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Anosov representations with Lipschitz limit set

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We study Anosov representations whose limit set has intermediate regularity, namely is a Lipschitz submanifold of a flag manifold. We introduce an explicit linear functional, the unstable Jacobian, whose orbit growth rate is integral on this class of representations. We prove that many interesting higher-rank representations, including Θ–positive representations, belong to this class, and establish several applications to rigidity results on the orbit growth rate in the symmetric space.

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1 Introduction

Let $\Gamma \subset \mathsf{PGL}_d(\mathbb{R})$ be a discrete subgroup. Following Guivarc'h, Benoist [4] has shown that if Γ contains a proximal element and acts irreducibly on $\mathbb{R}^d$ then its action on projective space $\mathbb{P}(\mathbb{R}^d)$ has a smallest closed invariant set. This is usually called *Benoist's limit set* or simply *the limit set* of Γ on $\mathbb{P}(\mathbb{R}^d)$ and denoted by L_Γ.

In contrast with the negatively curved situation, the limit set of a subgroup Γ whose Zariski closure has rank ≥ 2 need not be a fractal object. Examples of infinite covolume Zariski-dense groups whose limit set is a proper C^1–submanifold arise in the study of strictly convex divisible sets (see Benoist [5]) and of Hitchin representations (see Labourie [35]). Lately, more examples of subgroups with this property were found by Pozzetti, Sambarino and Wienhard [39] and Zhang and Zimmer [48].

Intermediate phenomena also occur. For example, the limit set of the direct sum $(\rho,\eta)\colon \pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R})$ of the holonomies of two hyperbolizations of a closed topological surface S is a Lipschitz circle that is never a C^1–submanifold of the product $\mathbb{S}^1 \times \mathbb{S}^1 = \mathsf{G}/\mathsf{B}$ unless the two hyperbolizations are conjugated. Lipschitz limit sets more generally occur for maximal representations (see Burger, Iozzi and Wienhard [12]), Quasi-Fuchsian AdS representations (see Barbot and Mérigot [3]), and $\mathbb{H}^{p,q}$–convex–cocompact representations; see Danciger, Guéritaud and Kassel [16].

We provide the first systematic investigation of this intermediate phenomenon — its main object are discrete groups whose limit set is a Lipschitz manifold. We will restrict our investigation to the class of Anosov subgroups, a robust and rich class of strongly undistorted subgroups of semisimple Lie groups; see Section 2.2 for the precise definition.

For discrete subgroups Γ of $\mathsf{SO}(1,n)$, Sullivan [46] established a beautiful relation between a geometric invariant of the limit set L_Γ, its Hausdorff dimension, and a dynamical invariant for the action of Γ on the symmetric space $\mathbb{H}^n$, the orbit growth rate. This was further used by Bowen [7] to prove a strong rigidity result: for fundamental groups of surfaces acting on $\mathbb{H}^3$, the Hausdorff dimension of the limit set is minimal if and only if the limit set is C^1 and Γ preserves a totally geodesic copy of $\mathbb{H}^2$ on which it acts cocompactly. When G has higher rank, the situation is more complicated as one can additionally consider orbit growth rates with respect to different linear functionals φ (as in, for example, Quint [41]). It is a challenging problem to understand which functionals φ have orbit growth rate that carries geometric information on the group Γ or on its limit set L_Γ.

Our main contribution is to single out an explicit linear functional, the *unstable Jacobian*, whose critical exponent is integral on Anosov subgroups whose limit set is a Lipschitz submanifold. In order to prove this we import ideas from nonconformal dynamics, such as the study of the affinity exponent, to the setting of Anosov groups, and use the Anosov property, together with ideas from geometric group theory, to establish a strengthening of the theory of Patterson–Sullivan densities developed by Quint; these two results are of independent interest. We then showcase the strength of our main result by applying it to several well-studied classes of representations: maximal representations, $\mathbb{H}^{p,q}$–convex–cocompact subgroups and Θ–positive representations.

The unstable Jacobian and the affinity exponent

We now introduce some notation useful to explain more precisely our results. We denote by

$$\mathsf{E} = \left\{ \underline{a} = (a_1, \dots, a_d) \in \mathbb{R}^d \;\middle|\; \sum_i a_i = 0 \right\}$$

the Cartan subspace of the Lie group $\mathsf{PGL}_d(\mathbb{R})$, by

$$\mathsf{a}_i(\underline{a}) = a_i - a_{i+1}$$

the i^{th} simple root and by $\mathsf{E}^+ \subset \mathsf{E}$ the Weyl chamber whose associated set of simple roots is $\Pi = \{\mathsf{a}_i : i \in [\![1, d-1]\!]\}$. Let $a: \mathsf{PGL}_d(\mathbb{R}) \to \mathsf{E}^+$ be the *Cartan projection* with respect to the choice of a scalar product τ. Concretely, $a(g) = (\log \sigma_1(g), \dots, \log \sigma_d(g))$, where the $\sigma_i(g)$ denote the *singular values* of the matrix g, the square roots of the eigenvalues of the matrix gg^*, where g^* is the adjoint operator of g with respect to τ.

Given a discrete subgroup $\Gamma < \mathsf{PGL}_d(\mathbb{R})$, the *critical exponent* of a linear form $\varphi \in \mathsf{E}^*$, denoted by $h_\Gamma(\varphi)$, is defined as

$$h_\Gamma(\varphi) := \lim_{T \to \infty} \frac{\log \#\{\gamma \in \Gamma \mid \varphi(a(\gamma)) < T\}}{T}.$$

We introduce the p^{th} *unstable Jacobian* $\mathcal{J}_p^u \in \mathsf{E}^*$, defined by

$$\mathcal{J}_p^u = (p+1)\omega_{\mathsf{a}_1} - \omega_{\mathsf{a}_{p+1}},$$

where $\omega_{\mathsf{a}_p}(\underline{a}) = \sum_1^p a_i$ is the fundamental weight relative to the p^{th} simple root a_p. Our main result is:

Theorem A *Let $\Gamma < \mathsf{PSL}_d(\mathbb{R})$ be a strongly irreducible, projective Anosov subgroup whose limit set $\mathsf{L}_\Gamma < \mathbb{P}(\mathbb{R}^d)$ is a Lipschitz submanifold of dimension p. Then*

$$h_\Gamma(\mathcal{J}_p^u) = 1.$$

If $p = 1$ the same holds, replacing strong irreducibility with weak irreducibility.[1]

[1] We say that a subgroup $\Gamma < \mathsf{PSL}_d(\mathbb{R})$ is *weakly irreducible* if the vector space $\text{span}(\mathsf{L}_\Gamma)$ is $\mathbb{R}^d$.

A similar result was proven, in the context of fundamental groups of compact strictly convex projective manifolds, by Potrie and Sambarino [38, Theorem B]; our approach is entirely different and, since we require less regularity, its scope of application is considerably broader. Note that, up to postcomposing with a suitable linear representation, any Anosov representation can be turned into a projective Anosov representation.

We prove the two inequalities in Theorem A as corollaries of two different results that are applicable in more general settings. We focus first on the lower bound on the critical exponent (Corollary 1.1) that follows from a general result on the Hausdorff dimension of limit sets (of projective Anosov representations).

An important step in the proof is the study, in the context of Anosov representations, of the *affinity exponent*, a key notion from nonconformal dynamics that first appeared in Kaplan and Yorke [31] and Douady and Oesterlé [19], and played a prominent role in Falconer's work [21]. More specifically, for a discrete subgroup $\Gamma < \mathsf{PSL}_d(\mathbb{R})$, we consider the *piecewise* Dirichlet series defined, for $p \in \mathbb{N}$ and $s \in [p-1, p]$, by

$$\Phi_\Gamma^{\mathsf{Aff}}(s) = \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\gamma) \cdots \frac{\sigma_p}{\sigma_1}(\gamma) \right) \left(\frac{\sigma_{p+1}}{\sigma_1}(\gamma) \right)^{s-(p-1)}.$$

The affinity exponent is the critical exponent of this series:

$$h_\Gamma^{\mathsf{Aff}} := \inf\{s : \Phi_\Gamma^{\mathsf{Aff}}(s) < \infty\} = \sup\{s : \Phi_\Gamma^{\mathsf{Aff}}(s) = \infty\} \in (0, \infty].$$

Our second main result is (see Section 3 for a statement for arbitrary local fields):

Theorem B *Let $\Gamma < \mathsf{PGL}_d(\mathbb{R})$ be projective Anosov, then*

$$\dim_{\mathrm{Hff}}(\mathsf{L}_\Gamma) \le h_\Gamma^{\mathsf{Aff}}.$$

It is easy to deduce from Theorem B relations between the Hausdorff dimension of the limit set of a projective Anosov subgroup and the orbit growth rate with respect to explicit linear functionals on the Weyl chamber. Since the quantity $h_\Gamma(\mathcal{J}_p^u)$ appearing in Theorem A is also the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\gamma \in \Gamma} \left(\frac{\sigma_1 \cdots \sigma_{p+1}}{\sigma_1^{p+1}}(\gamma) \right)^s,$$

we get:

Corollary 1.1 *Let $\Gamma < \mathsf{PGL}_d(\mathbb{R})$ be projective Anosov and assume furthermore that $\dim_{\mathrm{Hff}}(\mathsf{L}_\Gamma)) \ge p$. Then*

$$\dim_{\mathrm{Hff}}(\mathsf{L}_\Gamma) \le p h_\Gamma(\mathcal{J}_p^u).$$

Observe that $\mathcal{J}^u_1 = \mathsf{a}_1$ and thus, whenever $\dim_{\mathrm{Hff}}(\mathsf{L}_\Gamma) \geq 1$, we obtain as a consequence the results of Glorieux, Monclair and Tholozan [25, Theorem 4.1] and Pozzetti, Sambarino and Wienhard [39, Proposition 4.1].

Existence of Patterson–Sullivan measures

The second inequality in Theorem A follows from an improvement on a result by Quint [42, théorème 8.1] concerning the relation between critical exponents and the existence of (Γ, φ)–Patterson–Sullivan measures.

Given a set $\Theta \subset \Pi$ of simple roots, we denote by $\mathcal{F}_\Theta$ the associated partial flag manifold, which consists of the space of flags of subspaces of dimension indexed by Θ. We denote by E_Θ the Levi subspace of E defined by

$$\mathsf{E}_\Theta = \bigcap_{p \notin \Theta} \ker \mathsf{a}_p.$$

The restrictions of the fundamental weights $\{\omega_{\mathsf{a}_p}|_{\mathsf{E}_\Theta} : p \in \Theta\}$ span its dual $(\mathsf{E}_\Theta)^*$. Using the Iwasawa decomposition of $\mathsf{PGL}_d(\mathbb{R})$, Quint introduced an *Iwasawa cocycle*

$$\mathrm{b}_\Theta : \mathsf{PGL}_d(\mathbb{R}) \times \mathcal{F}_\Theta \to \mathsf{E}_\Theta$$

that is the higher-rank analog of the more-studied *Busemann cocycle* in negative curvature; see Quint [42, lemme 6.6] and Section 5.3 for the precise definition. With this notation at hand we can recall the definition of a (Γ, φ)–Patterson–Sullivan measure from [42]:

Definition 1.2 Given a discrete subgroup $\Gamma < \mathsf{PGL}_d(\mathbb{R})$ and $\varphi \in (\mathsf{E}_\Theta)^*$, a (Γ, φ)–*Patterson–Sullivan measure* on $\mathcal{F}_\Theta$ is a finite Radon measure μ such that, for every $g \in \Gamma$,

$$\frac{\mathrm{d} g_* \mu}{\mathrm{d}\mu}(x) = e^{-\varphi(\mathrm{b}_\Theta(g^{-1}, x))}.$$

Inspired by a classical result by Sullivan [46], Quint shows [42, théorème 8.1] that the existence of a (Γ, φ)–Patterson–Sullivan measure on $\mathcal{F}_\Theta$ gives an upper bound on a related critical exponent

$$h_\Gamma(\varphi + \rho_{\theta^c}) \leq 1. \tag{1-1}$$

Here ρ_{θ^c} is an explicit linear functional which is positive on the interior of the Weyl chamber and accounts for the possible growth along the fibers of the projection $\mathcal{F}_\Delta \to \mathcal{F}_\Theta$ [42, lemme 8.3]. In general, $h_\Gamma(\varphi + \rho_{\theta^c}) < h_\Gamma(\varphi)$, and thus Quint's result is not sharp enough for our purposes.

Using ideas from geometric group theory we show that, provided the group Γ is Anosov with respect to one of the roots in Θ, there is no contribution from the fibers.

Given $\Theta \subset \Pi$, define $\mathrm{i}\Theta = \{d-p : p \in \Theta\}$. Two points $(x, y) \in \mathcal{F}_\Theta \times \mathcal{F}_{\mathrm{i}\Theta}$ are *transverse* if, for every $p \in \Theta$, one has that $x^p \cap y^{d-p} = \{0\}$. A *complementary subspace of* $\mathcal{F}_\Theta$ is a subset of $\mathcal{F}_\Theta$ of the form

$$\{x \in \mathcal{F}_\Theta : x \text{ is not transverse to } y_0\}$$

for a given $y_0 \in \mathcal{F}_{\mathrm{i}\Theta}$. If $\Theta' \subset \Theta$ then we let $\pi_{\Theta,\Theta'} \colon \mathcal{F}_\Theta \to \mathcal{F}_{\Theta'}$ be the canonical projection.

Theorem C *Let $\Gamma < \mathsf{PGL}_d(\mathbb{R})$ be projective Anosov and consider $\Theta \subset \Pi$ such that $\mathsf{a}_1 \in \Theta$. Let $\varphi \in (\mathsf{E}_\Theta)^*$. If there exists a (Γ, φ)–Patterson–Sullivan measure on $\mathcal{F}_\Theta$ with support on $\pi^{-1}_{\Theta,\mathsf{a}_1}(\mathsf{L}_\Gamma)$ and not contained on a complementary subspace, then*

$$h_\Gamma(\varphi) \leq 1.$$

We refer the reader to Section 5 and Theorem 5.14 for a version of Theorem C where the target group is an arbitrary semisimple group over a local field.

We provide the link between Theorems C and A in Section 6, where we establish that, if $\Gamma < \mathsf{PSL}_d(\mathbb{R})$ is a projective Anosov subgroup whose limit set L_Γ is a Lipschitz submanifold of dimension p, then there exists a $(\Gamma, \mathcal{J}^u_p)$–Patterson–Sullivan measure on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_p\}}$. In fact we explicitly construct such a measure using Rademacher's theorem and an explicit volume form on the almost everywhere defined tangent space to L_Γ (Proposition 6.4).

Example 1.3 If $\rho\colon \pi_1 S \to \mathsf{PSp}(4, \mathbb{R})$ is a maximal representation (see Section 9 for the definition), the combination of Theorems B and C gives $h_{\rho(\pi_1 S)}(\mathsf{a}_2) = 1$, while Quint's result (1-1) becomes $h_{\rho(\pi_1 S)}(\omega_{\mathsf{a}_2}) \leq 1$. This latter inequality is implied by the former equality, and often far from being sharp: one can find representations ρ for which $h_{\rho(\pi_1 S)}(\omega_{\mathsf{a}_2})$ is arbitrarily small.

Theorem C is complementary to (and independent from) the Patterson–Sullivan theory for Anosov representations developed by Dey and Kapovich [18]. They only consider Patterson–Sullivan densities with respect to functionals φ that, as opposed to the unstable Jacobian, belong to $(\mathsf{E}_\theta)^*$, where the representation is assumed to be Anosov with respect to *all* elements of θ, and induce Finsler distances on the symmetric space; see also Ledrappier [36] for a different approach yielding similar results. A drawback of their approach is that they can only relate the critical exponent with a premetric induced from a Finsler distance on the symmetric space that is hard to compute. In contrast, we begin with a natural measure, supported on the limit set, which belongs to the Lebesgue measure class, find a suitable functional, the unstable Jacobian, turning the measure into a Patterson–Sullivan measure, and deduce from this geometric properties of the action of Γ on the symmetric space.

Intermediate regularity and C^1–dichotomy

The class of Anosov subgroups with Lipschitz limit sets is very rich, and includes the images of many well-studied classes of representations, such as maximal representations (see Burger, Iozzi and Wienhard [12] and Section 9), quasi-Fuchsian AdS representations (see Barbot and Mérigot [3]) and $\mathbb{H}^{p,q}$–convex–cocompact representations (see Danciger, Guéritaud and Kassel [16] and Section 8).

As another contribution of independent interest, we show that Θ–positive representations of fundamental groups of surfaces in $\mathsf{SO}(p,q)$ (see Guichard and Wienhard [27]) yield subgroups with this property. We refer the reader to Section 10 for the precise definition of Θ–positive representations. We will only[2] consider here the Θ–positive representations that are furthermore Θ–Anosov for $\Theta = \{\mathsf{a}_1, \dots, \mathsf{a}_{p-1}\}$. As a result, for each $k \in \Theta$, they admit a boundary map $\xi^k : \partial\Gamma \to \mathsf{Is}_k(\mathbb{R}^{p,q})$ parametrizing the limit set in the Grassmannian of k–dimensional isotropic subspaces. In Section 10 we prove:

Theorem D *Let $\rho\colon \Gamma \to \mathsf{SO}(p,q)$ be a Θ–Anosov representation that is Θ–positive. Then the images of the boundary maps $\xi^k : \partial\Gamma \to \mathsf{Is}_k(\mathbb{R}^{p,q})$ are C^1–submanifolds for each $1 \le k < p-1$; moreover $\xi^{p-1}(\partial\Gamma)$ is Lipschitz.*

We will prove the parts of Theorem D separately, in Corollary 10.4 and Proposition 10.5, respectively.

At least for representations of fundamental groups of surfaces, the regularity of the limit set on a given (maximal) flag space seems to be related to the position of the associated root among the Anosov roots. By definition, a simple root is an *Anosov root* (for a subgroup Γ) if its kernel intersects trivially the limit cone $\mathcal{L}_\Gamma$ of Γ. Among such roots one can consider the internal (every neighboring root in the Dynkin diagram is also an Anosov root) or boundary (connected to a root that nontrivially intersects $\mathcal{L}_\Gamma$) roots. For example, for a Θ–positive representation in $\mathsf{SO}(p,q)$, the roots $\{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}\}$ are internal while a_{p-1} is the only boundary root.

[2]Guichard, Labourie and Wienhard announced that all Θ–positive representations are Θ–Anosov, so this should not pose any restriction.

The intermediate regularity (Lipschitz but not C^1) of limit sets for surface groups seems only to occur for boundary roots. For internal roots, we can prove a C^1–dichotomy, ruling out intermediate regularity in several interesting cases. More specifically we consider fundamental groups Γ of compact surfaces and study small deformations of representations of the form

$$\Gamma \to \mathsf{PSL}_2(\mathbb{R}) \xrightarrow{R} \mathsf{PSL}_d(\mathbb{R})$$

that are $\{\mathsf{a}_1, \mathsf{a}_2\}$–Anosov (this latter assumption can be rephrased as a proximality assumption on the linear representation R). For any such representation we have an explicit dichotomy: the associated limit set is either C^1 or not even Lipschitz (Corollary 7.8). We refer the reader to Section 7 for the precise statement of the dichotomy.

Entropy rigidity results

We conclude the introduction by discussing three well-studied classes of representations to which Theorem A applies. Interestingly, in all these cases, the mere information on the critical exponent of the unstable Jacobian provided by Theorem A allows us to obtain a sharp upper bound on the critical exponent for the action on the symmetric space endowed with the Riemannian distance function. In the case of Θ–positive representations this is even sufficient to prove that the bound is rigid: it is attained only on the specific Fuchsian locus, the generalization, in our setting, of Bowen's aforementioned result.

Maximal representations

Maximal representations are well-studied representations of fundamental groups of surfaces in Hermitian Lie groups $\mathsf{G}_{\mathbb{R}}$ that were introduced by Burger, Iozzi and Wienhard [12] through a cohomological invariant, the Toledo invariant. For these representations Theorem A applies, and gives:

Theorem 1.4 *Let $\mathsf{G}_{\mathbb{R}}$ be a classical simple Hermitian Lie group of tube type. Let $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{R}}$ be a maximal representation, and let $\check{\mathsf{a}}$ denote the root associated to the stabilizer of a point in the Shilov boundary of $\mathsf{G}_{\mathbb{R}}$. Then $h_\rho(\check{\mathsf{a}}) = 1$.*

Concretely, in the case $\mathsf{G}_{\mathbb{R}} \in \{\mathsf{Sp}(2p, \mathbb{R}), \mathsf{SU}(p, p), \mathsf{SO}^*(4p)\}$, the root $\check{\mathsf{a}}$ computes the logarithm of the square of the middle eigenvalue, while for $\mathsf{G} = \mathsf{SO}_0(2, p)$ the root $\check{\mathsf{a}}$ is the first root, computing the logarithm of the first eigenvalue gap.

Theorem 1.4 also holds for the exceptional Hermitian Lie group of tube type if the representation is Zariski-dense, and we expect it to hold unconditionally. We refer the reader to Section 9 for a slightly more general statement, further explanations and consequences, in particular concerning a sharp upper bound on the exponential orbit growth rate for the action on the symmetric space (see Proposition 9.9).

$\mathbb{H}^{p,q}$–convex–cocompact representations

Generalizing work of Mess [37] and Barbot and Mérigot [3], Danciger, Guéritaud and Kassel [16] introduced a class of representations called $\mathbb{H}^{p,q}$*–convex–cocompact*. Here $\mathbb{H}^{p,q}$ is the *pseudo-Riemannian hyperbolic space*, consisting of negative lines in $\mathbb{P}(\mathbb{R}^d)$ for a fixed nondegenerate form Q of signature $(p, q+1)$. It follows then from [16, Theorem 1.11] that a projective Anosov subgroup $\Gamma < \mathsf{PO}(Q) = \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$*–convex–cocompact* if, for every pairwise distinct triple of points $x, y, z \in \mathsf{L}_\Gamma$, the restriction $Q|_{\langle x,y,z\rangle}$ has signature $(2, 1)$.

Consider a representation $\Lambda : \mathsf{PO}(p, 1) \to \mathsf{PO}(p, q+1)$ whose image stabilizes a $(p+1)$–dimensional subspace V of $\mathbb{R}^d$, where $Q|_V$ has signature $(p, 1)$. Endow the symmetric space $X_{p,q+1}$ with the $\mathsf{PO}(p, q+1)$–invariant Riemannian metric normalized so that the totally geodesic copy of $\mathbb{H}^p$ in $X_{p,q+1}$ stabilized by Λ has constant curvature -1.

