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BIRATIONAL GEOMETRY OLD AND NEW

ANTONELLA GRASSI

Abstract. A classical problem in algebraic geometry is to describe quantities that are invariants under birational equivalence as well as to determine some convenient birational model for each given variety, a minimal model. One such quantity is the ring of objects which transform like a tensor power of a differential of top degree, known as the canonical ring. The histories of the existence of minimal models and the finite generation of the canonical ring are intertwined; minimal models and canonical rings constitute the major building blocks for the birational classification of algebraic varieties. In this paper I will discuss some of the ideas involved, recent advances on the existence of minimal models, some applications, and the (algebro-geometric proof) of the finite generation of the canonical ring. These results have been long standing conjectures in algebraic geometry.

1. Plan

A central problem in many areas of geometry, is how to "best" complete an open variety, and also, to understand the relations between the different ways to complete one given variety. In the language of algebraic geometry this becomes constructing a *minimal* and *canonical* model and understanding *birational equivalence*; the construction of the canonical model relies on the *finite generation* of the canonical ring.

While the existence, and meaning, of minimal models turns out to be fairly clear in the case of curves, the finite generation is not always obvious (Section 2). In the first part of the last century minimal models were defined and shown to exists for surfaces; a complete classification in birational classes was then later understood (Section 3.1). The question of the finite generation of the canonical ring for surfaces was settled in the early 1960s (Section 3.1.2). The definition of minimal models which works so well for surfaces does not make sense in higher dimension: an appropriate generalization was provided in the '80s, and the "minimal model program" was successfully carried out in dimension 3 (Sections 3.2.1, 3.2.2, 3.3.2).

Since then many partial results have been obtained, and quite a few turn out to be essential to the proofs of the finite generation of the canonical ring in all dimension and the existence of minimal model for varieties of general type. It should be said that it is not possible to give here adequate

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credit to all the people who contributed to the results stated, but there is a brief historical summary at the end. A reference in brackets often merely indicates a paper where a proof could be found (Section 6).

In 2005 Siu [30] and in 2006 Birkar, Cascini, Hacon and McKernan [3] announced the proof of the finite generation of the canonical ring for a smooth variety of general type of any dimension. Birkar, Cascini, Hacon and McKernan also deduce that the canonical ring of any smooth variety is finitely generated. We do not discuss here Siu's proof, which is based on analysis. The proof of [3] follows from the authors' results on the existence of minimal models for varieties of general type. Actually, their proof is a clever round-about induction on the dimension of the variety, which builds on the assumption that finite generation and minimal models exist for varieties of lower dimension. In Section 4 we present an outline of the induction arguments. The proofs of [3] contains quite a few novel ideas and we try to outline some of them; these include, among others, the Minimal model program with scaling.

Section 2 contains examples, facts, and results to introduce algebraic geometry, in particular birational algebraic geometry. Section 3 is an introduction to the minimal model program: we state results, examples, motivations and lingo. Some newer results, and applications are in in Section 5.1 and 5.2, which contains application of the previous results to the classification of algebraic varieties.

In many branches of mathematics classification constitutes an important goal. In high dimensional topology classifications of manifolds of a given homotopy type (like a sphere or projective space), or the classification of three manifolds (like the classical Poincaré conjecture, the Thurston program), or the classification of finite simple groups are a few well known examples. In algebraic geometry one motivation for the minimal model program is that it provides a powerful tool for the classification of algebraic varieties (or function fields of algebraic varieties); this will be discussed in Section 5.2.

In this discussion, I will attempt to assume no previous familiarity with algebraic geometry.

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2. Origins of Birational Algebraic Geometry

2.1. **Some Algebraic Geometry.** The origin of algebraic geometry can perhaps be traced to René Descartes, who noticed that certain geometric objects, namely the graphs of equations, could be studied by combining techniques from algebra and geometry. The fundamental insight is that equations represent relations among different quantities and the representation of these equations in graph form results in either a curve, a surface or

an object of higher dimension. When the equations are given in polynomial terms, the corresponding graphs are called *algebraic varieties*.

The polynomials are taken with *complex coefficients*, so our varieties will be *complex algebraic varieties*. The complex numbers are in fact in many ways easier than the real numbers: a complex polynomial of degree n in one variable always has n roots (counted with multiplicity) and differentiable functions have a much tighter structure than infinitely differentiable real functions.

