-1}(0) \subset \mathbb{R}^{110}$ is a smooth 11-dimensional manifold near each f_t.

Notation. Let us fix $t \in (-1, 1)$. For simplicity, by an abuse of notation, in this and next section we denote $f_t(\boldsymbol{s}) \in \mathbb{R}^5$ by $\boldsymbol{s}$. In other words, in what follows $\boldsymbol{s} \in \mathbb{R}^5$ denotes a vector (of q_{-1}-norm 1 or -1 depending whether the corresponding hyperplane in $\mathbb{A}\mathrm{d}\mathbb{S}^4$ is timelike or spacelike, respectively) from Table 3, and is therefore implicitly considered as a function of t. Its derivative in t will be denoted by $\dot{\boldsymbol{s}}$. The symbol $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ will denote the corresponding element of $g_0^{-1}(0) \subset \mathbb{R}^{110}$, as a function from the standard generators of Γ_{22} to $\mathbb{R}^5$, while $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ will denote an element in the kernel of the differential of g_0 at $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$, and will be called an *infinitesimal deformation* of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$.

Observe that the vectors $\boldsymbol{A}, \dots, \boldsymbol{F}$ of Table 3 are constant in t, hence the derivative of the path in $g_0^{-1}(0)$ provided by Table 3 satisfies $\dot{\boldsymbol{X}} = 0$ for all $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$.

By Remark 3.5, the natural $\mathrm{O}(q_{-1})$-action on $g_0^{-1}(0)$ is given by $\boldsymbol{s} \mapsto A \cdot \boldsymbol{s}$ for $A \in \mathrm{O}(q_{-1})$. Therefore, the tangent space to the orbit of an element $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ of $g_0^{-1}(0)$ consists precisely of the elements of the kernel of dg_0 of the form

$$\boldsymbol{s} \mapsto \dot{\boldsymbol{s}} = \mathfrak{a} \cdot \boldsymbol{s}, \tag{8}$$

where $\boldsymbol{s}$ varies in $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ and $\mathfrak{a} = \frac{d}{dt}\big|_{t=0} A_t \in \mathfrak{so}(q_{-1})$, for any smooth path $t \mapsto A_t$ in $\mathrm{O}(q_{-1})$ with $A_0 = \mathrm{id}$.

The first step in the proof of Proposition 4.11 is to show that, up to this infinitesimal action, we can assume that *any* infinitesimal deformation vanishes at least on four elements of $\{\boldsymbol{A}, \dots, \boldsymbol{F}\}$.

Lemma 4.13. *Fix $t \in (-1, 1)$, and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$. Up to the action of $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ as in (8), we can assume that*

$$\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = \dot{\boldsymbol{D}} = 0, \tag{9}$$

and that

$$\dot{\boldsymbol{E}} = (0, 0, 0, 0, \epsilon) \quad \textit{and} \quad \dot{\boldsymbol{F}} = (0, 0, 0, 0, \phi) \tag{10}$$

for some $\epsilon, \phi \in \mathbb{R}$.

The analogous lemma in the hyperbolic case, for $t \neq 0$, has been proved in [15, Proposition 11.1], and in fact the arguments here follow roughly the same lines as their proof. However, the first part of their proof uses a nice geometric argument which would be complicated to adapt to AdS geometry. For this reason, we rather use a linear algebra argument here.

Notation. To simplify the notation, from here to the end of Section 4.7, we denote by $\langle \cdot, \cdot \rangle$ the bilinear form b_{-1}. If one wants to repeat the proof for $G = \mathrm{Isom}(\mathbb{H}^4)$, then $\langle \cdot, \cdot \rangle$ should denote b_1. The reader should pay attention that in Section 6 the bracket $\langle \cdot, \cdot \rangle$ will instead be used to denote the Minkowski bilinear form on $\mathbb{R}^4$.

Proof. The proof will follow from three claims.

First we claim that we can assume that $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$. Equivalently, given any infinitesimal deformation $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$, we want to show that there exists $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ such that

$$\mathfrak{a} \cdot \boldsymbol{A} = \dot{\boldsymbol{A}}, \quad \mathfrak{a} \cdot \boldsymbol{B} = \dot{\boldsymbol{B}}, \quad \mathfrak{a} \cdot \boldsymbol{C} = \dot{\boldsymbol{C}}. \tag{11}$$

Indeed, if (11) is true, we can then subtract to $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ the element in the tangent space to the orbit of the form (8) (i.e., given by $\dot{\boldsymbol{s}} = \mathfrak{a} \cdot \boldsymbol{s}$) and obtain a new infinitesimal deformation for which $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$.

To show the first claim, consider the basis $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, e_4\}$ of $\mathbb{R}^5$, where $e_4 = (0, 0, 0, 0, 1)$. Recall that matrices $\mathfrak{a}$ in the Lie algebra $\mathfrak{so}(q_{-1})$ are characterised by the condition that

$$\langle \mathfrak{a} \cdot u, w \rangle + \langle u, \mathfrak{a} \cdot w \rangle = 0 \tag{12}$$

for every u, w, and that it suffices in fact to check the condition for all pairs of elements u, w of our fixed basis. Moreover, to define the matrix $\mathfrak{a}$ in $\mathfrak{so}(q_{-1})$, it suffices to define it on four vectors of the basis of $\mathbb{R}^5$, such that (12) holds when u, w are chosen among these four vectors. The definition of $\mathfrak{a}$ on the last vector of the basis is then uniquely determined by (12).

Let us now apply these preliminary remarks. By differentiating the conditions

$$\langle \boldsymbol{A}, \boldsymbol{A} \rangle = \langle \boldsymbol{B}, \boldsymbol{B} \rangle = \langle \boldsymbol{C}, \boldsymbol{C} \rangle = 1$$

we obtain

$$\langle \boldsymbol{A}, \dot{\boldsymbol{A}} \rangle = \langle \boldsymbol{B}, \dot{\boldsymbol{B}} \rangle = \langle \boldsymbol{C}, \dot{\boldsymbol{C}} \rangle = 0. \tag{13}$$

By differentiating the asymptoticity conditions

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \langle \boldsymbol{A}, \boldsymbol{C} \rangle = \langle \boldsymbol{B}, \boldsymbol{C} \rangle = -1$$

we get the conditions

$$\langle \boldsymbol{A}, \dot{\boldsymbol{B}} \rangle + \langle \dot{\boldsymbol{A}}, \boldsymbol{B} \rangle = 0, \quad \langle \boldsymbol{A}, \dot{\boldsymbol{C}} \rangle + \langle \dot{\boldsymbol{A}}, \boldsymbol{C} \rangle = 0, \quad \langle \boldsymbol{B}, \dot{\boldsymbol{C}} \rangle + \langle \dot{\boldsymbol{B}}, \boldsymbol{C} \rangle = 0. \tag{14}$$

Equations (13) and (14) show that any linear transformation $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ sending $\boldsymbol{A}$ to $\dot{\boldsymbol{A}}$, $\boldsymbol{B}$ to $\dot{\boldsymbol{B}}$, and $\boldsymbol{C}$ to $\dot{\boldsymbol{C}}$ satisfies the conditions of (12) for all pairs of u, w chosen in $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$. It remains to define $\mathfrak{a}$ on the remaining two elements $\boldsymbol{D}$ and e_4 of the basis. Equation (12) imposes the value of $\langle \mathfrak{a} \cdot \boldsymbol{D}, u \rangle$ and $\langle \mathfrak{a} \cdot e_4, u \rangle$ for all $u \in \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$. Moreover, we must have $\langle \mathfrak{a} \cdot \boldsymbol{D}, \boldsymbol{D} \rangle = \langle \mathfrak{a} \cdot e_4, e_4 \rangle = 0$. Therefore, $\mathfrak{a} \cdot \boldsymbol{D}$ and $\mathfrak{a} \cdot e_4$ can be chosen with one degree of freedom given by the value of $\langle \mathfrak{a} \cdot \boldsymbol{D}, e_4 \rangle = -\langle \boldsymbol{D}, \mathfrak{a} \cdot e_4 \rangle$. This shows that we can find $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ satisfying equation (11), and our first claim is proved.

Second, we claim that we can further assume that

$$\langle \dot{\boldsymbol{D}}, e_4 \rangle = 0.$$

To see this second claim, by repeating the same reasoning as in the beginning of this proof, it suffices to find another $\mathfrak{a}' \in \mathfrak{so}(q_{-1})$ so that

$$\mathfrak{a}' \cdot \boldsymbol{A} = \mathfrak{a}' \cdot \boldsymbol{B} = \mathfrak{a}' \cdot \boldsymbol{C} = 0 \quad \text{and} \quad \mathfrak{a}' \cdot \boldsymbol{D} = \langle \dot{\boldsymbol{D}}, e_4 \rangle e_4. \tag{15}$$

Indeed, if (15) holds, then the conditions (12) are satisfied for u, w chosen in $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$, and we have already remarked that $\mathfrak{a}' \cdot e_4$ will then be uniquely determined by (12) in such a way that $\mathfrak{a}' \in \mathfrak{so}(q_{-1})$. This shows our second claim.

Finally, we claim that, under the above assumptions, necessarily

$$\dot{\boldsymbol{D}} = 0, \quad \dot{\boldsymbol{E}} = (0, 0, 0, 0, \epsilon), \quad \text{and} \quad \dot{\boldsymbol{F}} = (0, 0, 0, 0, \phi).$$

This part of the proof follows closely [15, Proposition 11.1].

As observed in the proof of Corollary 4.12, since $\langle \boldsymbol{A}, \boldsymbol{D} \rangle = -1$, the vectors $\boldsymbol{A}$ and $\boldsymbol{D}$ play the role of two non-commuting generators (reflections along two timelike hyperplanes that are asymptotic) of a cusp group generated by the images of $\boldsymbol{A}$, $\boldsymbol{D}$, $\mathbf{3}^+$, $\mathbf{3}^-$, $\mathbf{2}^+$, and $\mathbf{2}^-$. By the assumption that asymptoticity conditions are preserved (recall that we are in Hom_0), any deformation of $\boldsymbol{A}$ and $\boldsymbol{D}$ satisfies $\langle \boldsymbol{A}, \boldsymbol{D} \rangle = -1$. So, by differentiating and using $\dot{\boldsymbol{A}} = 0$, we obtain $\langle \boldsymbol{A}, \dot{\boldsymbol{D}} \rangle = 0$. Analogously, $\langle \boldsymbol{B}, \dot{\boldsymbol{D}} \rangle = 0$.

Together with $\langle \boldsymbol{D}, \dot{\boldsymbol{D}} \rangle = 0$ (which follows from $\langle \boldsymbol{D}, \boldsymbol{D} \rangle = 1$) and the assumption $\langle \dot{\boldsymbol{D}}, v \rangle = 0$, we have necessarily

$$\dot{\boldsymbol{D}} = (\sqrt{2}\delta, \delta, \delta, -\delta, 0)$$

for some δ. Similarly for $\dot{\boldsymbol{E}}$, using that $\langle \boldsymbol{A}, \dot{\boldsymbol{E}} \rangle = \langle \boldsymbol{C}, \dot{\boldsymbol{E}} \rangle = \langle \boldsymbol{E}, \dot{\boldsymbol{E}} \rangle = 0$, we find that

$$\dot{\boldsymbol{E}} = (\sqrt{2}\epsilon', \epsilon', -\epsilon', \epsilon', \epsilon).$$

For $\boldsymbol{F}$, from $\langle \boldsymbol{B}, \dot{\boldsymbol{F}} \rangle = \langle \boldsymbol{C}, \dot{\boldsymbol{F}} \rangle = \langle \boldsymbol{F}, \dot{\boldsymbol{F}} \rangle = 0$ we find that

$$\dot{\boldsymbol{F}} = (\sqrt{2}\phi', -\phi', \phi', \phi', \phi).$$

Now using that $\boldsymbol{D}$ and $\boldsymbol{E}$ remain asymptotic, and similarly for the pairs $\{\boldsymbol{D}, \boldsymbol{F}\}$ and $\{\boldsymbol{E}, \boldsymbol{F}\}$, we have the relations

$$\langle \boldsymbol{D}, \dot{\boldsymbol{E}} \rangle + \langle \dot{\boldsymbol{D}}, \boldsymbol{E} \rangle = 0, \quad \langle \boldsymbol{D}, \dot{\boldsymbol{F}} \rangle + \langle \dot{\boldsymbol{D}}, \boldsymbol{F} \rangle = 0, \quad \langle \boldsymbol{E}, \dot{\boldsymbol{F}} \rangle + \langle \dot{\boldsymbol{E}}, \boldsymbol{F} \rangle = 0,$$

which read as

$$2\sqrt{2}\,\delta + 2\sqrt{2}\,\epsilon' = 0, \quad 2\sqrt{2}\,\delta + 2\sqrt{2}\,\phi' = 0, \quad 2\sqrt{2}\,\epsilon' + 2\sqrt{2}\,\phi' = 0.$$

Hence $\delta = \epsilon' = \phi' = 0$, and this shows the claim. The proof of Lemma 4.13 is complete. ∎

4.7. Infinitesimal deformations of the positive and negative generators

We conclude in this section the proof of Proposition 4.11.

A direct computation from Table 3 shows that the tangent vector to our explicit path f_t in $g_0^{-1}(0)$ is given by

$$\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^-, \quad \dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+, \quad \dot{\boldsymbol{X}} = 0, \tag{16}$$

where

$$\lambda = \frac{1}{(1-t^2)^{3/2}},$$

for all $\boldsymbol{i} \in \{\boldsymbol{0}, \dots, \boldsymbol{7}\}$ and $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$. (In the hyperbolic case, from Table 2, one would instead obtain $\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^-$, $\dot{\boldsymbol{i}}^- = -\lambda \boldsymbol{i}^+$, and $\dot{\boldsymbol{X}} = 0$ for $\lambda = (1+t^2)^{-3/2}$.)

We shall now show that, under the assumption in the statement of Lemma 4.13, *every* infinitesimal deformation $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfies (16) for some λ. Again, the proof follows roughly the lines of [15, Section 12], with the necessary adaptations to the AdS setting, and some simplifications.

Lemma 4.14. *Fix $t \in (-1, 1)$, and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of the normalised vectors $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfying* (9) *and* (10). *Then*

$$\begin{aligned} \dot{\boldsymbol{0}}^+ &= \lambda \boldsymbol{0}^-, & \dot{\boldsymbol{0}}^- &= \lambda \boldsymbol{0}^+, \\ \dot{\boldsymbol{3}}^+ &= \lambda \boldsymbol{3}^-, & \dot{\boldsymbol{3}}^- &= \lambda \boldsymbol{3}^+ \end{aligned} \tag{17}$$

for some $\lambda \in \mathbb{R}$ (depending on t).

Proof. Using the assumptions $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$, the derivatives of the relations $\langle \mathbf{0}^+, \boldsymbol{A}\rangle = \langle \mathbf{0}^+, \boldsymbol{B}\rangle = \langle \mathbf{0}^+, \boldsymbol{C}\rangle = 0$ yield

$$\langle \dot{\mathbf{0}}^+, \boldsymbol{A}\rangle = \langle \dot{\mathbf{0}}^+, \boldsymbol{B}\rangle = \langle \dot{\mathbf{0}}^+, \boldsymbol{C}\rangle = 0. \tag{18}$$

Together with

$$\langle \dot{\mathbf{0}}^+, \mathbf{0}^+\rangle = 0, \tag{19}$$

we obtain $\dot{\mathbf{0}}^+ = \lambda_0^+ \mathbf{0}^-$ for some λ_0^+.

Indeed, the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\mathbf{0}^+$ are linearly independent, and $\mathbf{0}^-$ satisfies all the four linear conditions (18) and (19), hence $\mathbf{0}^-$ spans the space of solutions. Similarly for $\mathbf{0}^-$, we obtain $\dot{\mathbf{0}}^- = \lambda_0^- \mathbf{0}^+$, and repeating the same argument for $\mathbf{3}^+$ and $\mathbf{3}^-$ (replacing the role of $\boldsymbol{C}$ by $\boldsymbol{D}$) we find $\dot{\mathbf{3}}^+ = \lambda_3^+ \mathbf{3}^-$ and $\dot{\mathbf{3}}^- = \lambda_3^- \mathbf{3}^+$.