Definition 1.5 For a subgroup $\Gamma < \mathsf{SO}(p, q+1)$ and $x_0 \in X_{p,q+1}$, denote by $h_\rho^{X_{p,q+1}}$ the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\gamma\in\Gamma} e^{-sd(x_0,\rho(\gamma)x_0)}.$$

We have the upper bound:

Proposition 1.6 *Assume that $\partial\Gamma$ is homeomorphic to a $(p-1)$–dimensional sphere, and let $\Gamma < \mathsf{PO}(p, q+1)$ be strongly irreducible and $\mathbb{H}^{p,q}$–convex–cocompact. Then*

$$h_\rho^{X_{p,q+1}} \le p - 1.$$

We expect this upper bound to be rigid, namely the upper bound should only be attained at an inclusion of a cocompact lattice in $\mathsf{PO}(p, 1)$ preserving a totally geodesic copy of $\mathbb{H}^p$ of the type induced by Λ. However, only the case $p = 2$ is known; see Collier, Tholozan and Toulisse [15].

Section 8 contains more information on $\mathbb{H}^{p,q}$–convex–cocompact representations. In particular the relation with recent work by Glorieux and Monclair [24].

Θ–positive representations

Thanks to Theorem D, Theorem A also applies to Θ–positive representations of fundamental groups of surfaces in $\mathsf{SO}(p,q)$ and gives:

Corollary 1.7 *Let $\rho\colon \Gamma \to \mathsf{SO}(p,q)$ be a Θ–Anosov representation that is Θ–positive and weakly irreducible. Then $h_\rho(\mathsf{a}_k) = 1$ for every $k \le p-1$.*

Inspired by Potrie and Sambarino [38], we deduce from Corollary 1.7 a rigid upper bound for the critical exponent of the action of a positive representation on the Riemannian symmetric space $X_{p,q}$ (see Theorem 10.7). More precisely, we now normalize the $\mathsf{SO}(p,q)$–invariant Riemannian metric on $X_{p,q}$ so that the totally geodesic copy of $\mathbb{H}^2$ induced by the representation $\Lambda\colon \mathsf{SL}_2(\mathbb{R}) \to \mathsf{SO}(p,q)$ that stabilizes a subspace of $\mathbb{R}^d$ of signature $(p, p-1)$ has constant curvature -1. We consider the critical exponent in Definition 1.5 with this normalization of distance.

Theorem 1.8 *Let Γ be the fundamental group of a surface and let $\rho\colon \Gamma \to \mathsf{SO}(p,q)$ be Θ–positive. Then the critical exponent with respect to the Riemannian metric satisfies*

$$h_\rho^{X_{p,q}} \le 1.$$

Furthermore, if equality is achieved at a totally reducible representation η, then η splits as $W \oplus V$, W has signature $(p, p-1)$, $\eta|_W$ has Zariski closure the irreducible $\mathsf{PO}(2,1)$ in $\mathsf{PO}(p, p-1)$, and $\eta|_V$ lies in a compact group.

New arguments are needed with respect to [38], since the Anosov–Levi space of a Θ–positive representation has codimension one (instead of 0, which is the case treated in [38]); see Section 10.

Plan of the paper

In Section 2 we introduce some required preliminaries, and recall some needed results from Bochi, Potrie and Sambarino [6] and Pozzetti, Sambarino and Wienhard [39]. Section 3 deals with the affinity exponent and Hausdorff dimension for Anosov representations, and in it we prove Theorem B for any local field. Section 4 is a reminder of (more or less) standard definitions on semisimple algebraic groups over a local field. In Section 5 we recall objects from higher-rank Patterson–Sullivan theory and in Section 5.3 we prove Theorem 5.14 (a broader version of Theorem C). Section 6 completes the proof of Theorem A. The remaining sections deal with the applications of this result discussed in the introduction.

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2 Preliminaries

We recall in this section the notions we will need concerning Anosov representations and cone types. We refer the reader to [6; 39] for more details.

Throughout the paper $\mathbb{K}$ will denote a local field with absolute value $\|\cdot\|: \mathbb{K} \to \mathbb{R}^+$. If $\mathbb{K}$ is non-Archimedean, we require that $|\omega| = 1/q$ where ω denotes the *uniformizing element*, namely a generator of the maximal ideal of the valuation ring $\mathcal{O}$, and q is the cardinality of the residue field $\mathcal{O}/\omega\mathcal{O}$ (this is finite because $\mathbb{K}$ is, by assumption, local). This guarantees that the Hausdorff dimension of $\mathbb{P}^1(\mathbb{K})$ equals 1.

2.1 Singular values and Anosov representations into $\mathsf{PGL}_d(V_{\mathbb{K}})$

A $\mathbb{K}$–norm $\|\cdot\|$ on a $\mathbb{K}$ vector space $V_{\mathbb{K}}$ induces a norm on every exterior power of V; the angle between two vectors $\angle(v, w)$ is the unique number in $[0, \pi]$ such that

$$\sin \angle(v, w) := \frac{\|v \wedge w\|}{\|v\|\|w\|}.$$

Given two points $[v], [w] \in \mathbb{P}(V)$, we define their distance as

$$d([v], [w]) := \sin \angle(v, w),$$

and, given any two subspaces $P, Q < V$, we define their minimal angle as

$$\angle(P, Q) = \min_{v \in P \setminus \{0\}} \min_{w \in Q \setminus \{0\}} \angle(v, w).$$

An element $a \in \mathsf{GL}(V_{\mathbb{K}})$ is a *semihomothecy* (for a norm $\|\cdot\|$) if there exists an a–invariant $\mathbb{K}$–orthogonal[3] decomposition $V = V_1 \oplus \cdots \oplus V_k$ and $\sigma_1, \dots, \sigma_k \in \mathbb{R}_+$ such that, for every $i \in [\![1, k]\!]$ and every $v_i \in V_i$,

$$\|a v_i\| = \sigma_i \|v_i\|.$$

The numbers σ_i are called the ratios of the semihomothecy a.

Following Quint [40, théorème 6.1], we fix a maximal abelian subgroup of diagonalizable matrices $A \subset \mathsf{GL}(V_{\mathbb{K}})$, a compact subgroup $K \subset \mathsf{GL}(V_{\mathbb{K}})$ such that if N is the normalizer of A in $\mathsf{GL}(V_{\mathbb{K}})$ then $N = (N \cap K)A$, and a $\mathbb{K}$–norm $\|\cdot\|$ on V preserved by K and such that A acts on V by semihomothecies. Let $e_1 \oplus \cdots \oplus e_d$ be the eigenlines of A (here $d = \dim V$) and choose the Weyl chamber A^+ consisting of those elements $a \in A$ whose corresponding semihomothecy ratios verify $\sigma_1(a) \geq \cdots \geq \sigma_d(a)$.

For every $g \in \mathsf{GL}(V_{\mathbb{K}})$ we choose a Cartan decomposition $g = k_g a_g l_g$ with a_g in A^+ and $k_g, l_g \in K$, and denote by

$$\sigma_1(g) \geq \sigma_2(g) \geq \cdots \geq \sigma_d(g)$$

the semihomothecy ratios of the Cartan projection $a_g \in A^+$ (these do not depend on the choice of the Cartan decomposition once K and $\|\cdot\|$ are fixed). In order to simplify notation we will often write $(\sigma_i/\sigma_j)(g) = \sigma_i(g)/\sigma_j(g)$.

We define, for $p \in [\![1, d-1]\!]$,

$$u_p(g) = k_g \cdot e_p \in V.$$

The set $\{u_p(g) : p \in [\![1, d-1]\!]\}$ is an *arbitrary* orthogonal choice of the axes (ordered in decreasing length) of the ellipsoid $\{Av : \|v\| = 1\}$, and, by construction, for every $v \in g^{-1} u_p(g)$ one has $\|gv\| = \sigma_p(g)\|v\|$. Let

$$U_p(g) = u_1(g) \oplus \cdots \oplus u_p(g) = k_g \cdot (e_1 \oplus \cdots \oplus e_p).$$

If g is such that $\sigma_p(g) > \sigma_{p+1}(g)$, then we say that g *has a gap of index* p. In that case the decomposition

$$U_{d-p}(g^{-1}) \oplus g^{-1}(U_p(g))$$

is orthogonal (see [39, Remark 2.4]) and, if $\mathbb{K}$ is Archimedean, the p–dimensional space $U_p(g)$ is independent of the Cartan decomposition of g.

[3]Recall that for $\mathbb{K}$ non-Archimedean a decomposition $V = V_1 \oplus \cdots \oplus V_k$ is *orthogonal* if $\|\sum v_i\| = \max_i \|v_i\|$ for every $v_i \in V_i$.

We will denote by $\Pi=\{\mathsf{a}_1,\dots,\mathsf{a}_{d-1}\}$ the root system of $\mathsf{PGL}(V_{\mathbb{K}})$, and, given a subset $\theta\subset\Pi$, by $\mathcal{F}_\theta$ the associated partial flag manifold. Given $\theta\subset\Pi$ we also denote by $U^\theta(g)$ the partial flag $U^\theta(g)=\{U_p(g):\mathsf{a}_p\in\theta\}$. The *$\theta$–basin of attraction of g*

$$B_{\theta,\alpha}(g)=\Big\{x^\theta\in\mathcal{F}_\theta(\mathbb{K}^d):\min_{\mathsf{a}_p\in\theta}\angle(x^p,U_{d-p}(g^{-1}))>\alpha\Big\} \tag{2-1}$$

is the complement of the α–neighborhood of $U^{\theta^c}(g^{-1})$. When θ consists of a single root a, we will write $B_{\mathsf{a},\alpha}(g)$ instead of $B_{\{\mathsf{a}\},\alpha}(g)$.

Remark 2.1 If g has a gap of index p, then $U_{d-p}(g^{-1})$ is well defined if $\mathbb{K}$ is Archimedean, and any two possible choices have distance at most $(\sigma_{p+1}/\sigma_p)(g)$ if $\mathbb{K}$ is non-Archimedean. It follows that, also in the non-Archimedean case, $B_{\theta,\alpha}(g)$ only depends on K provided α is bigger than the minimal singular value gap.

We recall for later use the following lemma, which explains the choice of the term basin of attraction:

Lemma 2.2 (Bochi, Potrie and Sambarino [6, Lemma A.6]) *For every $g\in\mathsf{PGL}_d(\mathbb{K})$ and $x\in B_{\mathsf{a}_1,\alpha}(g)$,*

$$d(U_1(g),g\cdot x)\le\frac{1}{\sin(\alpha)}\frac{\sigma_2}{\sigma_1}(g).$$

2.2 Anosov representations

Let Γ be a word-hyperbolic group with identity element e, and fix a finite symmetric generating set S_Γ. For $\gamma\in\Gamma\setminus\{e\}$ denote by $|\gamma|$ the least number of elements of S_Γ needed to write γ as a word on S, and define the induced distance $d_\Gamma(\gamma,\eta)=|\gamma^{-1}\eta|$. A geodesic segment on Γ is a sequence $\{\alpha_i\}_0^k$ of elements in Γ such that $d_\Gamma(\alpha_i,\alpha_j)=|i-j|$.

Definition 2.3 A representation $\rho\colon\Gamma\to\mathsf{PGL}_d(\mathbb{K})$ is a_p*–Anosov*[4] if there exist positive constants c and μ, the a_p*–Anosov constants of* ρ, such that for all $\gamma\in\Gamma$,

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma))\le ce^{-\mu|\gamma|}. \tag{2-2}$$

An a_1–Anosov representation will be called *projective Anosov*.

The following result was proven in Bochi, Potrie and Sambarino [6] for $\mathbb{K}=\mathbb{R}$. The same arguments also give the result for any local field.

[4]In the language of Bochi, Potrie and Sambarino [6, Section 3.1], an a_p–Anosov representation is called *p–dominated*. It was proven by Kapovich, Leeb and Porti [33] that if a group Γ admits an Anosov representation, it is necessarily word hyperbolic. See also Bochi, Potrie and Sambarino [6] for a different approach.

Proposition 2.4 [6, Lemma 2.5] *Let $\rho\colon \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be a projective Anosov representation. Then there exists $\eta_\rho > 0$ and $L \in \mathbb{N}$ such that, for every geodesic segment $\{\alpha_i\}_0^k$ in Γ through e with $|\alpha_0|, |\alpha_k| \geq L$,*

$$\angle\big(U_1(\rho(\alpha_k)), U_{d-1}(\rho(\alpha_0))\big) > \eta_\rho.$$

Proposition 2.4 is a key ingredient in the construction of boundary maps:

Proposition 2.5 [6, Lemma 4.9] *Let $\rho\colon \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be projective Anosov and $(\alpha_i)_0^\infty \subset \Gamma$ a geodesic ray based at the identity converging to $x \in \partial\Gamma$. Then*

$$\xi^1_\rho(x) := \lim_{i\to\infty} U_1(\rho(\alpha_i)) \quad \text{and} \quad \xi^{d-1}_\rho(x) := \lim_{i\to\infty} U_{d-1}(\rho(\alpha_i))$$

exist, do not depend on the ray, and define continuous ρ–equivariant transverse maps $\xi^1_\rho\colon \partial\Gamma \to \mathbb{P}(\mathbb{K}^d)$ and $\xi^{d-1}_\rho\colon \partial\Gamma \to \mathbb{P}((\mathbb{K}^d)^*)$. Furthermore, there are positive constants C and μ depending only on ρ such that

$$d\big(U_1(\rho(\alpha_k)), \xi^1_\rho(x)\big) \leq Ce^{-\mu k}.$$

The next lemma, concerning properties of boundary maps, will be valuable in Section 3.1:

Lemma 2.6 (Bochi, Potrie and Sambarino [6, Lemma 3.9]) *Let $\rho\colon \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be projective Anosov. Then there exist constants $\nu \in (0,1)$, $a_0 > 0$ and $a_1 > 0$ such that, for every $\gamma, \eta \in \Gamma$,*

$$d_\Gamma(\gamma,\eta) \geq \nu(|\gamma| + |\eta|) - a_0 - a_1\big|\log d\big(U_1(\rho(\gamma)), U_1(\rho(\eta))\big)\big|.$$

3 Hausdorff dimension of limit sets and the affinity exponent

Generalizing the definition given in Section 1, we define the affinity exponent h^{Aff}_ρ of a projective Anosov representation $\rho\colon \Gamma \to \mathsf{PGL}(V_\mathbb{K})$ as the critical exponent of the broken Dirichlet series

$$\Phi^{\mathsf{Aff}}_\rho(s) = \sum_{\gamma\in\Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \cdots \frac{\sigma_{p-1}}{\sigma_1}(\rho(\gamma))\right)^{d_\mathbb{K}} \left(\frac{\sigma_p}{\sigma_1}(\rho(\gamma))\right)^{s-d_\mathbb{K}(p-2)}$$

for $s \in [d_\mathbb{K}(p-2), d_\mathbb{K}(p-1)]$, where the dimension $d_\mathbb{K}$ of $\mathbb{P}^1(\mathbb{K})$ is 1 unless $\mathbb{K} = \mathbb{C}$ in which case $d_\mathbb{C} = 2$.

Recall furthermore that, for a metric space (Λ, d) and for $s > 0$, one defines its *s–capacity* as

$$\mathcal{H}^s(\Lambda) = \inf_\varepsilon \left\{ \sum_{U\in\mathfrak{U}} \operatorname{diam} U^s \;\middle|\; \mathfrak{U} \text{ is a covering of } \Lambda \text{ with } \sup_{U\in\mathfrak{U}} \operatorname{diam} U < \varepsilon \right\},$$

and that the *Hausdorff dimension* of Λ is defined by

(3-1) $$\dim_{\mathrm{Hff}}(\Lambda) = \inf\{s : \mathcal{H}^s(\Lambda) = 0\} = \sup\{s : \mathcal{H}^s(\Lambda) = \infty\}.$$

The goal of the section is to prove:

Theorem 3.1 *Let $\mathbb{K}$ be a local field. If $\rho\colon \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ is a_1–Anosov, then*

$$\dim_{\mathrm{Hff}}(\xi^1_\rho(\partial\Gamma)) \le h^{\mathsf{Aff}}_\rho.$$

The proof of Theorem 3.1 is elementary and based on the construction of a good cover of the image of the limit map (explicitly constructed in Section 3.1), which we show in Section 3.2 to be contained in ellipses of controlled axis.

3.1 Coarse cone types

In Pozzetti, Sambarino and Wienhard [39, Section 2.3.1] we used cone types at infinity to construct well-behaved coverings of the boundary of the group. We now introduce a coarse version of these sets, which will be more useful for our purposes.

Recall that a sequence $(\alpha_j)_0^\infty$ is a (c_0, c_1)–quasigeodesic if, for every pair j, l,

$$\frac{1}{c_0}|j-l| - c_1 \le d_\Gamma(\alpha_j, \alpha_l) \le c_0|j-l| + c_1.$$

We associate to every element γ a *coarse cone type at infinity*, consisting of endpoints at infinity of quasigeodesic rays based at γ^{-1} passing through the identity:

$$\mathcal{C}^{c_0,c_1}_\infty(\gamma) = \bigl\{[(\alpha_j)_0^\infty] \in \partial\Gamma \mid (\alpha_i)_0^\infty \text{ is a } (c_0,c_1)\text{–quasigeodesic, } \alpha_0 = \gamma^{-1}, e \in \{\alpha_j\}\bigr\}.$$

Hyperbolicity of Γ lets us understand the overlaps of coarse cone types; this will be crucial in Section 5.3 to guarantee bounded overlap of suitable covers of the limit set.

Proposition 3.2 *Let Γ be word hyperbolic. For every c_0 and c_1 there exists $C > 0$ such that if*

$$\gamma\mathcal{C}^{c_0,c_1}_\infty(\gamma) \cap \eta\mathcal{C}^{c_0,c_1}_\infty(\eta) \ne \varnothing$$

then

$$d_\Gamma(\gamma, \eta) \le \bigl||\gamma| - |\eta|\bigr| + C.$$

Proof Assume that $x \in \gamma\mathcal{C}^{c_0,c_1}_\infty(\gamma) \cap \eta\mathcal{C}^{c_0,c_1}_\infty(\eta)$. Since Γ is hyperbolic, by the Morse lemma there exists $K > 0$ (only depending on c_0, c_1 and the hyperbolicity constant of Γ) such that γ is at distance at most K from a geodesic ray from e to x. The same holds then for η, and, using the hyperbolicity of Γ again, we can assume up to enlarging the constant K (still depending on c_0 and c_1 only) that the two rays agree. This implies

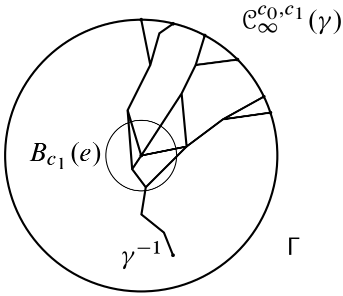

Figure 1: The coarse cone type at infinity. The black broken lines are (c_0, c_1)–quasigeodesics. All endpoints of geodesic rays from γ^{-1} intersecting the ball $B_{c_1}(e)$ clearly belong to $\mathcal{C}_\infty^{c_0,c_1}(\gamma)$.

that there exist g_0 and g_1 on a geodesic ray from e to x such that $d(\gamma, g_0) \le K$ and $d(\eta, g_1) \le K$. Since g_0 and g_1 lie in a geodesic $d(g_0, g_1) \le \big||g_0| - |g_1|\big|$, and thus

$$d(\gamma, \eta) \le 4K + \big||\gamma| - |\eta|\big|. \qquad \square$$

Our next goal is to show that, for an Anosov representation, the intersections of Cartan's basins of attraction $B_{\theta,\alpha}(\rho(\gamma))$ with the image of the boundary map are contained in the image of a suitably big coarse cone type of γ. Let $\theta \subset \Pi$ be a subset containing the first root a_1. We will denote by $\pi_{\theta,1}\colon \mathcal{F}_\theta(\mathbb{K}^d) \to \mathbb{P}(\mathbb{K}^d)$ the canonical projection. Recall from (2-1) that, for every α, we associate to each $g \in \mathsf{PGL}(V_{\mathbb{K}})$ a basin of attraction $B_{\theta,\alpha}(g) \subset \mathcal{F}_\theta$. We will now use Lemma 2.6 to show that, for every α, there exist c_0 and c_1 such that the intersection of a θ–basin of attraction $B_{\theta,\alpha}(\rho(\gamma))$ with the image of the boundary map is contained in a (c_0, c_1)–coarse cone type.