If $f(z_1, z_2)$ is a complex polynomial of degree 2,

$$C = \{(z,_1, z_2) \in \mathbb{C}^2 \text{ such that } f(z_1, z_2) = 0\}$$

is an algebraic curve (and a plane conic).
We will consider here *projective varieties*, that is varieties defined by *homogenous* polynomials in the projective space $\mathbb{P}^N$:

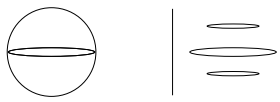
Definition 1. $\mathbb{P}^N = (\mathbb{C}^{N+1} \setminus 0) / \mathbb{C}^*$, where $[a_0, \cdots, a_N] = [\lambda a_0, \cdots, \lambda a_N]$, $a_j \in \mathbb{C}$, not all zero, and $\lambda \in \mathbb{C}^*$; recall that $\mathbb{C}^* \stackrel{def}{=} \mathbb{C} \setminus \{0\}$.

To visualize the complex projective spaces one can look at the moment map:

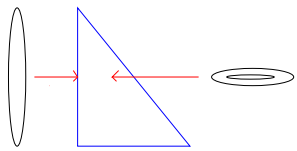
Example 2. (The moment map for $\mathbb{P}^1$.) Let $\mu : \mathbb{P}^1 \to \mathbb{R}^2$ be defined as:

$$[z_0, z_1] \to \frac{1}{|z_0|^2 + |z_1|^2}(|z_0|^2, |z_1|^2).$$

The image of μ is the line segment $I \subset \mathbb{R}^2$ (with coordinates (x, y)) defined by the equation $x + y = 1, \quad 0 \leq x \leq 1$. More precisely, $\mu : \mathbb{P}^1 \twoheadrightarrow I$.
It is easy to see that the fibers are circles on the points in the interior of I and a single point over $(0, 1)$ and $(1, 0)$; $\mathbb{P}^1$ is then a topological sphere S^2:



Example 3. $\mathbb{P}^2 \stackrel{def}{=} (\mathbb{C}^3 \setminus 0) / \mathbb{C}^*$. Similarly, let $\mu : \mathbb{P}^2 \to \mathbb{R}^3$ be defined as: $[z_0, z_1, z_2] \to \frac{1}{|z_0|^2+|z_1|^2+|z_2|^2}(|z_0|^2, |z_1|^2, |z_2|^2)$. The image of μ is a triangle $T \subset \mathbb{R}^3$. More precisely, $\mu : \mathbb{P}^2 \twoheadrightarrow T$.
It is easy to see that the fibers are tori on the points in the interior of T, circles on interior of the three edges a single point over the corners:



Note that $C \subset \mathbb{P}^2$, defined by $\{z_0 = 0\}$, can be visualized as the S^2 on the left; C is the locus of coordinates $[0, z_1, z_2]$, $(z_1, z_2) \in \mathbb{C}^2$, which can be identified with the projective line $\mathbb{P}^1$. C is a codimension 1 subvariety, a *divisor* (see Section 2.1.2, below).

If X is a projective and algebraic variety, then it is also compact (see [31]): the meromorphic functions on X are rational functions and we denote by $\mathbb{C}(X)$, the *function field* of X, the field of all rational functions. The function field of X carries a lot of information about X; for example, if a function field of a *smooth* variety X has a meromorphic function with only a simple pole, then X is necessarily $\mathbb{P}^1$. [An algebraic variety is *smooth* (non singular) at a point P if the rank of the Jacobian of the defining polynomials (locally around P) is maximal. A theorem of Zariski shows that this definition does not depend on the embedding given by the defining polynomials.]

In the *Zariski topology* the closed sets are the algebraic subvarieties. For example the algebraic variety $Z = [0, z_1, z_2] \in \mathbb{P}^2$ defined by $z_0 = 0$ is a closed set, its complement $\mathcal{U} = \mathbb{P}^2 \setminus Z$ is an *open set* which can be identified with $\mathbb{C}^2$, via the map $[1, z_2, z_2] \mapsto (z_1, z_2) \in \mathbb{C}^2$. Note also that Z can be identified with $\mathbb{P}^1$. These identifications are examples of *isomorphisms*:

Definition 4. Let X and $Y \subset \mathbb{P}^N$ be projective varieties. $f : X \dashrightarrow Y$ is a rational map if f can be represented by rational functions $(f_0, \cdots, f_N)$. Note that f does not need to be well defined on the whole variety X.

Definition 5. $f : X \to Y$ is a morphism if f is a rational map defined on everywhere on X.

Note that a rational map $f : X \to Y$ gives a morphism $f_{|\mathcal{U}} : \mathcal{U} \to Y$ where $\mathcal{U} \subset X$ is a Zariski open set.

Example 6. The map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $[z_0, z_1, z_2] \mapsto [z_0, z_1]$ is a rational map but not a morphism. In fact, $f([0, 0, 1]) = [0, 0]$ is not a point of $\mathbb{P}^1$, so f is not defined at $P = [0, 0, 1]$; on the other hand, f is well defined on the open set $\{\mathbb{P}^2 \setminus P\}$ and it is also a morphism.