Now, differentiating the relation $\langle \mathbf{0}^+, \mathbf{0}^-\rangle = 0$, we get

$$0 = \langle \dot{\mathbf{0}}^+, \mathbf{0}^-\rangle + \langle \mathbf{0}^+, \dot{\mathbf{0}}^-\rangle = \lambda_0^+ \langle \mathbf{0}^-, \mathbf{0}^-\rangle + \lambda_0^- \langle \mathbf{0}^+, \mathbf{0}^+\rangle = \lambda_0^+ - \lambda_0^-$$

which implies that $\lambda_0^+ = \lambda_0^-$. Similarly, we have $\lambda_3^+ = \lambda_3^-$. Finally, by differentiating $\langle \mathbf{3}^+, \mathbf{0}^-\rangle = 0$ we find that

$$0 = \langle \dot{\mathbf{3}}^+, \mathbf{0}^-\rangle + \langle \mathbf{3}^+, \dot{\mathbf{0}}^-\rangle = \lambda_3^+ \langle \mathbf{3}^-, \mathbf{0}^-\rangle + \lambda_0^- \langle \mathbf{3}^+, \mathbf{0}^+\rangle = \lambda_0^- - \lambda_3^+$$

whence $\lambda_0^- = \lambda_3^+$. This concludes the proof. ■

We remark that in the hyperbolic case the same computation shows that $\dot{\mathbf{0}}^+ = \lambda \mathbf{0}^-$, $\dot{\mathbf{0}}^- = -\lambda \mathbf{0}^+$, $\dot{\mathbf{3}}^+ = \lambda \mathbf{3}^-$, and $\dot{\mathbf{3}}^- = -\lambda \mathbf{3}^+$ for some $\lambda \in \mathbb{R}$, as the only differences with respect to the AdS argument is that $\langle \mathbf{0}^+, \mathbf{0}^+\rangle = \langle \mathbf{3}^+, \mathbf{3}^+\rangle = 1$ and $\langle \mathbf{3}^+, \mathbf{0}^+\rangle = -1$ from Table 2.

So, using the assumption $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = \dot{\boldsymbol{D}} = 0$, we have proved that (16) holds for $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \mathbf{3}^+\}$ and $\boldsymbol{i}^- \in \{\mathbf{0}^-, \mathbf{3}^-\}$. If we knew that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$, we could repeat a similar argument to show that (16) holds also for the remaining $\boldsymbol{i}^\pm$'s. It thus essentially remains to show that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$.

Lemma 4.15. *Fix $t \in (-1, 1)$ and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of the normalised vectors $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfying (9) and (10). Then $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$ and there exists $\lambda \in \mathbb{R}$ such that, for every $\boldsymbol{i} \in \{\mathbf{0}, \ldots, \mathbf{7}\}$,*

$$\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^- \quad \textit{and} \quad \dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+.$$

Proof. Let $\lambda \in \mathbb{R}$ be as in the conclusion of Lemma 4.14. Let us first focus on the variations of $\mathbf{1}$ and $\mathbf{2}$, similarly to the proof of Lemma 4.14. Taking the derivatives of the relations $\langle \mathbf{1}^+, \boldsymbol{A}\rangle = \langle \mathbf{1}^+, \boldsymbol{C}\rangle = 0$ and using $\dot{\boldsymbol{A}} = \dot{\boldsymbol{C}} = 0$, we have

$$\langle \dot{\mathbf{1}}^+, \boldsymbol{A}\rangle = \langle \dot{\mathbf{1}}^+, \boldsymbol{C}\rangle = 0$$

whereas from $\langle \mathbf{1}^+, \mathbf{1}^+\rangle = -1$ we derive

$$\langle \dot{\mathbf{1}}^+, \mathbf{1}^+\rangle = 0.$$

Here we do not know that $\dot{\boldsymbol{E}} = 0$ yet, hence we cannot argue that $\langle \dot{\mathbf{1}}^+, \boldsymbol{E}\rangle = 0$, which would imply that $\dot{\mathbf{1}}^+$ is a multiple of $\mathbf{1}^-$. However, observing that $\boldsymbol{A}$, $\boldsymbol{C}$, and $\mathbf{1}^+$ are linearly independent, and that the linear system for the $\dot{\mathbf{1}}^+$ given by the above three conditions is satisfied by the vectors $\mathbf{0}^-$ and $\mathbf{1}^-$ (which are linearly independent), by a dimension argument $\dot{\mathbf{1}}^+$ is necessarily a linear combination of $\mathbf{0}^-$ and $\mathbf{1}^-$. Analogously, one gets that $\dot{\mathbf{1}}^-$ is necessarily a linear combination of $\mathbf{0}^+$ and $\mathbf{1}^+$, and replacing $\boldsymbol{C}$ by $\boldsymbol{D}$, and $\mathbf{0}$ by $\mathbf{3}$, we find similar relations for $\dot{\mathbf{2}}^-$ and $\dot{\mathbf{2}}^+$. Let us summarise them here:

$$\begin{aligned}
\dot{\mathbf{1}}^- &= \lambda_1^- \mathbf{1}^+ + \mu_1^- \mathbf{0}^+,\\
\dot{\mathbf{1}}^+ &= \lambda_1^+ \mathbf{1}^- + \mu_1^+ \mathbf{0}^-,\\
\dot{\mathbf{2}}^- &= \lambda_2^- \mathbf{2}^+ + \mu_2^- \mathbf{3}^+,\\
\dot{\mathbf{2}}^+ &= \lambda_2^+ \mathbf{2}^- + \mu_2^+ \mathbf{3}^-.
\end{aligned}$$

We claim here that, as expected from (16), $\lambda_1^- = \lambda_1^+ = \lambda_2^- = \lambda_2^+ = \lambda$ and $\mu_1^- = \mu_1^+ = \mu_2^- = \mu_2^+ = 0$, and moreover $\dot{\boldsymbol{E}} = 0$. In fact, it will suffice to show that $\mu_1^+ = 0$. Indeed, recalling the assumption $\dot{\boldsymbol{E}} = (0,0,0,0,\epsilon)$, the derivative of the relation $\langle \boldsymbol{E}, \mathbf{1}^+\rangle = 0$ gives

$$0 = \langle \dot{\boldsymbol{E}}, \mathbf{1}^+\rangle + \langle \boldsymbol{E}, \dot{\mathbf{1}}^+\rangle = \frac{1}{\sqrt{1-t^2}}(\epsilon - 2\sqrt{2}\mu_1^+), \tag{20}$$

hence we will obtain $\epsilon = 0$, namely $\dot{\boldsymbol{E}} = 0$. Once we have $\dot{\boldsymbol{E}} = 0$, we can proceed exactly as in Lemma 4.14 to deduce that $\mu_1^- = \mu_1^+ = \mu_2^- = \mu_2^+ = 0$ and then $\lambda_1^- = \lambda_1^+ = \lambda_2^- = \lambda_2^+ = \lambda$ (which also follows from equation (21) below).

We shall need one more intermediate step. By differentiating the relation $\langle \mathbf{0}^-, \mathbf{1}^+\rangle = 0$, we find that

$$0 = \langle \dot{\mathbf{0}}^-, \mathbf{1}^+\rangle + \langle \mathbf{0}^-, \dot{\mathbf{1}}^+\rangle = \lambda\langle \mathbf{0}^+, \mathbf{1}^+\rangle + \lambda_1^+\langle \mathbf{0}^-, \mathbf{1}^-\rangle + \mu_1^+\langle \mathbf{0}^-, \mathbf{0}^-\rangle = \lambda - \lambda_1^+ + \mu_1^+.$$

Using similarly the relations $\langle \mathbf{0}^+, \mathbf{1}^-\rangle = \langle \mathbf{2}^-, \mathbf{3}^+\rangle = \langle \mathbf{2}^+, \mathbf{3}^-\rangle = 0$, we find three analogous identities. We summarise these four important identities here:

$$\lambda = \lambda_1^- - \mu_1^- = \lambda_1^+ - \mu_1^+ = \lambda_2^- - \mu_2^- = \lambda_2^+ - \mu_2^+. \tag{21}$$

We can now focus on proving that $\mu_1^+ = 0$. Differentiating $\langle \mathbf{1}^+, \mathbf{2}^-\rangle = 0$, we see that

$$0 = \lambda_1^+\langle \mathbf{1}^-, \mathbf{2}^-\rangle + \mu_1^+\langle \mathbf{0}^-, \mathbf{2}^-\rangle + \lambda_2^-\langle \mathbf{1}^+, \mathbf{2}^+\rangle + \mu_2^-\langle \mathbf{1}^+, \mathbf{3}^+\rangle.$$

Using $\langle \mathbf{1}^-, \mathbf{2}^-\rangle = -1$, $\langle \mathbf{1}^+, \mathbf{2}^+\rangle = 1$ and an explicit computation for the other two terms, we obtain

$$\lambda_2^- - \lambda_1^+ = \frac{3+t^2}{1-t^2}\mu_1^+ + \frac{1+3t^2}{1-t^2}\mu_2^-.$$

On the other hand, from equation (21) we have $\lambda_2^- - \lambda_1^+ = \mu_2^- - \mu_1^+$, whence

$$\mu_1^+ + t^2\mu_2^- = 0. \tag{22}$$

If $t = 0$, we are done. Otherwise, we will combine (22) with the derivative of the relation $\langle \boldsymbol{E}, \boldsymbol{2}^- \rangle = 0$, namely

$$\frac{t}{\sqrt{1-t^2}}(-\epsilon - 2\sqrt{2}\mu_2^-) = 0,$$

which together with equation (20) gives $\mu_1^+ + \mu_2^- = 0$. Together with (22), this shows that $\mu_1^+ = 0$.

Having proved that $\dot{\boldsymbol{E}} = 0$, the proof that $\dot{\boldsymbol{F}} = 0$ follows exactly the same lines, with $\mathbf{4}$ and $\mathbf{5}$ playing the role of $\mathbf{1}$ and $\mathbf{2}$. Arguing as in Lemma 4.14, one then shows that $\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^-$ and $\dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+$ for all $\boldsymbol{i} \in \{\mathbf{1}, \dots, \mathbf{7}\}$. ∎

This provides the conclusion of the proof of Proposition 4.11.

Proof of Proposition 4.11. Let us fix $t \in (-1, 1)$. We now show that the kernel of the differential of $g_0 \colon \mathbb{R}^{110} \to \mathbb{R}^{138}$ is 11-dimensional at $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\}) \in g_0^{-1}(0)$.

Lemmas 4.13, 4.14, and 4.15 showed that every element in the kernel of dg_0 is of the form (16) up to adding an element of the form (8), that is an element tangent to the orbit of the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action. It is also easy to see that such element in the tangent space of the orbit is unique, for if two elements $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have this property, it follows that $\mathfrak{a} := \mathfrak{a}_1 - \mathfrak{a}_2$ satisfies $\mathfrak{a} \cdot \boldsymbol{X} = 0$ for $\boldsymbol{X} = \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and the characterising conditions (12) (already used in Lemma 4.13) show that $\mathfrak{a} = 0$. The very same argument shows that the map defined in (8) from the Lie algebra $\mathfrak{isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ into the kernel of the differential of g_0 (whose image is the tangent space to the orbit of the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action) is injective.

In other words, the 10-dimensional tangent space of the orbit has a 1-dimensional complement, consisting precisely of the elements of the form (16), hence the kernel of the differential of g_0 has dimension 11. By the constant rank theorem, $g_0^{-1}(0)$ is a manifold of dimension 11 near the elements in the orbit of ρ_t. ∎

4.8. Topology of the neighbourhood $\mathcal{U}$

We now state a weaker version of Theorem 1.1:

Theorem 4.16. *Let G be* $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ *or* G_{HP^4}. *Then* $[\rho_0]$ *has a neighbourhood* $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$ *in* $X(\Gamma_{22}, G)$ *homeomorphic to* $\mathcal{S} = \{(x_1^2 + \cdots + x_{12}^2) \cdot x_{13} = 0\} \subset \mathbb{R}^{13}$, *so that*

- $[\rho_0]$ *corresponds to the origin;*
- $\mathcal{V}$ *corresponds to the* x_{13}*-axis, and consists of the conjugacy classes of the holonomy representations* ρ_t^G*;*
- $\mathcal{H}$ *corresponds to* $\{x_{13} = 0\}$, *identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$ *acts on* $\mathcal{S}$ *by changing sign to the last coordinate* x_{13}.

This statement is weaker than Theorem 1.1 because it gives a purely topological description of $\mathcal{U}$, while the smoothness and transversality of its components will be proved in Section 7.

We prove here Theorem 4.16 in the AdS case. The proof in the hyperbolic case is completely analogous, so we omit it, while the proof in the HP case will be given later in Section 6.4. We decided to give a proof only in the AdS case, since the fact that the points $[\rho_t]$ for $t > 0$ form a smooth curve (Proposition 4.7) has already been proved in [15], while its AdS counterpart is completely new. The proof for the hyperbolic case is analogous (recall Remark 4.10). Moreover, the description of the collapse (namely, at the representation ρ_0) is also new in both (hyperbolic and AdS) cases.

Proof of Theorem 4.16—AdS case. We split the proof into several steps.

Step 1. As a first step, let us define $\widetilde{\mathcal{V}} \subset \mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ as the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-orbit of the curve $\{\rho_t\}_{t\in(-1,1)}$. Let us also observe that $\widetilde{\mathcal{V}}$ is contained in the subset $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ introduced in Definition 4.8.

Since by Lemma 4.3 the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action by conjugation is free on $\{\rho_t\}_{t\in(-1,1)}$, the map $(g, t) \mapsto g \cdot \rho_t$ defines a continuous injection

$$\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4) \times (-1, 1) \to \mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)),$$

where by Proposition 4.11 the latter is a smooth 11-dimensional manifold. By the invariance of domain, this injection is a homeomorphism onto its image, which is $\widetilde{\mathcal{V}}$. By Lemmas 4.2 and 4.3, the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action by conjugation is free and proper on $\widetilde{\mathcal{V}}$ thus the projection in the quotient $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ is

$$\mathcal{V} := \big\{[\rho_t] \mid t \in (-1, 1)\big\},$$

which is homeomorphic to a line.

Step 2. The second component $\mathcal{H}$ is defined as follows.

Recall from Section 4.3 that we have a fixed copy $\mathbb{H}^3 \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$ defined by $x_4 = 0$ and $x_0 > 0$, fixed by the reflection r. Its stabiliser G_0 is $\mathrm{Isom}(\mathbb{H}^3) \times \langle r\rangle$, where we consider $\mathrm{Isom}(\mathbb{H}^3)$ as a subgroup of $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

Consider the reflection group Γ_{co} of the ideal right-angled cuboctahedron. We define the map

$$\Psi\colon \mathrm{Hom}\left(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)\right) \to \mathrm{Hom}\left(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)\right)$$

that associates to $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$ the representation $\Psi_\eta\colon \Gamma_{22} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ sending each of the generators $\mathbf{0}^+, \dots, \mathbf{7}^+$ of Γ_{22} to the reflection r, and each of the generators $\mathbf{0}^-, \dots, \mathbf{7}^-, \boldsymbol{A}, \dots, \boldsymbol{F}$ to the corresponding element of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ through η. It is then straightforward to check that

(1) the map Ψ is well defined and equivariant for the conjugacy action of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$,

(2) the following induced map

$$\hat{\Psi}\colon \bar{X}\left(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)\right) \to X\left(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)\right),$$

where $\bar{X}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)) := \mathrm{Hom}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))/\,\mathrm{Isom}(\mathbb{H}^3)$, is injective.

Indeed, (1) holds because, using that r commutes with the elements of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the images of the generators in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ through Ψ_η satisfy the relations of Γ_{22}, so that Ψ_η is indeed a representation of Γ_{22} in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. The equivariance of Ψ is clear using again that r commutes with $\mathrm{Isom}(\mathbb{H}^3)$. It also follows that $\widehat{\Psi}$ is well defined. Indeed, by the equivariance of Ψ, if η_1 and η_2 are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$, then Ψ_{η_1} and Ψ_{η_1} are conjugate in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. Up to composing with r, which commutes with both Ψ_{η_1} and Ψ_{η_2}, the latter are conjugate in $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

Moreover, (2) holds because if two representations Ψ_{η_1} and Ψ_{η_2} in the image of Ψ are conjugate by some $g \in \mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, then, since $\Psi_{\eta_1}(\boldsymbol{i}^+) = \Psi_{\eta_2}(\boldsymbol{i}^+) = r$, the isometry g must fix $\mathbb{H}^3 \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$, and therefore $g \in G_0 \cong \mathrm{Isom}(\mathbb{H}^3) \times \langle r \rangle$. Moreover, up to composing with r, which commutes with both Ψ_{η_i}, we can also assume that g belongs to the subgroup $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, hence η_1 and η_2 are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$.

The set $\bar{X}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is a 12-dimensional manifold in a neighbourhood (say $\mathcal{H}_0$) of $[\eta_0]$, since it corresponds to a neighbourhood of the complete hyperbolic orbifold structure of the right-angled cuboctahedron in its deformation space. To show this, the same proofs of [15, Proposition 5.2] apply (see also the related discussion in [15, Section 5]) as a well-known "reflective" orbifold version of Thurston's hyperbolic Dehn filling [29] (note that the ideal cuboctahedron has 12 cusps).