Proposition 3.3 *Let $\rho\colon \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ be projective Anosov and consider $\alpha > 0$. There exist c_0 and c_1 only depending on α and ρ such that, for every $\theta \subset \Pi$ containing a_1 and every $\gamma \in \Gamma$,*

$$(\xi^1)^{-1}\big(\pi_{\theta,1}\big(B_{\theta,\alpha}(\rho(\gamma))\big)\big) \subset \mathcal{C}_\infty^{c_0,c_1}(\gamma).$$

Proof It is enough to show that if $\xi^1(x) \in \pi_{\theta,1}(B_{\theta,\alpha}(\rho(\gamma)))$ and $|\gamma|$ is big enough, then there is a quasigeodesic ray from γ^{-1} to x that passes through the identity and whose constants only depend on α and ρ. Consider a quasigeodesic ray $\{\alpha_j\}$ converging to x, and fix $1 > \alpha' > \alpha$. Since by assumption $\xi^1(x) \in B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$, we can find a constant L depending only on ρ such that, for every $j > L$, it holds that $U_1(\rho(\alpha_i)) \in B_{\mathsf{a}_1,\alpha'}(\rho(\gamma))$. The uniformity of L follows from the last statement in

Proposition 2.5. By definition we have $\angle(U_1(\rho(\alpha_j)), U_{d-1}(\rho(\gamma^{-1}))) > \alpha'$, and thus, in particular, $d(U_1(\rho(\alpha_j)), U_1(\rho(\gamma^{-1})) > \alpha'$. Now let $(\alpha_j)_{i=0}^{-|\gamma|_S}$ be a geodesic segment with $\alpha_0 = e$ and $\alpha_{-|\gamma|_S} = \gamma$. Up to further enlarging α' and L (depending on the representation only), $d(U_1(\rho(\alpha_{-L})), U_1(\rho(\alpha_L)) > \alpha'$. Lemma 2.6 implies that the sequence $(\alpha_j)_{i=-|\gamma|_S}^{\infty}$, obtained as concatenation of the geodesic between γ^{-1} and the identity and the ray from the identity to x, is a quasigeodesic ray, thus the result. □

Corollary 3.4 *Let $\rho\colon \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ be projective Anosov and consider $\alpha > 0$. There exists C only depending on α and ρ such that, for every $\theta \subset \Pi$ containing a_1, if*

$$\xi^1(\partial\Gamma) \cap \pi_{\theta,1}\big(\rho(\gamma)\cdot B_{\theta,\alpha}(\rho(\gamma)) \cap \rho(\eta)\cdot B_{\theta,\alpha}(\rho(\eta))\big) \neq \varnothing$$

then

$$d(\gamma,\eta) \le \big||\gamma| - |\eta|\big| + C.$$

Proof This follows immediately by combining Propositions 3.3 and 3.2. □

In particular, we can use basins of attraction to construct coverings of the image of the boundary map with bounded overlap:

Proposition 3.5 (see Pozzetti, Sambarino and Wienhard [39, Lemma 2.22]) *Let $\rho\colon \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ be projective Anosov. There exists α small enough such that, for every $T > 0$, the family of open sets*

$$\mathcal{U}_T := \{\rho(\gamma)\cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

defines an open covering of $\xi^1(\partial\Gamma)$. Furthermore there exists a constant C depending on α (and ρ) such that, for every $x \in \partial\Gamma$ and every T, $\xi(x)$ is contained in at most C elements of $\mathcal{U}_T$.

Proof Let $x \in \partial\Gamma$ and let $\{\gamma_j\}$ be a geodesic ray based at the identity representing x. Propositions 2.4 and 2.5 guarantee that there exists $\alpha = \alpha_\rho$ such that

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1})) > \alpha;$$

therefore $\xi^1(x) \in \rho(\gamma_T)B_{\mathsf{a}_1,\alpha}(\rho(\gamma_T))$. The second statement is a direct consequence of Corollary 3.4. □

3.2 Ellipses

The purpose of this section is to prove that, for a projective Anosov representation, the set $\rho(\gamma)\cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ is coarsely contained in an ellipsoid with axes of size

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Definition 3.6 Let V be a d–dimensional $\mathbb{K}$–vector space with $\mathbb{K}$–norm $\|\cdot\|$. Let

$$u_1 \oplus \cdots \oplus u_d$$

be a $\mathbb{K}$–orthogonal decomposition and let $v = \sum v_j u_j$ be the associated decomposition of $v \in V$ for suitable $v_j \in \mathbb{K}$. Choose positive real numbers $a_2 \geq \cdots \geq a_d \geq 1$. If $\mathbb{K}$ is Archimedean, an *ellipsoid* about $\mathbb{K}u_1$ is the projectivization of

$$\left\{ v \in V \;\middle|\; |v_1|^2 \geq \sum_2^d (a_j |v_j|)^2 \right\}$$

for some $a_i > 0$. If instead $\mathbb{K}$ is non-Archimedean, an *ellipsoid* about $\mathbb{K}u_1$ is the projectivization of

$$\left\{ v \in V : |v_1| \geq \max_{2 \leq i \leq d} (a_j |v_j|) \right\}.$$

The vector spaces $u_1 \oplus u_j$ are *the axes* of the ellipsoid, and the *size* of the axis $u_1 \oplus u_j$ is $1/a_j$. We need the following covering lemma:

Lemma 3.7 *Let E be an ellipsoid with axis of size $1 \geq \beta_2 \geq \cdots \geq \beta_d$. For every $p \in [\![2, d]\!]$, E can be covered by*

$$2^{2p} \left(\frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}} \right)^{d_{\mathbb{K}}}$$

balls of radius $\sqrt{d}\,\beta_p$.

Proof We consider the affine chart of $\mathbb{P}(V)$ corresponding to $u_1 = 1$. The ellipsoid E is contained in the product of the balls $\{|v_i| \leq \beta_i\} \subset \mathbb{K}$ (it agrees with such a product if $\mathbb{K}$ is non-Archimedean). If $\mathbb{K}$ is Archimedean, the ball $\{|v_j| \leq \beta_j\}$ is contained in the union of $\lceil \beta_j / \beta_p \rceil^{d_{\mathbb{K}}}$ balls of radius β_p. Since the product of d balls of radius β_p is contained in a ball of radius $\sqrt{d}\,\beta_p$, we obtain that E can be covered by

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil^{d_{\mathbb{K}}} \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil^{d_{\mathbb{K}}}$$

balls of radius $\sqrt{d}\,\beta_p$.

If instead $\mathbb{K}$ is non-Archimedean, the ball $\{|v_j| \leq \beta_j\}$ can be decomposed into $q^{\lfloor \log_q(\beta_j/\beta_p) \rfloor}$ balls of radius β_p, and hence E can be covered with

$$q^{\lfloor \log_q(\beta_2/\beta_p) \rfloor} \cdots q^{\lfloor \log_q(\beta_{p-1}/\beta_p) \rfloor}$$

balls of radius β_p. □

Proposition 3.8 *Consider $\alpha > 0$. For $g \in \mathsf{PGL}(V_{\mathbb{K}})$, the image of the corresponding Cartan basin of attraction $g \cdot B_{\mathsf{a}_1,\alpha}(g)$ is contained in the ellipsoid about $U_1(g)$ with axes $u_1(g) \oplus u_j(g)$ of size*

$$\frac{1}{\sin\alpha}\frac{\sigma_j}{\sigma_1}(g).$$

Proof Assume first that $\mathbb{K}$ is Archimedean. By definition of $B_{\mathsf{a}_1,\alpha}(g)$, for every $v \in \mathbb{K}^d$ with $\mathbb{K} \cdot v \in B_{\mathsf{a}_1,\alpha}(g)$,

$$|v_1|^2 \geq (\sin\alpha)^2 \sum_1^d |v_j|^2,$$

where $(v_1, \dots, v_d)$ are the coefficients in the decomposition of v with respect to the orthogonal splitting $V = \bigoplus g^{-1}u_j(g)$.

Since the coefficients w_j of gv in the decomposition induced by the orthogonal decomposition $V = \bigoplus u_j(g)$ satisfy $|w_j| = \sigma_j(g)|v_j|$,

$$|w_1|^2 = \sigma_1(g)^2|v_1|^2 \geq \sigma_1(g)^2(\sin\alpha)^2 \sum_{j=2}^d |v_j|^2 = \sigma_1(g)^2(\sin\alpha)^2 \sum_{j=2}^d \frac{1}{\sigma_j(g)^2}|w_j|^2.$$

One concludes that gv lies on the corresponding ellipsoid. The non-Archimedean case follows analogously. □

3.3 The lower bound on the affinity exponent

We now have all the ingredients needed to prove Theorem 3.1:

Proof For each $T > 0$, denote by $\mathcal{U}_T$ the covering of $\xi^1(\partial\Gamma)$ given by Proposition 3.5. By definition, $U = U_\gamma \in \mathcal{U}_T$ is of the form $\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ for some γ satisfying $|\gamma| = T$. Proposition 3.8 applied to $\rho(\gamma)$ implies that $\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ is contained in an ellipsoid about $\mathbb{K}u_1(\rho(\gamma))$ with axes of sizes

$$\frac{1}{\sin\alpha}\frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin\alpha}\frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Furthermore, since ρ is Anosov, we deduce from Lemma 2.2 that $\sup_{U \in \mathcal{U}_T} \operatorname{diam} U$ is arbitrarily small as T goes to infinity. Recall that the s–capacity $\mathcal{H}^s$ was defined by (3-1). Applying Lemma 3.7 to these ellipses and any $p \in [\![2, d]\!]$, we obtain

$$\mathcal{H}^s(\xi(\partial\Gamma)) \\ \leq 2^{2p}\left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \inf_T \sum_{|\gamma| \geq T} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \cdots \frac{\sigma_{p-1}}{\sigma_1}(\rho(\gamma))\right)^{d_{\mathbb{K}}} \left(\frac{\sigma_p}{\sigma_1}(\rho(\gamma))\right)^{s-d_{\mathbb{K}}(p-2)}.$$

By definition of the affinity exponent h_ρ^{Aff}, for all $s > h_\rho^{\mathsf{Aff}}$ the broken Dirichlet series

$$\sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\cdots\frac{\sigma_{p-1}}{\sigma_1}(\rho(\gamma))\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_p}{\sigma_1}(\rho(\gamma))\right)^{s-d_{\mathbb{K}}(p-2)}, \quad s\in[d_{\mathbb{K}}(p-2), d_{\mathbb{K}}(p-1)],$$

is convergent, and thus, for all $s > h_\rho^{\mathsf{Aff}}$,

$$2^p\left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \inf_T \sum_{|\gamma|\geq T}\left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\cdots\frac{\sigma_{p-1}}{\sigma_1}(\rho(\gamma))\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_p}{\sigma_1}(\rho(\gamma))\right)^{s-d_{\mathbb{K}}(p-2)} = 0.$$

As a result we conclude that for all $s > h_\rho^{\mathsf{Aff}}$ the s–capacity $\mathcal{H}^s(\xi(\partial\Gamma))$ vanishes; hence,

$$h_\rho^{\mathsf{Aff}} \geq \dim_{\mathrm{Hff}}(\xi(\partial\Gamma)). \qquad \square$$

The following generalization of Corollary 1.1 is also immediate:

Corollary 3.9 *If $\rho\colon\Gamma\to\mathsf{PGL}(V_{\mathbb{K}})$ is projective Anosov and $\dim_{\mathrm{Hff}}(\xi(\partial\Gamma))\geq pd_{\mathbb{K}}$, then*

$$\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) \leq p h_\rho(\mathcal{J}_p^u).$$

Proof Observe that, for every $s\in[d_{\mathbb{K}}p, d_{\mathbb{K}}(p+1)]$, the value of the broken Dirichlet series defining the affinity exponent

$$\Phi_\rho^{\mathsf{Aff}}(s) = \sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\cdots\frac{\sigma_{p+1}}{\sigma_1}(\rho(\gamma))\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_{p+2}}{\sigma_1}(\rho(\gamma))\right)^{s-d_{\mathbb{K}}p}$$

is smaller than or equal to the value of the series associated to the p^{th} unstable Jacobian divided by p:

$$\Phi_\rho^{\mathcal{J}_p^u/p}(s) = \sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\cdots\frac{\sigma_{p+1}}{\sigma_1}(\rho(\gamma))\right)^{\frac{s}{p}}.$$

Indeed,

$$\begin{aligned}\left(\frac{\sigma_{p+2}}{\sigma_1}(\rho(\gamma))\right)^{s-d_{\mathbb{K}}p} &= \left(\frac{\sigma_{p+2}}{\sigma_1}(\rho(\gamma))\right)^{p\left(\frac{s}{p}-d_{\mathbb{K}}\right)}\\ &\leq \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\right)^{\frac{s}{p}-d_{\mathbb{K}}}\cdots\left(\frac{\sigma_{p+1}}{\sigma_1}(\rho(\gamma))\right)^{\frac{s}{p}-d_{\mathbb{K}}}.\end{aligned}$$

As a result, if $d_{\mathbb{K}}p \leq h_\rho^{\mathsf{Aff}} \leq d_{\mathbb{K}}(p+1)$, then $p h_\rho(\mathcal{J}_p^u) \geq h_\rho^{\mathsf{Aff}}$.

The result follows as, for all $k\in[\![1, d-1]\!]$ and $v\in\mathsf{E}^+$,

$$\frac{\mathcal{J}_{k-1}^u(v)}{k-1} \leq \frac{\mathcal{J}_k^u(v)}{k},$$

which implies $k h_\rho(\mathcal{J}_k^u) \leq (k-1) h_\rho(\mathcal{J}_{k-1}^u)$. $\square$

4 Semisimple algebraic groups

Let G be a connected semisimple $\mathbb{K}$–group, $\mathsf{G}_{\mathbb{K}}$ the group of its $\mathbb{K}$–points, A a maximal $\mathbb{K}$–split torus and $\boldsymbol{X}(\mathsf{A})$ the group of its $\mathbb{K}^*$–characters. Consider the real vector space $\mathsf{E}^* = \boldsymbol{X}(\mathsf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ and E its dual. For every $\chi \in \boldsymbol{X}(\mathsf{A})$, we denote by χ^{ω} the corresponding linear form on E.

4.1 Restricted roots and parabolic groups

Let Σ be the set of restricted roots of A in $\mathfrak{g}$. Then the set Σ^{ω} is a root system of E^*. Let Σ^+ be a system of positive roots and Π the associated subset of simple roots. Let E^+ be the Weyl chamber determined by the positive roots $(\Sigma^{\omega})^+$.

Let W be the Weyl group of Σ. It is isomorphic to the quotient of the normalizer $N_{\mathsf{G}_{\mathbb{K}}}(\mathsf{A}_{\mathbb{K}})$ of $\mathsf{A}_{\mathbb{K}}$ in $\mathsf{G}_{\mathbb{K}}$ by its centralizer $Z_{\mathsf{G}_{\mathbb{K}}}(\mathsf{A}_{\mathbb{K}})$. Let $\mathrm{i}\colon \mathsf{E} \to \mathsf{E}$ be the opposition involution: if $u\colon \mathsf{E} \to \mathsf{E}$ is the unique element in the Weyl group with $u(\mathsf{E}^+) = -\mathsf{E}^+$, then $\mathrm{i} = -u$.

A subset $\Theta \subset \Pi$ determines a pair of opposite parabolic subgroups P_Θ and $\check{\mathsf{P}}_\Theta$ whose Lie algebras are defined by

$$\mathfrak{p}_\Theta = \bigoplus_{\mathsf{a}\in\Sigma^+\cup\{0\}} \mathfrak{g}_{\mathsf{a}} \oplus \bigoplus_{\mathsf{a}\in\langle\Pi-\Theta\rangle} \mathfrak{g}_{-\mathsf{a}} \quad\text{and}\quad \check{\mathfrak{p}}_\Theta = \bigoplus_{\mathsf{a}\in\Sigma^+\cup\{0\}} \mathfrak{g}_{-\mathsf{a}} \oplus \bigoplus_{\mathsf{a}\in\langle\Pi-\Theta\rangle} \mathfrak{g}_{\mathsf{a}}.$$

The group $\check{\mathsf{P}}_\Theta$ is conjugate to the parabolic group $\mathsf{P}_{\mathrm{i}\Theta}$. Let

$$\mathfrak{l}_\Theta = \mathfrak{p}_\Theta \cap \check{\mathfrak{p}}_\Theta$$

be the Lie algebra of the associated Levi group.

The $\mathbb{K}$–*flag space* associated to Θ is $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}}) = \mathsf{G}_{\mathbb{K}}/\mathsf{P}_{\Theta,\mathbb{K}}$, and the $\mathsf{G}_{\mathbb{K}}$ orbit of the pair $([\mathsf{P}_{\Theta,\mathbb{K}}], [\check{\mathsf{P}}_{\Theta,\mathbb{K}}])$ is the unique open orbit for the action of $\mathsf{G}_{\mathbb{K}}$ in the product $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}}) \times \mathcal{F}_{\mathrm{i}\Theta}(\mathsf{G}_{\mathbb{K}})$. This orbit is denoted by $\mathcal{F}^{(2)}_\Theta(\mathsf{G}_{\mathbb{K}})$.

For $y \in \mathcal{F}_{\mathrm{i}\Theta}(\mathsf{G}_{\mathbb{K}})$ denote by

(4-1) $$\operatorname{Ann}(y) = \{x \in \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}}) : (x,y) \notin \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})^{(2)}\}$$

the closed submanifold of flags in $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$ that are not transverse to y.

Denote by $(\cdot,\cdot)$ a W–invariant inner product on E and also the induced inner product on E^*, define

$$\langle \chi, \psi\rangle = \frac{2(\chi,\psi)}{(\psi,\psi)},$$

and let $\{\omega_{\mathsf{a}}\}_{\mathsf{a}\in\Pi}$ be the *dual basis* of Π, ie $\langle \omega_{\mathsf{a}}, \mathsf{b}\rangle = d_{\mathsf{a}}\delta_{\mathsf{ab}}$, where $d_{\mathsf{a}} = 1$ if $2\mathsf{a} \notin (\Sigma^\omega)^+$ and $d_{\mathsf{a}} = 2$ otherwise. The linear form ω_{a} is *the fundamental weight* associated to a.

4.2 Cartan decomposition

Let $\nu\colon \mathsf{A}_{\mathbb{K}} \to \mathsf{E}$ be defined, for $z \in \mathsf{A}_{\mathbb{K}}$, as the unique vector in E such that for every $\chi \in \boldsymbol{X}(\mathsf{A})$,

$$\chi^\omega(\nu(z)) = \log|\chi(z)|.$$

Define $\mathsf{A}_{\mathbb{K}}^+ = \nu^{-1}(\mathsf{E}^+)$.

Let $K \subset \mathsf{G}_{\mathbb{K}}$ be a compact group that contains a representative for every element of the Weyl group W; that is, $N_{\mathsf{G}_{\mathbb{K}}}(\mathsf{A}_{\mathbb{K}}) = (N_{\mathsf{G}_{\mathbb{K}}}(\mathsf{A}_{\mathbb{K}}) \cap K)\mathsf{A}_{\mathbb{K}}$, where $N_{\mathsf{G}_{\mathbb{K}}}$ is the normalizer. One has $\mathsf{G}_{\mathbb{K}} = K\mathsf{A}_{\mathbb{K}}^+K$, and if $z, w \in \mathsf{A}_{\mathbb{K}}^+$ are such that $z \in KwK$, then $\nu(z) = \nu(w)$. There exists thus a function

$$a\colon \mathsf{G}_{\mathbb{K}} \to \mathsf{E}^+,$$

such that for every $g_1, g_2 \in \mathsf{G}_{\mathbb{K}}$ one has that $g_1 \in Kg_2K$ if and only if $a(g_1) = a(g_2)$. It is called the *Cartan projection* of $\mathsf{G}_{\mathbb{K}}$.

In the case of $G_{\mathbb{K}} = \mathsf{PGL}(V_{\mathbb{K}})$ this is nothing but the ordered list of semihomothecy ratios defined in Section 2.1.

4.3 Representations of $\mathsf{G}_{\mathbb{K}}$

Let $\Lambda\colon \mathsf{G} \to \mathsf{PGL}(V)$ be a finite-dimensional irreducible representation that is also a rational map between algebraic varieties and denote by $\phi_\Lambda\colon \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Λ. Then the *weight space* associated to $\chi \in \boldsymbol{X}(\mathsf{A})$ is the vector space

$$V_\chi = \{v \in V : \phi_\Lambda(a)v = \chi(a)v \text{ for all } a \in \mathsf{A}_{\mathbb{K}}\},$$

and if $V_\chi \neq 0$ then we say that $\chi^\omega \in \mathsf{E}^*$ is a *restricted weight* of Λ. Theorem 7.2 of Tits [47] states that the set of weights has a unique maximal element with respect to the order $\chi \geq \psi$ if $\chi - \psi$ is positive on E^+. This is called *the highest weight* of Λ and denoted by χ_Λ.

Definition 4.1 Let Θ_Λ be the set of simple roots $\mathsf{a} \in \Pi$ such that $\chi_\Lambda - \mathsf{a}$ is still a weight of Λ.

Remark 4.2 The subset Θ_Λ is the subset of simple roots such that, for $\mathsf{a} \in \Sigma^+$, $n \in \mathfrak{g}_{-\mathsf{a}}$ and $v \in \chi_\Lambda$, we have $\phi_\Lambda(n)v = 0$ if and only if $\mathsf{a} \in \langle \Pi - \Theta_\Lambda \rangle$.

Definition 4.3 We denote by $\|\cdot\|_\Lambda$ a good norm on V, invariant under ΛK, and such that $\Lambda \mathsf{A}_\mathbb{K}$ consists of semihomothecies; if $\mathbb{K}$ is Archimedean the existence of such a norm is classical, and if $\mathbb{K}$ is non-Archimedean then this is the content of Quint [40, théorème 6.1].

For every $g \in \mathsf{G}_\mathbb{K}$,

$$\log \|\Lambda g\|_\Lambda = \chi_\Lambda(a(g)). \tag{4-2}$$

If $g = k_g z_g l_g$ with $k, l \in K$ and $z_g \in \mathsf{A}_\mathbb{K}^+$, then for all $v \in \Lambda(l_g^{-1}) V_{\chi_\Lambda}$ one has $\|\Lambda g(v)\|_\Lambda = \|\Lambda g\|_\Lambda \|v\|_\Lambda$.

Denote by W_{χ_Λ} the $\Lambda \mathsf{A}_\mathbb{K}$–invariant complement of V_{χ_Λ}. Note that the stabilizer in $\mathsf{G}_\mathbb{K}$ of W_{χ_Λ} is $\check{\mathsf{P}}_{\Theta,\mathbb{K}}$, and thus one has a map of flag spaces

$$(\xi_\Lambda, \xi_\Lambda^*) \colon \mathcal{F}^{(2)}_{\Theta_\Lambda}(\mathsf{G}_\mathbb{K}) \to \mathcal{G}^{(2)}_{\dim V_{\chi_\Lambda}}(V), \tag{4-3}$$

a proper embedding which is a homeomorphism onto its image. Here $\mathcal{G}^{(2)}_{\dim V_{\chi_\Lambda}}(V)$ is the open $\mathsf{PGL}(V_\mathbb{K})$–orbit in the product of the Grassmannian of $(\dim V_{\chi_\Lambda})$–dimensional subspaces and the Grassmannian of $(\dim V - \dim V_{\chi_\Lambda})$–dimensional subspaces.

Proposition 4.4 (Tits [47]; see also Humphreys [30, Chapter XI]) *For each* $\mathsf{a} \in \Pi$ *there exists a finite-dimensional rational irreducible representation* $\Lambda_\mathsf{a} \colon \mathsf{G} \to \mathsf{PSL}(V_\mathsf{a})$ *such that* $\chi_{\Lambda_\mathsf{a}}$ *is an integer multiple of the fundamental weight* ω_a *and* $\dim V_{\chi_{\Lambda_\mathsf{a}}} = 1$. *All other weights of* Λ_a *are of the form*

$$\chi_\mathsf{a} - \mathsf{a} - \sum_{\mathsf{b} \in \Pi} n_\mathsf{b} \mathsf{b},$$

where $n_\mathsf{b} \in \mathbb{N}$.