Definition 7. $f : X \to Y$ is a *birational map* if there exist $\mathcal{U} \subset X$ and $\mathcal{V} \subset Y$ Zariski open sets, such that $f_{|\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is an invertible morphism and the inverse is also a morphism. In this case we write $X \sim Y$, and we say X and Y are *birational*.
$f : X \to Y$ is a *birational morphism* if f is as above and $\mathcal{U} = X$.
A birational morphism f is called an *isomorphism* if also $\mathcal{V} = Y$; in this case we write $X \simeq Y$, and we say X and Y are *isomorphic*.

So X and Y are birational if they have isomorphic open sets.

Definition-Proposition 8. (See for example [12, 31]) The following are equivalent:

(1) X and Y are birational (birationally equivalent);
(2) There exist $\mathcal{U}_X \subset X$ and $\mathcal{U}_Y \subset Y$, Zariski open sets, such that $\mathcal{U}_X \simeq \mathcal{U}_Y$;
(3) X and Y have the same function fields.

Example 9. Let $C \subset \mathbb{P}^2$ be the cubic curve defined as $z_3 z_2^2 - z_1^3 = 0$; then $f : \mathbb{P}^1 \dashrightarrow C$ as $[1, t] \mapsto [t^2, t^3, 1]$ is a birational morphism, but it can be shown (see for example, [31]) that it is not an isomorphism.

Example-Definition 10. (A *blow up*.) Let us consider $\mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $((x_0, x_1), [z_0, z_1])$ and $B = \{((x_0, x_1), [z_0, z_1]) \in \mathbb{C}^2 \times \mathbb{P}^1$ such that $x_j z_i - x_i z_j = 0\}$.
Then $f : B \to \mathbb{C}^2$ with $((x_0, x_1), [z_0, z_1]) \mapsto (x_0, x_1)$ is a birational morphism. Note also that

$$f_{|_{B \setminus f^{-1}(0,0)}} : B \setminus f^{-1}(0, 0) \to \mathbb{C}^2 \setminus (0, 0)$$

is an isomorphism and that $f^{-1}(0, 0) \simeq \mathbb{P}^1$.

$E \stackrel{def}{=} f^{-1}(0, 0)$ is called the *exceptional divisor* of the birational morphism f (see also the introduction to Section 3).

The morphism $f : B \to \mathbb{C}^2$ in Example 10 can be extended to a birational morphism between compact varieties. Let $\widehat{B}$ be the (Zariski) closure of B in $\mathbb{P}^2 \times \mathbb{P}^1$. Then f extends to a morphism $\hat{f} : \widehat{B} \to \mathbb{P}^2$, which restricts to an isomorphism:

$$\hat{f}_{|_{\widehat{B} \setminus \hat{f}^{-1}([0,0,1])}} : \widehat{B} \setminus \hat{f}^{-1}([0, 0, 1]) \to \mathbb{P}^2 \setminus [0, 0, 1].$$

$\widehat{B}$ and $\mathbb{P}^2$ are *birational*.

Example 11. The blow up of $\mathbb{P}^2$ at a point in the previous example can be visualized via the image of the moment map on the right in the picture below. The triangle on the left is the image of the moment map of $\mathbb{P}^2$, as in Example 3, the picture on the right is the image of $\widehat{B}$ under the moment map. The the blue segment on the right is image, under the moment map, of the exceptional divisor E, which we see as a topological sphere S^2.

The blow up construction gives two different ways to "*complete*" the open variety $\mathbb{P}^2 \setminus [0, 0, 1]$ to two different compact projective varieties.

Since the above construction is local around $(0, 0) \in \mathbb{C}^2$, it can be carried through for other surfaces. A similar construction works in all dimensions.

The search for the *"minimal model"* can be thought as the search for the simplest way to birationally complete an open variety, and also, understand the relations between the different minimal models, if more than one model exists. As we will see below and in Section 3, in case of curve and surfaces, it was shown that every birational class has a *smooth* minimal model[1]. It is still not known if any variety has a minimal model; it has been showed that in the dimension 3 and higher the minimal model cannot be smooth; the recent paper of [3] shows that a minimal model always exists for a certain type of varieties (varieties of general type).

A related question is to describe quantities which are invariant under birational morphisms. Before defining some of these invariants (the Kodaira dimension $\kappa(X)$, the plurigenera, the canonical ring), let us see their significance in the case of complex curves:

2.1.1. *The case of complex curves: Kodaira dimension and some birational properties.* Riemann showed that any compact connected orientable (real) surface can be embedded in a projective space $\mathbb{P}^N$, where the image, a complex curve C, is defined by a finite set of polynomial equations. The Riemann surfaces are determined up to homeomorphisms by the number g of holes, which is defined to be the *genus* of the complex curve C. It turns out that the genus is a birational invariant.