Therefore, a neighbourhood $\mathcal{H}_0$ of $[\eta_0]$ $\bar{X}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to $\mathbb{R}^{12}$, and we can also assume that $\widehat{\Psi}|_{\mathcal{H}_0}$ is a homeomorphism onto its image. Then let us define $\mathcal{H} := \widehat{\Psi}(\mathcal{H}_0)$.

Step 3. We claim that the intersection of $\mathcal{H}$ and $\mathcal{V}$ consists only of the point $[\rho_0]$.

Indeed, suppose that $[\rho] \in \mathcal{H} \cap \mathcal{V}$, for ρ in $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. On the one hand, $\rho = \Psi(\eta)$, where $\eta \in \mathrm{Hom}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is a deformation of the orbifold fundamental group of the cuboctahedron. On the other hand, ρ lies in $\widetilde{\mathcal{V}} \subset \mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. In particular, η maps each peripheral subgroup of Γ_{co} to a cusp group.

By the Mostow–Prasad rigidity, η is conjugate to the holonomy representation η_0 of the complete right-angled ideal cuboctahedron. Since both ρ and ρ_0 send each of the generators $\mathbf{0}^+, \ldots, \mathbf{7}^+$ to r, which commutes with $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the representations ρ and ρ_0 are also conjugate in $\mathrm{Isom}(\mathbb{H}^3)$, and therefore $[\rho] = [\rho_0]$.

Step 4. Let us now show that the point $[\rho_0] \in X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ has a neighbourhood $\mathcal{U}$ which is contained in the union of the two components $\mathcal{V}$ and $\mathcal{H}$.

To see this, let ρ be a representation nearby ρ_0. We claim that if two generators which are sent by ρ_0 to the same reflection r (hence necessarily of the form $\boldsymbol{i}^+$ and $\boldsymbol{j}^+$) are sent to reflections in coinciding hyperplanes also by ρ, then all generators $\mathbf{0}^+, \ldots, \mathbf{7}^+$ are sent by ρ to the same reflection. That is, if $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$ for some $\boldsymbol{i}, \boldsymbol{j}$, then $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$ for all $\boldsymbol{i}, \boldsymbol{j}$. This will show our thesis by the rigidity property of Proposition 3.17: if $[\rho]$ is not on the "horizontal" component $\mathcal{H}$, then no two letter generators are sent to the same reflection, and thus all the collapsed cusp groups of ρ_0 are cusp groups for ρ. That is, ρ lies in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ and thus in the "vertical" component $\widetilde{\mathcal{V}}$, since $\widetilde{\mathcal{V}}$ is open in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$.

To prove the claim, suppose that two generators $\boldsymbol{i}^+$ and $\boldsymbol{j}^+$ are such that $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$. By the symmetries of the polytope $\mathcal{P}_t$ (see [26, Lemma 7.6]) and Proposition 3.17, we can assume that the two generators are $\mathbf{0}^+$ and $\mathbf{1}^+$. Up to conjugation in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, we can also assume that $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+) = r$. To simplify the notation, let f be a preimage of ρ in $g^{-1}(0)$, which associates to each generator of Γ_{22} a vector in $\mathbb{R}^5$ of square norm 1 or -1 with respect to q_{-1}.

Up to changing the sign if necessary, $f(\mathbf{0}^+) = f(\mathbf{1}^+) = e_4 = (0,0,0,0,1)$. From the relations in Γ_{22}, the vector $f(\mathbf{2}^+)$ is necessarily orthogonal to $f(\mathbf{1}^-)$, $f(\mathbf{2}^-)$, $f(\mathbf{3}^-)$, and $f(\boldsymbol{A})$. But by the assumption $f(\mathbf{0}^+) = f(\mathbf{1}^+) = e_4$ and the relations involving $\mathbf{0}^+$, the vector e_4 is orthogonal to $f(\mathbf{1}^-)$, $f(\mathbf{3}^-)$, and $f(\boldsymbol{A})$, while from the relations involving $\mathbf{1}^+$, the vector e_4 is orthogonal to $f(\mathbf{2}^-)$.

For a small deformation of ρ_0, the vectors $f(\mathbf{1}^-)$, $f(\mathbf{2}^-)$, $f(\mathbf{3}^-)$, and $f(\boldsymbol{A})$ are linearly independent, because they are for ρ_0 (see Table 3). Hence the conditions of being orthogonal to these four vectors define a linear system of four independent equations, which are satisfied by e_4. Hence $f(\mathbf{2}^+)$, which is a solution of the system, coincides with e_4 up to rescaling. Since $q(f(\mathbf{2}^+)) = -1$, we can assume that $f(\mathbf{2}^+) = e_4$. Namely, $\rho(\mathbf{2}^+) = r$. By arguing similarly for $\mathbf{3}^+$ and then for all the other generators, one easily finds sufficiently many relations to show that $\rho(\boldsymbol{i}^+) = r$ for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and therefore ρ is in $\mathcal{H}$. This proves the claim.

Step 5. Summarising the previous steps, we have shown that the class $[\rho_0]$ has a neighbourhood $\mathcal{U}$ which only consists of points of $\mathcal{H}$ and $\mathcal{V}$. Since we already know that $\mathcal{H}$ and $\mathcal{V}$ are smooth manifolds outside of ρ_0, it is harmless to enlarge $\mathcal{U}$ so that it contains entirely $\mathcal{H}$ and $\mathcal{V}$.

We have therefore obtained a neighbourhood $\mathcal{U}$ of $[\rho_0]$ in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ homeomorphic to

$$\big(\{0\} \times \mathbb{R}\big) \cup \big(\mathbb{R}^{12} \times \{0\}\big) \subset \mathbb{R}^{13},$$

where the two components are precisely $\mathcal{H}$ and $\mathcal{V}$.

Step 6. It remains to prove the last sentence about the action of the group

$$\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)/\,\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4) \cong \mathbb{Z}/2\mathbb{Z}$$

generated by the coset of the reflection r.

This is now simple: on the one hand, as observed after Definition 4.1, conjugation by r acts on $\mathcal{V}$, which is homeomorphic to $(-1,1)$, by $[\rho_t] \mapsto [\rho_{-t}]$. On the other hand, by construction of $\mathcal{H}$, conjugation by r fixes pointwise the elements in $\mathcal{H}$, which are of the form Ψ_η for some $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$. This concludes the proof. ∎

We conclude the section with a couple of observations on the nature of the fixed points for the action of G on $\mathrm{Hom}(\Gamma_{22}, G)$.

Lemma 4.2 shows that the stabiliser of each point ρ_t in $\mathrm{Hom}(\Gamma_{22}, G)$, for the conjugacy action of G, is trivial, except ρ_0 which has stabiliser $\langle r \rangle$. In fact, a small adaptation of the proof shows that, in a neighbourhood of ρ_0, the stabiliser of all points in the horizontal component $\mathcal{H}$ is as well the group $\mathbb{Z}/2\mathbb{Z}$ generated by r. This is because we can find that a neighbourhood of ρ is in the image of Ψ such that, for a lift f of ρ, the vectors $f(\boldsymbol{A}), f(\boldsymbol{B}), f(\boldsymbol{C}), f(\boldsymbol{D}) \in \mathbb{R}^5$ are linearly independent. Indeed, the vectors $f_0(\boldsymbol{A})$, $f_0(\boldsymbol{B})$, $f_0(\boldsymbol{C})$, $f_0(\boldsymbol{D})$ are linearly independent, and being independent is an open condition. By the structure of the group Γ_{22}, the vectors $f(\boldsymbol{A})$, $f(\boldsymbol{B})$, $f(\boldsymbol{C})$, $f(\boldsymbol{D})$ are necessarily orthogonal to $(0,0,0,0,1)$, since ρ maps each generator $\boldsymbol{i}^+$ to r. Hence one can repeat the proof of Lemma 4.2 and see that an element in the stabiliser of ρ must necessarily fix $\{x_4 = 0\}$ setwise, and moreover must act trivially on $\{x_4 = 0\}$. Hence the only possible candidates are the identity and r, both of which fix ρ by definition of Ψ.

In conclusion, let us consider the full quotient $\mathrm{Hom}(\Gamma_{22}, G)/G$, which is a $\mathbb{Z}/2\mathbb{Z}$-quotient of $X(\Gamma_{22}, G)$, where $\mathbb{Z}/2\mathbb{Z} \cong G/G^+$. A local picture of this full quotient is given in Figure 1 (right), as a consequence of the fact that the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by changing sign to the x_{13}-coordinate, hence as a "reflection" with respect to the horizontal component $\mathcal{H}$. The "horizontal" component (which is the projection of $\mathcal{H}$ to the full quotient $\mathrm{Hom}(\Gamma_{22}, G)/G$) entirely consists of points with associated group $\mathbb{Z}/2\mathbb{Z}$. They are "double" points in a suitable sense, which recalls "mirror" points in the language of orbifolds.

5. Reflections and cusp groups in HP geometry

In this section, we introduce HP geometry, discuss its relations with Minkowski geometry, and prove the HP version of the flexibility and rigidity statements for right-angled cusp groups.

5.1. Half-pipe geometry

Let us denote by q_0 the following degenerate bilinear form on $\mathbb{R}^{n+1}$:

$$q_0(x) = -x_0^2 + x_1^2 + \cdots + x_{n-1}^2.$$

The n-dimensional *HP space* is defined as

$$\mathsf{HP}^n = \left\{x \in \mathbb{R}^{n+1} \mid q_0(x) = -1,\ x_0 > 0\right\},$$

and the group of *HP transformations* is

$$G_{\mathsf{HP}^n} = \left\{A \in \mathrm{O}(q_0) \mid A(\mathsf{HP}^n) = \mathsf{HP}^n,\ Ae_n = \pm e_n\right\}$$

(here $e_0, \dots, e_n$ is the canonical basis of $\mathbb{R}^{n+1}$).

Explicitly, an element $A \in G_{\mathsf{HP}^n}$ has the form

$$A = \left(\begin{array}{ccc|c} & & & 0 \\ & \widehat{A} & & \vdots \\ & & & 0 \\ \hline \star & \cdots & \star & \pm 1 \end{array}\right)$$

for some n-by-n matrix $\widehat{A}$ which preserves the bilinear form of signature $(-,+,\dots,+)$ on $\mathbb{R}^n$ and the upper sheet $\mathbb{H}^{n-1}$ of the hyperboloid $\{q_0=-1\}\subset\mathbb{R}^n\times\{0\}$, where the stars denote the entries of any vector in $\mathbb{R}^n$. Hence there is an obvious epimorphism $G_{\mathsf{HP}^n}\to\mathrm{Isom}(\mathbb{H}^{n-1})$, given by $A\mapsto\widehat{A}$. Despite that an inequality is involved in the definition of G_{HP^n}, this group is naturally an algebraic Lie group via the isomorphism between G_{HP^n} and $\mathrm{O}(1,n-1)\ltimes\mathbb{R}^n$ of Lemma 5.1, and the fact that the latter group can be defined as an algebraic subgroup of $\mathrm{Aff}(\mathbb{R}^n)\subset\mathrm{GL}(n+1,\mathbb{R})$.

The *boundary at infinity* of HP^n is

$$\partial\mathsf{HP}^n=\{x\in\mathbb{R}^{n+1}\mid q_0(x)=0\}/\mathbb{R}^*,$$

and can be visualised as the union of a cylinder constituted by those $[x]\in\partial\mathsf{HP}^n$ such that $(x_0,\dots,x_{n-1})$ does not vanish, and the point at infinity $[e_n]\in\partial\mathsf{HP}^n$. The latter is a distinguished point, since it is preserved by the action of every element of G_{HP^n} on $\partial\mathsf{HP}^n$.

As usual, we consider $\mathsf{HP}^n\cup\overline{\mathsf{HP}}^n$ as a subset of $\mathbb{RP}^n$, the *ideal closure* of a subset $A\subset\mathsf{HP}^n$ that is closed in HP^n is its closure $\overline{A}$ in $\mathbb{RP}^n$, and we have $\overline{\mathsf{HP}}^n=\mathsf{HP}^n\cup\partial\mathsf{HP}^n$.

There is a natural map from HP^n to $\{x\in\mathsf{HP}^n: x_n=0\}$, which is a copy of $\mathbb{H}^{n-1}$, given simply by $(x_0,\dots,x_n)\mapsto(x_0,\dots,x_{n-1},0)$. We shall call this map the *projection*

$$\pi\colon\mathsf{HP}^n\to\mathbb{H}^{n-1}.$$

The map π is equivariant with respect to the obvious epimorphism $G_{\mathsf{HP}^n}\to\mathrm{Isom}(\mathbb{H}^{n-1})$, and extends to a map $\overline{\pi}\colon\overline{\mathsf{HP}}^n\setminus\{[e_n]\}\to\overline{\mathbb{H}}^{n-1}$. The fibres of $\overline{\pi}$ are called *degenerate lines*, since they extend to projective lines in $\mathbb{RP}^n$ by adding the point $[e_n]$ at infinity, and the restriction of the bilinear form b_0 associated to q_0 is degenerate. Degenerate lines are preserved by the action of G_{HP^n}.

5.2. Duality with Minkowski space

We will find comfortable to exploit the well-known duality between HP and Minkowski geometry. We will not provide details of the proofs here; see [2,11,26] for a more complete treatment.

The fundamental observation is that HP^n is identified to the space of spacelike affine hyperplanes in Minkowski space $\mathbb{R}^{1,n-1}:=(\mathbb{R}^n,q_1)$, where q_1 is the non-degenerate bilinear form on $\mathbb{R}^n$ introduced in Section 2.1. The correspondence is given by associating to a point $x\in\mathsf{HP}^n$ the affine hyperplane of $\mathbb{R}^{1,n}$ defined by the equation

$$b_1\big((x_0,\dots,x_{n-1}),(y_0,\dots,y_{n-1})\big)+x_n=0, \tag{23}$$

for b_1 the bilinear form associated to q_1.

The isometry group $\mathrm{Isom}(\mathbb{R}^{1,n-1})\cong\mathrm{O}(q_1)\ltimes\mathbb{R}^n$ acts naturally on the space of spacelike affine hyperplanes, and the correspondence is also well behaved with respect to the group actions, as we summarise in the following lemma (see, for instance, [2, 11] or [26, Lemma 2.8]).

Lemma 5.1. *There is a "duality" homeomorphism*

$$\{\textit{spacelike affine hyperplanes in } \mathbb{R}^{1,n-1}\} \cong \mathsf{HP}^n$$

which is equivariant with respect to a group isomorphism

$$\phi\colon \mathrm{Isom}(\mathbb{R}^{1,n-1}) \to G_{\mathsf{HP}^n}.$$

In this work, we will adopt almost entirely this "dual" point of view for HP geometry. In this setting, the boundary $\partial\mathsf{HP}^n$ has a natural identification:

$$\partial\mathsf{HP}^n \cong \{\text{lightlike affine hyperplanes in } \mathbb{R}^{1,n-1}\} \cup \{\infty\}, \tag{24}$$

where the point $[e_n]$ in $\partial\mathsf{HP}^n$ corresponds to ∞ on the right-hand side, while $\partial\mathsf{HP}^n \setminus \{[e_n]\}$ identifies to the space of lightlike affine hyperplanes using again (23). Geometrically, the decomposition on the right-hand side of (24) reflects the fact that, up to taking a subsequence, a sequence of spacelike affine hyperplanes in $\mathbb{R}^{1,n-1}$ may either converge to a lightlike hyperplane or escape from all compact subsets.

The projection π is interpreted in this dual setting as the map which associates to a spacelike affine hyperplane in $\mathbb{R}^{1,n-1}$ its unique parallel linear hyperplane. Equivalently, thinking of π with values in $\mathbb{H}^{n-1}$, it associates to a spacelike affine hyperplane its normal direction with respect to the Minkowski product b_1. Of course, π extends to the complement of ∞ in $\partial\mathsf{HP}^n$, with values in $\partial\mathbb{H}^{n-1}$.

5.3. Hyperplanes

Let us now consider hyperplanes in HP geometry.

Definition 5.2 (HP hyperplane). An *HP hyperplane* is the intersection of HP^n with a linear hyperplane in $\mathbb{R}^{n+1}$. It is called *degenerate* if it contains a degenerate line of HP^n; *non-degenerate* otherwise.

From now on, we will always think of HP^n dually as the space of spacelike affine hyperplanes in $\mathbb{R}^{1,n-1}$, using Lemma 5.1. For more details on the proofs of the following statements, see [26, Section 4.3].