We will fix from now on such a set of representations and call them, for each $\mathsf{a} \in \Pi$, the *Tits representation associated to* a.

4.4 The center of the Levi group $\mathsf{P}_{\Theta,\mathbb{K}} \cap \check{\mathsf{P}}_{\Theta,\mathbb{K}}$

We now consider the vector subspace

$$\mathsf{E}_\Theta = \bigcap_{\mathsf{a} \in \Pi - \Theta} \ker \mathsf{a}^\omega,$$

together with the unique projection $\pi_\Theta\colon \mathsf{E} \to \mathsf{E}_\Theta$ that is invariant under the subgroup W_Θ of the Weyl group spanned by reflections associated to roots in $\Pi - \Theta$:

$$W_\Theta = \{w \in W : w(v) = v \text{ for all } v \in \mathsf{E}_\Theta\}.$$

The dual space $(\mathsf{E}_\Theta)^*$ is canonically the subspace of E^* of π_Θ–invariant linear forms and it is spanned by the fundamental weights of roots in Θ:

$$(\mathsf{E}_\Theta)^* = \{\varphi \in \mathsf{E}^* : \varphi \circ \pi_\Theta = \varphi\} = \langle \omega_{\mathsf{a}} : \mathsf{a} \in \Theta\rangle.$$

Since $\pi_\Theta^2 = \pi_\Theta$, precomposition with π_Θ induces a projection $\mathsf{E}^* \to (\mathsf{E}_\Theta)^*$ denoted by

$$\varphi \mapsto \varphi^\Theta := \varphi \circ \pi_\Theta.$$

Examples 4.5 and 4.6 will be relevant in Sections 7 and 8, respectively.

Example 4.5 Let $\mathsf{G}_{\mathbb{K}} = \mathsf{PGL}(V_{\mathbb{K}})$ and, as above, denote by $\mathsf{a}_k \in \mathsf{E}^*$ the k^{th} simple root, so that $\mathsf{a}_k(a_1, \dots, a_d) = a_k - a_{k+1}$. We then choose $p \in [\![2, d-2]\!]$ and let $\Theta = \{\mathsf{a}_1, \mathsf{a}_p, \mathsf{a}_{d-1}\}$, so that

$$\mathsf{E}_\Theta = \{(a_1, \dots, a_d) \in \mathsf{E} : a_2 = \cdots = a_p \text{ and } a_{p+1} = \cdots = a_{d-1}\}$$

is three-dimensional. Using the fact that the fundamental weights ω_i (for $i = 1, p, d-1$) belong to $(\mathsf{E}_\Theta)^*$, one checks that the projection is

$$\begin{aligned}
&\varepsilon_1(\pi_\Theta(a)) = a_1,\\
&\varepsilon_i(\pi_\Theta(a)) = \frac{a_2 + \cdots + a_p}{p-1} = \frac{\omega_p - \omega_1}{p-1}(a) && \text{for every } i \in [\![2, p]\!],\\
&\varepsilon_i(\pi_\Theta(a)) = \frac{a_{p+1} + \cdots + a_{d-1}}{d-p-1} = \frac{\omega_{d-1} - \omega_p}{d-p-1}(a) && \text{for every } i \in [\![p+1, d-1]\!],\\
&\varepsilon_d(\pi_\Theta(a)) = a_d.
\end{aligned}$$

Then

$$\mathsf{a}_p^\Theta = \frac{\omega_p - \omega_1}{p-1} - \frac{\omega_{d-1} - \omega_p}{d-p-1}$$

and $\mathsf{a}_p^\Theta|_{\mathsf{E}^+\setminus\{0\}} \geq \mathsf{a}_p|_{\mathsf{E}^+\setminus\{0\}}$.

Example 4.6 Consider the group $\mathsf{SO}(p,q)$ of transformations in $\mathsf{PSL}_{p+q}(\mathbb{R})$ preserving a signature (p,q) bilinear form with $p < q$. One has

$$\mathsf{E} = \{(a_1, \dots, a_p) : a_i \in \mathbb{R}\}$$

equipped with the root system

$$\Sigma^\omega = \{\varepsilon_i : i \in [\![1, p]\!]\} \cup \{a \mapsto a_i - a_j : i, j \in [\![1, p]\!]\}.$$

A Weyl chamber can be chosen as

$$\mathsf{E}^+ = \{a \in \mathsf{E} : a_i \geq a_{i+1} \text{ for all } i \in [\![1, p-1]\!] \text{ and } a_p \geq 0\},$$

with the associated set of simple roots

$$\Pi = \{\mathsf{a}_i : i \in [\![1, p-1]\!]\} \cup \{\varepsilon_p\}.$$

Consider then $\Theta = \{\mathsf{a}_i : i \in [\![1, p-1]\!]\}$, so that $\mathsf{E}_\Theta = \ker \varepsilon_p$ and thus $\mathsf{a}_i \in (\mathsf{E}_\Theta)^*$ for $i \in [\![1, p-2]\!]$. Moreover,

$$\mathsf{a}_{p-1}^{\Theta} = \varepsilon_{p-1},$$

and one has that $\mathsf{a}_{p-1}^{\Theta}|_{\mathsf{E}^+\setminus\{0\}} \geq \mathsf{a}_{p-1}|_{\mathsf{E}^+\setminus\{0\}}$.

4.5 Gromov product

Recall from Sambarino [45] that the *Gromov product*[5] based at K is the map

$$(\cdot\,|\,\cdot)_K : \mathcal{F}_\Theta^{(2)}(\mathsf{G}_\mathbb{K}) \to \mathsf{E}_\Theta,$$

defined to be the unique vector $(x|y)_K \in \mathsf{E}_\Theta$ such that

$$\chi_\mathsf{a}((x|y)_K) = -\log \sin \angle_{\|\,\|_{\Lambda_\mathsf{a}}}(\xi_{\Lambda_\mathsf{a}} x, \xi^*_{\Lambda_\mathsf{a}} y)$$

for all $\mathsf{a} \in \Theta$, where χ_a is the fundamental weight associated to the Tits representation Λ_a of a. Note that

$$\max_{\mathsf{a}\in\Theta} \chi_\mathsf{a}((x|y)_K) = \max_{\mathsf{a}\in\Theta} \bigl|\chi_\mathsf{a}((x|y)_K)\bigr| = -\log \min_{\mathsf{a}\in\Theta} \sin \angle_{\|\,\|_{\Lambda_\mathsf{a}}}(\xi_{\Lambda_\mathsf{a}} x, \xi^*_{\Lambda_\mathsf{a}} y). \tag{4-4}$$

One has the following remark from Bochi, Potrie and Sambarino [6]:

Remark 4.7 [6, Remark 8.11] Let $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ be a finite-dimensional rational irreducible representation. If $(x, y) \in \mathcal{F}^{(2)}_{\Theta_\Lambda}(\mathsf{G}_\mathbb{K})$ then

$$(\xi_\Lambda x | \xi^*_\Lambda y)_{\|\,\|_\Lambda} = \chi_\Lambda((x|y)_K),$$

where $\|\,\|_\Lambda$ denotes the (stabilizer of the) inner product on V such that ΛK is orthogonal (see Definition 4.3).

4.6 Iwasawa cocycle and its relation to representations of G

Another important decomposition of Lie groups that will play a role in our work is the Iwasawa decomposition

$$\mathsf{G}_\mathbb{K} = K \mathsf{A}_\mathbb{K} \mathsf{U}_{\Pi,\mathbb{K}},$$

[5] This is the negative of the product defined in [45].

where $\mathsf{P}_{\Pi,\mathbb{K}}$ is the minimal parabolic subgroup and $\mathsf{U}_{\Pi,\mathbb{K}}$ is its unipotent radical. For a general local field $\mathbb{K}$ the decomposition of an element is not necessarily unique, but if $z_1, z_2 \in \mathsf{A}_{\mathbb{K}}$ are such that $z_1 \in K z_2 \mathsf{U}_{\Pi,\mathbb{K}}$, then $\nu(z_1) = \nu(z_2)$.

Quint used the Iwasawa decomposition to define the Iwasawa cocycle

$$\mathrm{b}_\Pi(g, x) = \nu(z),$$

where $x = k[\mathsf{P}_{\Theta,\mathbb{K}}] \in \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$ with $k \in K$ and $g \in \mathsf{G}_{\mathbb{K}}$, and gk has Iwasawa decomposition $gk = lzu$.

Lemma 4.8 (Quint [42, lemmes 6.1 et 6.2]) *The map $p_\Theta \circ \mathrm{b}_\Pi$ factors through a map $\mathrm{b}_\Theta \colon \mathsf{G}_{\mathbb{K}} \times \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}}) \to \mathsf{E}_\Theta$. The map b_Θ verifies the cocycle relation: for every $g, h \in \mathsf{G}_{\mathbb{K}}$ and $x \in \mathcal{F}_{\Theta,\mathbb{K}}(\mathsf{G}_{\mathbb{K}})$,*

$$\mathrm{b}_\Theta(gh, x) = \mathrm{b}_\Theta(g, hx) + \mathrm{b}_\Theta(h, x).$$

One also has the following behavior of b_Θ under the representations of G:

Lemma 4.9 (Quint [42, lemme 6.4]) *Suppose $\Lambda \colon \mathsf{G} \to \mathsf{PGL}(V)$ is a proximal irreducible representation. Then for every $x \in \mathcal{F}_{\Theta_\Lambda}(\mathsf{G}_{\mathbb{K}})$ and $g \in \mathsf{G}_{\mathbb{K}}$,*

$$\chi_\Lambda(\mathrm{b}_{\Theta_\Lambda}(g, x)) = \log \frac{\|\Lambda(g)v\|_\Lambda}{\|v\|_\Lambda},$$

where $v \in \xi_\Lambda(x) \setminus \{0\}$.

4.7 Cartan attractors and Cartan's attracting basins

Consider $g \in \mathsf{G}_{\mathbb{K}}$ and let $g = k_g z_g l_g$ be a Cartan decomposition. Given $\Theta \subset \Pi$, the *Cartan attractor of g in $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$* is defined by

$$U_\Theta(g) = U_\Theta^K(g) = k_g[\mathsf{P}_{\Theta,\mathbb{K}}],$$

and the *Cartan basin of g* is defined, for $\alpha > 0$, by

$$B_{\Theta,\alpha}(g) = \{x \in \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}}) : (x | U_{\mathrm{i}\Theta}(g^{-1}))_K < \alpha\}.$$

Remark 4.10 If $\Lambda \colon \mathsf{G} \to \mathsf{PGL}(V)$ is a rational irreducible representation with $\Theta_\Lambda \subset \Theta$, then

$$\xi_\Lambda(U_\Theta(g)) = U^{\|\|_\Lambda}_{\dim V_{\chi_\Lambda}}(\Lambda(g)).$$

Notice that the flag $U_\Theta(g)$ is an arbitrary choice of a "most expanding" flag of type Θ for g. However, it is clear from the definition that given $\alpha > 0$ there exists a constant K_α such that, if $y \in \mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$ belongs to $B_{\Theta,\alpha}(g)$, then, for all $\mathsf{a} \in \Theta$,

$$|\chi_{\mathsf{a}}(a(g) - \mathrm{b}_\Theta(g, y))| \le K_\alpha. \tag{4-5}$$

4.8 The $\mathsf{PSL}_d(\mathbb{K})$ case

Given a good norm τ on $\mathbb{K}^d$, and considering the exterior power representations of $\mathsf{PSL}_d(\mathbb{K})$, one sees that Lemma 4.9 provides the following computation for the Iwasawa cocycle $\mathrm{b}\colon \mathsf{PSL}_d(\mathbb{K})\times\mathcal{F}(\mathbb{K}^d)\to\mathsf{E}$ associated to a maximal compact group stabilizing τ. For $p\in[\![1,d]\!]$ and given $g\in\mathsf{PSL}_d(\mathbb{K})$ and $x\in\mathcal{F}(\mathbb{K}^d)$,

$$\omega_p(\mathrm{b}(g,x)) = \log\frac{\|gv_1\wedge\cdots\wedge gv_p\|}{\|v_1\wedge\cdots\wedge v_p\|}, \tag{4-6}$$

where $\{v_1,\ldots,v_p\}$ is any basis of the p–dimensional space x^p of x and $\|\cdot\|$ is the norm on $\bigwedge^p\mathbb{K}^d$ induced by τ.

Notice that, by definition, the number $\omega_p(\mathrm{b}(g,x))$ only depends on x^p, so in order to simplify notation we will also denote it by $\omega_p(\mathrm{b}(g,x^p))$.

5 Patterson–Sullivan measures in non-Anosov directions

An interesting quantity associated to a discrete subgroup $\Gamma<\mathsf{G}_{\mathbb{K}}$ is h_Γ^X, its critical exponent, which measures the exponential growth rate of orbit points in balls (in the symmetric space of $\mathsf{G}_{\mathbb{K}}$) as the radius grows. The theory of Quint's growth indicator function, which we briefly recall in Section 5.1, allows us to deduce information on h_Γ^X from information on the critical exponent of linear forms ϕ on the Weyl chamber E, which is often easier to handle with the aid of Patterson–Sullivan measures. When the discrete group $\Gamma<\mathsf{G}_{\mathbb{K}}$ is the image of an Anosov representation $\rho\colon\Gamma\to\mathsf{G}_{\mathbb{K}}$, and the form ϕ belongs to the dual of the Levi–Anosov subspace E_{θ_ρ}, the thermodynamical formalism applies (see Theorem 5.12).

In this section we will instead be interested in studying forms ϕ that do not belong to $(\mathsf{E}_{\theta_\rho})^*$. Our main result is Theorem 5.14, in which we show that, provided a representation ρ is Anosov with respect to some root, the existence of a Patterson–Sullivan measure in any flag manifold — and thus also in non-Anosov directions ϕ — has strong implications for the critical exponent of ϕ.

5.1 Quint's growth indicator

We recall here some definitions from Quint [41; 42].

Let $\Gamma\subset\mathsf{G}_{\mathbb{K}}$ be a discrete subgroup; its *Quint growth indicator function* [41]

$$\Psi_\Gamma\colon\mathsf{E}^+\to\mathbb{R}_+\cup\{-\infty\}$$

is defined as follows. Given a norm $\|\cdot\|$ on E and an open cone $\mathscr{C} \subset \mathsf{E}^+$, let $h_{\mathscr{C}}^{\|\|}$ be the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\{g \in \Gamma : a(g) \in \mathscr{C}\}} e^{-s\|a(g)\|}$$

and define $\Psi_\Gamma \colon \mathsf{E}^+ \to \{-\infty\} \cup [0,\infty)$ by

$$\Psi_\Gamma(v) = \|v\| \inf_{v \in \mathscr{C}} h_{\mathscr{C}}^{\|\|},$$

where the infimum is taken over all open cones containing v. One can easily check that Ψ_Γ does not depend on the chosen norm $\|\cdot\|$ and is 1–positively homogenous.

Dually, one considers the growth on linear forms. The *limit* (*or Benoist* [4]) *cone* $\mathcal{L}_\Gamma$ of Γ is defined as the limit points of sequences $t_n a(g_n)$ where $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ converges to 0 and $(g_n)_{n\in\mathbb{N}} \subset \Gamma$. Denote its dual cone by

$$(\mathcal{L}_\Gamma)^* = \{\varphi \in \mathsf{E}^* : \varphi|_{\mathcal{L}_\Gamma \setminus \{0\}} \geq 0\},$$

and for $\varphi \in (\mathcal{L}_\Gamma)^*$ let $h_\Gamma(\varphi)$ be the critical exponent of the Dirichlet series

$$\sum_{g\in\Gamma} e^{-s\varphi(a(g))},$$

that is,

$$h_\Gamma(\varphi) = \limsup_{t\to\infty} \frac{1}{t} \log \#\{g \in \Gamma \mid \varphi(a(g)) < t\}.$$

Lemma 5.1 $h_\Gamma(\min\{\phi_1, \dots, \phi_k\}) = \max\{h_\Gamma(\phi_1), \dots, h_\Gamma(\phi_k)\}.$

Proof One inequality is clear. For the other one,

$$\begin{aligned} h_\Gamma(\min\{\phi_1, \dots, \phi_k\}) &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sum_{i=1}^{k} \#\{\gamma \in \Gamma \mid \phi_i(a(\rho(\gamma))) < t\} \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log k \max_i \#\{\gamma \in \Gamma \mid \phi_i(a(\rho(\gamma))) < t\} \\ &= \max\{h_\Gamma(\phi_1), \dots, h_\Gamma(\phi_k)\} \end{aligned}$$ □

One can then define the subset

$$\mathcal{D}_\Gamma = \{\varphi \in (\mathcal{L}_\Gamma)^* : h_\Gamma(\varphi) \in (0,1]\}.$$

The next lemma is clear from the definitions, but is very useful in applications:

Lemma 5.2 *If ϕ belongs to $\mathcal{D}_\Gamma$, then $\phi + \psi \in \mathcal{D}_\Gamma$ for every $\psi \in (\mathcal{L}_\Gamma)^*$.*

The following result from Quint [41] allows one to deduce information on the critical exponent of various norms in terms of growth of linear functions, which are often easier to compute:

Proposition 5.3 (Quint [41]) *One has that*

$$\mathcal{D}_\Gamma = \{\varphi \in \mathsf{E}^* : \varphi(v) \geq \Psi_\Gamma(v) \text{ for all } v \in \mathsf{E}^+\},$$

and thus it is a convex set. Moreover, for any 1*–positively homogenous function* $\Theta\colon \mathsf{E}^+ \to \mathbb{R}$, *the critical exponent* $h_\Gamma(\Theta)$ *of the Dirichlet series*

$$s \mapsto \sum_{g\in\Gamma} e^{-s\Theta(a(g))}$$

can be computed as $h_\Gamma(\Theta) = \sup_{v\in\mathsf{E}^+} \Psi_\Gamma(v)/\Theta(v)$.

A useful property of the set $\mathcal{D}_\Gamma$ is provided by the next theorem.

Theorem 5.4 (Quint [41]) *If the Zariski closure of* Γ *is semisimple then* Ψ_Γ *is concave. Consequently, for every norm* $\|\cdot\|$ *on* E,

$$h_\Gamma^{\|\|} = \inf\{\|\varphi\|^* : \varphi \in \mathcal{D}_\Gamma\},$$

where $\|\cdot\|^*$ *is the induced operator norm on* E^*.

Remark 5.5 Recall that, if we endow the symmetric space (or the affine building) X associated to $\mathsf{G}_\mathbb{K}$ with a $\mathsf{G}_\mathbb{K}$–invariant Riemannian metric, there exists an Euclidean norm $\|\cdot\|_X$ on E such that, for every $g \in \mathsf{G}_\mathbb{K}$,

$$d_X([K], g[K]) = \|a(g)\|_X.$$

So Theorem 5.4 provides the following formula for the critical exponent of a discrete group with reductive Zariski closure in the symmetric space X:

$$h_\Gamma^X = \inf\{\|\phi\|_X^* : \phi \in \mathcal{D}_\Gamma\}.$$

The topological boundary $\mathcal{Q}_\Gamma$ of $\mathcal{D}_\Gamma$ will be called *Quint's indicator set of* Γ. We will also write

$$\mathcal{Q}_{\Gamma,\Theta} = \mathcal{Q}_\Gamma \cap (\mathsf{E}_\Theta)^*.$$

Let us record here a useful direct consequence of the convexity of $\mathcal{D}_\Gamma$:

Lemma 5.6 *Let* $\phi, \varphi \in (\mathcal{L}_\Gamma)^*$. *Then*

$$h_\Gamma(\phi + \varphi) \leq \frac{h_\Gamma(\phi) h_\Gamma(\varphi)}{h_\Gamma(\phi) + h_\Gamma(\varphi)}.$$

We end this subsection with a definition from Quint [42]:

Definition 5.7 Given $\Theta \subset \Pi$ and $\varphi \in (\mathsf{E}_\Theta)^*$, a (Γ, φ)–*Patterson–Sullivan measure* on $\mathcal{F}_\Theta(\mathsf{G}_\mathbb{K})$ is a finite Radon measure μ such that, for every $g \in \Gamma$,

$$\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x) = e^{-\varphi(\mathrm{b}_\Theta(g^{-1},x))}.$$

5.2 Anosov representations with values in $\mathsf{G}_\mathbb{K}$

Let Γ be a discrete group and fix $\Theta \subset \Pi$.

Definition 5.8 A representation $\rho\colon \Gamma \to \mathsf{G}_\mathbb{K}$ is Θ–*Anosov* if there exist constants $c \geq 0$ and $\mu > 0$ such that, for every $\gamma \in \Gamma$ and $\mathsf{a} \in \Theta$,

$$\mathsf{a}(a(\rho(\gamma))) \geq \mu|\gamma| - c.$$

If $\rho\colon \Gamma \to \mathsf{G}_\mathbb{K}$ is Θ–Anosov and Λ_a is as in Proposition 4.4, then $\Lambda_\mathsf{a}\rho\colon \Gamma \to \mathsf{PGL}(V_\mathbb{K})$ is projective Anosov. In particular, Section 2.2 applies to arbitrary $\mathsf{G}_\mathbb{K}$ and one obtains the following result:

Theorem 5.9 (Kapovich, Leeb and Porti [34]) *If $\rho\colon \Gamma \to \mathsf{G}_\mathbb{K}$ is Θ–Anosov then Γ is word hyperbolic and there exist continuous equivariant maps $\xi^\Theta_\rho\colon \partial\Gamma \to \mathcal{F}_\Theta(\mathsf{G}_\mathbb{K})$ and $\xi^{\mathrm{i}\Theta}_\rho\colon \partial\Gamma \to \mathcal{F}_{\mathrm{i}\Theta}(\mathsf{G}_\mathbb{K})$ such that the product map $(\xi^\Theta_\rho, \xi^{\mathrm{i}\Theta}_\rho)\colon \partial^{(2)}\Gamma \to \mathcal{F}^{(2)}_\Theta(\mathsf{G}_\mathbb{K})$ is transverse.*

We will sometime use the notation introduced in [39] and, if $x \in \partial\Gamma$ is a point, denote by

$$x^\Theta_\rho := \xi^\Theta_\rho(x) \in \mathcal{F}_\Theta(\mathsf{G}_\mathbb{K})$$

the image of x via the boundary map. If $\theta = \{\mathsf{a}_k\}$ consists of a single root, we will also write ξ^k_ρ and x^k_ρ instead of $\xi^{\{\mathsf{a}_k\}}_\rho$ and $x^{\{\mathsf{a}_k\}}_\rho$.