Theorem 12. $\dim(X) = 1$*: If X and Y are smooth and birational, then X and Y are isomorphic.*

A couple of remarks are needed here: the theorem does not hold if X and Y are not both smooth, as we saw in Example 9; the theorem does not directly generalize to higher dimension, as we saw in Example 10.

The topological Euler characteristic of smooth curves is also a birational invariant; it turns out to be $\chi_{top}(C) = 2 - 2g$, which is also the negative of the *degree* of the *canonical divisor* K_C (see Example-Definition 14). It also turns out that $2g - 2 = degK_C$ is related to the *curvature* of the real surface: if the degree is negative, the curvature of the real surface (a sphere) is positive. If the degree is zero, the surface is flat (a torus); if the degree of the canonical divisor is positive, the curvature is negative. Equivalently the area of a circle of radius r is respectively smaller, equal or bigger than πr^2. The *Kodaira dimension* is the algebraic analogue of the curvature, and in the above cases we say it is respectively negative, zero or positive.

[1]We consider *normal* varieties, see for example [12]. If X is normal, then the singular locus has at most codimension 2. For example all normal curves are also smooth, normal surfaces are singular at most in a set of points.

2.1.2. *Divisors, canonical divisors; canonical rings, finite generation; Kodaira dimension.*

Definition 13. An effective *Weil divisor* D on a variety X is a finite sum $D = \sum_i a_i D_i$ with $a_i \in \mathbb{Z}$, where $D_i = [V_i]$ are irreducible codimension 1 subvarieties V_i[2]. If $a_i \geq 0, \forall\ i$, D is an *effective divisor*.

Example-Definition 14. Let X be a complex curve: a divisor $D = \sum_i a_i D_i$ with $a_i \in \mathbb{Z}$ is a collection of points. Then $\sum a_i$ is the *degree of* D.

Similarly we can define the degree of a divisor in $\mathbb{P}^N$; if a variety $V \subset \mathbb{P}^N$ is defined by a polynomial of degree d, d is the degree of the corresponding divisor $[V]$ in $\mathbb{P}^N$.
A rational function f on X defines a divisor (f), namely the locus of zeroes and poles counted with multiplicity and sign (positive the zeroes, negative the poles); these are the *principal* divisors. Two divisors are *linearly equivalent* if their difference is a principal divisor.

Locally on an open set $\mathcal{U}_j$ a Weil divisor on a smooth variety is defined by a function f_j. One could also define a divisor as the collection of these open sets and functions $\{(\mathcal{U}_j, f_j)\}$ with obvious compatibility conditions and equivalence relations. These are called *Cartier divisors*. One can also view Cartier divisors as invertible sheaves (or line bundles), see for example [12]. Multiplication of functions induces tensor powers of the line bundles and sums of divisors. We will use the additive notation, say mD represents the m-th tensor power of the line bundle D.

When X is not smooth Weil divisors are not always Cartier divisors, (see Example 58). This is indeed a feature of varieties which are relevant to birational geometry, as we will see in the next Sections.

Example 15. If $C \subset \mathbb{P}^2$ is a curve, and $H \subset \mathbb{P}^2$ a hyperplane, the intersection of H with C, denoted as $H \cdot C$, defines a divisor on C, which is called *very ample*.

Before defining a very ample divisor in general we need some more properties: Let D be an effective divisors: $H^0(X, D)$ is the set of *global sections* of D, that is the collection of rational functions with at most poles on D; $H^0(X, D)$ is a finite dimensional vector space. Let $\{s_0, \cdots s_N\}$ be a basis of $H^0(X, D)$; we can define the rational map:

$$\begin{aligned} \phi_{|D|} : X \dashrightarrow & \quad \mathbb{P}^N \\ p \mapsto & \quad [s_0(p), \cdots, s_N(p)] \end{aligned}$$

This map depends only on the linear equivalence class of the divisor D.

[2] V_i is irreducible if it is not *reducible*. A variety A is reducible if there exist varieties B and C such that $B \cup C = A$ with $B \neq A$ and $C \neq A$.

Definition 16. Let X be an algebraic variety and D a divisor on X; D is *very ample* if $\phi_{|D|} : X \hookrightarrow \mathbb{P}^N$ is an embedding[3]. In particular D is the restriction of an hyperplane $H \subset \mathbb{P}^N$ to X; we write $D \stackrel{def}{=} \phi^*(H)$.