Lemma 5.3. *Any non-degenerate hyperplane of HP^n is dual to the set of spacelike affine hyperplanes going through a given point $p \in \mathbb{R}^{1,n-1}$.*

We will refer to the point p as the *dual* point to the non-degenerate hyperplane, and conversely we will make reference to the hyperplane *dual* to a point of $\mathbb{R}^{1,n-1}$. With this duality approach, it is very easy to describe the relative position of non-degenerate hyperplanes:

Lemma 5.4. *Given two points $p, q \in \mathbb{R}^{1,n-1}$, their dual hyperplanes*

- *intersect in HP^n if and only if $p - q$ is spacelike,*

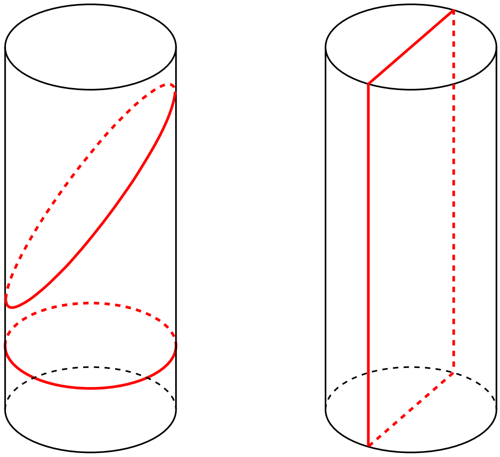

Figure 6. Hyperplanes in the affine (cylindric) model of HP^n: on the left, two spacelike hyperplanes, on the right, a degenerate hyperplane.

- *are disjoint in* HP^n *but their ideal closures intersect in* $\partial\mathsf{HP}^n$ *if and only if* $p-q$ *is lightlike,*
- *have disjoint ideal closures in* $\overline{\mathsf{HP}}^n$ *if and only if* $p-q$ *is timelike.*

In HP geometry, the situation for degenerate and non-degenerate hyperplanes (see Figure 6) is qualitatively different, as we shall see also in Section 5.4. Let us first characterise degenerate hyperplanes in terms of Minkowski geometry:

Lemma 5.5. *Any degenerate hyperplane of* HP^n *is the preimage of a hyperplane in* $\mathbb{H}^{n-1}$ *by the projection map* $\pi\colon \mathsf{HP}^n \to \mathbb{H}^{n-1}$. *That is, it is dual to the set of spacelike affine hyperplanes having normal direction in a given hyperplane of* $\mathbb{H}^{n-1}$.

There are three possibilities for the relative position of two degenerate hyperplanes $H_1 = \pi^{-1}(S_1)$ and $H_2 = \pi^{-1}(S_2)$ in HP^n:

- if S_1 and S_2 intersect in $\mathbb{H}^{n-1}$, then H_1 and H_2 intersect in HP^n in the subset $\pi^{-1}(S_1 \cap S_2)$;
- if $\overline{S}_1$ and $\overline{S}_2$ intersect in $\partial\mathbb{H}^{n-1}$, then $\overline{H}_1$ and $\overline{H}_2$ intersect in a degenerate line of $\partial\mathsf{HP}^n$;
- if $\overline{S}_1$ and $\overline{S}_2$ are disjoint in $\overline{\mathbb{H}}^{n-1}$, then $\overline{H}_1$ and $\overline{H}_2$ only intersect in $\infty \in \partial\mathsf{HP}^n$.

Asymptoticity of two hyperplanes and asymptoticity of a hyperplane to a point at infinity are defined similarly to the hyperbolic and AdS settings (see Section 2.3).

5.4. Reflections

Like in pseudo-Riemannian geometry, a *reflection* in HP^n is a non-trivial involution in G_{HP^n} that fixes pointwise a hyperplane.

We shall again distinguish two cases:

Proposition 5.6. *There exists a unique reflection in G_{HP^n} fixing a given non-degenerate hyperplane in HP^n.*

Proof. By Lemma 5.3, a reflection in G_{HP^n} is induced by an element of $\mathrm{Isom}(\mathbb{R}^{1,n-1})$ that fixes setwise all the spacelike hyperplanes going through a point $p \in \mathbb{R}^{1,n-1}$. The involution $\phi(-\mathrm{id}, 2p)$ therefore has such a property. It is the only reflection fixing the hyperplane dual to p. Indeed, for a transformation $\phi(A, v)$ with this property, the linear part A must fix all the timelike directions in $\mathbb{R}^{1,n-1}$, hence $A = \pm\mathrm{id}$, but the choice $A = \mathrm{id}$ implies necessarily that $v = 0$ because $\phi(A, v)$ has order two, and therefore gives a trivial transformation. ∎

Let us now consider degenerate hyperplanes:

Proposition 5.7. *There exists a one-parameter family of reflections in G_{HP^n} fixing a given degenerate hyperplane in HP^n.*

Proof. From Lemma 5.5, a degenerate hyperplane in HP^n has the form $\pi^{-1}(H_X)$ where, using the notation of Section 2.2, X denotes a vector in $\mathbb{R}^{1,n-1}$ such that $q_1(X) = 1$ and H_X is the hyperplane in $\mathbb{H}^{n-1}$ induced by the orthogonal complement $X^{\perp_1}$. Any reflection in G_{HP^n} fixing $\pi^{-1}(H_X)$ pointwise must be of the form $\phi(A, v)$, where the linear part A fixes $X^{\perp_1}$ pointwise. Hence the only possible candidates for A are the identity and the Minkowski reflection in H_X, which we denote by r_X. Since (A, v) is assumed to be an involution, $A = \mathrm{id}$ only gives the trivial transformation (i.e., $v = 0$). On the other hand, imposing the involutive condition for the choice $A = r_X$, we obtain the reflections $\phi(r_X, v)$ for any $v \in \mathrm{Span}(X)$. These are indeed reflections in the HP hyperplane $\pi^{-1}(H_X)$, since they fix setwise all spacelike hyperplanes of $\mathbb{R}^{1,n-1}$ with normal direction in H_X. ∎

Finally, it is necessary to analyse conditions which assure that two reflections commute. From Proposition 5.6, it is clear that two reflections $\phi(-\mathrm{id}, 2p)$ and $\phi(-\mathrm{id}, 2q)$ in non-degenerate hyperplanes do not commute unless $p = q$, i.e., unless the hyperplanes of reflection coincide.

By Proposition 5.7, reflections in degenerate hyperplanes are induced by Minkowski reflections in timelike hyperplanes. Hence two reflections $\phi(r_{X_1}, v_1)$ and $\phi(r_{X_2}, v_2)$ commute if and only if their linear parts commute.

The remaining case is considered in the following lemma, which is straightforward:

Lemma 5.8. *Let v, w, X be vectors in $\mathbb{R}^{1,n-1}$, with $q_1(X) = 1$ and $v \in \mathrm{Span}(X)$. The Minkowski isometries (r_X, v) and $(-\mathrm{id}, w)$ commute if and only if $w = v + u$ with $u \in X^{\perp_1}$.*

Proof. An easy computation shows that (r_X, v) and $(-\mathrm{id}, w)$ commute if and only if

$$(\mathrm{id} - r_X)(w) = 2v. \tag{25}$$

Writing $w = \lambda X + u$ for $\lambda \in \mathbb{R}$ and $u \in X^{\perp_1}$, we have $r_X(w) = -\lambda X + u$, hence the condition (25) is equivalent to $\lambda X = v$. ■

5.5. Right-angled cusp groups

Let us now discuss the properties of flexibility and rigidity of cusp representations for HP geometry, similarly to what we did for hyperbolic and AdS geometry in Section 3. The statements will be completely analogous, but the proofs simpler than their AdS (and hyperbolic) counterparts above.

The definitions of cusp groups and collapsed cusp groups are parallel to the AdS case:

Definition 5.9 (Cusp groups for HP^3). The image of a representation of Γ_{rect} into G_{HP^3} is called:

- a *cusp group* if the four generators are sent to reflections in four distinct planes asymptotic to a common point in $\partial\mathsf{HP}^3$;
- a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes, two degenerate and one non-degenerate, asymptotic to a common point in $\partial\mathsf{HP}^3$.

It follows from the discussion of the previous section that a cusp group representation must necessarily map two generators corresponding to opposite sides of the rectangle to reflections in degenerate hyperplanes, and the other two generators to reflections in non-degenerate hyperplanes.

The following example describes the structure of a (possibly collapsed) cusp group in HP geometry. By the non-uniqueness of HP reflections in a degenerate plane (Proposition 5.7), we need to describe not only the planes fixed by the reflections associated to each generator, but also the reflections themselves.

Example 5.10. Let the image of $\rho\colon \Gamma_{\text{rect}} \to G_{\mathsf{HP}^3}$ be a cusp group or collapsed cusp group, let $\boldsymbol{s}_1, \boldsymbol{s}_2$ be the generators such that $\rho(\boldsymbol{s}_1), \rho(\boldsymbol{s}_2)$ are reflections in a non-degenerate plane, and let $\boldsymbol{t}_1, \boldsymbol{t}_2$ be those such that $\rho(\boldsymbol{t}_1), \rho(\boldsymbol{t}_2)$ are reflections in a degenerate plane. Up to conjugacy, we can assume that $\rho(\boldsymbol{s}_1) = \phi(-\mathrm{id}, 0)$, that is, $\rho(\boldsymbol{s}_1)$ is the unique reflection in the dual plane to the origin of $\mathbb{R}^{1,2}$.

Using Lemma 5.8, $\rho(\boldsymbol{t}_1)$ and $\rho(\boldsymbol{t}_2)$ are necessarily of the form $\phi(r_{X_i}, 0)$, for X_i a unit spacelike vector in $\mathbb{R}^{1,2}$. This means that the two degenerate planes fixed by $\rho(\boldsymbol{t}_i)$ are of the form $\pi^{-1}(H_{X_i})$, for $i = 1, 2$. Since the ideal closures of the four planes are assumed to meet in a single point in $\partial\mathsf{HP}^3$, necessarily the ideal closure of the geodesics H_{X_1} and H_{X_2} of $\mathbb{H}^2$ meet in $\partial\mathbb{H}^2$. This means that $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a lightlike line in $\mathbb{R}^{1,2}$.

Finally, by Lemma 5.8 $\rho(\boldsymbol{s}_2)$ must be of the form $\phi(-\mathrm{id}, w)$ for some $w \in X_1^{\perp_1} \cap X_2^{\perp_1}$. This means that the non-degenerate plane fixed by $\rho(\boldsymbol{s}_2)$ is the dual of the point $w/2 \in \mathbb{R}^{1,2}$. If $w = 0$, then we have a collapsed cusp group, otherwise a cusp group. See the left side of Figure 7.

Let us now prove the HP analogue of Propositions 3.8 and 3.10 (see Figure 7).

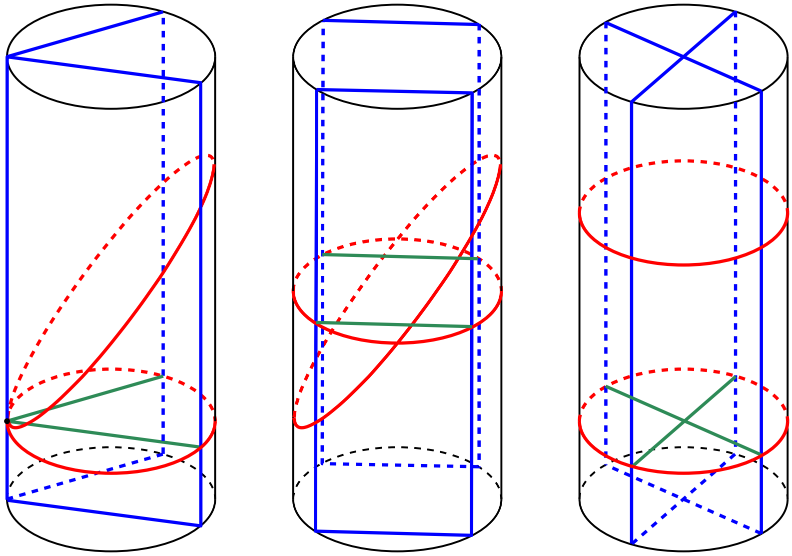

Figure 7. Three possibilities for a representation of Γ_{rect} in G_{HP^3}, as in the proof of Proposition 5.11. Two non-degenerate planes (in red), two degenerate planes (in blue), and their intersections (in green), which are geodesics in a copy of $\mathbb{H}^2$. On the left, the green geodesics are asymptotic and we have a cusp group. In the middle, they are ultraparallel, so the degenerate blue planes of HP^3 are disjoint, while the non-degenerate red planes intersect. On the right, the green geodesics intersect, so do the blue (degenerate) planes, while the red (non-degenerate) planes are disjoint.

Proposition 5.11. *Let $\rho\colon \Gamma_{\text{rect}} \to G_{\mathsf{HP}^3}$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) *if s_1 and s_2 are generators such that $\rho(s_1) = \rho(s_2)$, then $\rho'(s_1) = \rho'(s_2)$;*

(2) *the image of ρ' is a cusp group;*

(3) *a pair of opposite planes intersect in HP^3, while the other pair of opposite planes have disjoint ideal closures in $\overline{\mathsf{HP}}^3$.*

Proof. Let $\rho'\colon \Gamma_{\text{rect}} \to G_{\mathsf{HP}^3}$ be a representation nearby ρ. As in Example 5.10, we can assume that the reflection associated to one of the generators s_1 of Γ_{rect} is $\rho'(s_1) = \phi(-\text{id}, 0)$, so that its fixed plane is the dual plane to the origin of $\mathbb{R}^{1,2}$. Repeating the argument of Example 5.10, we have $\rho'(t_i) = \phi(r_{X_i}, 0)$ for some unit spacelike vectors X_i, and $\rho'(s_2) = \phi(-\text{id}, w)$ for some $w \in X_1^{\perp_1} \cap X_2^{\perp_1}$.

If $w = 0$, we are in case (1). Let us therefore assume that $w \neq 0$. If the ideal closures of the geodesics H_{X_1} and H_{X_2} intersect in $\partial\mathbb{H}^2$, then $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a lightlike geodesic, hence the image of ρ' is a cusp group as in Example 5.10 and we are in case (2).

If H_{X_1} and H_{X_2} intersect in $\mathbb{H}^2$, then $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a timelike geodesic, hence w is timelike. By Lemma 5.4, the fixed planes of $\rho'(s_1)$ and $\rho'(s_2)$ are disjoint, while the degenerate hyperplanes fixed by $\rho'(\boldsymbol{t}_1)$ and $\rho'(\boldsymbol{t}_2)$, namely $\pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{X_2})$, intersect in HP^3 (along a degenerate line). Hence point (3) is fulfilled.

Finally, if H_{X_1} and H_{X_2} are ultraparallel geodesics, then the closures of $\pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{X_2})$ only intersect in $\{\infty\}$. In this case, $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a spacelike geodesic, hence by Lemma 5.4 the fixed planes of $\rho'(\boldsymbol{s}_1)$ and $\rho'(\boldsymbol{s}_2)$ intersect. Therefore, point (3) is fulfilled again. ∎

Moving to dimension four, we define cusp groups in HP geometry:

Definition 5.12 (Cusp groups for HP^4). The image of a representation of Γ_{cube} into G_{HP^4} is called

- a *cusp group* if the six generators are sent to reflections in six distinct hyperplanes asymptotic to a common point at infinity;
- a *collapsed cusp group* if the six generators are sent to reflections along five distinct hyperplanes, four degenerate and one spacelike, asymptotic to a common point at infinity.

The HP version of Propositions 3.17 and 3.19 is now proved along the same lines:

Proposition 5.13. *Let $\rho\colon \Gamma_{\text{cube}} \to G_{\mathsf{HP}^4}$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) *if $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$ is a reflection in a non-degenerate hyperplane, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$;*

(2) *the image of ρ' is a cusp group.*

Proof. Let us denote by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ the generators of Γ_{cube} (corresponding to opposite faces of the cube) that are sent by ρ to reflections in a non-degenerate hyperplane; by $\boldsymbol{t}_1, \boldsymbol{t}_2$ and $\boldsymbol{u}_1, \boldsymbol{u}_2$ the other two pairs of opposite generators, which are necessarily sent to reflections in degenerate hyperplanes. By continuity, the same holds for ρ'.

Up to conjugation, we can assume that $\rho'(\boldsymbol{s}_1) = \phi(-\mathrm{id}, 0)$, and therefore by Lemma 5.8 $\rho'(\boldsymbol{t}_i) = \phi(r_{X_i}, 0)$ and $\rho'(\boldsymbol{u}_i) = \phi(r_{Y_i}, 0)$, for X_i, Y_i unit spacelike vectors. The restriction of ρ' to the subgroup generated by these four elements gives a representation of Γ_{rect} in a copy of $\mathrm{Isom}(\mathbb{H}^3)$, and is nearby a 3-dimensional cusp group.