If $\Theta \subset \Pi$ contains the root a, we denote by $\pi_\mathsf{a}\colon \mathcal{F}_\theta(\mathsf{G}_\mathbb{K}) \to \mathcal{F}_\mathsf{a}(\mathsf{G}_\mathbb{K})$ the natural projection. It is easy to deduce from Corollary 3.4 the following more general statement:

Corollary 5.10 *Let $\rho\colon \Gamma \to \mathsf{G}_\mathbb{K}$ be a–Anosov and consider $\alpha > 0$. There exists C only depending on α and ρ such that, for every $\theta \subset \Pi$ containing a, if*

$$\xi^\mathsf{a}_\rho(\partial\Gamma) \cap \pi_\mathsf{a}\big(\rho(\gamma) \cdot B_{\theta,\alpha}(\rho(\gamma)) \cap \rho(\eta) \cdot B_{\theta,\alpha}(\rho(\eta))\big) \neq \varnothing$$

then

$$d(\gamma, \eta) \leq \big||\gamma| - |\eta|\big| + C.$$

Definition 5.11 Given a representation $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{K}}$ we define its *Anosov–Levi space* as $(\mathsf{E}_{\Theta_\rho})^*$, where

$$\Theta_\rho = \{\mathsf{a} \in \Pi : \rho \text{ is } \mathsf{a}\text{–Anosov}\}.$$

It is spanned by the fundamental weights $\{\omega_{\mathsf{a}} : \mathsf{a} \in \Theta_\rho\}$.

A more precise description of the indicator set of ρ can be given on its Anosov–Levi space. The following is a combination of Bridgeman, Canary, Labourie and Sambarino [9, Theorem 1.3], Potrie and Sambarino [38, Proposition 4.11] and Sambarino [44]:

Theorem 5.12 *Let $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{K}}$ be a representation. Then $\mathcal{Q}_{\rho(\Gamma),\Theta_\rho}$ is an analytic codimension-*1 *embedded submanifold of $(\mathsf{E}_{\Theta_\rho})^*$ that varies analytically with ρ. Moreover, its restriction to the dual of the vector space spanned by the periods is strictly convex.*

5.3 When some wall is not attained

The purpose of this subsection is to explore $\mathcal{Q}_{\rho(\Gamma)}$ in directions that are not controlled by the roots with respect to which ρ is Anosov.

Definition 5.13 Let $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{K}}$ be an a–Anosov representation. Consider $\Theta \subset \Pi$ with $\mathsf{a} \in \Theta$ and let μ^φ be a $(\rho(\Gamma), \varphi)$–Patterson–Sullivan measure on $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$ for some $\varphi \in (\mathsf{E}_\Theta)^*$. We say that ρ is *μ^φ–irreducible* if, for every $y \in \mathcal{F}_{\mathrm{i}\Theta}(\mathsf{G}_{\mathbb{K}})$,

$$\mu^\varphi(\mathrm{Ann}(y)) < \mu^\varphi(\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})).$$

It is clear that if $\rho(\Gamma)$ is Zariski-dense in $\mathsf{G}_{\mathbb{K}}$ then it is μ^φ–irreducible for any Patterson–Sullivan measure. Even assuming Zariski-density, the following result is a refinement of Quint [42, théorème 8.1] when Θ contains a root with respect to which ρ is Anosov. Indeed, in the general case treated by Quint, one needs to control the mass of shadows on the flag space associated to $\Pi \setminus \Theta$, and, as a result, the existence of a $(\rho(\Gamma), \varphi)$–Patterson–Sullivan measure only ensures that $\varphi + \rho_{\theta^c}$ is in $\mathcal{D}_{\rho(\Gamma)}$, where ρ_{θ^c} is a suitable form that is nonnegative on the Weyl chamber. In our case, the Anosov condition with respect to one root in Θ permits us to control φ directly.

Theorem 5.14 *Let $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{K}}$ be an a–Anosov representation. Consider $\Theta \subset \Pi$ with $\mathsf{a} \in \Theta$ and let μ^φ be a $(\rho(\Gamma), \varphi)$–Patterson–Sullivan measure on $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$ for some $\varphi \in (\mathsf{E}_\Theta)^*$. Assume ρ is μ^φ–irreducible, and that* $\operatorname{supp}\mu \subset \pi_{\mathsf{a}}^{-1}(\xi_\rho^{\mathsf{a}}(\partial\Gamma))$*. Then*

$$\varphi \in \mathcal{D}_{\rho(\Gamma)}.$$

The rest of the section is devoted to the proof of this result. We begin with the following lemma from Quint [42], who assumes that the representation is Zariski-dense, a hypothesis that is too strong for the applications we have in mind. We observe however that for the proof to work only μ^φ–irreducibility is needed. We sketch the proof for completeness.

Lemma 5.15 [42, lemme 8.2] *Let $\rho\colon \Gamma \to \mathsf{G}_{\mathbb{K}}$ be a representation and μ^φ be a $(\rho(\Gamma), \varphi)$–Patterson–Sullivan measure on $\mathcal{F}_\Theta(\mathsf{G}_{\mathbb{K}})$. Assume ρ is μ^φ–irreducible. Then there exists $\alpha_0 > 0$ such that, for every given $0 < \alpha < \alpha_0$, there exists $k > 0$, only depending on α, such that, for every $\gamma \in \Gamma$,*

$$k^{-1} e^{-\varphi(a(\rho(\gamma)))} \leq \mu^\varphi\big(\rho(\gamma) B_{\Theta,\alpha}(\rho(\gamma))\big) \leq k e^{-\varphi(a(\rho(\gamma)))}.$$

Proof Observe that μ^φ–irreducibility guarantees that there exist $\alpha, k > 0$ such that for every $\gamma \in \Gamma$ we have $\mu^\varphi(B_{\theta,\alpha}(\rho(\gamma))) \geq k$. Indeed, otherwise there would be a sequence of reals $\alpha_n \to 0$ and elements $\gamma_n \in \Gamma$ with $\mu^\varphi(B_{\theta,\alpha_n}(\rho(\gamma_n))) \leq 1/n$. We can assume, up to extracting a subsequence, that the complement of $B_{\theta,\alpha_n}(\rho(\gamma))$ converges to $\operatorname{Ann}(y)$ for some $y \in \mathcal{F}_{\mathrm{i}\theta}$, and this contradicts μ^φ–irreducibility.

The result then follows from the definition of the $(\rho(\Gamma), \phi)$–Patterson–Sullivan measure using (4-5). □

The rest of the proof of Theorem 5.14 is similar to the argument showing that if there exists a Patterson–Sullivan density of a given exponent, then this exponent must be greater than the critical exponent; see for example Sullivan [46] and Quint's notes [43, Theorem 4.11].

Proof of Theorem 5.14 We have to show that, for every $s > 0$,

$$\sum_{\gamma \in \Gamma} e^{-(1+s)\varphi(a(\rho(\gamma)))} < \infty.$$

Corollary 5.10 implies that given $\alpha > 0$ there exists $N \in \mathbb{N}$ such that, if $t > 0$ and

$$\Gamma_t = \{\gamma \in \Gamma : t \leq |\gamma| \leq t+1\},$$

then, for every $x \in \partial\Gamma$,

$$\#\{\gamma \in \Gamma_t : \pi_{\mathsf{a}}^{-1}(\xi_\rho^{\mathsf{a}}(x)) \cap \rho(\gamma) B_{\Theta,\alpha}(\rho(\gamma)) \neq \varnothing\} \leq N.$$

Lemma 5.15 now yields, for every $t \geq 0$,

$$\infty > \mu^\varphi(\pi_{\mathsf{a}}^{-1}(\xi_\rho^{\mathsf{a}}(\partial\Gamma))) \geq C \sum_{\gamma \in \Gamma_t} e^{-\varphi(a(\rho(\gamma)))}, \tag{5-1}$$

where C is independent of t. This is to say, there exists $K > 0$ independent of $t \in \mathbb{R}_+$ such that

$$\sum_{\gamma \in \Gamma_t} e^{-\varphi(a(\rho(\gamma)))} < K.$$

Since ρ is a–Anosov, for any norm N on E there exist positive δ and C such that

$$N(a(\rho(\gamma))) \geq \delta|\gamma| - C.$$

One concludes that, for every $s > 0$,

$$\sum_{\gamma \in \Gamma} e^{-\varphi(a(\rho(\gamma)) - sN(a(\rho(\gamma))}) \leq \sum_{n=0}^{\infty} \sum_{\gamma \in \Gamma_n} e^{-\varphi(a(\rho(\gamma)))} e^{-sN(\rho(\gamma))} \leq Ke^{C} \sum_{n=0}^{\infty} e^{-\delta sn} < \infty.$$

Consider now the counting measure ν on E defined by

$$\nu(B) = \#\{a(\Gamma) \cap B\}.$$

The above implies that the measure $\nu' = e^{-\varphi}\nu$ has growth indicator $\Psi_{\nu'} \leq 0$, and so [41, Corollary 3.1.5] gives

$$0 \geq \Psi_{e^{-\varphi}\nu} = \Psi_\nu - \varphi = \Psi_{\rho(\Gamma)} - \varphi,$$

as desired. □

6 Anosov representations with Lipschitz limit set

In this section we will prove Theorem A. We will hence fix some notation throughout this section.

Assumption 6.1 The group Γ will be a word-hyperbolic group whose boundary $\partial\Gamma$ is homeomorphic to a sphere of dimension d_Γ.[6] We will also fix a projective Anosov representation $\rho\colon \Gamma \to \mathsf{PSL}_d(\mathbb{R})$ such that the sphere $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$, ie it is locally the graph of a Lipschitz map. Note that we have restricted ourselves to $\mathbb{K} = \mathbb{R}$.

6.1 The p^{th} Jacobian

Given a line ℓ contained in a $(p+1)$–dimensional subspace V of $\mathbb{R}^d$, the space of infinitesimal deformations of ℓ inside V

$$\mathsf{T}_\ell\mathbb{P}(V) \subset \mathsf{T}_\ell\mathbb{P}(\mathbb{R}^d)$$

[6] It follows from [32, Theorem 4.4] that this is the case as soon as $\partial\Gamma$ has an open subset homeomorphic to $\mathbb{R}^{d_\Gamma}$.

carries a natural volume form induced by the choice of a scalar product τ on $\mathbb{R}^d$. Namely, if one considers the τ–orthogonal decomposition $V = \ell \oplus \ell^{\perp}_V$, then one canonically identifies $\mathsf{T}_\ell \mathbb{P}(V) = \hom(\ell, \ell^{\perp}_V)$ and thus one can define $\Omega_{\ell,V} \in \bigwedge^p(\mathsf{T}_\ell \mathbb{P}(V))$ by

$$\Omega_{\ell,V}(\varphi_1, \dots, \varphi_p) = \frac{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)}{\|v\|^{p+1}}$$

for any $v \in \ell \setminus \{0\}$.

Definition 6.2 The linear form $\mathcal{J}^u_p \in (\mathsf{E}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}})^*$, defined by

$$\mathcal{J}^u_p = (p+1)\omega_1 - \omega_{p+1},$$

is called *the p^{th} unstable Jacobian*.

Lemma 6.3 *Given $g \in \mathsf{PSL}_d(\mathbb{R})$ and a partial flag $(\ell, V) \in \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(\mathbb{R}^d)$,*

$$g^*\Omega_{g\ell,gV} = \exp\bigl(-\mathcal{J}^u_p\bigl(\mathsf{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,(\ell,V))\bigr)\bigr)\Omega_{\ell,V}.$$

Proof This is an explicit computation using (4-6) and the definition of $\Omega_{\ell,V}$.

Indeed, whenever $\varphi_1, \dots, \varphi_p \in \hom(\ell, \ell^{\perp}_V)$ are linearly independent, for any $v \in \ell \setminus \{0\}$ the vectors $\{v, \varphi_1(v), \dots, \varphi_p(v)\}$ form a basis of V, and thus

$$\begin{aligned}
&g^*\Omega_{g\ell,gV}(\varphi_1, \dots, \varphi_p) \\
&\qquad = \Omega_{g\ell,gV}(g\varphi_1, \dots, g\varphi_p) \\
&\qquad = \frac{gv \wedge (g\varphi_1)(gv) \wedge \cdots \wedge (g\varphi_p)(gv)}{\|gv\|^{p+1}} = \frac{gv \wedge g(\varphi_1(v)) \wedge \cdots \wedge g(\varphi_p(v))}{\|gv\|^{p+1}} \\
&\qquad = \frac{gv \wedge g(\varphi_1(v)) \wedge \cdots \wedge g(\varphi_p(v))}{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)} \frac{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)}{\|v\|^{p+1}} \frac{\|v\|^{p+1}}{\|gv\|^{p+1}} \\
&\qquad = \exp\bigl(\omega_{p+1}(\mathsf{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,V)) - (p+1)\omega_1(\mathsf{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,\ell))\bigr)\Omega_{\ell,V}. \qquad \square
\end{aligned}$$

6.2 Existence of a $\mathcal{J}^u_{d_\Gamma}$–Patterson–Sullivan measure

Proposition 6.4 *Under Assumption 6.1, there exists a $(\rho(\Gamma), \mathcal{J}^u_{d_\Gamma})$–Patterson–Sullivan measure on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{d_\Gamma}\}}$, which we will denote by ν_ρ.*

Proof It follows from Rademacher's theorem [20, Theorem 3.2] that $\xi^1_\rho(\partial\Gamma)$ has a well-defined Lebesgue measure class (see [22, Section 3.2]), and that Lebesgue almost every point $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ has a well-defined tangent space. This defines a $(d_\Gamma+1)$–dimensional vector subspace $x^{d_\Gamma+1}_\rho \in \mathcal{F}_{\{\mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$ such that

$$\mathsf{T}_{\xi^1_\rho(x)}(\xi^1_\rho(\partial\Gamma)) = \hom(\xi^1_\rho(x), x^{d_\Gamma+1}_\rho / \xi^1_\rho(x)). \tag{6-1}$$

Consider the ρ–equivariant measurable map $\zeta_\rho\colon \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$ defined by

$$\zeta_\rho(\xi^1_\rho(x)) = (\xi^1_\rho(x), x^{d_\Gamma+1}_\rho). \tag{6-2}$$

We can then define a volume form on $\xi^1_\rho(\partial\Gamma)$ via

$$\xi^1_\rho(x) \mapsto \Omega_{\zeta_\rho(\xi^1_\rho(x))}.$$

This form is defined Lebesgue almost everywhere and thus defines a Lebesgue measure on $\xi^1_\rho(\partial\Gamma)$, which we will denote by ν_ρ. Lemma 6.3 implies directly that the pushforward $(\zeta_\rho)_*\nu_\rho$ is the desired measure. □

6.3 When $\partial\Gamma$ is a circle

Recall from Section 1 that we say that ρ is *weakly irreducible* if the vector space $\operatorname{span}(\xi^1_\rho(\partial\Gamma))$ is the whole space.

Lemma 6.5 *Under Assumption 6.1 together with weakly irreducibility of ρ and $d_\Gamma = 1$, one has that ρ is μ^φ–irreducible for any $(\rho(\Gamma),\varphi)$–Patterson–Sullivan measure on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\mathbb{R}^d)$ whose projection is absolutely continuous with the measure ν_ρ constructed in Proposition 6.4.*

Proof If this were not the case, there would exist $(W_0, P_0) \in \mathcal{F}_{\{\mathsf{a}_{d-2},\mathsf{a}_{d-1}\}}(\mathbb{R}^d)$ such that $\operatorname{Ann}(W_0, P_0)$ would have full μ^φ–mass; as ρ is projective Anosov we can furthermore assume that $P_0 = \xi^{d-1}_\rho(x)$ for some $x \in \partial\Gamma$ and thus the condition $\xi^1_\rho(y) \subset P_0$ only occurs for $y = x$.

Hence, since the projection of μ^φ is absolutely continuous with respect to ν_ρ, one has that, for μ^φ–almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$, the vector space x^2_ρ from Section 6.2 intersects W_0.

Let us choose a scalar product τ on $\mathbb{R}^d$, and the induced distance function of $\mathbb{P}(\mathbb{R}^d)$. Let us denote by $[W_0]$ the quotient vector space $\mathbb{R}^d/W_0$. It is a 2–dimensional vector space and every line $\ell \notin W_0$ defines a line $[\ell \oplus W_0]$ in $[W_0]$. Moreover, for every $\delta > 0$, the double quotient projection

$$\pi\colon \{\ell \in \mathbb{P}(\mathbb{R}^d) : \angle_\tau(\ell, W_0) > \delta\} \to \mathbb{P}([W_0]),$$

defined by $\pi(\ell) = [[\ell \oplus W_0]]$, is Lipschitz.

We denote by $U_\delta \subset \xi^1_\rho(\partial\Gamma)$ the relative open subset defined by

$$U_\delta = \{\ell \in \xi^1_\rho(\partial\Gamma) : \angle_\tau(\ell, W_0) > \delta\}$$

and consider the Lipschitz map $\pi|_{U_\delta}: U_\delta \to \mathbb{P}([W_0])$. Since, by assumption, for μ^φ–almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ the plane x^2_ρ intersects W_0, one concludes from (6-1) that $\pi|_{U_\delta}$ has zero derivative ν_ρ–almost everywhere.

Since Lipschitz maps are absolutely continuous, and in particular satisfy the fundamental theorem of calculus, we deduce that $\pi|_{\xi^1_\rho(\partial\Gamma)}$ is constant. This implies that

$$\xi^1_\rho(\partial\Gamma) \subset W_0 \oplus \xi^1_\rho(x)$$

for any $x \in \partial\Gamma$, which contradicts the weak irreducibility assumption. □

We can now prove Theorem A when $d_\Gamma = 1$:

Corollary 6.6 *Let Γ be a word-hyperbolic group such that $\partial\Gamma$ is homeomorphic to a circle. Let $\rho: \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a weakly irreducible a_1–Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz curve. Then*

$$\mathsf{a}_1 \in \mathcal{Q}_{\rho(\Gamma)}.$$

Proof Note that $\mathsf{a}_1 = \mathcal{J}^u_1$ is the first unstable Jacobian. Since $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz circle, it has Hausdorff dimension 1, and thus Corollary 1.1 implies that $h^{\mathsf{a}_1}_\rho \geq 1$.

On the other hand, Proposition 6.4 provides a $(\rho(\Gamma), \mathcal{J}^u_1)$–Patterson–Sullivan measure $\mu^{\mathcal{J}^u_1}$ on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(V_\mathbb{R})$ that projects to the Lebesgue measure on $\xi^1_\rho(\partial\Gamma)$. Since ρ is weakly irreducible, Lemma 6.5 implies that it is $\mu^{\mathcal{J}^u_1}$–irreducible, thus Theorem 5.14 applies to give

$$\mathsf{a}_1 = \mathcal{J}^u_1 \in \mathcal{D}_{\rho(\Gamma)}.$$

This is to say, $h_\rho(\mathsf{a}_1) \leq 1$. □

Before proceeding to arbitrary d_Γ we record a direct consequence of Corollary 6.6. Let us say that ρ is *coherent* if the first root arising in $\operatorname{span}(\xi^1_\rho(\partial\Gamma))$ is a_1.

Corollary 6.7 *Let Γ be a word-hyperbolic group such that $\partial\Gamma$ is homeomorphic to a circle. Let $\rho: \Gamma \to \mathsf{G}_\mathbb{K}$ be an a–Anosov representation and assume there exists a proximal, real representation $\Lambda: \mathsf{G}_\mathbb{K} \to \mathsf{PGL}(V_\mathbb{R})$ with first root a such that $\Lambda \circ \rho$ is coherent. Then*

$$\mathsf{a} \in \mathcal{Q}_{\rho(\Gamma)}.$$

6.4 When $\partial\Gamma$ has arbitrary dimension

Recall that a subgroup $\Gamma \subset \mathsf{PGL}(V_\mathbb{K})$ is *strongly irreducible* if any finite-index subgroup acts irreducibly. It is well known that this is equivalent to the fact that the connected component of the identity of the Zariski closure of Γ acts irreducibly on $\mathbb{K}^d$.

We will need the following lemma (which does not require Assumption 6.1):

Lemma 6.8 *Let $\eta\colon \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a strongly irreducible a_1–Anosov representation, and assume that there exists $p \in [\![1, d-1]\!]$ and a measurable η–equivariant section $\zeta\colon \partial\Gamma \to \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_p\}}(\mathbb{R}^d)$. Then η is μ^φ–irreducible for any $(\rho(\Gamma),\varphi)$–Patterson–Sullivan measure on $\mathcal{F}_\Theta(\mathbb{K}^d)$.*

Proof Otherwise, we would be able to find a subspace $W_0 \in \mathcal{F}_{\{\mathsf{a}_{d-p}\}}(\mathbb{R}^d)$ such that for almost every[7] $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ one has $\zeta(x)^p \cap W_0 \neq \{0\}$. Since ζ is η–equivariant, we would find a p–dimensional subspace V such that, for every $\gamma \in \Gamma$,

$$\eta(\gamma)V \cap W_0 \neq \{0\}.$$

This implies that for every g in the Zariski closure of $\eta(\Gamma)$ it holds that $\dim gV \cap W_0 \geq 1$. The contradiction comes from Labourie [35, Proposition 10.3]: if G is an algebraic subgroup of $\mathsf{SL}(n,\mathbb{R})$, $C \in \mathcal{G}_k(\mathbb{R}^d)$, $B \in \mathcal{G}_{d-k}(\mathbb{R}^d)$ and $\dim(gC \cap B) \geq 1$ for every $g \in \mathsf{G}$, then the connected component of the identity of G is not irreducible. $\square$

We can now prove Theorem A for arbitrary d_Γ.