It is a crucial problem in this setting (see Section 3.2) to identify a priori which divisors (or their multiples) are very ample and which divisors (or their multiples) determine morphisms, not necessarily embeddings:

Definition 17. Let X be an algebraic variety and D a divisor on X; D is *base point free* if there exists a morphism $\phi_{|D|} : X \to \mathbb{P}^N$.

Definition 18. Let X be an algebraic variety and D a divisor on X; D is *semiample* if there exists some $m > 0$ such that mD is base point free.

A *canonical form* on $\mathbb{C}^n$, with coordinates $(z_1, \cdots, z_n)$ is a top differential form, for example $dz_1 \wedge \cdots \wedge dz_n$. On a general smooth variety X, the canonical form is a collection of volume forms on open sets with obvious compatibility conditions. The *canonical divisor* is the divisor naturally associated to it and denoted by K_X; the canonical divisor is defined up to linear equivalence.

In some cases the canonical divisor can be defined even when a variety X is singular, as in the surface below:

Example 19. (*Kummer surface*) Let $X = (E \times E)/ < \imath, id >$, where E is the torus in Example 28 and $\imath : (z_1, z_2) \mapsto (-z_1, -z_2)$ with $(z_1, z_2) \in \mathbb{C}^2$. X has 16 singular points, and the canonical differential on $E \times E$ is preserved by the group action and induces a canonical divisor on X. The canonical divisor on X is still trivial: $H^0(X, K_X) \simeq \mathbb{C}$.

It is of particular interest, in the context of birational geometry, to understand under which conditions on a variety X the canonical divisor K_X is semiample, as we will see in Example 29 below, Theorem 37 and Section 5.2. Also of importance are the space of sections $H^0(mK_X)$.

Definition 20. $dim H^0(X, mK_X), m \geq 0$ is called the *m-plurigenus* of X.

The following proposition holds:

Proposition 21. *The plurigenera $dim H^0(X, mK_X)$ are birational invariants.*

The algebraic definition of the genus of a smooth curve X is $g(X) = \dim H^0(X, K_X)$, which coincides with the topological definition of the genus, as we will see in Examples 27, 28 and 29 below.

Definition 22. (Mumford, [27]) $R(X, K_X) = \oplus_{m \geq 0} H^0(mK_X)$ is the *canonical ring*.

[3]That is, $\phi(X)$ is a closed subvarieties of $\mathbb{P}^N$ isomorphic (via ϕ) to X.

The previous proposition implies that the canonical ring is a birational invariant. We assume X to be connected, hence $H^0(X, 0 \cdot K_X) = \mathbb{C}$.

Definition 23. The *Kodaira dimension* $\kappa(X)$ of a variety X is defined as follows:
$\kappa(X) < 0$ if $H^0(X, mK_X) = 0,\ \forall m > 0$;
$\kappa(X) = k$, if $\dim H^0(X, mK_X) \sim m^k$ as $m \to \infty$.

Note that $\kappa(X) < 0$ is also denoted by $\kappa(X) = -1$ or $\kappa(X) = -\infty$. Obviously $\kappa(X) \leq \dim(X)$; it turns out that

$$\kappa(X) = transc. \deg R(X, K_X) - 1.$$

Definition 24. X is of *general type* if $\kappa(X) = \dim(X)$. If X is a variety of general type, K_X is *big*.

Similarly, one defines $\kappa(X, D)$, the Kodaira dimension of a divisor D. If $\kappa(X, D) = \dim(X)$, D is called *big* and D is of *general type*.

There are many varieties of general type, for example the curves of general types are the curves with genus $g \geq 2$, see the Examples below. Any hypersurface defined by polynomials of degree greater than $d + 1$ in $\mathbb{P}^d$ is also of general type, this can be seen from the *adjunction formula*:

Proposition 25. *If $V \subset X$ is a smooth hypersurface in a smooth variety X, then:*

$$K_V = (K_X + V)_{|V}. \tag{2.1}$$

We now see some examples of the results discussed above in the case of curves:

Remark 26. If X is a smooth curve

$$K_X = \Omega^1_X = T^*_X \text{ and } \chi_{top}(X) = 2 - 2g = -deg(K_X),$$

where T_X, T^*_X denote the tangent and cotangent bundle respectively. Note also that $c_1(X) = -K_X$. Then K_X is non effective (we write $K_X < 0$) if and only if $g = 0$, K_X is trivial (we write $K_X \simeq 0$) if and only if $g = 1$ and K_X is effective (we write $K_X > 0$) if and only if $g \geq 2$.