Suppose first that $\rho'|_{\Gamma_{\text{rect}}}$ is a cusp group in $\mathrm{Isom}(\mathbb{H}^3)$. This means that $X_1^{\perp_1} \cap X_2^{\perp_1} \cap Y_1^{\perp_1} \cap Y_2^{\perp_1}$ is a lightlike line in $\mathbb{R}^{1,3}$. Then $\rho'(\boldsymbol{s}_2)$ is of the form $\phi(-\mathrm{id}, w)$ and by Lemma 5.8 $w \in X_1^{\perp_1} \cap X_2^{\perp_1} \cap Y_1^{\perp_1} \cap Y_2^{\perp_1}$. Hence ρ gives a cusp group in G_{HP^4} and we are in point (2).

If $\rho'|_{\Gamma_{\text{rect}}}$ does not give a cusp group in $\mathbb{H}^3$, by Proposition 3.8 two planes intersect in $\mathbb{H}^3$, while the ideal closures of the other two are disjoint in $\overline{\mathbb{H}}^3$. We will derive a contradiction. Up to relabelling, we can assume that the planes H_{X_1} and H_{X_2} intersect in $\mathbb{H}^3$, while the closures of H_{Y_1} and H_{Y_2} are disjoint. Hence in the degenerate subspace $\pi^{-1}(H_{X_1})$ (which is a copy of HP^3), the sets $\pi^{-1}(H_{Y_1}) \cap \pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{Y_2}) \cap \pi^{-1}(H_{X_1})$ are disjoint. Applying Proposition 5.11 to the restriction of ρ' to the subgroup generated by s_1, s_2, u_1, u_2, the fixed planes of $\rho'(s_1)$ and $\rho'(s_2)$ intersect in $\pi^{-1}(H_{X_1})$ (and thus in HP^4).

On the other hand, in $\pi^{-1}(H_{Y_1})$ (which is again a copy of HP^3), $\pi^{-1}(H_{X_1}) \cap \pi^{-1}(H_{Y_1})$ and $\pi^{-1}(H_{X_2}) \cap \pi^{-1}(H_{Y_1})$ intersect. In fact, H_{X_1} and H_{X_2} intersect in $\mathbb{H}^3$, hence also in H_{Y_1} since H_{X_1} and H_{X_2} are orthogonal to H_{Y_1}. By Proposition 5.11 again, the fixed planes of $\rho'(s_1)$ and $\rho'(s_2)$ are disjoint in HP^4, which contradicts the conclusion of the previous paragraph. ■

6. Group cohomology and the HP character variety of Γ_{22}

The goal of this section is to prove the HP part of Theorem 4.16. An essential step is an explicit computation of the first cohomology group $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ in Proposition 6.5, a result for which we will give other applications in Section 7.

6.1. Preliminaries on group cohomology

We recall here a few notions of group cohomology.

Let Γ be a group, V a finite-dimensional real vector space, and $\varrho\colon \Gamma \to \mathrm{GL}(V)$ a representation. The *first cohomology group* of Γ associated to ϱ is the quotient

$$H^1_\varrho(\Gamma, V) = Z^1_\varrho(\Gamma, V) \big/ B^1_\varrho(\Gamma, V),$$

where

- the space of *cocycles* is

$$Z^1_\varrho(\Gamma, V) = \big\{\tau\colon \Gamma \to V \mid \forall \gamma,\ \eta \in \Gamma\ \tau(\gamma\eta) = \varrho(\gamma)\tau(\eta) + \tau(\gamma)\big\},$$

- the space of *coboundaries* is

$$B^1_\varrho(\Gamma, V) = \big\{\tau\colon \Gamma \to V \mid \exists v \in V\ \forall \gamma \in \Gamma\ \tau(\gamma) = \varrho(\gamma)v - v\big\}.$$

The space $Z^1_\varrho(\Gamma, V)$ coincides with the space of *affine deformations* of ϱ, namely the functions $\tau\colon \Gamma \to V$ such that (ϱ, τ) gives a representation of Γ to $\mathrm{GL}(V) \ltimes V$. The difference $\tau - \tau'$ of two cocycles is a coboundary if and only if the corresponding representations (ϱ, τ) and (ϱ, τ') are conjugate in V. We have in summary the following.

Lemma 6.1. *The vector space $H^1_\varrho(\Gamma, V)$ parameterises the representations of Γ in $\mathrm{GL}(V) \ltimes V$ having linear part ϱ, up to conjugation.*

When Γ is a right-angled Coxeter group, the space $Z^1_\varrho(\Gamma, V)$ has the following description in terms of generators and relations.

Lemma 6.2. *Let Γ be a right-angled Coxeter group as in Definition 3.1, and $\varrho\colon \Gamma \to \mathrm{GL}(V)$ a representation. Then $Z^1_\varrho(\Gamma, V)$ is isomorphic to the vector space of functions $\tau\colon S \to V$ such that*

- $\tau(s) \in \mathrm{Ker}(\mathrm{id} + \varrho(s))$ *for all $s \in S$, and*
- $(\mathrm{id} - \varrho(s_i))\tau(s_j) = (\mathrm{id} - \varrho(s_j))\tau(s_i)$ *for all $(s_i, s_j) \in R$.*

Proof. Clearly a cocycle in $Z^1_\varrho(\Gamma, V)$ is determined by its values on the generators. The conditions that have to be satisfied by τ for each relation are $0 = \tau(\boldsymbol{s}^2) = \varrho(\boldsymbol{s})\tau(\boldsymbol{s}) + \tau(\boldsymbol{s})$, from which we get the first point, and $\tau(\boldsymbol{s}_i\boldsymbol{s}_j) = \tau(\boldsymbol{s}_j\boldsymbol{s}_i)$ for every $(\boldsymbol{s}_i, \boldsymbol{s}_j) \in R$. Expanding $\tau(\boldsymbol{s}_i\boldsymbol{s}_j) = \varrho(\boldsymbol{s}_i)\tau(\boldsymbol{s}_j) + \tau(\boldsymbol{s}_i)$, we obtain the second point. ■

6.2. A curve of geometric representations

We now introduce the HP representations of our interest, which have been computed in [26, Remark 7.16] by applying a rescaling argument to the hyperbolic or AdS holonomy representations ρ_t.

Notation. Throughout the following, we will denote by $\langle\cdot,\cdot\rangle$ the Minkowski product of $\mathbb{R}^{1,3}$ (previously denoted by b_1) and by $v^\perp \subset \mathbb{R}^{1,3}$ the orthogonal complement of $v \in \mathbb{R}^{1,3}$ with respect to the Minkowski product.

Recall that by Lemma 5.1 the transformation group G_{HP^4} is isomorphic to

$$\mathrm{Isom}(\mathbb{R}^{1,3}) \cong \mathrm{O}(1,3) \ltimes \mathbb{R}^4.$$

We will exhibit the HP holonomies as representations in $\mathrm{Isom}(\mathbb{R}^{1,3})$.

Definition 6.3 (The HP representation ρ_λ). Given $\lambda \in \mathbb{R}$, we define a representation

$$\rho_\lambda = (\varrho_0, \tau_\lambda)\colon \Gamma_{22} \to \mathrm{O}(1,3) \ltimes \mathbb{R}^4$$

on the standard generators of Γ_{22} as follows. The linear part ϱ_0 is independent of λ and is defined by

$$\begin{aligned} \varrho_0(\boldsymbol{i}^+) &= -\mathrm{id} && \text{for each } \boldsymbol{i} \in \{\boldsymbol{0},\dots,\boldsymbol{7}\},\\ \varrho_0(\boldsymbol{i}^-) &= r_{v_{\boldsymbol{i}}} && \text{for each } \boldsymbol{i} \in \{\boldsymbol{0},\dots,\boldsymbol{7}\},\\ \varrho_0(\boldsymbol{X}) &= r_{v_{\boldsymbol{X}}} && \text{for each } \boldsymbol{X} \in \{\boldsymbol{A},\dots,\boldsymbol{F}\}, \end{aligned} \tag{26}$$

while the translation part is

$$\begin{aligned} \tau_\lambda(\boldsymbol{i}^+) = \tau_\lambda(\boldsymbol{i}^-) &= (-1)^i\lambda v_{\boldsymbol{i}} && \text{for each } \boldsymbol{i} \in \{\boldsymbol{0},\dots,\boldsymbol{7}\},\\ \tau_\lambda(\boldsymbol{X}) &= 0 && \text{for each } \boldsymbol{X} \in \{\boldsymbol{A},\dots,\boldsymbol{F}\}, \end{aligned} \tag{27}$$

where the vectors $v_{\boldsymbol{s}}$ are defined in Table 4.

Recall that the vectors in Table 4 define the bounding planes of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. Moreover, r_v denotes the reflection in $\mathrm{O}(1,3)$ in the hyperplane $v^\perp$, namely, the linear transformation acting on $v^\perp$ as the identity and on the subspace generated by v as minus the identity.

Remark 6.4. When $\lambda = 0$, the representation ρ_0 is naturally identified to those introduced in Definition 4.1 for $t = 0$. Indeed, recall from Section 4.3 that in the hyperbolic and AdS case ρ_0 takes value in the stabiliser G_0 of the hyperplane $\{x_4 = 0\}$, and G_0 is a common subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, both seen as subgroups of $\mathrm{GL}(5,\mathbb{R})$.

Now, the representation $\rho_0 = (\varrho_0, 0)$ introduced in Definition 6.3 also takes value in the stabiliser of $\{x_4 = 0\}$ in G_{HP^4} which coincides again with the subgroup G_0 of $\mathrm{GL}(5,\mathbb{R})$. Under the isomorphism with $\mathrm{Isom}(\mathbb{R}^{1,3})$, the group G_0 is dually identified with the stabiliser of the origin in $\mathbb{R}^{1,3}$, namely the linear subgroup $\mathrm{O}(1,3) < \mathrm{Isom}(\mathbb{R}^{1,3})$. The explicit isomorphism $\mathrm{O}(1,3) \cong G_0$ is given by

$$\mathrm{O}(1,3) \ni A \mapsto \pm \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_0, \tag{28}$$

where the sign $\pm$ is positive if A preserves $\mathbb{H}^3 \subset \mathbb{R}^{1,3}$, and negative otherwise.

Under the isomorphism (28), the representation $\rho_0 = (\varrho_0, 0)$ of Definition 6.3 (with zero translation part) coincides with the "collapsed" representation expressed in (6). This justifies that in the statement of Theorem 1.1 we refer to the *same* representation ρ_0 in all three geometries.

The goal of the following section is to compute the first cohomology group associated to the representation $\varrho_0 \colon \Gamma_{22} \to \mathrm{O}(1,3)$ of Definition 6.3. Applications of the result will then be given in Sections 6.4 and 7.2.

6.3. The geometric cocycle is a generator

Recall Definition 6.3. The goal of this section is to prove the following:

Proposition 6.5. *The vector space $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has dimension one.*

To prove Proposition 6.5, we will show that every cohomology class in $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ is represented by a cocycle τ_λ of the form (27), for some $\lambda \in \mathbb{R}$.

We already know from [26, Remark 7.16] that $\rho_\lambda = (\varrho_0, \tau_\lambda)$ of Definition 6.3 is a representation of Γ_{22}, hence τ_λ is a cocycle. This can however be checked directly from (26) and (27) using Lemma 6.2. Let us introduce some additional notation:

Definition 6.6 (The subspace U_0). We denote by U_0 the 1-dimensional vector subspace of $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ composed of cocycles of the form (27), for some $\lambda \in \mathbb{R}$.

Let us observe that τ_λ vanishes on all the letter generators and that $\tau_\lambda(\boldsymbol{i}^-)$ and $\tau_\lambda(\boldsymbol{i}^+)$ are all vectors of norm $|\lambda|$ for the Minkowski product on $\mathbb{R}^{1,3}$, since all the $v_{\boldsymbol{i}}$ have unit Minkowski norm.

The ultimate goal will be to show that any cocycle $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has a unique decomposition $\tau = \tau_\lambda - \eta$, for some $\eta \in B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ and $\tau_\lambda \in U_0$. The proof will follow from a sequence of computational lemmas.

Lemma 6.7. *Let $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. Then,*

$$\tau(\boldsymbol{i}^-) \in \mathrm{Span}(v_{\boldsymbol{i}}) \quad \textit{for each } \boldsymbol{i}^- \in \{\boldsymbol{0}^-, \dots, \boldsymbol{7}^-\}$$

and

$$\tau(\boldsymbol{X}) \in \mathrm{Span}(v_{\boldsymbol{X}}) \quad \textit{for each } \boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}.$$

Proof. By Lemma 6.2, we get $\tau(\boldsymbol{i}^-) \in \operatorname{Ker}(\mathrm{id} + \varrho_0(\boldsymbol{i}^-))$. This kernel equals the subspace generated by $v_{\boldsymbol{i}}$ since $\varrho_0(\boldsymbol{i}^-)$ is the Minkowski reflection fixing the hyperplane $v_{\boldsymbol{i}}^{\perp}$. The proof for the letter generators is the same. ■

The following step is a first reduction of the problem.

Lemma 6.8. *Let $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. Then there exists a unique $\eta \in B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ such that, if $\hat{\tau} = \tau - \eta$, then*

$$\hat{\tau}(\boldsymbol{A}) = \hat{\tau}(\boldsymbol{B}) = \hat{\tau}(\boldsymbol{C}) = \hat{\tau}(\boldsymbol{D}) = 0. \tag{29}$$

After the proof of Lemma 6.8, we will show that if $\hat{\tau}$ satisfies (29), then it is of the form (27) for some $\lambda \in \mathbb{R}$. Together with Lemma 6.8, this will imply that

$$Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = U_0 \oplus B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \tag{30}$$

and therefore that $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ is 1-dimensional.

Proof of Lemma 6.8. Let τ be any cocycle. By Lemma 6.7, we have that $\tau(\boldsymbol{X}) \in \operatorname{Span}(v_{\boldsymbol{X}})$ for all $\boldsymbol{X} \in \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$. Define the linear map

$$L \colon \mathbb{R}^{1,3} \to \operatorname{Span}(v_{\boldsymbol{A}}) \oplus \operatorname{Span}(v_{\boldsymbol{B}}) \oplus \operatorname{Span}(v_{\boldsymbol{C}}) \oplus \operatorname{Span}(v_{\boldsymbol{D}})$$

by

$$L(w) = \big(\varrho_0(\boldsymbol{A})w - w, \varrho_0(\boldsymbol{B})w - w, \varrho_0(\boldsymbol{C})w - w, \varrho_0(\boldsymbol{D})w - w\big).$$

The proof follows if we show that L is invertible.

Let us write the matrix associated to L in the basis $\{v_{\boldsymbol{A}}, v_{\boldsymbol{B}}, v_{\boldsymbol{C}}, v_{\boldsymbol{D}}\}$ on the source and on the target. Recalling that the $v_{\boldsymbol{X}}$ are all unit vectors for the Minkowski product $\langle\cdot,\cdot\rangle$ and that $\rho_0(\boldsymbol{X})$ is the reflection in $v_{\boldsymbol{X}}^{\perp}$, we have

$$\varrho_0(\boldsymbol{X})v_{\boldsymbol{Y}} - v_{\boldsymbol{Y}} = \varrho_0(\boldsymbol{X})\big(\langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}}\big) - \langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}} = -2\langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}}.$$

This shows that the associated matrix of L is

$$-2\begin{pmatrix} \langle v_{\boldsymbol{A}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{B}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{C}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{D}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{D}}\rangle \end{pmatrix},$$

which is invertible by the non-degeneracy of the Minkowski product. ■

Remark 6.9. The proof of Lemma 6.8 also shows that $B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has dimension four. Indeed, we have a surjective linear map $\mathbb{R}^{1,3} \to B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ that sends $w \in \mathbb{R}^{1,3}$ to the coboundary $\tau(\gamma) = \varrho_0(\gamma)w - w$. This map is injective, by the injectivity of the map L introduced in the proof of Lemma 6.8. Hence $\dim B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = 4$.

Let us now compute the cocycle condition which arises from any orthogonality condition in Γ_{22}.