Corollary 6.9 *Under Assumption 6.1 together with strong irreducibility of ρ,*

$$\mathcal{J}^u_{d_\Gamma} \in \mathcal{Q}_\rho(\Gamma).$$

Proof Since $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz sphere it has Hausdorff dimension d_Γ, and thus Corollary 1.1 implies that $h_\rho(\mathcal{J}^u_{d_\Gamma}) \geq 1$. Proposition 6.4 guarantees the existence of a $(\rho(\Gamma),\mathcal{J}^u_{d_\Gamma})$–Patterson–Sullivan measure. Moreover, the equivariant map from (6-2) allows us to apply Lemma 6.8, and thus we have the hypothesis of Theorem 5.14. Consequently, $h_\rho(\mathcal{J}^u_{d_\Gamma}) \leq 1$. $\square$

7 $(1,1,p)$–hyperconvex representations and a C^1–dichotomy for surface groups

In this section we will consider projective Anosov representations whose image of the boundary map is a C^1–submanifold. In the second part of the section we will prove Corollary 7.8, providing a C^1–dichotomy for surface groups.

[7]This is with respect to the pushed-forward measure $\pi_*\mu^\varphi$, where $\pi\colon \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_p\}}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$ consists of forgetting the p^{th} coordinate.

7.1 $(1,1,p)$–hyperconvex representations

Definition 7.1 We say a $\{\mathsf{a}_1,\mathsf{a}_p\}$–Anosov representation $\rho\colon\Gamma\to\mathsf{PGL}_d(\mathbb{R})$ is $(1,1,p)$–*hyperconvex* if, for every pairwise distinct $x,y,z\in\partial\Gamma$, the sum

$$\xi^1(x)+\xi^1(y)+\xi^{d-p}(z)$$

is direct.

Example 7.2 Zariski-dense hyperconvex representations can be obtained by deforming $\mathsf{S}^k\circ\iota$, where S^k denotes the k^{th} symmetric power and $\iota\colon\Gamma\to\mathsf{PO}(1,p)$ is the inclusion of a cocompact lattice; see Pozzetti, Sambarino and Wienhard [39, Corollary 7.6].

Hyperconvex representations were introduced by Labourie [35] for surface groups, and further studied by Zhang and Zimmer [48] when the boundary of Γ is topologically a sphere and by Pozzetti, Sambarino and Wienhard [39] for arbitrary hyperbolic groups. In both [39, Proposition 7.4] and [48, Theorem 1.1] one finds:

Theorem 7.3 *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho\colon\Gamma\to\mathsf{PGL}_d(\mathbb{R})$ be a $(1,1,p)$–hyperconvex representation. Then $\xi^1_\rho(\partial\Gamma)$ is a C^1–sphere.*

Theorem A then gives:

Corollary 7.4 *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho\colon\Gamma\to\mathsf{PSL}_d(\mathbb{R})$ be strongly irreducible and $(1,1,p)$–hyperconvex. Then $h_\rho(\mathfrak{J}^u_p)=1$.*

Remark 7.5 This generalizes Potrie and Sambarino [38, Corollary 7.1]. Observe however that, since the limit set $\xi^1(\partial\Gamma)$ is a C^1–submanifold of $\mathbb{P}(\mathbb{R}^d)$, the arguments of [38] adapt directly to give a version of Corollary 7.4 without requiring strong irreducibility.

Theorem 7.6 (Glorieux, Monclair and Tholozan [25]) *Let $\rho\colon\Gamma\to\mathsf{PGL}_d(\mathbb{R})$ be an a_1–Anosov representation that preserves a properly convex domain. Then*

$$2h_\rho(\omega_1+\omega_{d-1})\le\dim_{\mathrm{Hff}}((\xi^1,\xi^{d-1})(\partial\Gamma)),$$

where $(\xi^1,\xi^{d-1})\colon\partial\Gamma\to\mathbb{P}(\mathbb{R}^d)\times\mathbb{P}((\mathbb{R}^d)^*)$.

As an application of Corollary 7.4 we show that, for $(1,1,p)$–hyperconvex representations with $p<d-1$, such a bound can never be achieved:

Proposition 7.7 *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho\colon\Gamma\to\mathsf{PGL}_d(\mathbb{R})$ be strongly irreducible and $(1,1,p)$–hyperconvex. If $p<d-1$, then*

$$2h_\rho(\omega_1+\omega_{d-1})<(1-\varepsilon)(p-1),$$

where $\varepsilon>0$ only depends on the $\{\mathsf{a}_1,\mathsf{a}_p\}$–Anosov constants of ρ.

Proof Since $p<d-1$ the functional $\phi\in\mathsf{E}^*$ given by

$$\phi=\frac{\omega_p-\omega_1}{p-1}-\frac{\omega_{d-1}-\omega_1}{d-2}$$

is nonzero. Moreover observe that, for every $v\in\mathsf{E}^+$,

$$\phi(v)\geq\frac{d-p-1}{d-2}\mathsf{a}_p(v).$$

Since ρ is a_p–Anosov, the last computation implies $\ker\phi\cap\mathcal{L}_{\rho(\Gamma)}=\{0\}$. This is to say that $\phi\in(\mathcal{L}_{\rho(\Gamma)})^*$, in particular ϕ has a well-defined entropy $h_\rho(\phi)\in(0,\infty)$. Then

$$h_\rho\left(\frac{p-1}{d-2}((d-1)\omega_1-\omega_{d-1})\right)=h_\rho(\mathcal{J}^u_{p-1}+(p-1)\phi)\leq\frac{h_\rho(\phi)}{h_\rho(\phi)+p-1},\tag{7-1}$$

where the equality comes from the equality between the corresponding linear forms and the inequality follows from Lemma 5.6 together with Corollary 7.4 stating that $h_\rho(\mathcal{J}^u_{p-1})=1$.

Finally, observe that

$$\begin{aligned}\tfrac{1}{2}(p-1)(\omega_1-\omega_{d-1})&=\frac{1}{2}\left(\frac{p-1}{d-2}((d-1)\omega_1-\omega_{d-1})+\frac{p-1}{d-2}((d-1)\omega_{d-1}-\omega_1)\right)\\&=\tfrac{1}{2}\big(\mathcal{J}^u_{p-1}+(p-1)\phi+(\mathcal{J}^u_{p-1}+(p-1)\phi)\circ\mathrm{i}\big),\end{aligned}$$

where $\mathrm{i}\colon\mathsf{E}\to\mathsf{E}$ is the opposition involution. Together with (7-1) and Lemma 5.6, this yields

$$\begin{aligned}\frac{2}{p-1}h_\rho(\omega_1-\omega_{d-1})&\leq2\frac{h_\rho(\mathcal{J}^u_{p-1}+(p-1)\phi)h_\rho\big((\mathcal{J}^u_{p-1}+(p-1)\phi)\circ\mathrm{i}\big)}{h_\rho(\mathcal{J}^u_{p-1}+(p-1)\phi)+h_\rho\big((\mathcal{J}^u_{p-1}+(p-1)\phi)\circ\mathrm{i}\big)}\\&=h_\rho(\mathcal{J}^u_{p-1}+(p-1)\phi)\leq\frac{h_\rho(\phi)}{h_\rho(\phi)+p-1}<1,\end{aligned}$$

since entropy is i–invariant.

To conclude the proof we observe that the functional ϕ belongs to the Anosov–Levi space of every $\{\mathsf{a}_1,\mathsf{a}_p\}$–Anosov representation. Its entropy thus varies continuously

(Theorem 5.12), and hence
$$\eta \mapsto \frac{h_\eta(\phi)}{h_\eta(\phi)+p-1}$$
is bounded away from 1 on compact subsets of $\mathfrak{X}_{\{\mathsf{a}_1,\mathsf{a}_p\}}(\Gamma,\mathsf{PGL}_d(\mathbb{R}))$. □

C^1–dichotomy

Now we prove the C^1–dichotomy announced in Section 1. As we will later see (Sections 9 and 10) there are many projective Anosov representations of surface groups where the image of the boundary map is Lipschitz. However, when we embed the surface group into $\mathsf{PSL}_2(\mathbb{R})$ and look at small deformations of representations
$$\Gamma \to \mathsf{PSL}_2(\mathbb{R}) \xrightarrow{R} \mathsf{PSL}_d(\mathbb{R}),$$
where R satisfies additional proximality assumptions ensuring that the representation is $\{\mathsf{a}_1,\mathsf{a}_2\}$–Anosov, then the image of the boundary map is never Lipschitz.

Recall that $g \in \mathsf{PGL}_d(\mathbb{R})$ is *proximal* if the generalized eigenspace associated to its greatest eigenvalue (in modulus) has dimension 1. A representation $R\colon G \to \mathsf{PGL}_d(\mathbb{R})$ of a given group G is *proximal* if its image contains a proximal element.

Corollary 7.8 *Let $R\colon \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ be a (possibly reducible) proximal representation such that $\bigwedge^2 R$ is also proximal. Let S be a closed connected surface of genus ≥ 2 and let $\rho_0\colon \pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ be discrete and faithful. Then we have the following dichotomy:*

- (i) *If the top two weight spaces of R belong to the same irreducible factor, then for every small deformation $\rho\colon \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $R\rho_0$ the curve $\xi^1_\rho(\partial\pi_1 S)$ is C^1.*
- (ii) *Otherwise, for every weakly irreducible small deformation $\rho\colon \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $R\rho_0$ the curve $\xi^1_\rho(\partial\pi_1 S)$ is not Lipschitz.*

Proof By the proximality assumptions on R, the representation
$$\rho := R\rho_0\colon \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$$
is $\{\mathsf{a}_1,\mathsf{a}_2\}$–Anosov: indeed, $\mathsf{PSL}(2,\mathbb{R})$ has rank one. This implies, on the one hand, that the discrete and faithful representation ρ_0 is Anosov, and on the other hand that the composition of ρ_0 with any proximal representation is a_1 Anosov.

Furthermore, if the first two weights of R belong to the same irreducible factor, the representation ρ is also $(1,1,2)$–hyperconvex [39, Proposition 6.16]. Hyperconvexity is an open property in $\mathfrak{X}(\pi_1 S, \mathsf{PSL}_d(\mathbb{R}))$ (Pozzetti, Sambarino and Wienhard [39]) and thus Theorem 7.3 implies that every small deformation of ρ has C^1 limit set.

If instead the two top weights of R belong to different irreducible factors, then it follows from the representation theory of $\mathsf{SL}_2(\mathbb{R})$ that

$$h_\rho(\mathsf{a}_1) = h_\rho(\mathscr{J}_1^u) = 2.$$

Note that the entropy of $\mathscr{J}_1^u$ is continuous on $\mathfrak{X}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\pi_1 S, \mathsf{PSL}_d(\mathbb{R}))$; see Theorem 5.12. In particular there exists a neighborhood $\mathcal{U}$ of ρ such that $h_\eta(\mathscr{J}_1^u) > 1$ for every $\eta \in \mathcal{U}$. Theorem A implies that no weakly irreducible representation in $\mathcal{U}$ can have Lipschitz limit set. □

The regular case, Corollary 7.8(i), is inspired by Labourie [35], who treated the case (of arbitrary deformations) of the irreducible representations, and was proven in Pozzetti, Sambarino and Wienhard [39, Proposition 9.4]. The novelty of this paper is item (ii), inspired by Barbot [1], who proved it for $d = 3$. We believe both items placed together give a clearer picture.

It is easy to obtain similar results for other groups G by considering suitable linear representations. On the other hand, the double proximality assumption is necessary: the composition of a maximal representation not in the Hitchin component and the irreducible linear representation of $\mathsf{Sp}(2n, \mathbb{R})$ of highest weight w_n is proximal, but its second exterior power is not proximal. It is possible to check that no small Zariski-dense deformation satisfies either (i) or (ii).

Along the same lines we can deduce that some natural Anosov representations of hyperbolic lattices do not have Lipschitz boundary maps:

Corollary 7.9 *Let $\Gamma < \mathsf{PO}(1, n)$ be a lattice, $n \geq 3$ and $\rho_1 : \Gamma \to \mathsf{PO}(1, m)$ strictly dominated by the lattice embedding ρ_0. Then, for any Zariski-dense small deformation of $\rho_0 \oplus \rho_1^{n-1}$, the limit set $\xi_\rho^1(\partial\Gamma)$ is not Lipschitz.*

Examples of lattices Γ admitting such representations were constructed by Danciger, Guéritaud and Kassel [17, Proposition 1.8].

8 $\mathbb{H}^{p,q}$ convex–cocompact representations

Generalizing work of Mess [37] and Barbot and Mérigot [3], Danciger, Guéritaud and Kassel [16] introduced a class of representations called $\mathbb{H}^{p,q}$*–convex–cocompact*.

These form another interesting class of representations with Lipschitz boundary map where Theorem A applies.

Let $d = p + q$ with $p, q \geq 1$ and let Q be a symmetric bilinear form on $\mathbb{R}^d$ of signature (p, q). The subspace of $\mathbb{P}(\mathbb{R}^d)$ consisting of negative definite lines is called the *pseudo-Riemannian hyperbolic space* and denoted by

$$\mathbb{H}^{p,q-1} = \{\ell \in \mathbb{P}(\mathbb{R}^d) : Q|_{\ell\setminus\{0\}} < 0\}.$$

The cone of isotropic lines is usually denoted by $\partial\mathbb{H}^{p,q-1}$.

Instead of the original definition of convex–cocompactness, we recall the characterization given by [16, Theorem 1.11]:

Definition 8.1 An a_1–Anosov representation $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ is $\mathbb{H}^{p,q-1}$*–convex–cocompact* if, for every pairwise distinct triple of points $x, y, z \in \partial\Gamma$, the restriction $Q|_{\xi^1_\rho(x)\oplus\xi^1_\rho(y)\oplus\xi^1_\rho(z)}$ has signature $(2, 1)$.

When Γ_0 is a cocompact lattice in $\mathsf{SO}(p, 1)$, $\mathbb{H}^{p,1}$–convex–cocompact representations of Γ_0 are usually referred to as AdS–*quasi-Fuchsian groups*. Barbot [2] proved that these groups form connected components of the character variety $\mathfrak{X}(\Gamma_0, \mathsf{SO}(p, 2))$ only consisting of Anosov representations. In [23] Glorieux and Monclair prove that the limit set of an AdS–quasi-Fuchsian group is never a C^1–submanifold, except for Fuchsian groups.

The following is well known and easy to verify; see for example Glorieux and Monclair [24, Proposition 5.2].

Proposition 8.2 *Assume that $\partial\Gamma$ is homeomorphic to a $(p-1)$–dimensional sphere. If $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ is $\mathbb{H}^{p,q}$–convex–cocompact, then $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q-1}$.*

Proof The space $\partial\mathbb{H}^{p,q-1}$ admits a twofold cover that splits as $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$. It is immediate to verify that, since for every pairwise distinct triple $(x, y, z) \in \partial\Gamma$ we have that $Q|_{\xi^1_\rho(x)\oplus\xi^1_\rho(y)\oplus\xi^1_\rho(z)}$ has signature $(2, 1)$, each one of the two lifts of $\xi^1_\rho(\partial\Gamma)$ to $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ is the graph of a 1–Lipschitz function $f : \mathbb{S}^{p-1} \to \mathbb{S}^{q-1}$, and, as such, is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q-1}$. □

Theorem A then yields:

Corollary 8.3 *Assume that $\partial\Gamma$ is homeomorphic to a $(p-1)$–dimensional sphere and let $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ be $\mathbb{H}^{p,q-1}$–convex–cocompact. Then*

- *for $p = 2$ and ρ weakly irreducible, $h_\rho(\mathfrak{J}^u_1) = 1$;*
- *for $p \geq 3$ and ρ strongly irreducible, $h_\rho(\mathfrak{J}^u_{p-1}) = 1$.*

One concludes the following upper bound for the entropy of the spectral radius inspired by Glorieux and Monclair [24].

Corollary 8.4 *Assume that $\partial\Gamma$ is homeomorphic to a $(p-1)$–dimensional sphere and let $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ be $\mathbb{H}^{p,q-1}$–convex–cocompact. Then*

- *for $p = 2$ and ρ weakly irreducible, $h_\rho(\omega_1) \leq 1$;*
- *for $p \geq 3$ and ρ strongly irreducible, $h_\rho(\omega_1) \leq p-1$.*

Proof Assume first that $p \leq q$ and note that, for every $g \in \mathsf{PO}(p,q)$,

$$(\omega_p - \omega_1)(\lambda(g)) = \lambda_2(g) + \cdots + \lambda_p(g) \geq 0.$$

By definition, $\mathfrak{J}^u_{p-1} = p\omega_1 - \omega_p$, and thus

$$\frac{h_\rho(\omega_1)}{p-1} = h_\rho((p-1)\omega_1) \leq h_\rho(\mathfrak{J}^u_{p-1}) = 1$$

by Corollary 8.3. The only difference in the case where $q < p$ is that $\mathfrak{J}^u_{p-1} = p\omega_1 - \omega_q$, but the same argument applies verbatim. □

The entropy for the first fundamental weight has a particular meaning for projective Anosov representations into $\mathsf{PO}(p,q)$, notably for $q \geq 2$. Fix $o \in \mathbb{H}^{p,q-1}$ and consider

$$S^o = \{W < \mathbb{R}^d : o \subset W, \dim W = q \text{ and } Q|_W \text{ is negative definite}\}.$$

This is a totally geodesic embedding of the symmetric space $X_{p,q-1}$ of $\mathsf{PO}(p,q-1)$ in the symmetric space $X_{p,q}$.

Given a projective Anosov representation $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ one defines the open subset of $\mathbb{H}^{p,q-1}$

$$\Omega_\rho = \{o \in \mathbb{H}^{p,q-1} : Q(o, \xi^1_\rho(x)) \neq 0 \text{ for all } x \in \partial\Gamma\}.$$

Carvajales [13] shows that, assuming $\Omega_\rho \neq \varnothing$, for every $o \in \Omega_\rho$ one has

$$\lim_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : d_{X_{p,q}}(S^o, \rho(\gamma)S^o)\}}{t} = h_\rho(\omega_1),$$

and provides an asymptotic for this counting function; see [13, Theorem A].

When ρ is also $\mathbb{H}^{p,q-1}$–convex–cocompact, Glorieux and Monclair [24, Section 1.2] introduce a *pseudo-Riemannian critical exponent* δ_ρ, and show, in particular, that

$$\delta_\rho \leq p-1$$

[24, Theorem 1.2]. Carvajales proves [13, Remarks 6.9 and 7.15] that $\delta_\rho = h_\rho(\omega_1)$, so Corollary 8.4 provides a different proof of [24, Theorem 1.2] when Γ is assumed to have boundary homeomorphic to a $(p-1)$–dimensional sphere.

We finish the section with a direct application of Theorem 5.4 and Corollary 8.3 allowing us to get a bound for the Riemannian critical exponent. We use freely the notation from Remark 5.5.

Consider a representation $\Lambda\colon \mathsf{PO}(p,1) \to \mathsf{PO}(p,q)$ such that its image stabilizes a $(p+1)$–dimensional subspace V of $\mathbb{R}^d$ where $Q|_V$ has signature $(p,1)$. Endow the symmetric space $X_{p,q}$ with a $\mathsf{PO}(p,q)$–invariant Riemannian metric such that the totally geodesic copy of $\mathbb{H}^p$ in $X_{p,q}$ induced by Λ has constant curvature -1. In particular, if $\iota\colon \Gamma \to \mathsf{PO}(p,1)$ is the lattice embedding, $h^X_{\Lambda\circ\iota} = p-1$. We show that this is an upper bound for any strongly irreducible, $\mathbb{H}^{p,q-1}$–convex–cocompact representation:

Proposition 8.5 *Assume that $\partial\Gamma$ is homeomorphic to a $(p-1)$–dimensional sphere, and let $\rho\colon \Gamma \to \mathsf{PO}(p,q)$ be strongly irreducible and $\mathbb{H}^{p,q-1}$–convex–cocompact. Then*

$$h^X_\rho \leq p-1.$$

Proof In view of Theorem 5.4 (or more precisely Remark 5.5), it suffices to recall that $\mathcal{D}_{\rho(\Gamma)}$ is convex (Proposition 5.3) and that, by Corollary 8.3,

$$\mathcal{J}^u_{p-1} \in \mathcal{Q}_{\rho(\Gamma)}.$$

See Potrie and Sambarino [38, Section 1.1] for more details. □

9 Maximal representations

An important class of representations that are in general only Anosov with respect to one maximal parabolic subgroup but admit boundary maps with Lipschitz image are maximal representations into Hermitian Lie groups. In this case the Lipschitz property for the image of the boundary map is a consequence of a positivity/causality property of the boundary map. We first describe the causal structure on the Shilov boundary of a Hermitian symmetric space of tube type, introduce the notion of a positive curve and show that the image of any positive curve (that is not necessarily equivariant with respect to a representation) is a Lipschitz submanifold. We then show how this applies to maximal representations and allows us to prove Theorem 9.8, the main result of this section. We also deduce consequences for the orbit growth rate on the symmetric space.

9.1 Causal structure and positive curves

Let $\mathsf{G}_\mathbb{R}$ be a simple Hermitian Lie group of tube type. Examples to keep in mind are the symplectic group $\mathsf{G}_\mathbb{R} = \mathsf{Sp}(2n, \mathbb{R})$ or the orthogonal group $\mathsf{G}_\mathbb{R} = \mathsf{SO}_0(2, n)$. The Shilov boundary $\check{S}$ of the bounded domain realization of the symmetric space associated to $\mathsf{G}_\mathbb{R}$ is a flag variety $\mathsf{G}_\mathbb{R}/\check{P}$, where $\check{P}$ is a maximal parabolic subgroup determined by a specific simple root $\{\check{\mathsf{a}}\}$. In the first of our main examples, $\mathsf{G}_\mathbb{R} = \mathsf{Sp}(2n, \mathbb{R})$, the parabolic subgroup $\check{P}$ in question is the stabilizer of a Lagrangian subspace $L \in \mathscr{L}(\mathbb{R}^{2n})$ so $\check{\mathsf{a}} = \mathsf{a}_n$, and in the second, $\mathsf{G}_\mathbb{R} = \mathsf{SO}_0(2, n)$, $\check{P}$ is the stabilizer of an isotropic line $l \in \mathsf{Is}_1(\mathbb{R}^{2,n})$, so $\check{\mathsf{a}} = \mathsf{a}_1$.