Example 27. (Genus $g = 0$.) If $X = \mathbb{P}^1$, on the open set $\mathcal{U}_1 \simeq \{[s, 1], s \in \mathbb{C}\}$ the local volume form is ds, if $\mathcal{U}_2 \simeq \{[1, t]\}$, on $\mathcal{U}_1 \cap \mathcal{U}_2$ $s = 1/t$ and the local volume form is $ds = d(1/t) = -1/t^2 dt$. The canonical divisor (Weil) corresponds to a pole with multiplicity 2; it is not an effective divisor, but its inverse is. In this case, $H^0(\mathbb{P}^1, K_X) = 0$, $g(X) = 0$, $\kappa(X) < 0$ and the canonical ring is trivial.

Example 28. (Genus $g = 1$.) The torus $E = \mathbb{C}/\mathbb{Z} \oplus \imath\mathbb{Z}$ can be given the structure of an algebraic curve X, which can be identified with a smooth cubic in $\mathbb{P}^2$. It is easy to see that the volume form on $\mathbb{C}$, dz_1 induces a global volume form X. The divisor associated to it does not have poles nor zeroes,

it is the trivial divisor. Here $H^0(E, K_X) = \mathbb{C}$, $g(X) = 1$, $\kappa(X) = 0$, and the canonical ring is $R(X, K_X) = \oplus_m \mathbb{C}$.

Example-Theorem 29. (*The tricanonical embedding*, [Riemann]) (See for example [12] or [31].)
Riemann showed that if C is a curve of genus $g \geq 2$, then $3K_C$ is *very ample*, that is there is an embedding $\phi_{|3K_C|} : C \hookrightarrow \mathbb{P}^N$ with the following property: if $H \subset \mathbb{P}^N$ is an hyperplane, then $H \cdot C$ is $3K_C$. Then the canonical ring $R(C, K_C)$ can be reconstructed[4] from the coordinate ring of $\mathbb{P}^N$. In particular R is *finitely generated* (see below). Note that the *pluricanonical* embedding determined by $3K_C$ gives us an explicit description of C as an algebraic subvariety in a projective space; in fact, $C \simeq \operatorname{Proj} R(C, K_C)$ (see the Remark 30 below). Note also that X is of general type.

Remark 30. We refer to [12] for a formal definition of $\operatorname{Proj}(R)$, for R a finitely generated ring. If R is a finitely generated ring over $\mathbb{C}$, the relations among the generators in $\mathbb{C}[x_0, \cdots, x_N]$ (or better the ideal generated by the relations) define an algebraic variety $X \subset \mathbb{P}^N$. Then $X \stackrel{def}{=} \operatorname{Proj}(R)$; R is the *homogeneous coordinate ring* of X (see also [12]).

A nice discussion of Proj in the context of birational geometry can be found in Section 6 of [19]. The Examples 27, 28, 29 show the following theorem:

Theorem 31. *The canonical ring R of a smooth algebraic curve X is always finitely generated. In addition, if X is of general type, then $X = \operatorname{Proj}(R)$. $X = \operatorname{Proj}(R)$ is called the* canonical model.

In the case of surfaces, it is much harder to show that the canonical ring is finitely generated; this will be discussed in the next section.

3. Minimal models? Canonical models? An introduction.

We have seen that a birational morphism between algebraic varieties X and Y is a map which is an isomorphism between some Zariski open sets. Zariski open sets are dense in algebraic varieties, and the existence of a birational morphism between X and Y implies strong relations between them. In the case of curves, there is a unique smooth projective curve in each birational equivalence class and the canonical ring R is finitely generated. In addition if a curve is of general type, R gives an explicit embedding of the curve, the canonical model.

It is natural to ask if similar statements also holds for surfaces, threefolds and higher dimensional varieties. We have seen in Example 10 that $\mathbb{P}^2$ and $\widehat{B}$ are smooth and birational surfaces, and that one can construct infinitely many smooth projective surfaces in each birational equivalence class. So it is

[4]In fact Serre's vanishing theorem for the first cohomology of the ideal sheaf of C in $\mathbb{P}^N$ (twisted by m) shows that there is a surjection $H^0(\mathbb{P}^N, mH) \to H^0(C, 3mK_C)$.

natural to ask if in each birational class we can identify a simpler model(s), a *minimal model*, and how to produce it. It is also natural to ask if the canonical ring is finite generated and it gives an embedding for varieties of general type.

Zariski [36] first studied the ring $R = \oplus_m H^0(mD)$, for a divisor D on a surface X; he also showed that R is not always finitely generated, for some particular divisors D. Much later, Wilson [35] gave examples of varieties where the canonical ring is not finitely generated: in these examples the varieties are not algebraic or they have worse singularities than the one considered here.