Lemma 6.10. *Let $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$.*

- *For any relation in Γ_{22} of the form $\boldsymbol{i}^+\boldsymbol{j}^- = \boldsymbol{j}^-\boldsymbol{i}^+$, we have that $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{j}^-) \in v_{\boldsymbol{j}}^\perp$.*
- *For any relation in Γ_{22} of the form $\boldsymbol{i}^+\boldsymbol{X} = \boldsymbol{X}\boldsymbol{i}^+$, we have that $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{X}) \in v_{\boldsymbol{X}}^\perp$.*

Proof. Let us show the first point, the second being completely analogous. By Lemma 6.2

$$\big(\mathrm{id} - \varrho_0(\boldsymbol{j}^-)\big)\tau(\boldsymbol{i}^+) = \big(\mathrm{id} - \varrho_0(\boldsymbol{i}^+)\big)\tau(\boldsymbol{j}^-) = 2\tau(\boldsymbol{j}^-) = \big(\mathrm{id} - \varrho_0(\boldsymbol{j}^-)\big)\tau(\boldsymbol{j}^-),$$

where we have used that $\varrho_0(\boldsymbol{i}^+) = -\mathrm{id}$, that $\varrho_0(\boldsymbol{j}^-)$ is the reflection in the Minkowski hyperplane $v_{\boldsymbol{j}}^\perp$, and that $\tau(\boldsymbol{j}^-) \in \mathrm{Span}(v_{\boldsymbol{j}})$ by Lemma 6.7. Hence $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{j}^-)$ is in the kernel of $\mathrm{id} - \varrho_0(\boldsymbol{j}^-)$, namely in $v_{\boldsymbol{j}}^\perp$. ∎

Let us now go back to showing that a cocycle $\hat{\tau}$ as in Lemma 6.8 is of the form (27). Our next step is the following lemma.

Lemma 6.11. *Suppose that $\hat{\tau} \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ satisfies (29). Then $\hat{\tau}(\mathbf{0}^+) = \hat{\tau}(\mathbf{0}^-) = \lambda v_{\mathbf{0}}$ and $\hat{\tau}(\mathbf{3}^+) = \hat{\tau}(\mathbf{3}^-) = -\lambda v_{\mathbf{3}}$ for some $\lambda \in \mathbb{R}$.*

Proof. It follows from Lemma 6.7 that $\hat{\tau}(\mathbf{0}^-) = \mu_0 v_{\mathbf{0}}$, and similarly $\hat{\tau}(\mathbf{3}^-) = \mu_3 v_{\mathbf{3}}$. We remark that we have no similar condition on the $\boldsymbol{i}^+$ coming from the relation that $\boldsymbol{i}^+$ squares to the identity.

However, we claim that in our assumption also $\hat{\tau}(\mathbf{0}^+) \in \mathrm{Span}(v_{\mathbf{0}})$ and $\hat{\tau}(\mathbf{3}^+) \in \mathrm{Span}(v_{\mathbf{3}})$. Indeed, applying Lemma 6.10 to the relation $\mathbf{0}^+\boldsymbol{A} = \boldsymbol{A}\mathbf{0}^+$ and using that $\hat{\tau}(\boldsymbol{A}) = 0$ by hypothesis, we get $\hat{\tau}(\mathbf{0}^+) \in v_{\boldsymbol{A}}^\perp$. Similarly, from $\mathbf{0}^+\boldsymbol{B} = \boldsymbol{B}\mathbf{0}^+$ and $\mathbf{0}^+\boldsymbol{C} = \boldsymbol{C}\mathbf{0}^+$, we obtain that $\hat{\tau}(\mathbf{0}^+)$ is in $v_{\boldsymbol{B}}^\perp$ and $v_{\boldsymbol{C}}^\perp$. Now, $v_{\boldsymbol{A}}$, $v_{\boldsymbol{B}}$, and $v_{\boldsymbol{C}}$ are linearly independent, hence $v_{\boldsymbol{A}}^\perp \cap v_{\boldsymbol{B}}^\perp \cap v_{\boldsymbol{C}}^\perp$ is 1-dimensional and therefore coincides with $\mathrm{Span}(v_{\mathbf{0}})$, since $v_{\mathbf{0}}$ is orthogonal to all of them. By applying the same argument to $\hat{\tau}(\mathbf{3}^+)$ and the letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$ (since by hypothesis $\hat{\tau}$ vanishes on $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$), we obtain that $\hat{\tau}(\mathbf{3}^+) \in \mathrm{Span}(v_{\mathbf{3}})$.

Hence we have shown that $\hat{\tau}(\mathbf{0}^+) = \lambda_0 v_{\mathbf{0}}$ and $\hat{\tau}(\mathbf{3}^+) = \lambda_3 v_{\mathbf{3}}$. We have to show that $\lambda_0 = \mu_0 = -\lambda_3 = -\mu_3$. Let us apply Lemma 6.10 to the relation $\mathbf{0}^+\mathbf{0}^- = \mathbf{0}^-\mathbf{0}^+$. We obtain

$$\hat{\tau}(\mathbf{0}^+) - \hat{\tau}(\mathbf{0}^-) \in v_{\mathbf{0}}^\perp,$$

that is,

$$0 = \langle \lambda_0 v_{\mathbf{0}} - \mu_0 v_{\mathbf{0}}, v_{\mathbf{0}} \rangle = \lambda_0 - \mu_0,$$

hence $\lambda_0 = \mu_0$. Analogously, $\lambda_3 = \mu_3$. If we now apply Lemma 6.10 to the relation $\mathbf{0}^+\mathbf{3}^- = \mathbf{3}^-\mathbf{0}^+$, we get

$$\hat{\tau}(\mathbf{0}^+) - \hat{\tau}(\mathbf{3}^-) \in v_{\mathbf{3}}^\perp,$$

which in turn gives

$$0 = \langle \lambda_0 v_{\mathbf{0}} - \lambda_3 v_{\mathbf{3}}, v_{\mathbf{3}} \rangle = \lambda_0 \langle v_{\mathbf{0}}, v_{\mathbf{3}} \rangle - \lambda_3 \langle v_{\mathbf{3}}, v_{\mathbf{3}} \rangle = -\lambda_0 - \lambda_3.$$

We conclude by setting $\lambda := \lambda_0 = -\lambda_3$. ∎

Remark 6.12. The proof of Lemma 6.11 only worked for $\boldsymbol{i} = \mathbf{0}, \mathbf{3}$ because we used that $\hat{\tau}$ vanishes on $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, and we needed to pick three linearly independent vectors among these four. Once we show that $\hat{\tau}$ also vanishes on $\boldsymbol{E}$ and $\boldsymbol{F}$ (Lemma 6.13 below), the same argument will apply exactly in the same way to show that

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-) = \lambda v_{\boldsymbol{i}},$$

for $\boldsymbol{i}$ odd, and

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-) = -\lambda v_{\boldsymbol{i}},$$

for $\boldsymbol{i}$ even. This will therefore conclude the proof that $\hat{\tau}$ is in the form (27).

Lemma 6.13. *If $\hat{\tau} \in Z^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ satisfies (29), then $\hat{\tau}(\boldsymbol{E}) = \hat{\tau}(\boldsymbol{F}) = 0$.*

Proof. From Lemma 6.7, we know that

$$\hat{\tau}(\boldsymbol{E}) = e v_{\boldsymbol{E}} \quad \text{and} \quad \hat{\tau}(\boldsymbol{F}) = f v_{\boldsymbol{F}}.$$

We wish to show that $e = f = 0$. Let us first prove that $e = 0$.

Observe that $v_{\boldsymbol{A}}$, $v_{\boldsymbol{C}}$, and $v_{\boldsymbol{E}}$ are linearly independent, and they are all orthogonal to $v_{\mathbf{1}}$. Hence $\{v_{\mathbf{1}}, v_{\boldsymbol{A}}, v_{\boldsymbol{C}}, v_{\boldsymbol{E}}\}$ is a (non-orthogonal!) basis of unit vectors and we can decompose

$$\hat{\tau}(\mathbf{1}^+) = \lambda_1 v_{\mathbf{1}} + \alpha v_{\boldsymbol{A}} + \gamma v_{\boldsymbol{C}} + \epsilon v_{\boldsymbol{E}}.$$

(We ultimately will get, at the end of the proof, that $\lambda_1 = -\lambda$ and $\alpha = \gamma = \epsilon = 0$, but we do not know this yet.) As a preliminary remark, observe that $\hat{\tau}(\mathbf{1}^-) = \lambda_1 v_{\mathbf{1}}$, since from the relation $\mathbf{1}^+\mathbf{1}^- = \mathbf{1}^-\mathbf{1}^+$ we obtain

$$\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\mathbf{1}^-) \in v_{\mathbf{1}}^{\perp},$$

and comparing with the above decomposition, necessarily $\hat{\tau}(\mathbf{1}^-) = \lambda_1 v_{\mathbf{1}}$.

Since $\hat{\tau}(\boldsymbol{A}) = 0$, from the relation $\mathbf{1}^+\boldsymbol{A} = \boldsymbol{A}\mathbf{1}^+$ we obtain

$$\hat{\tau}(\mathbf{1}^+) \in v_{\boldsymbol{A}}^{\perp},$$

namely,

$$0 = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{A}} \rangle = \alpha - \gamma - \epsilon. \tag{31}$$

From the same computation for the relation $\mathbf{1}^+\boldsymbol{C} = \boldsymbol{C}\mathbf{1}^+$, we derive

$$0 = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{C}} \rangle = -\alpha + \gamma - \epsilon. \tag{32}$$

Finally, the relation $\mathbf{1}^+\boldsymbol{E} = \boldsymbol{E}\mathbf{1}^+$ implies that $\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\boldsymbol{E}) \in v_{\boldsymbol{E}}^{\perp}$, whence

$$e = \langle \hat{\tau}(\boldsymbol{E}), v_{\boldsymbol{E}} \rangle = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{E}} \rangle = -\alpha - \gamma + \epsilon. \tag{33}$$

From (31), (32), and (33) together we find that

$$\alpha = \gamma = -\frac{e}{2}, \quad \epsilon = 0. \tag{34}$$

On the other hand, consider the relation $\mathbf{1}^+\mathbf{2}^- = \mathbf{2}^-\mathbf{1}^+$. It implies that

$$\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\mathbf{2}^-) \in v_{\mathbf{2}}^{\perp},$$

where we already know that $\hat{\tau}(\mathbf{2}^-) = \lambda_2 v_{\mathbf{2}}$. A direct computation gives

$$0 = \langle \lambda_1 v_{\mathbf{1}} + \alpha v_{\boldsymbol{A}} + \gamma v_{\boldsymbol{C}} + \epsilon v_{\boldsymbol{E}} - \lambda_2 v_{\mathbf{2}}, v_{\mathbf{2}} \rangle = -\lambda_1 - 2\sqrt{2}\gamma - \lambda_2.$$

If we show that $\lambda_1 = -\lambda_2$, we are done for $\hat{\tau}(\boldsymbol{E})$, since $\gamma = 0$ implies that $e = 0$ from (34).

To see this last point, recall that $\hat{\tau}(\mathbf{0}^+) = \lambda v_{\mathbf{0}}$ and $\hat{\tau}(\mathbf{3}^+) = -\lambda v_{\mathbf{3}}$ as proved in Lemma 6.11. Now, from the orthogonality relation $\mathbf{0}^+\mathbf{1}^- = \mathbf{1}^-\mathbf{0}^+$ we find that $\hat{\tau}(\mathbf{0}^+) - \hat{\tau}(\mathbf{1}^-) \in v_{\mathbf{1}}^{\perp}$. Using the preliminary remark at the beginning of the proof,

$$0 = \lambda\langle v_{\mathbf{0}}, v_{\mathbf{1}}\rangle - \lambda_1 \langle v_{\mathbf{1}}, v_{\mathbf{1}}\rangle = -\lambda - \lambda_1.$$

Thus $\lambda_1 = -\lambda$. Repeating the same argument to the relation $\mathbf{3}^+\mathbf{2}^- = \mathbf{2}^-\mathbf{3}^+$, one finds that $\lambda_2 = \lambda$, and therefore $\lambda_1 = -\lambda_2$.

The proof that $f = 0$ follows the same lines, applied to $\mathbf{4}^+$ in place of $\mathbf{1}^+$, with the letters $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{F}$, and in the final part to $\mathbf{5}^-$ in place of $\mathbf{2}^-$. ■

Having shown that $\hat{\tau}(\boldsymbol{X}) = 0$ for every $\boldsymbol{X}$, it remains to show that $\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-)$ has the form of (27). For $\boldsymbol{i} = \mathbf{0}, \mathbf{3}$, this is the content of Lemma 6.11. Following the same proof, one shows first that

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-)$$

for every $\boldsymbol{i}$ (it suffices to modify the proof by picking three letters $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ so that $v_{\boldsymbol{X}}$, $v_{\boldsymbol{Y}}$, and $v_{\boldsymbol{Z}}$ are orthogonal to $v_{\boldsymbol{i}}$). Then using the crossed relations $\boldsymbol{i}^+\boldsymbol{j}^- = \boldsymbol{j}^-\boldsymbol{i}^+$—it is easy to see that there are indeed enough of such relations—one mimics the second part of Lemma 6.11 and obtains that

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-) = (-1)^i \lambda v_{\boldsymbol{i}}.$$

This concludes the proof of Proposition 6.5, namely that $\dim H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = 1$.

6.4. Topology of the neighbourhood $\mathcal{U}$

We are ready to conclude the proof of our weak version of Theorem 1.1, namely Theorem 4.16, in the HP case. The proof of Theorem 1.1 will be completed in Section 7.

As a preliminary setup, recalling that $G_{\mathsf{HP}^4} \cong \mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$, one has a natural map

$$\mathcal{L}\colon X(\Gamma, G_{\mathsf{HP}^4}) \to X\big(\Gamma, \mathrm{O}(1,3)\big)$$

which associates to the conjugacy class of a representation $\rho\colon \Gamma \to G_{\mathsf{HP}^4}$ the conjugacy class of the linear part of ρ. Recalling Lemma 6.1, one has the identification

$$\mathcal{L}^{-1}\big([\varrho]\big) \cong H^1_{\varrho}(\Gamma, \mathbb{R}^{1,3}). \tag{35}$$

Observe that if $\varrho' = h \circ \varrho \circ h^{-1}$ for $h \in \mathrm{O}(1,3)$, then $H^1_{\varrho}(\Gamma, \mathbb{R}^{1,3})$ and $H^1_{\varrho'}(\Gamma, \mathbb{R}^{1,3})$ are isomorphic by means of the map $\tau \mapsto h \circ \tau$.

Proof of Theorem 4.16—HP case. The proof follows a similar strategy to the AdS (and hyperbolic) case, so we will split again the proof in several steps which are parallel to those given in Section 4.8. Most steps are much simpler here.

Step 1. Let us define the vertical component $\mathcal{V}$ in $X(\Gamma_{22}, G_{\mathsf{HP}^4})$ as $\mathcal{L}^{-1}([\varrho_0])$, namely, $\mathcal{V}$ consists of all the conjugacy classes of representations with linear part in $[\varrho_0]$. By (35), $\mathcal{V}$ is identified to $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$, hence is homeomorphic to a line by Proposition 6.5. By construction, $\mathcal{V}$ contains the holonomy of the HP orbifold structures we built in [26].

Step 2. The second component $\mathcal{H}$ is defined similarly to the AdS case. We define the map

$$\Psi\colon \mathrm{Hom}\left(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)\right) \to \mathrm{Hom}(\Gamma_{22}, G_{\mathsf{HP}^4})$$

sending a representation $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$ to the representation $\Psi_\eta\colon \Gamma_{22} \to \mathrm{O}(1,3) < \mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$ (hence with trivial translation part, which we omit) which sends each of the generators $\mathbf{0}^-, \dots, \mathbf{7}^-, \boldsymbol{A}, \dots, \boldsymbol{F}$ to the corresponding element of $\mathrm{O}(1,3)$, and each $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$ to $-\mathrm{id}$.

Again, it is straightforward to check that the induced map

$$\widehat{\Psi}\colon \bar{X}\left(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)\right) \to X(\Gamma_{22}, G_{\mathsf{HP}^4})$$

is well defined and injective.

The representation ρ_0 is clearly in the image of Ψ, since $\rho_0 = \Psi_{\eta_0}$, where η_0 is the holonomy representation of the complete hyperbolic orbifold structure of the cuboctahedron. As in the AdS case, $[\eta_0]$ has a neighbourhood $\mathcal{H}_0$ in $\bar{X}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ homeomorphic to $\mathbb{R}^{12}$ and on which $\widehat{\Psi}$ is a homeomorphism onto its image, and we define $\mathcal{H}$ to be the image of $\mathcal{H}_0$.

Step 3. Clearly, the intersection of $\mathcal{H}$ and $\mathcal{V}$ consists only of the point $[\rho_0]$, since any element in $\mathcal{H}$ has a trivial translation part (up to conjugacy).