In general, for a simple Hermitian Lie group of rank n, there is a special set of n *strongly orthogonal roots* $\mathsf{b}_1, \dots, \mathsf{b}_n$ of the complexification $\mathfrak{g}_\mathbb{C}$; see [29, pages 582–583]. The set of strongly orthogonal roots give rise to a (holomorphic) embedding of a maximal polydisk. If the symmetric space is of tube type, the simple root $\check{\mathsf{a}}$ is the smallest strongly orthogonal root $\check{\mathsf{a}} = \mathsf{b}_n$. All the other strongly orthogonal roots are of the form $\mathsf{b}_i = \mathsf{b}_n + \varphi$, where $\varphi \in \mathsf{E}^*$ is nonnegative on the Weyl-chamber. We record the following for later use:

Lemma 9.1 *Let $a \in \mathsf{E}^+$. Then $\check{\mathsf{a}}(a) = \min_{i=1,\dots,n} \mathsf{b}_i(a)$.*

For Hermitian groups of tube type, the Shilov boundary carries a natural causal structure: for every $p \in \check{S}$ there is an open convex acute cone $C_p \subset \mathsf{T}_p\check{S}$, which we now define.

Recall that $\mathsf{G}_\mathbb{R}/\check{P}$ can be identified as the space of parabolic subgroups of $\mathsf{G}_\mathbb{R}$ that are conjugate to $\check{P}$. Let us fix a point $\check{p} = \check{P} \in \check{S}$, which one should think of as a point at infinity. Then, at any point $p = P \in \check{S}$ that is transverse to $\check{p}$, ie such that the parabolic groups P and $\check{P}$ are opposite, the tangent space $\mathsf{T}_p\check{S}$ is identified with the Lie algebra $\check{\mathfrak{n}}$ of the unipotent radical of $\check{P}$, and the cone C_p is an open convex acute cone $\check{C} \subset \check{\mathfrak{n}}$, invariant under the action of the connected component of $P \cap \check{P}$.

In the case of $\mathsf{Sp}(2n, \mathbb{R})$ this is the cone of positive definite symmetric matrices, and in the case of $\mathsf{SO}_0(2, n)$ it is the cone of vectors with positive first entry that are positive for the induced conformal class of Lorentzian inner products on $\mathsf{T}_P\,\mathsf{Is}_1(\mathbb{R}^{2,n})$.

This invariant cone $\check{C} \subset \check{\mathfrak{n}}$ in fact also gives rise to the notion of maximal triples in $\check{S}$ via the exponential map. A triple $(P, Q, \check{P})$ is said to be maximal if there exists an $s \in \check{C}$ such that $Q = \exp s \cdot P$. Extending this by the action of G leads to a notion of maximal triples in $\check{S}$, which actually coincides exactly with those triples which have maximal (generalized) Maslov index as introduced by Clerc and Ørsted [14].

Definition 9.2 Let $\check{S}$ be the Shilov boundary of a Hermitian symmetric space of tube type. A curve $\xi\colon \mathbb{S}^1 \to \check{S}$ is *positive* if the image of any positively oriented triple is a maximal triple.

Proposition 9.3 *Let $\xi\colon \mathbb{S}^1 \to \check{S}$ be a positive curve. Then $\xi(\mathbb{S}^1)$ is a Lipschitz submanifold of $\check{S}$.*

Proof Note that whenever we pick two points $p_1 = P_1$ and $p_2 = P_2$ on the image of ξ, the image $\xi(\mathbb{S}^1)$ can be covered by the two charts consisting of parabolic subgroups that are transverse to p_1 and p_2, respectively.

In any of these charts the inverse image of ξ, under the exponential map

$$\mathfrak{n}_i \to \mathsf{G}_{\mathbb{R}}/P_i, \quad s \mapsto \exp(s)\check{P}_j,$$

gives a map $\bar{\xi}\colon \mathbb{R} \to \mathfrak{n}_i$ such that, for every $t_1 < t_2$, we have that $\bar{\xi}(t_2) - \bar{\xi}(t_1)$ is contained in the open convex acute cone $\check{C}$. It then follows (see for example Burger, Iozzi, Labourie and Wienhard [10, Lemma 8.10]) that the restriction of $\bar{\xi}$ to any bounded interval has finite length. As a result, $\xi(\mathbb{S}^1) \subset \check{S}$ is rectifiable. It is thus possible to reparametrize $\mathbb{S}^1$ so that ξ is a Lipschitz map. □

Remark 9.4 We did not assume that the positive map is equivariant with respect to a representation. This will be important in Section 10, where we will apply Proposition 9.3 in this generality.

9.2 Maximal representations

Let G denote a Hermitian semisimple Lie group and let Γ denote the fundamental group of a closed hyperbolic surface S. We consider representations $\rho\colon \Gamma \to \mathsf{G}$ that are maximal, in the sense that they maximize the Toledo invariant, whose definition was recalled in Section 1. Important for us is that they can be characterized in terms of boundary maps:

Theorem 9.5 (Burger, Iozzi and Wienhard [12, Theorem 8]) *A representation $\rho\colon \Gamma \to \mathsf{G}$ is maximal if and only if there exists a continuous, ρ–equivariant, positive map $\phi\colon \partial\Gamma \to \check{S}$.*

In order to apply Corollary 6.7 we need to verify some weak irreducibility assumptions. Let us first treat the case when the Zariski closure of $\rho(\Gamma)$ is simple.

Corollary 9.6 *Let* G *be a simple Hermitian Lie group of tube type and let* $\check{\mathsf{a}}$ *be the root associated the Shilov boundary of* G. *If* $\rho\colon \Gamma \to \mathsf{G}$ *is a Zariski-dense maximal representation*, *then*

$$\check{\mathsf{a}} \in \mathcal{Q}_{\rho(\Gamma)},$$

or equivalently $h_\rho(\check{\mathsf{a}}) = 1$.

Proof The proof follows from Corollary 6.7 and Proposition 9.3 by considering the representation $\Lambda_{\check{a}}$ from Proposition 4.4. $\square$

In the remainder of this section we show how the case of maximal representations with semisimple target group that are not necessarily Zariski-dense can be reduced to Corollary 9.6. To this end we will use a result from Burger, Iozzi and Wienhard [11] describing the Zariski closure H of a maximal representation: H splits as $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$, each factor is Hermitian, and the inclusion in $\mathsf{H} \to \mathsf{G}$ is *tight*. In the following we will not need the definition of a tight homomorphism, and therefore refer the interested reader to [11, Definition 1].

The following lemma will then be useful:

Lemma 9.7 *Let* G *be a classical simple Hermitian Lie group of tube type and consider a tight embedding* $\iota\colon \mathsf{H} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathsf{G}$. *If we denote by* $\iota_*\colon \mathsf{E}^+_{\mathsf{H}} \to \mathsf{E}^+_{\mathsf{G}}$ *the induced map*, *then*

$$\check{\mathsf{a}}_{\mathsf{G}} \circ \iota_* = \min_i \check{\mathsf{a}}_{\mathsf{H}_i}.$$

Proof Denote by $\pi : \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n \to \mathfrak{g}$ the associated Lie algebra homomorphism. Let E_i be a Cartan subspace of H_i and E_{G} a Cartan subspace of G such that $\pi(\mathsf{E}_i) \subset \mathsf{E}_{\mathsf{G}}$.

As ι is tight and G is classical, the classification of Hamlet and Pozzetti [28] applies and gives that we have an orthogonal decomposition $\mathsf{E}_{\mathsf{G}} = \mathsf{B}_1 \oplus \cdots \oplus \mathsf{B}_k$ such that $\pi|_{\bigoplus \mathsf{E}_i}$ is a direct sum of maps $\pi_i : \mathsf{E}_i \to \mathsf{B}_i$. Furthermore, there are only a few possibilities for the linear map π_i. If H_i has rank greater than one, then $\mathsf{B}_i = \mathsf{E}_i^{m_i}$ for some m_i and π_i is a diagonal inclusion; if instead E_i is one-dimensional, or equivalently $\mathsf{H}_i \cong \mathsf{PSL}_2(\mathbb{R})$, then π_i is induced from a direct sum of nontrivial irreducible representations (of varying degrees). It is easy to check that the subspace B_i is then the span of the real vectors in $\mathfrak{p}$ associated to the strongly orthogonal roots that do not vanish on $\pi(\mathsf{E}_i)$. Setting $\mathsf{b}_i = \min_{j, \mathsf{b}_j|_{\mathsf{E}_i} \neq 0} \mathsf{b}_j$, we have $\mathsf{b}_i|_{\pi(\mathsf{E}_i)} = \check{\mathsf{a}}_{\mathsf{H}_i}$. And hence, with Lemma 9.1, we have $\check{\mathsf{a}}_{\mathsf{G}} = \min_i (\check{\mathsf{a}}_{\mathsf{H}_i})$. $\square$

Theorem 9.8 *Let G be a Hermitian semisimple Lie group such that all factors of G that are of tube type are classical. Let $\theta \subset \Delta$ be the subset of simple roots associated to the Shilov boundary of G. Then for every maximal representation $\rho\colon \Gamma \to \mathsf{G}$,*

$$\theta \subset \mathcal{Q}_{\rho(\Gamma)}.$$

Proof If $\mathsf{G} = \mathsf{G}_1 \times \cdots \times \mathsf{G}_n$ then $\check{S} = \check{S}_1 \times \cdots \times \check{S}_n$, and so $\theta = \{\check{\mathsf{a}}_{\mathsf{G}_1}, \dots, \check{\mathsf{a}}_{\mathsf{G}_n}\}$; see Burger, Iozzi and Wienhard [11, Lemma 3.2(1)]. Furthermore $\rho\colon \Gamma \to \mathsf{G}$ is maximal if and only if all $\rho_i : \Gamma \to \mathsf{G}_i$ are maximal (see Burger, Iozzi and Wienhard [12, Lemma 6.1(3)]). Therefore we can restrict to the case that G is simple.

Since every maximal representation factors through a representation into the normalizer of a maximal tube type subgroup $\mathsf{H} < \mathsf{G}$ (Burger, Iozzi and Wienhard [12, Theorem 5(3)]), which is simple, has the same rank as G, and is such that $\check{\mathsf{a}}_{\mathsf{G}} = \check{\mathsf{a}}_{\mathsf{H}}$, we can restrict to the tube type case as the limit set in $\check{S}_{\mathsf{G}}$ is contained in $\check{S}_{\mathsf{H}}$ and coincides with the limit set in $\check{S}_{\mathsf{H}}$. The maximal tube type domains are always classical Hermitian symmetric spaces, except for the one exceptional Hermitian symmetric space of tube type.

If now ρ is not Zariski-dense, then the Zariski closure is reductive and of tube type, so it is of the form $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ and the representations into H_i are Zariski-dense and maximal. Therefore we have $h_\rho(\check{\mathsf{a}}_{\mathsf{H}_i}) = 1$ for all i. As the inclusion $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathsf{G}$ is tight, the result follows from Lemmas 9.7 and 5.1. ∎

9.3 Application to the Riemannian critical exponent

Any simple Hermitian Lie group G admits a diagonal embedding $\iota^\Delta\colon \mathsf{SL}_2(\mathbb{R}) \to \mathsf{G}$, which is equivariant with the inclusion of a diagonal disk in a maximal polydisk. We say that a representation $\rho\colon \Gamma \to \mathsf{G}$ is *diagonal-Fuchsian* if it has the form $\rho = \iota^\Delta \circ \rho_0$, where $\rho_0\colon \Gamma \to \mathsf{SL}_2(\mathbb{R})$ is the lift of the holonomy of a hyperbolization.

Let $K_\Delta < \mathsf{G}$ be the centralizer of the image of ι^Δ, which is compact. Then a diagonal Fuchsian representation ρ can be twisted by a representation $\chi\colon \Gamma \to K_\Delta$. We call the corresponding representation $\rho_\chi : \Gamma \to \mathsf{G}$ a *twisted diagonal* representation. Observe that the Riemannian critical exponent h^X is constant on twisted diagonal representations (the exact value h^X_{diag} depends on the choice of the normalization of the Riemannian metric)

Proposition 9.9 *Let Γ be the fundamental group of a closed surface and let $\rho\colon \Gamma \to \mathsf{G}$ be a maximal representation. Then $h^X_\rho \le h^X_{\mathrm{diag}}$.*

Proof Let $\mathsf{b}_1, \dots, \mathsf{b}_n$ be the set of strongly orthogonal roots for $\mathsf{G}_{\mathbb{C}}$. It is immediate to verify that the limit cone $\mathcal{L}_{\rho_0(\Gamma)}$ of a representation ρ_0 in the Fuchsian locus is concentrated in the span of the vertex of the Weyl chamber $\sum_{i=1}^n \mathsf{b}_i^*$, where b^* is the basis of E dual to $\{\mathsf{b}_1, \dots, \mathsf{b}_n\}$. We know from Corollary 9.6 that, for every ρ, the growth rate $h_\rho(\check{\mathsf{a}})$ equals 1. Thus, if we denote by $(\mathsf{E}^+)^*$ the cone of functionals that are nonnegative on the Weyl chamber, we get that $\check{\mathsf{a}} + (\mathsf{E}^+)^* \subset \mathcal{D}_{\rho(\Gamma)}$, and in particular all the strongly orthogonal roots are in $\mathcal{D}_{\rho(\Gamma)}$. A simple computation shows that the affine simplex determined by the strongly orthogonal roots meets the ray $\mathbb{R}\sum_{i=1}^n \mathsf{b}_i$ orthogonally in a point (it is just the diagonal in a positive quadrant meeting the span of the basis vectors), whose norm has to compute the Riemannian orbit growth rate of any representation ρ_0 in the Fuchsian locus: $\mathcal{Q}_{\rho_0(\Gamma)}$ is the affine hyperplane orthogonal to $\mathbb{R}\sum_{i=1}^n \mathsf{b}_i$ that contains $\check{\mathsf{a}}$. Remark 5.5 concludes the proof. □

Remark 9.10 When G is $\mathsf{Sp}(4,\mathbb{R})$, or more generally $\mathsf{SO}_\circ(2,n)$, it follows from Collier, Tholozan and Toulisse [15] that the bound is furthermore rigid: the equality is strict unless ρ is equal to ρ_0 up to a character in the compact centralizer of its image.

Note that for maximal representations into $\mathsf{Sp}(2n,\mathbb{R})$, for $n \geq 3$, every connected component of the space of maximal representations contains a twisted diagonal representation. However for $\mathsf{Sp}(4,\mathbb{R})$ there are exceptional components, discovered by Gothen, where every representation is Zariski-dense; see Bradlow, García-Prada and Gothen [8] and Guichard and Wienhard [26]. In these components it is easy to verify that the bound we provide is sharp, despite not being achieved.

In the special case of the Hitchin component of $\mathsf{Sp}(2n,\mathbb{R})$ the bound of Proposition 9.9 is never attained, as the irreducible representations provide a better bound that is furthermore rigid; see Potrie and Sambarino [38].

10 Θ–positive representations

Throughout this section we will write

$$\mathsf{G} = \mathsf{SO}(p,q),$$

with $p < q$. We consider the subset $\Theta = \{\mathsf{a}_1, \dots, \mathsf{a}_{p-1}\}$ of the simple roots discussed in Example 4.6 and denote by P_Θ the corresponding parabolic group, by L_Θ its Levi factor and by U_Θ its unipotent radical.

The group G admits a Θ–*positive structure* as defined by Guichard and Wienhard [27]. This means that for every $\mathsf{b} \in \Theta$ there exists an L^0_Θ–invariant sharp convex cone c_b in

$$\mathfrak{u}_\mathsf{b} = \sum_{\substack{\mathsf{a} \in \Sigma^+_\Theta \\ \mathsf{a} = \mathsf{b} \bmod \mathrm{Span}(\Pi \setminus \Theta)}} \mathfrak{g}_\mathsf{a}.$$

Here $\Sigma^+_\Theta = \Sigma^+ \setminus \mathrm{Span}(\Pi \setminus \Theta)$. For $\mathsf{b} \in \{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}\}$, the space $\mathfrak{u}_\mathsf{b}$ is one-dimensional and the sharp convex cone $c_\mathsf{b} = \mathbb{R}^+ \subset \mathbb{R}$ consists of the positive elements, while $\mathfrak{u}_{\mathsf{a}_{p-1}} = \mathbb{R}^{q-p+2}$ is endowed with a form q_J of signature $(1, q-p+1)$ preserved by the action of $\mathsf{L}^0_\Theta = \mathbb{R}^{p-2} \times \mathsf{SO}^0(1, q-p+1)$. The cone $c_{\mathsf{a}_{p-1}}$ consists precisely of the positive vectors for q_J whose first entry is positive.

Following [27, Section 4.3] we denote by $W(\Theta)$ the subgroup of the Weyl group W generated by the reflections $\{\sigma_i\}_{i=1}^{p-2}$ together with the longest element σ_{p-1} of the Weyl group $W_{\mathsf{a}_{p-1},\mathsf{a}_p}$ of the subroot system generated by the last two simple roots. $W(\Theta)$ is, in our case, a Weyl group of type B_{p-1}. We denote by w^0_Θ the longest element of $W(\Theta)$, and choose a reduced expression $w^0_\Theta = \sigma_{i_1} \cdots \sigma_{i_l}$. Of course every reflection σ_i appears at least once among the σ_{i_k}. We consider the map

$$F_{\sigma_{i_1} \cdots \sigma_{i_l}} : c^0_{\mathsf{a}_{i_1}} \times \cdots \times c^0_{\mathsf{a}_{i_l}} \to U_\Theta, \quad (v_1, \dots, v_l) \mapsto \exp(v_1) \cdots \exp(v_l).$$

The Θ–positive semigroup U^+_Θ is defined as the image of the map $F_{\sigma_{i_1} \cdots \sigma_{i_l}}$, and doesn't depend on the choice of the reduced expression [27, Theorem 4.5].

A θ–positive structure on G gives rise to the notion of a positive triple in G/P_Θ.

Definition 10.1 A pairwise transverse triple in $(\mathsf{G}/P_\Theta)^3$ is Θ–*positive* if it lies in the G–orbit of a triple of the form $(F_1, u \cdot F_1, F_3)$, where $\mathrm{Stab}(F_3) = P_\Theta$, F_1 is transverse to F_3 and $u \in U^+_\Theta$ [27, Definition 4.6].

Remark 10.2 The stabilizer in $\mathsf{SO}^0(1, q-p+1)$ of a vector $v \in c_{\mathsf{a}_{p-1}}$ is compact. As a result one readily checks that the stabilizer in G of a Θ–positive triple is compact.

Let Γ_g be the fundamental group of a hyperbolic surface. A representation $\rho\colon \Gamma_g \to \mathsf{G}$ is Θ–*positive* if there exists a ρ–equivariant map $\partial\Gamma_g \to \mathsf{G}/P_\Theta$ sending positive triples to Θ–positive triples [27, Definition 5.3]. Guichard, Labourie and Wienhard show that every Θ–positive representation is necessarily Θ–Anosov [27, Conjecture 5.4], but since the proof has not yet appeared in print, in this section we will freely add this last assumption, and only discuss Θ–positive Anosov representations.

Theorem 10.3 *Let $\rho\colon \Gamma_g \to \mathsf{SO}(p,q)$ be Θ–positive and Θ–Anosov. For every $1 \le k \le p-2$ the representation $\bigwedge^k \rho$ is $(1,1,2)$–hyperconvex.*

Proof We denote by $\xi\colon \partial\Gamma_g \to \mathsf{G}/P_\Theta$ the Θ–positive continuous equivariant boundary map, and by $\xi^i : \partial\Gamma_g \to \mathsf{Is}_i(\mathbb{R}^{p,q})$ the induced maps. Let $(x,y,z) \in \partial^3\Gamma$ be a positively oriented triple. By assumption, $\xi(y) = s\cdot\xi(x)$ for some element s in the positive semigroup of the unipotent radical of the stabilizer of $\xi(z)$. In turn, $s = \exp(v_1)\cdots\exp(v_l)$ with $v_t \in c^{\mathbf{0}}_{\mathsf{a}_{i_t}}$ (recall that $i_t \in \{1,\dots,p-1\}$).

We set $d = p+q$. It follows from [39, Proposition 8.11] that, in order to check that $\bigwedge^k \rho$ is $(1,1,2)$–hyperconvex, it is enough to verify that the sum

$$\xi^k_\rho(x) + (\xi^k_\rho(y) \cap \xi^{d-k+1}_\rho(z)) + \xi^{d-k-1}_\rho(z)$$

is direct, or equivalently that the sum

$$\xi^k_\rho(x) + s\cdot(\xi^k_\rho(x) \cap \xi^{d-k+1}_\rho(z)) + \xi^{d-k-1}_\rho(z)$$

is direct (recall that s belongs to the stabilizer of $\xi_\rho(z)$). Without loss of generality we can assume that the form Q defining the group $\mathsf{SO}(p,q)$ is represented by

$$Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ K^t & 0 & 0 \end{pmatrix},$$

with

$$K = \begin{pmatrix} 0 & 0 & (-1)^{p-1} \\ 0 & \iddots & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad\text{and}\quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\mathrm{Id}_{q-p} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can furthermore assume that $\xi^l(z) = \langle e_1,\dots,e_l\rangle$ and $\xi^l(x) = \langle e_d,\dots,e_{d-l+1}\rangle$, so that $\xi^k(x) \cap \xi^{d-k+1}(z) = e_{d-k+1}$. In order to check that the representation is $(1,1,2)$–hyperconvex, we only have to verify that, given s as above, writing $s\cdot e_{d-k+1} = \sum \alpha_i e_i$, the coefficient α_{d-k} never vanishes. We claim that such coefficient is just $\sum_{i_t=k} v_t > 0$. Indeed, by construction, if $v_t \in c^{\mathbf{0}}_{\mathsf{a}_m}$ with $m \in \{1,\dots,p-2\}$, then $\exp(v_t) \in \mathsf{SO}(p,q)$ differs from the identity only in the positions $(t,t+1)$ and $(d-t,d-t+1)$ where it is equal to v_t (see [27, Section 4.5]), while if $v_t \in c^{\mathbf{0}}_{\mathsf{a}_{p-1}}$,

$$\exp(v_t) = \begin{pmatrix} \mathrm{Id}_{p-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & v^t & q_J(v) & 0 \\ 0 & 0 & \mathrm{Id}_{q-p+2} & Jv & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{p-2} \end{pmatrix}. \qquad \square$$

In particular, we deduce from [39, Proposition 7.4]:

Corollary 10.4 *Let $\rho\colon \Gamma_g \to \mathsf{SO}(p,q)$ be Θ–positive Anosov. For every $1 \le k \le p-2$ the image of $\xi^k_\rho(\partial\Gamma)$ is a C^1 submanifold of $\mathsf{Is}_k(\mathbb{R}^{p,q})$.*

Proof Since $\bigwedge^k \rho$ is $(1,1,2)$–hyperconvex, by [39, Proposition 7.4] its limit set is a C^1 submanifold of $\mathbb{P}(\bigwedge^k \mathbb{R}^{p,q})$. Since the inclusion $\bigwedge^k\colon \mathsf{Is}_k(\mathbb{R}^{p,q}) \to \mathbb{P}(\bigwedge^k \mathbb{R}^{p,q})$ is analytic, and the limit set of $\bigwedge^k \rho$ is the image under this inclusion of the limit set of ρ, the result follows. $\square$

We now turn to the proof of the last statement in Theorem D. Instead of directly verifying that the map ξ^{p-1}_ρ has Lipschitz image, we will study properties of the map $\xi^{\Theta_0}_\rho\colon \partial\Gamma_g \to \mathsf{G}/P_{\Theta_0}$, where

$$\Theta_0 = \{\mathsf{a}_{p-2}, \mathsf{a}_{p-1}\}.$$

The flag manifold G/P_{Θ_0} consists of nested pairs of isotropic subspaces of dimension $p-2$ and $p-1$.