We will see that in the case of surfaces the canonical ring R and its finite generation can be instrumental to the understanding of a given variety (see Remark 40). The proof of the finite generation of the canonical ring for surfaces was one of the motivations for the birational classification of surfaces, which was then completed on the basis of the existence of minimal models. Also, the proof of finite generation relies on the existence of minimal models.

In higher dimension the canonical model $\mathrm{Proj}(R)$ turns out to be a crucial ingredient to the understanding of birational geometry of varieties, and the existence of minimal models and canonical models become interdependent and intertwined. In particular, in the arguments of [3] and [11], the existences of canonical and minimal models depend on each other in the ascending induction arguments.

In the first part of the 20th century Castelnuovo, Enriques and Severi gave an algorithm to construct a *minimal model* of a given smooth surface; the definition and the algorithm are specific to dimension 2. By the middle 1980s a proposal for a *Minimal Model Program* in higher dimension had been formed and also carried out in many aspects, by work of Clemens, Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov, Viehweg and many others (see [19] and [23]). The Minimal Model Program was successfully completed, by Mori [26] for $\dim X = 3$, by the end of the 1980s. It was also understood how different minimal models of threefolds are related (see again [19] and [23]).

3.1. **The case of surfaces.** The results stated in this section are by now called *classical*; we discuss them in detail because they illustrate some of the basic ideas which recur in higher dimension, without many of the complications. On the other hand the presentation we give here is not the classical one developed by the Italian school, which is in fact specific to dimension two only.

3.1.1. *Minimal models, following the Italian school.*

Proofs for the results stated in this section can be found in many books; here we mostly follow [1] and [23].

In Example 10 we noted that any smooth point on a surface can be blown up to obtain another smooth, birationally equivalent, surface. The "Italian school" came up with the following definition and proved the theorems below:

Definition 32. *(Classical)* A smooth surface $\bar{S}$ is <u>*minimal*</u> if any birational morphism $\psi : S \to T$ to another smooth surface T is an isomorphism.

Theorem 33. *Any smooth surface S has a minimal model.*

Proof. If any birational morphism $S \to T$, with T a smooth surface is an isomorphism, then S is minimal and we are done. If S is not minimal, we can find a sequence of birationally equivalent smooth surfaces $\{S_j\},\ j = 1, \cdots, \ell$ with S_ℓ minimal. In fact, if S is not minimal by definition there exists a smooth surface S_1 and a birational morphism $\varphi_1 : S \to S_1$ which is not an isomorphism; it is not hard to see that $b_2(S_1) < b_2(S)$, where b_2 is the second betti number (see, for example, [1]). If S_1 is minimal we are done, otherwise, we find another smooth surface S_2 birationally equivalent to S and $b_2(S_2) < b_2(S)$. Thus we get a collection of morphisms $\varphi_j : S_{j-1} \to S_j$ and smooth surfaces S_j, birationally equivalent to S. The process must stop after a ℓ number of steps, since $b_2(S_j) < b_2(S_{j+1})$, $\forall j$. □

Furthermore the following holds:

Theorem 34. *Let S be a smooth surface:*

- **a:** *If $\kappa(S) \geq 0$, then the minimal model of S is unique.*
- **b:** *If $\kappa(S) < 0$, the minimal model is not unique. In this case S is birationally equivalent to $\mathbb{P}^1 \times C$, for some smooth curve C, and we say that S is* <u>birationally ruled</u>.

It is not hard to see, for example, that $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $T = \mathbb{P}^2$ are both minimal surfaces, according to Definition 32, with $\kappa(S) = \kappa(T) < 0$. S and T are also birational, as the blow up of S at one point is isomorphic to the blow up of T at two different points.

Definition 32 and Theorem 33 do not provide an explicit algorithm for constructing a minimal model for a given smooth surface; the basic tool for such an algorithm is given by <u>*Castelnuovo's contraction criterion*</u>, which is the building block of the classical birational theory of surfaces. Castelnuovo's theorem provides a necessary and sufficient condition for a smooth curve E on a smooth surface S_1 to be the exceptional divisor of a blow up $\varphi : S_1 \to S_2$:

Theorem 35. *[Castelnuovo] Let S_1 be a smooth surface and E a smooth rational curve on S_1 with $E^2 = -1$. Then there exists a birational morphism $\varphi : S_1 \to S_2$, where S_2 is a smooth surface; φ is the blow up of S_2 with exceptional divisor E (as in Example 10).*

The statement of this theorem is specific to surfaces: the exceptional divisors are curves only in the case of surfaces and the "negative self-intersection" makes sense only in dimension 2. In Section 3.2.1 we revisit

these Theorems (and Definition), in light of the Minimal Model Algorithm, which makes sense in all dimensions.