Step 4. We now show that the point $[\rho_0] \in X(\Gamma_{22}, G_{\mathsf{HP}^4})$ has a neighbourhood $\mathcal{U}$ which is contained in the union of the two components $\mathcal{V}$ and $\mathcal{H}$.

Let ρ be a nearby representation, with linear part $\mathcal{L}\rho$ and translation part $\tau\colon \Gamma_{22} \to \mathbb{R}^{1,3}$. Observe that, since $-\mathrm{id}$ is an isolated point in the representations of $\mathbb{Z}/2\mathbb{Z}$ into $\mathrm{O}(1,3)$, for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$ we have $\mathcal{L}\rho(\boldsymbol{i}^+) = -\mathrm{id}$.

We claim that if two distinct generators which are sent by ρ_0 to $-\mathrm{id}$ (hence necessarily of the form $\boldsymbol{i}^+$ or $\boldsymbol{j}^+$) are sent by ρ to the same reflection, then all the generators $\mathbf{0}^+, \dots, \mathbf{7}^+$ are sent by ρ to the same reflection. In other words, if $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$ for some $\boldsymbol{i}^+ \neq \boldsymbol{j}^+$, then $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$ for all $\boldsymbol{i}^+, \boldsymbol{j}^+$.

Assuming the claim, the proof then follows by the following argument. We can assume (up to conjugation) that $\tau(\boldsymbol{i}^+) = 0$ for all $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$. By Proposition 5.13, if some of the collapsed cusp groups of ρ_0 is not deformed to a cusp group, then up to conjugation ρ has the property that $\rho(\boldsymbol{i}^+) = (-\mathrm{id}, 0)$ for all $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and therefore $[\rho] \in \mathcal{H}$. On the other hand, if all the collapsed cusp groups of ρ_0 are deformed in ρ to cusp groups, then the linear part of ρ is of the form $\mathcal{L}\rho = \Psi_\eta$ for a representation $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$

which sends all peripheral groups to (3-dimensional) cusp groups in $\mathbb{H}^3$, and therefore η is conjugate to η_0 in $\mathrm{Isom}(\mathbb{H}^3)$ by the Mostow–Prasad rigidity. Thus $[\mathcal{L}\rho] = [\mathcal{L}\rho_0]$, which means that $[\rho] \in \mathcal{V}$.

To prove the claim, suppose that $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$. We can assume that $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+) = 0$ by conjugation. Analogously to the same step in the AdS case, by symmetry (see [26, Lemma 7.6]) and Proposition 3.17, we can assume that the two generators are $\mathbf{0}^+$ and $\mathbf{1}^+$. Hence we have $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+) = (-\mathrm{id}, 0)$. We see from the relations involving $\mathbf{0}^+$ that $\rho(\mathbf{0}^+)$ commutes with $\rho(\mathbf{1}^-)$, $\rho(\mathbf{3}^-)$, and $\rho(\boldsymbol{A})$, which all have a reflection in $\mathbb{H}^3$ as linear part. By Lemma 5.8, $\rho(\mathbf{1}^-)$, $\rho(\mathbf{3}^-)$, and $\rho(\boldsymbol{A})$ have zero translation part. Additionally, we see from the relations involving $\mathbf{1}^+$ that $\rho(\mathbf{2}^-)$ has zero translation part. Now, from the relations involving $\mathbf{2}^+$, we get that $\rho(\mathbf{2}^+)$ commutes with $\rho(\mathbf{1}^-)$, $\rho(\mathbf{2}^-)$, $\rho(\mathbf{3}^-)$ and $\rho(\boldsymbol{A})$. Observe that the linear part of $\rho(\mathbf{2}^+)$ is necessarily $-\mathrm{id}$, in a neighbourhood of ρ_0. Hence, by applying Lemma 5.8 again, the translation part of $\rho(\mathbf{2}^+)$ is in the intersection of the hyperplanes of $\mathbb{R}^{1,3}$ fixed by $\rho(\mathbf{1}^-)$, $\rho(\mathbf{2}^-)$, $\rho(\mathbf{3}^-)$, and $\rho(\boldsymbol{A})$. The hyperplanes fixed by $\rho_0(\mathbf{1}^-)$, $\rho_0(\mathbf{2}^-)$, $\rho_0(\mathbf{3}^-)$, and $\rho_0(\boldsymbol{A})$ are $v_{\mathbf{1}}^{\perp}$, $v_{\mathbf{2}}^{\perp}$, $v_{\mathbf{3}}^{\perp}$, and $v_{\boldsymbol{A}}^{\perp}$, where the vectors $v_{\mathbf{1}}$, $v_{\mathbf{2}}$, $v_{\mathbf{3}}$, and $v_{\boldsymbol{A}}$ are listed in Table 4 and are linearly independent. Hence they remain linearly independent for ρ, that is a deformation of ρ_0 in a small neighbourhood. This means that the translation part of $\rho(\mathbf{2}^+)$ is zero, since the only solution of the linear system which imposes the orthogonality to these four linearly independent vectors is the trivial solution. This shows that $\rho(\mathbf{2}^+) = (-\mathrm{id}, 0)$, which therefore coincides with $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+)$.

Similarly to the AdS case, one argues similarly for $\mathbf{3}^+$ and then for all the other generators, to show that $\rho(\boldsymbol{i}^+) = (-\mathrm{id}, 0)$ for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and this concludes the claim.

Step 5. In summary, we showed that $[\rho_0]$ has a neighbourhood $\mathcal{U}$ in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ which only consists of points of $\mathcal{H}$ and $\mathcal{V}$. Additionally, one can repeat the same reasoning in the first part of the previous step to show that for any other $[\rho_0']$ in $\mathcal{V}$ (hence having the same linear part as ρ_0 and non-vanishing translation part) a neighbourhood of $[\rho_0']$ is contained in $\mathcal{V}$, as a consequence of the HP cusp rigidity of Proposition 5.13 (the non-collapsed case). Hence, by taking the union of all these neighbourhoods, one finds a $\mathcal{U}$ containing $[\rho_0]$ such that $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$.

Step 6. For the last statement, it is evident that conjugation by $\mathbb{Z}/2\mathbb{Z} \cong G_{\mathsf{HP}^4}/G^+_{\mathsf{HP}^4}$ acts by switching sign to the x_{13}-coordinate, since conjugation by $(-\mathrm{id}, 0)$, whose class generates $\mathbb{Z}/2\mathbb{Z}$, acts on $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ by changing the sign. This concludes the proof. ■

7. Smoothness and transversality

In this section, we complete the proof of Theorem 1.1, showing the smoothness and transversality of the two components $\mathcal{V}$ and $\mathcal{H}$ of the neighbourhood $\mathcal{U}$ of $[\rho_0]$ in $X(\Gamma_{22}, G)$. To that purpose, we first need to study the cohomology group $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$, complementing and using the results of Section 6.3. Then we conclude the proof by an application of the implicit function theorem.

7.1. Preliminaries

Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or G_{HP^4}, and let $\mathfrak{g}$ be its Lie algebra. We shall apply the definition of first cohomology group given in Section 6.1 to the representation

$$\mathrm{Ad}\ \rho_0 \colon \Gamma_{22} \to \mathrm{GL}(\mathfrak{g}),$$

which is the composition of our $\rho_0 \colon \Gamma_{22} \to G$ and the adjoint representation

$$\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g}).$$

In general, for a finitely presented group Γ with a given presentation with s generators and r relations, the set $\mathrm{Hom}(\Gamma, G)$ is identified to a subset of G^s defined by the vanishing of r conditions given by the relations. If we encode these conditions by $F \colon G^s \to G^r$, so as to identify $\mathrm{Hom}(\Gamma, G)$ with $F^{-1}(0)$, then it is known from [31] that $Z^1_{\mathrm{Ad}\ \rho}(\Gamma, \mathfrak{g})$ is naturally identified with the kernel of dF at ρ. The isomorphism between these two vector spaces essentially associates to a germ of paths at ρ represented by $t \mapsto \rho_t$ the cocycle τ defined by

$$\tau(\gamma) = \frac{d}{dt}\bigg|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1}, \tag{36}$$

which is therefore interpreted as an infinitesimal deformation of ρ.

Remark 7.1. In general, the Zariski tangent space at ρ of the real variety associated to $\mathrm{Hom}(\Gamma, G)$ is only a subspace of $Z^1_{\mathrm{Ad}\ \rho}(\Gamma, \mathfrak{g})$. We will show that in our situation for Γ_{22} they coincide at the point ρ_0.

Let us now look at the coboundaries. It was observed in [31] (see also [14, Lemma 2.2]) that the subspace of $\mathrm{Ker}(dF)$ corresponding to the tangent space to the G^+-orbit of ρ identifies to $B^1_{\mathrm{Ad}\ \rho}(\Gamma, \mathfrak{g})$ under the correspondence (36). Indeed, by a straightforward computation, the differential of the orbit map $G \to \mathrm{Hom}(\Gamma, G)$ defined by $g \mapsto g\rho g^{-1}$ maps an element $X \in \mathfrak{g}$ to the coboundary $\tau(\gamma) = X - \mathrm{Ad}\,\rho(\gamma)X$. Observe that in our setting, by Lemma 4.2, the action of G is not free at ρ; but the action of the identity component of G, namely G^+, is indeed free. As we will see (Lemma 7.4), this implies by a standard argument that $B^1_{\mathrm{Ad}\ \rho_0}(\Gamma_{22}, \mathfrak{g}) \cong \mathfrak{g}$.

We conclude by stating a smoothness criterion used by Weil [31, Lemma 1], essentially consisting of an application of the implicit function theorem. We refer to [15, Section 2.2] for more details.

Let $\mathcal{C}$ be an algebraic subset of $\mathrm{Hom}(\Gamma, G)$ containing ρ, say obtained by adding k extra polynomial equations. We identify $\mathcal{C}$ with $\tilde{F}^{-1}(0)$, for some $\tilde{F} \colon G^s \to G^{r+k}$ compatible with F. Suppose that a neighbourhood of ρ in $\mathcal{C}$ is a smooth submanifold of G^s of the same dimension of the kernel K of $d\tilde{F}$ at ρ. Then the following hold: at the point ρ, the real variety associated to $\mathcal{C}$ is smooth, its Zariski tangent space is isomorphic to K (and not to a proper subspace, compare with Remark 7.1) and is naturally identified with the tangent space of $\mathcal{C}$ as a submanifold.

7.2. The first cohomology group

Let us now go back to the representation $\rho_0\colon \Gamma_{22} \to G_0 = \mathrm{Stab}_G(\mathbb{H}^3)$. In this subsection, we analyse the vector space $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$.

There is a well-known splitting

$$\mathfrak{g} \cong \mathfrak{isom}(\mathbb{H}^{n-1}) \oplus \mathbb{R}^n. \tag{37}$$

When $G = \mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, the splitting is given by writing an element $\mathfrak{a}$ of $\mathfrak{g}$ as

$$\mathfrak{a} = \left(\begin{array}{ccc|c} & & & \vdots \\ & \mathfrak{a}_0 & & \mp w \\ & & & \vdots \\ \hline \cdots & w^T J & \cdots & 0 \end{array}\right), \tag{38}$$

where $J = \mathrm{diag}(-1, 1, \dots, 1)$, for $\mathfrak{a}_0 \in \mathfrak{so}(1, n-1)$ and $w \in \mathbb{R}^n$. When $G = G_{\mathsf{HP}^n}$, the splitting (37) is even simpler to obtain, by using the isomorphism

$$G_{\mathsf{HP}^n} \cong \mathrm{O}(1, n-1) \ltimes \mathbb{R}^{1,n-1}.$$

The splitting (37) is equivariant with respect to the three natural actions of G_0: the adjoint action on $\mathfrak{g}$, the adjoint action on $\mathfrak{isom}(\mathbb{H}^3)$ by means of the isomorphism $G_0 \cong \mathrm{Isom}(\mathbb{H}^3) \times (\mathbb{Z}/2\mathbb{Z})$, and the action on $\mathbb{R}^{1,3}$ by means of the isomorphism $G_0 \cong \mathrm{O}(1,3)$ of (28). We thus have natural decompositions:

$$Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) = Z^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \tag{39}$$

and

$$B^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) = B^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}), \tag{40}$$

hence

$$H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) = H^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}). \tag{41}$$

Recall that ρ_0 coincides with the composition of the representation $\varrho_0\colon \Gamma_{22} \to G_0$ defined in (26) with the inclusion $G_0 \to G$.

Let us look at the first factor of the decomposition (41) of $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$.

Proposition 7.2. *The vector space* $H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$ *has dimension* 12.

Proof. First, we claim that the vector space $H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$ is isomorphic to $H^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3))$, where Γ_{co} is the reflection group of the right-angled cuboctahedron and ι is its inclusion into $\mathrm{Isom}(\mathbb{H}^3)$. The latter has dimension 12, since the character variety of Γ_{co} in $\mathrm{Isom}(\mathbb{H}^3)$ is smooth and 12-dimensional near $[\iota]$. We have already mentioned (in Section 4.8, Step 2) that this last fact is true by "reflective hyperbolic Dehn filling".

To show the claim, we will show that the restriction morphism

$$H^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \to H^1_{\mathrm{Ad}\,\iota}\big(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3)\big)$$

is invertible by explicitly constructing an inverse. Define a map

$$\psi\colon Z^1_{\mathrm{Ad}\,\iota}\big(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3)\big) \to Z^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big),$$

identifying Γ_{co}, with the subgroup of Γ_{22} generated by $\mathbf{0}^-, \ldots \mathbf{7}^-, \boldsymbol{A}, \ldots, \boldsymbol{F}$, by

$$\psi(\tau)(\boldsymbol{X}) = \tau(\boldsymbol{X}), \quad \psi(\tau)(\boldsymbol{i}^-) = \tau(\boldsymbol{i}^-), \quad \psi(\tau)(\boldsymbol{i}^+) = 0,$$

for every $\tau \in Z^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3))$. We recall from Definition 6.3 that ϱ_0 satisfies

$$\varrho_0(\boldsymbol{X}) = \iota(\boldsymbol{X}), \quad \varrho_0(\boldsymbol{i}^-) = \iota(\boldsymbol{i}^-), \quad \varrho_0(\boldsymbol{i}^+) = -\mathrm{id}.$$

It follows that

$$\mathrm{Ad}\,\varrho_0(\boldsymbol{i}^+) = \mathrm{id}. \tag{42}$$

Let us now show that the map ψ is well defined and induces a well-defined map in cohomology. First, using Lemma 6.2 and (42), one checks easily that

$$\psi(\tau)(\boldsymbol{i}^+\boldsymbol{X}) = \psi(\tau)(\boldsymbol{X}\boldsymbol{i}^+) \quad \text{and} \quad \psi(\tau)(\boldsymbol{i}^+\boldsymbol{j}^-) = \psi(\tau)(\boldsymbol{j}^-\boldsymbol{i}^+).$$

Hence ψ maps cocycles to cocycles. Moreover, it maps coboundaries to coboundaries, for if $\tau(\boldsymbol{s}) = \mathrm{Ad}\,\iota(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for some $\mathfrak{a} \in \mathfrak{isom}(\mathbb{H}^3)$, then of course $\psi(\tau)(\boldsymbol{s}) = \mathrm{Ad}\,\varrho_0(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for $\boldsymbol{s} = \boldsymbol{i}^-$ or $\boldsymbol{X}$. As $\mathrm{Ad}\,\varrho_0(\boldsymbol{i}^+) = \mathrm{id}$ by (42), the identity holds trivially also for $\boldsymbol{s} = \boldsymbol{i}^+$.

Thus ψ induces a map

$$\psi\colon H^1_{\mathrm{Ad}\,\iota}\big(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3)\big) \to H^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big).$$

We claim that it is a vector space isomorphism. To see that it is injective, suppose that $\psi(\tau)$ is a coboundary, namely $\psi(\tau)(\boldsymbol{s}) = \mathrm{Ad}\,\varrho_0(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for all generators $\boldsymbol{s}$. From (42), we have $\psi(\tau)(\boldsymbol{i}^+) = 0$, and from the definition of ϱ_0, $\tau(\boldsymbol{s}) = \mathrm{Ad}\,\iota(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for $\boldsymbol{s} = \boldsymbol{i}^-$ or $\boldsymbol{X}$. Hence τ is a coboundary, which concludes injectivity. It remains to show that it is surjective. To see this, given any cocycle $\sigma \in Z^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$, Lemma 6.2 implies that $\sigma(\boldsymbol{i}^+)$ is in the kernel of $\mathrm{id} + \mathrm{Ad}\,\varrho(\boldsymbol{i}^+)$, which in fact equals 2id by (42). Hence $\sigma(\boldsymbol{i}^+) = 0$ and σ is in the image of ψ. This concludes the proof. ∎

The second factor of the decomposition (41) of $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$ has already been computed in Proposition 6.5. We have, in particular, the following corollary.