Proposition 10.5 *Let $\rho\colon \Gamma_g \to \mathsf{SO}(p,q)$ be Θ–positive Anosov. The image of the map $\xi^{\Theta_0}_\rho\colon \partial\Gamma_g \to \mathsf{G}/P_{\Theta_0}$ is a Lipschitz submanifold of G/P_{Θ_0}.*

Proof Fix a point $z \in \partial\Gamma$ and assume without loss of generality that $\xi^k_\rho(z) = \langle e_1, \dots, e_k\rangle$. We denote by $\mathcal{A} \subset G/P_{\Theta_0}$ the set of points transverse to $\xi^{p-2,p-1}_\rho(z)$. We will show that the image of $\xi^{\Theta_0}_\rho|_{\partial\Gamma\setminus\{z\}}$ is a Lipschitz submanifold of $\mathcal{A}$. Denote by $\mathcal{A}_{p-2} \subset G/P_{\mathsf{a}_{p-2}}$ the set of isotropic subspaces of dimension $p-2$ transverse to $\xi^{p-2}_\rho(z) = \langle e_1, \dots, e_{p-2}\rangle$, by Z_{p-1} the $(p-1)$–isotropic subspace $Z_{p-1} := \xi^{p-1}_\rho(z) = \langle e_1, \dots, e_{p-1}\rangle$ and by $Z^\perp_{p-1}$ its orthogonal with respect to the form Q defining $\mathsf{SO}(p,q)$. Observe that we have a smooth map

$$\mathcal{A} \mapsto \mathcal{A}_{p-2} \times \mathsf{Is}_1(Z^\perp_{p-2}/Z_{p-2}), \quad (Y_{p-2}, Y_{p-1}) \mapsto (Y_{p-2}, [Y_{p-1} \cap Z^\perp_{p-2}]),$$

whose image is the product of $\mathcal{A}_{p-2}$ with the set $\mathcal{I}_Z$ of isotropic lines transverse to the image of $Z^\perp_{p-1}$. Indeed, for every pair $(Y_{p-2}, v) \in \mathcal{A}_{p-2} \times \mathcal{I}_Z$, the subspace $v + Z_{p-2}$ has dimension $p-1$ and $\dim((v+Z_{p-2}) \cap Y^\perp_{p-2}) = 1$ as $Y^\perp_{p-2}$ and Z_{p-2} are transverse. We then have $Y_{p-1} = Y_{p-2} + ((v + Z_{p-2}) \cap Y_{p-2})$.

Denote by $\xi_Z\colon \partial\Gamma\setminus\{z\} \to \mathcal{I}_Z$ the composition of the map $\xi^{p-2,p-1}$ and the projection to the second factor in the product decomposition. The form Q induces a form of signature $(2, q-p+2)$ on $Z^\perp_{p-2}/Z_{p-2}$, which gives rise to the notion of positive curves (as introduced in Section 9). We claim that ξ_Z is a positive curve. This amounts to showing that if $(x,y,z)\in\partial\Gamma$ is positively oriented then $\xi_Z(y)=s^Z\xi_Z(x)$ for some positive element s^Z in the unipotent radical of the stabilizer of $[Z_{p-1}]\in\mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$. Since the representation ρ is Θ–positive we know that $\xi(y)=s\cdot\xi(x)$ for some element in the positive semigroup U_Θ^+ and, as in the proof of Theorem 10.3, we can write $s=\exp(v_1)\cdots\exp(v_l)$ with $v_t\in c^0_{\mathsf{a}_{i_t}}$. Observe that, for every $v_t\in c^0_{\beta_{i_t}}$, $\exp(v_t)$ induces an element $\exp(v_t)^Z$ in the unipotent radical of the stabilizer of $[Z_{p-1}]\in\mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$, and the element $\exp(v_t)^Z$ is trivial unless $\beta_{i_t}=\mathsf{a}_{p-1}$, in which case $\exp(v_t)^Z$ belongs to the positive semigroup of the unipotent radical of the stabilizer of $[Z_{p-1}]$. As at least one of the v_t in the decomposition of s belongs to such a subgroup, we deduce that ξ_Z is positive, as we claimed. It follows from Proposition 9.3 that $\xi_Z(\partial\Gamma\setminus\{z\})$ is a Lipschitz submanifold of $\mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$.

As we know from Theorem 10.3 that ξ^{p-2} is a C^1–curve, we deduce that the curve $\xi^{p-2,p-1}$ is Lipschitz, being the image of a monotone map between a C^1–submanifold and a Lipschitz submanifold. □

10.1 The critical exponent on the symmetric space is rigid

Let $\iota_{2p-1}\colon\mathsf{PO}(1,2)\to\mathsf{PO}(p,p-1)\to\mathsf{PO}(p,q)$ be the composition of the irreducible representation of dimension $2p-1$ with the standard embedding of $\mathsf{PO}(p,p-1)$ into $\mathsf{PO}(p,q)$. We call any representation $\rho\colon\Gamma\to\mathsf{PO}(p,q)$ which is the composition of a Fuchsian representation with ι_{2p-1} a $(p,p-1)$*–Fuchsian* representation.

Lemma 10.6 *Let $\rho\colon\Gamma_g\to\mathsf{PO}(p,q)$ be Θ–positive Anosov. The barycenter of the affine simplex in E^*_Θ determined by $\{\mathsf{a}_1,\dots,\mathsf{a}_{p-2},\varepsilon_{p-1}\}$ belongs to $\mathcal{D}_{\rho(\Gamma),\Theta}$.*

Proof Recall that, in the case of Θ–positive representations in $\mathsf{PO}(p,q)$, the Levi–Anosov subspace is $\mathsf{E}_\Theta:=\ker(\mathsf{a}_p)$. In particular, for every $k\le p-2$ we have that a_k belongs to the dual of E_Θ, and belongs to the boundary of $\mathcal{D}_{\rho(\Gamma),\Theta}$ by Corollary 10.4. Furthermore $\varepsilon_{p-1}=\mathsf{a}_{p-1}+\mathsf{a}_p$ belongs to $\mathcal{D}_{\rho(\Gamma),\Theta}$, being the sum of a linear form with entropy one (the form a_{p-1} has entropy one by Proposition 10.5) and a linear form positive on the Weyl chamber (the root a_p). In particular, the form corresponding to the barycenter of the affine simplex they determine in E^*_Θ belongs to $\mathcal{D}_{\rho(\Gamma),\Theta}$. □

Theorem 10.7 *Let Γ be the fundamental group of a surface and let $\rho\colon\Gamma\to\mathsf{PO}(p,q)$ be Θ–positive Anosov. Then $h^{\mathcal{X}}_\rho\le h^{\mathcal{X}}_{\rho_0}$ for any $(p,p-1)$–Fuchsian representation ρ_0.*

If equality is achieved at a totally reducible representation η, then η splits as $W\oplus V$, where

(1) *W has signature $(p, p-1)$ and $\eta|_W$ has Zariski closure the irreducible* $\mathsf{PO}(2,1)$ *in* $\mathsf{PO}(p, p-1)$,

(2) *$\eta|_V$ lies in a compact group.*

Proof The inequality follows from Lemma 10.6, together with the convexity of $\mathcal{D}_{\rho(\Gamma),\Theta}$, established by Theorem 5.12.

Assume now that η is a totally reducible representation such that equality holds. We can assume that $p \geq 3$, as the result for $p = 2$ was proven by Collier, Tholozan and Toulisse [15, Theorem 4].

Let $\mathsf{G} = \overline{\eta(\Gamma)}^Z$ be the Zariski closure. By definition, G is a real reductive group. We consider G as an abstract group, and denote by $\Lambda\colon \mathsf{G} \to \mathsf{SO}(p,q)$ the inclusion representation and by

$$\phi\colon \mathfrak{g} \to \mathfrak{so}(p,q)$$

the associated Lie algebra morphism. Denote by $\mathfrak{a}_{\mathsf{G}}$ a Cartan subspace of $\mathfrak{g}$.

Since $h_\eta^{\mathcal{X}}$ attains it maximal value, Theorem 5.12 forces the Quint indicator set $\mathcal{Q}_{\eta(\Gamma),\Theta}$ to be the affine hyperplane of $(\mathsf{E}_\Theta)^*$ spanned by Δ. The strict convexity guaranteed by Theorem 5.12 implies that G has real rank at most 2. Moreover, we have that $\phi(\mathfrak{a}_{\mathsf{G}}) = \langle (2(p-1), 2(p-2), \dots, 2, 0), (0, \dots, 0, 1) \rangle$.

Denote by $T = \langle \xi_\eta^1(\partial\Gamma) \rangle$ the vector space spanned by the projective limit curve of η. Since η is totally reducible, the action of $\eta(\Gamma)$, and hence that of G, on T is irreducible.

Fix a Weyl chamber $\mathfrak{a}_{\mathsf{G}}^+$ and let $\chi \in \mathfrak{a}_{\mathsf{G}}^*$ be the highest weight of $\phi(\mathfrak{g})|_T$. Since η is a_1–Anosov, the attracting eigenvector of every element in $\eta(\Gamma)$, and hence of every purely loxodromic element of G, is in V. We therefore conclude that, for every $a \in \mathfrak{a}_{\mathsf{G}}^+$,

$$\chi(a) = \lambda_1(\phi(a)).$$

We denote by $\mathcal{L}_\eta^{\mathsf{G}} \subset \mathfrak{a}_{\mathsf{G}}^+$ Benoist's limit cone of $\eta(\Gamma)$ in G. As the representation η is a_2–Anosov, and thus $\mathcal{L}_\eta^{\mathsf{G}}$ avoids the only wall not orthogonal to the kernel of a_1, there exists a linear form $\mu \in \mathfrak{a}_{\mathsf{G}}^*$ such that, for every $a \in \mathcal{L}_\eta^{\mathsf{G}}$,

$$\mu(a) = \mathsf{a}_1(\lambda(\phi(a))).$$

Furthermore, as η is $(1,1,2)$–hyperconvex, for every $x \in \partial\Gamma$ the 2–dimensional space $\xi^{\mathsf{a}_2}(x)$ lies in T, and therefore $(\chi - \mu)(a) = \lambda_2(\phi(a))$, which implies that μ is a simple root and $\chi = (p-1)\mu$.

For a weight ψ of the representation $\phi(\mathfrak{g})|_T$, or of an irreducible factor of $\phi(\mathfrak{g})|_{T^\perp}$, denote by V^ψ the associated weight space. We obtain from the description of $\phi(\mathfrak{a}_{\mathsf{G}})$ that the weight spaces $V^{\chi-i\mu}$ for $i \in [\![0, 2p-2]\!]$ are also 1–dimensional and contained in T. The weight space decomposition of T thus has the form

$$T = \bigoplus_{i=0}^{2p-2} V^{\chi-i\mu} \oplus V^0 \oplus V^q \oplus V^{-q},$$

where V^0 consists of vectors in the kernel of $\phi(\mathfrak{a}_{\mathsf{G}}^+)$ (except $V^{\chi-(p-1)\mu}$) and V^q corresponds to the eigenvalue $\varepsilon_p(\lambda(\phi(a)))$. Here V^0 as well as V^q and V^{-q} could be instead contained in $T^\perp$, and therefore not appear in the decomposition.

Now let W denote the Weyl group of $\mathfrak{g}$. As the weight lattice of $\eta|_T$ is W–invariant, and there is no other weight of $\eta|_T$ at distance $p-1$ from the origin, we deduce that W is reducible, and $\mathfrak{g}$ splits as $\mathfrak{g}_1 + \mathfrak{g}_2$. If μ is the root associated to $\mathfrak{g}_1$, we deduce from the fact that $V^{\chi-\mu}$, and thus $\mathfrak{g}_\mu$, is one-dimensional that $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{R})$. As the actions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute and the highest weight space for the restricted action of $\mathfrak{g}_1$ is one-dimensional, we furthermore deduce that $\mathfrak{g}_2$ acts trivially on T. In particular, T is an irreducible $\mathfrak{sl}(2,\mathbb{R})$–module of dimension $2p-1$ and the signature of $T^\perp$ of the (p,q)–quadratic form preserved by $\mathfrak{so}(p,q)$ is thus either negative or $(1, q-p)$. In the first case we conclude that $\phi(\mathfrak{g})|_{T^\perp}$ is compact, which is the desired result.

In order to conclude the proof we need to exclude the second case. We know from Theorem 10.3 that for every $1 \leq k \leq p-2$ and for every distinct $x, y, z \in \partial\Gamma$ the sum

$$\xi^k(x) + (\xi^k(y) \cap \xi^{d-k+1}(z)) + \xi^{d-k-1}(z)$$

is direct. With an inductive argument we deduce that, for every $1 \leq k \leq p-2$ and for every $\gamma \in \Gamma$, the k^{th} eigenline belongs to T and therefore the Anosov map ξ would be the boundary of a Fuchsian representation composed with an embedding of $\mathsf{PO}(1,2) \to \mathsf{PO}(p-1,p) \to \mathsf{PO}(p,q)$. However, such an embedding can never be positive because it has noncompact centralizer (compare with Remark 10.2). $\square$

References

[1] **T Barbot**, *Three-dimensional Anosov flag manifolds*, Geom. Topol. 14 (2010) 153–191 MR Zbl

[2] **T Barbot**, *Deformations of Fuchsian AdS representations are quasi-Fuchsian*, J. Differential Geom. 101 (2015) 1–46 MR Zbl

[3] **T Barbot**, **Q Mérigot**, *Anosov AdS representations are quasi-Fuchsian*, Groups Geom. Dyn. 6 (2012) 441–483 MR Zbl

[4] **Y Benoist**, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997) 1–47 MR Zbl

[5] **Y Benoist**, *Convexes divisibles*, *I*, from "Algebraic groups and arithmetic" (S G Dani, G Prasad, editors), Tata Inst. Fund. Res., Mumbai (2004) 339–374 MR Zbl

[6] **J Bochi**, **R Potrie**, **A Sambarino**, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. 21 (2019) 3343–3414 MR Zbl

[7] **R Bowen**, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. 50 (1979) 11–25 MR Zbl

[8] **S B Bradlow**, **O García-Prada**, **P B Gothen**, *Deformations of maximal representations in* $\mathrm{Sp}(4,\mathbb{R})$, Q. J. Math. 63 (2012) 795–843 MR Zbl

[9] **M Bridgeman**, **R Canary**, **F Labourie**, **A Sambarino**, *The pressure metric for Anosov representations*, Geom. Funct. Anal. 25 (2015) 1089–1179 MR Zbl

[10] **M Burger**, **A Iozzi**, **F Labourie**, **A Wienhard**, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. 1 (2005) 543–590 MR Zbl

[11] **M Burger**, **A Iozzi**, **A Wienhard**, *Tight homomorphisms and Hermitian symmetric spaces*, Geom. Funct. Anal. 19 (2009) 678–721 MR Zbl

[12] **M Burger**, **A Iozzi**, **A Wienhard**, *Surface group representations with maximal Toledo invariant*, Ann. of Math. 172 (2010) 517–566 MR Zbl

[13] **L Carvajales**, *Counting problems for special-orthogonal Anosov representations*, Ann. Inst. Fourier (Grenoble) 70 (2020) 1199–1257 MR Zbl

[14] **J L Clerc**, **B Ørsted**, *The Maslov index revisited*, Transform. Groups 6 (2001) 303–320 MR Zbl

[15] **B Collier**, **N Tholozan**, **J Toulisse**, *The geometry of maximal representations of surface groups into* $\mathrm{SO}_0(2,n)$, Duke Math. J. 168 (2019) 2873–2949 MR Zbl

[16] **J Danciger**, **F Guéritaud**, **F Kassel**, *Convex cocompactness in pseudo-Riemannian hyperbolic spaces*, Geom. Dedicata 192 (2018) 87–126 MR Zbl

[17] **J Danciger**, **F Guéritaud**, **F Kassel**, *Proper affine actions for right-angled Coxeter groups*, Duke Math. J. 169 (2020) 2231–2280 MR Zbl

[18] **S Dey**, **M Kapovich**, *Patterson–Sullivan theory for Anosov subgroups*, Trans. Amer. Math. Soc. 375 (2022) 8687–8737 MR Zbl

[19] **A Douady**, **J Oesterlé**, *Dimension de Hausdorff des attracteurs*, C. R. Acad. Sci. Paris Sér. A-B 290 (1980) 1135–1138 MR Zbl

[20] **L C Evans**, **R F Gariepy**, *Measure theory and fine properties of functions*, CRC, Boca Raton, FL (2015) MR Zbl

[21] **K J Falconer**, *The Hausdorff dimension of self-affine fractals*, Math. Proc. Cambridge Philos. Soc. 103 (1988) 339–350 MR Zbl

[22] **H Federer**, *Geometric measure theory*, Grundl. Math. Wissen. 153, Springer (1969) MR Zbl

[23] **O Glorieux**, **D Monclair**, *Regularity of limit sets of AdS quasi-Fuchsian groups*, preprint (2018) arXiv 1809.10639

[24] **O Glorieux**, **D Monclair**, *Critical exponent and Hausdorff dimension in pseudo-Riemannian hyperbolic geometry*, Int. Math. Res. Not. 2021 (2021) 13661–13729 MR Zbl

[25] **O Glorieux**, **D Monclair**, **N Tholozan**, *Hausdorff dimension of limit sets for projective Anosov representations*, preprint (2019) arXiv 1902.01844

[26] **O Guichard**, **A Wienhard**, *Topological invariants of Anosov representations*, J. Topol. 3 (2010) 578–642 MR Zbl

[27] **O Guichard**, **A Wienhard**, *Positivity and higher Teichmüller theory*, from "European Congress of Mathematics" (V Mehrmann, M Skutella, editors), Eur. Math. Soc., Zürich (2018) 289–310 MR Zbl

[28] **O Hamlet**, **M B Pozzetti**, *Classification of tight homomorphisms*, preprint (2014) arXiv 1412.6398

[29] **Harish-Chandra**, *Representations of semisimple Lie groups, VI: Integrable and square-integrable representations*, Amer. J. Math. 78 (1956) 564–628 MR Zbl

[30] **J E Humphreys**, *Linear algebraic groups*, Graduate Texts in Math. 21, Springer (1975) MR Zbl

[31] **J L Kaplan**, **J A Yorke**, *Chaotic behavior of multidimensional difference equations*, from "Functional differential equations and approximation of fixed points" (H-O Peitgen, H-O Walther, editors), Lecture Notes in Math. 730, Springer (1979) 204–227 MR Zbl

[32] **I Kapovich**, **N Benakli**, *Boundaries of hyperbolic groups*, from "Combinatorial and geometric group theory" (S Cleary, R Gilman, A G Myasnikov, V Shpilrain, editors), Contemp. Math. 296, Amer. Math. Soc., Providence, RI (2002) 39–93 MR Zbl

[33] **M Kapovich**, **B Leeb**, **J Porti**, *Morse actions of discrete groups on symmetric space*, preprint (2014) arXiv 1403.7671

[34] **M Kapovich**, **B Leeb**, **J Porti**, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol. 22 (2018) 3827–3923 MR Zbl

[35] **F Labourie**, *Anosov flows, surface groups and curves in projective space*, Invent. Math. 165 (2006) 51–114 MR Zbl

[36] **F Ledrappier**, *Structure au bord des variétés à courbure négative*, Sémin. Théor. Spectr. Géom. 13 (1995) 97–122 MR Zbl

[37] **G Mess**, *Lorentz spacetimes of constant curvature*, Geom. Dedicata 126 (2007) 3–45 MR Zbl

[38] **R Potrie**, **A Sambarino**, *Eigenvalues and entropy of a Hitchin representation*, Invent. Math. 209 (2017) 885–925 MR Zbl

[39] **M B Pozzetti**, **A Sambarino**, **A Wienhard**, *Conformality for a robust class of non-conformal attractors*, J. Reine Angew. Math. 774 (2021) 1–51 MR Zbl

[40] **J-F Quint**, *Cônes limites des sous-groupes discrets des groupes réductifs sur un corps local*, Transform. Groups 7 (2002) 247–266 MR Zbl

[41] **J-F Quint**, *Divergence exponentielle des sous-groupes discrets en rang supérieur*, Comment. Math. Helv. 77 (2002) 563–608 MR Zbl

[42] **J-F Quint**, *Mesures de Patterson–Sullivan en rang supérieur*, Geom. Funct. Anal. 12 (2002) 776–809 MR Zbl

[43] **J-F Quint**, *An overview of Patterson–Sullivan theory* (2006) Available at `http://www.math.u-bordeaux.fr/~jquint/publications/courszurich.pdf`

[44] **A Sambarino**, *Hyperconvex representations and exponential growth*, Ergodic Theory Dynam. Systems 34 (2014) 986–1010 MR Zbl

[45] **A Sambarino**, *The orbital counting problem for hyperconvex representations*, Ann. Inst. Fourier (Grenoble) 65 (2015) 1755–1797 MR Zbl

[46] **D Sullivan**, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. 50 (1979) 171–202 MR Zbl

[47] **J Tits**, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. 247 (1971) 196–220 MR Zbl

[48] **T Zhang**, **A Zimmer**, *Regularity of limit sets of Anosov representations*, preprint (2019) arXiv 1903.11021

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