Corollary 7.3. *The vector space $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$ has dimension* 13.

We deduce the dimensions of the spaces of cocycles and coboundaries from the following simple lemma, which will also be used in the next section.

Lemma 7.4. *The orbit map $G^+ \to \mathrm{Hom}(\Gamma_{22}, G)$ is an embedding.*

Proof. The orbit map is injective as a consequence that the G^+-action is free (Lemma 4.2). To see that its differential at any point is injective, suppose by contradiction that a non-zero vector is in the kernel of the differential. Acting by left multiplication on G^+, one then finds a nonvanishing vector field on G^+ which is, at any point, in the kernel of the differential. Hence the orbit map would be constant on any integral path of this vector field, thus contradicting injectivity. ∎

Corollary 7.5. *The vector spaces* $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22},\mathfrak{g})$ *and* $B^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22},\mathfrak{g})$ *have dimension* 23 *and* 10*, respectively.*

Proof. By Lemma 7.4, $B^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22},\mathfrak{g})$, which is the tangent space to the orbit, is isomorphic to $\mathfrak{g}$ and thus has dimension 10. Combining this with Corollary 7.3, we conclude that $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22},\mathfrak{g})$ has dimension 10. ∎

Remark 7.6. One could also check directly that $\dim B^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22},\mathfrak{g})=10$: from Remark 6.9, the second factor in the decomposition (40) has dimension four, and by a similar argument one can prove that the first factor has dimension six.

7.3. Conclusion of the proofs

We can now prove the main result of the section.

Recall that G is $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or G_{HP^4}, and $\mathfrak{g}$ is its Lie algebra. We denote as usual by $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{H}}\subset \mathrm{Hom}(\Gamma_{22},G)$ the preimages of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{H}\subset X(\Gamma_{22},G)$, respectively. All these sets are defined and studied in the proof of Theorem 4.16 in Sections 4.8 and 6.4. Recall that $\mathcal{U}=\mathcal{V}\cup\mathcal{H}$ and $\mathcal{V}\cap\mathcal{H}=\{[\rho_0]\}$.

For brevity, in the following, given a real affine algebraic set $\mathcal{S}$, by "Zariski tangent space to" (resp. "component of") $\mathcal{S}$ we refer to the Zariski tangent space to (resp. a component of) the real variety associated to $\mathcal{S}$.

Theorem 7.7. *The sets* $\widetilde{\mathcal{V}}$ *and* $\widetilde{\mathcal{H}}$ *are smooth components of* $\widetilde{\mathcal{U}}$*, of dimension* 11 *and* 22*, respectively. Moreover,* $\widetilde{\mathcal{V}}\cap\widetilde{\mathcal{H}}$ *is the* G*-orbit of* ρ_0*, and the Zariski tangent spaces of* $\widetilde{\mathcal{V}}$ *and* $\widetilde{\mathcal{H}}$ *at* ρ_0 *intersect transversely in the Zariski tangent space of* $\mathrm{Hom}(\Gamma_{22},G)$ *at* ρ_0.

Proof. We first show that $\widetilde{\mathcal{V}}$ is a smooth component of $\widetilde{\mathcal{U}}$ by applying the smoothness criterion stated at the end of Section 7.1. Note that $\widetilde{\mathcal{V}}$ is the intersection of $\widetilde{\mathcal{U}}$ with the algebraic subset $\mathrm{Hom}_0(\Gamma_{22},G)$ of $\mathrm{Hom}(\Gamma_{22},G)$ (see Definition 4.8 and the discussion below). In particular, $\widetilde{\mathcal{V}}$ is a neighbourhood of ρ_0 in $\mathrm{Hom}_0(\Gamma_{22},G)$.

Moreover, $\widetilde{\mathcal{V}}$ is a smooth 11-dimensional manifold. This is true in the hyperbolic or AdS case by Proposition 4.11, and in the HP case since there $\widetilde{\mathcal{V}}$ is the total space of a smooth vector bundle whose fibre is $Z^1_{\varrho_0}(\Gamma_{22},\mathbb{R}^{1,3})$ and whose base is the $\mathrm{Isom}(\mathbb{H}^3)$-orbit of ϱ_0 (hence a rank-5 bundle over a 6-manifold). Recall indeed that $Z^1_{\varrho_0}(\Gamma_{22},\mathbb{R}^{1,3})$ has dimension five by Proposition 6.5 and Remark 6.9.

The tangent space $T_{\rho_0}\widetilde{\mathcal{V}}$ to the smooth manifold $\widetilde{\mathcal{V}}$ is contained in

$$B^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22},\mathfrak{isom}(\mathbb{H}^3))\oplus Z^1_{\varrho_0}(\Gamma_{22},\mathbb{R}^{1,3})$$

under the identification (36) and the splitting (39). Indeed, in the hyperbolic and AdS case, by a direct computation one can see that the derivative $\frac{d}{dt}|_{t=0}\rho_t \circ \rho_0^{-1}$ is in the second factor in the decomposition (39), and gives a nonzero cocycle of the form (27) in $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ whose class generates $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ (see (30)). In the HP case, the HP representations with linear part ϱ_0 are themselves identified to the vector space $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$, and their derivatives are in $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ itself, seen as the second factor of (39).

As a consequence of Proposition 6.5 and Corollary 7.5,

$$B^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$$

has dimension 11. Since $T_{\rho_0}\widetilde{\mathcal{V}}$ and $B^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)) \oplus Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ have the same dimension, they are equal, and we conclude that $\widetilde{\mathcal{V}}$ is a smooth component of $\widetilde{\mathcal{U}}$ by the smoothness criterion.

Let us now look at $\widetilde{\mathcal{H}}$. Note that it is the intersection of $\widetilde{\mathcal{U}}$ with an algebraic subset of $\mathrm{Hom}(\Gamma_{22}, G)$. Combining our study of $\mathcal{H}$ in Sections 4.8 and 6.4 with Lemma 7.4, we get that $\widetilde{\mathcal{H}}$ is a smooth 22-dimensional manifold. Moreover, $T_{\rho_0}\widetilde{\mathcal{H}}$ is contained in $Z^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)) \oplus B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ under (36) and (39).

The proof of Proposition 7.2 shows that the first factor is indeed isomorphic to $Z^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3))$. The latter is the space of infinitesimal deformations of the right-angled cuboctahedron, and has dimension $18 = \sharp\,\mathrm{cusps} + \dim(\mathrm{Isom}(\mathbb{H}^3))$ for general and well-known reasons around 3-dimensional hyperbolic Dehn filling (namely, roughly speaking, "half" of the infinitesimal deformations of the peripheral subgroups extend to the whole group; see the discussion in [15, Section 5]). One can even compute this number directly as indicated in Section 7.1 by means of Table 4.

Moreover, since the second factor $B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has dimension four by Remark 6.9 (or Lemma 7.4), we conclude as similarly done for $\widetilde{\mathcal{V}}$ that $\widetilde{\mathcal{H}}$ is a smooth 22-dimensional component of $\widetilde{\mathcal{U}}$.

By the previous considerations, the integrable vectors arising from $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ generate $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$, and the Zariski tangent space at ρ_0 to $\mathrm{Hom}(\Gamma_{22}, G)$ is the whole $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$ (recall Remark 7.1). Moreover, $T_{\rho_0}\widetilde{\mathcal{V}}$ and $T_{\rho_0}\widetilde{\mathcal{H}}$, which are identified to the corresponding Zariski tangent spaces, are transverse in $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$. Finally, $\widetilde{\mathcal{V}} \cap \widetilde{\mathcal{H}}$ is the G-orbit of ρ_0 thanks to Theorem 4.16. ■

The proof of Theorem 1.1 is complete combining Theorems 4.16 and 7.7.

Remark 7.8. We have shown in the proof of Theorem 7.7 that the Zariski tangent space at ρ_0 to $\mathrm{Hom}(\Gamma_{22}, G)$ is isomorphic to the whole $Z^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$. Moreover, integrable vectors to $\widetilde{\mathcal{U}}$ at ρ_0 correspond precisely to those in

$$B^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \cup Z^1_{\mathrm{Ad}\,\varrho_0}\big(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)\big) \oplus B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}),$$

since $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ are the only components of $\widetilde{\mathcal{U}}$ by Theorem 4.16.

Remark 7.9. Without entering into details on the semialgebraic structure of $X(\Gamma_{22}, G)$ when G is reductive, our analysis implies that, roughly speaking, $\mathcal{V}$ and $\mathcal{H}$ can be thought as "smooth and transverse components" of $\mathcal{U}$ in a reasonable and satisfactory sense. See, in particular, Propositions 6.5 and 7.2 and the splitting (41), and compare with the topological description of $\mathcal{U}$ in Theorem 4.16, the algebraic description of $\widetilde{\mathcal{U}}$ in Theorem 7.7, and the differential geometric one in Remark 7.8.

Acknowledgements. We are grateful to Francesco Bonsante and Bruno Martelli for interesting discussions, useful advices, and encouragement. We also thank Jeffrey Danciger and Gye-Seon Lee for interest in this work and related discussions, and Clément Guérin and Andrea Maffei for ad-hoc explanations about character varieties. We are grateful to an anonymous referee whose suggestions highly improved the exposition and results of this paper.

We thank the mathematics departments of Pavia, Luxembourg, and Neuchâtel, for the warm hospitality during the respective visits while part of this work was done. The stage of this collaboration was set during the workshop "Moduli spaces", held in Ventotene in September 2017: we are grateful to the organisers for this opportunity.

Funding. The authors were partially supported by FIRB 2010 project "Low dimensional geometry and topology" (RBFR10GHHH003), by the 2020 Germaine de Staël project "Deformations of geometric structures on higher-dimensional manifolds", and are members of the national research group GNSAGA. The first author was supported by the Mathematics Department of the University of Pisa (research fellowship "Deformazioni di strutture iperboliche in dimensione quattro"), and by the Swiss National Science Foundation (project no. PP00P2-170560).

References

[1] N. A'Campo and A. Papadopoulos, Transitional geometry. In *Sophus Lie and Felix Klein: the Erlangen program and its impact in mathematics and physics*, pp. 217–235, IRMA Lect. Math. Theor. Phys. 23, Eur. Math. Soc., Zürich, 2015 MR 3362217

[2] T. Barbot and F. Fillastre, Quasi-Fuchsian co-Minkowski manifolds. In *In the tradition of Thurston—geometry and topology*, pp. 645–703, Springer, Cham, 2020 Zbl 07444894 MR 4264589

[3] T. Barbot et al., Some open questions on anti-de Sitter geometry. 2012, arXiv:1205.6103

[4] M. Boileau, B. Leeb, and J. Porti, Geometrization of 3-dimensional orbifolds. *Ann. of Math. (2)* **162** (2005), no. 1, 195–290 Zbl 1087.57009 MR 2178962

[5] S. Choi, G.-S. Lee, and L. Marquis, Deformations of convex real projective manifolds and orbifolds. In *Handbook of group actions. Vol. III*, pp. 263–310, Adv. Lect. Math. (ALM) 40, Int. Press, Somerville, MA, 2018 Zbl 1410.57012 MR 3888622

[6] D. Cooper, J. Danciger, and A. Wienhard, Limits of geometries. *Trans. Amer. Math. Soc.* **370** (2018), no. 9, 6585–6627 Zbl 1395.57022 MR 3814342

[7] D. Cooper, C. D. Hodgson, and S. P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds. With a postface by Sadayoshi Kojima*. MSJ Memoirs 5, Mathematical Society of Japan, Tokyo, 2000 Zbl 0955.57014 MR 1778789

[8] J. Danciger, *Geometric transitions: from hyperbolic to AdS geometry*. Ph.D. thesis, Stanford University, 2011

[9] J. Danciger, A geometric transition from hyperbolic to anti-de Sitter geometry. *Geom. Topol.* **17** (2013), no. 5, 3077–3134 Zbl 1287.57020 MR 3190306

[10] J. Danciger, Ideal triangulations and geometric transitions. *J. Topol.* **7** (2014), no. 4, 1118–1154 Zbl 1308.57006 MR 3286899

[11] F. Fillastre and A. Seppi, Spherical, hyperbolic, and other projective geometries: convexity, duality, transitions. In *Eighteen essays in non-Euclidean geometry*, pp. 321–409, IRMA Lect. Math. Theor. Phys. 29, Eur. Math. Soc., Zürich, 2019 Zbl 1417.51015 MR 3965127

[12] M. Heusener, J. Porti, and E. Suárez, Regenerating singular hyperbolic structures from Sol. *J. Differential Geom.* **59** (2001), no. 3, 439–478 Zbl 1042.57008 MR 1916952

[13] C. D. Hodgson, *Degeneration and regeneration of hyperbolic structures on three-manifolds (foliations, Dehn surgery)*. Ph.D. thesis, Princeton University, 1986

[14] D. Johnson and J. J. Millson, Deformation spaces associated to compact hyperbolic manifolds. In *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, pp. 48–106, Progr. Math. 67, Birkhäuser, Boston, MA, 1987 Zbl 0664.53023 MR 900823

[15] S. P. Kerckhoff and P. A. Storm, From the hyperbolic 24-cell to the cuboctahedron. *Geom. Topol.* **14** (2010), no. 3, 1383–1477 Zbl 1213.57023 MR 2653730

[16] K. Kozai, *Singular hyperbolic structures on pseudo-Anosov mapping tori*. Ph.D. thesis, Stanford University, 2013

[17] K. Kozai, Hyperbolic structures from Sol on pseudo-Anosov mapping tori. *Geom. Topol.* **20** (2016), no. 1, 437–468 Zbl 1339.57025 MR 3470718

[18] M. T. Lozano and J. M. Montesinos-Amilibia, Geometric conemanifold structures on $\mathbb{T}_{p/q}$, the result of p/q surgery in the left-handed trefoil knot $\mathbb{T}$. *J. Knot Theory Ramifications* **24** (2015), no. 12, 1550057, 38 Zbl 1333.57031 MR 3415239

[19] M. T. Lozano and J. M. Montesinos-Amilibia, On the degeneration of some 3-manifold geometries via unit groups of quaternion algebras. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **109** (2015), no. 2, 669–715 Zbl 1325.53069 MR 3383442

[20] B. Martelli and S. Riolo, Hyperbolic Dehn filling in dimension four. *Geom. Topol.* **22** (2018), no. 3, 1647–1716 Zbl 1422.57045 MR 3780443

[21] J. Porti, Regenerating hyperbolic and spherical cone structures from Euclidean ones. *Topology* **37** (1998), no. 2, 365–392 Zbl 0897.58042 MR 1489209

[22] J. Porti, Regenerating hyperbolic cone structures from Nil. *Geom. Topol.* **6** (2002), 815–852 Zbl 1032.57015 MR 1943382

[23] J. Porti, Regenerating hyperbolic cone 3-manifolds from dimension 2. *Ann. Inst. Fourier (Grenoble)* **63** (2013), no. 5, 1971–2015 Zbl 1293.57012 MR 3186514

[24] J. Porti and H. Weiss, Deforming Euclidean cone 3-manifolds. *Geom. Topol.* **11** (2007), 1507–1538 Zbl 1159.57007 MR 2326950

[25] R. W. Richardson and P. J. Slodowy, Minimum vectors for real reductive algebraic groups. *J. London Math. Soc. (2)* **42** (1990), no. 3, 409–429 Zbl 0675.14020 MR 1087217

[26] S. Riolo and A. Seppi, Geometric transition from hyperbolic to anti-de Sitter structures in dimension four. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **23** (2022), no. 1, 115–176 Zbl 07524672 MR 4407187

[27] A. Seppi, Examples of geometric transition in low dimensions. In *Séminaire de théorie spectrale et géométrie (Grenoble)*, pp. 163–196, 35, 2017–2019
[28] C. Series, Limits of quasi-Fuchsian groups with small bending. *Duke Math. J.* **128** (2005), no. 2, 285–329 Zbl 1081.30038 MR 2140265
[29] W. P. Thurston, The geometry and topology of three-manifolds. 1980, lecture notes (electronic version 1.1), http://library.msri.org/nonmsri/gt3m
[30] S. J. Trettel, *Families of geometries, real algebras, and transitions*. Ph.D. thesis, University of California, Santa Barbara, 2019
[31] A. Weil, Remarks on the cohomology of groups. *Ann. of Math. (2)* **80** (1964), 149–157 Zbl 0192.12802 MR 169956

Received 29 June 2020.

Stefano Riolo
Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy; stefano.riolo@unibo.it

Andrea Seppi
Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble Cedex 9, France; andrea.seppi@univ-grenoble-alpes.fr