3.1.2. *Canonical models, finite generation of the canonical ring.*

The Castelnuovo-Enriques-Kodaira classification then provides a finer structure theory for the surfaces which are not birationally ruled, which we discuss briefly at the end of this Section. The classification is used in the proof of the Theorem below, following [1]:

Theorem 36. *(Mumford, 1962) The canonical ring of an algebraic surface Z is finitely generated.*

Proof. (Sketch) Let S be a minimal model of Z (see Section 3.1.1); if $\kappa(S) < 0$, then the canonical ring of S is finitely generated, as in the case of $\mathbb{P}^1$ (Example 27). Recall that $\kappa(S) \leq \dim(S) = 2$. If $\kappa(S) = 0$, then the birational classification implies that $nK_S \simeq \mathcal{O}_{\mathcal{S}}$ for some n, so the canonical ring is again finite generated, as in the case of elliptic curves (Example 28). If $\kappa(S) = 1, 2$ then it can be shown that K_S is semiample (see Definition 18) and we can deduce that the canonical ring is finitely generated, as in the case of curves of general type (Example 29). □

The proof sketched above is not Mumford's original proof, as it uses the classification of surfaces; the completion of this classification was in fact spurred by Mumford's theorem.

Following Mumford, Bombieri showed:

Theorem 37. *([2]) Let S be a minimal surface of general type. Then mK_S is base point free (see Definition 17) for $m \geq 4$ and if $m \geq 5$ the morphism is birational.*

Remark 38. Let $\phi_{|mK_S|} : S \to W_m$ be the morphism determined by mK_S, $m \geq 5$ as above. It follows that the $W_m \simeq \mathrm{Proj}(R(S, K_S))$, for all the relevant m; $X^{can} \stackrel{def}{=} W_m$ is called the <u>*canonical model*</u> of S. The canonical model is unique.

Remark 39. If $\phi_{|mK_S|} : S \to X^{can}$ is not an isomorphism, then S can be seen as a minimal resolution of the canonical model X^{can}. The singularities on X^{can} are called *canonical* (see Theorem-Definition 61).

The two-dimensional canonical singularities are also known as rational double points, and they are locally of the form $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is a finite subgroup. The singularities in Example 19 are canonical.

Remark 40. The morphisms determined by the canonical divisor K_S (or its multiples mK_S) on surfaces of general type were studied by Enriques [7], as a crucial tool to understand these surfaces; see a nice exposition in [4].

Another strategy to construct the minimal model for a surface S of $\kappa(S) = 2$ could be to consider the canonical model X^{can} and its minimal smooth resolution (which is knows to exist).

Part (**b**) in Theorem 34, together with Theorem 41 below provides a "structure" theorem for surfaces of Kodaira dimension negative and $\kappa(S) = 1$:

Theorem 41. *Let S be a minimal surface with $\kappa(S) = 1$. The canonical divisor K_S is semiample and, for some $m >> 0$ there is a morphism $\phi_{|mK_S|} : S \to B$, where $B = \mathrm{Proj}(R(S, K_S))$; the general fiber is a curve C with $\kappa(C) = 0$ (an elliptic curve).*

The fibration in the above theorem is called the *Iitaka's fibration*, see Theorem 90.

3.2. **Towards higher dimensions.** The Castelnuovo-Enriques classification was completed by the middle of the 19th century; for a long time the proof of the finite generation of the canonical ring and a birational classification in higher dimension seemed out of reach. One of the major obstacles was the lack of a "good" analogue of the notion of minimal model; in fact the classical Definition 32 does not makes sense in three-dimension (see Example 86).

3.2.1. *Minimal models for surfaces, towards higher dimension.*

The starting point is the following observation:

Remark 42. Castelnuovo's Theorem 35 says that a birational morphism of smooth surfaces $f : S \to \bar{S}$ which is not an isomorphism contracts (to points) rational curves E on S such that $K_S \cdot E < 0$. This follows from the adjunction formula on surfaces, 25.

Definition 43. A divisor D on a variety S is *nef* if $D \cdot C \geq 0$, for all the effective curves C in S.

The following holds:

Theorem 44. *If S is not birationally ruled and K_S is not nef, then there exists a rational exceptional curve as in Castelnuovo's criterion.*

We will not discuss here the proof of this results, which is related to the existence of rational curves when K_S is not nef and the "*bend and break*" arguments (see [19], Chapter 1.1).

Definition 45. (Revised): A smooth surface S is *minimal* if K_S is nef.

This definition makes sense in all dimension. So the key idea of Mori's algorithm in all dimension is to contract all curves C such that $K_X \cdot C < 0$. We will see that contracting such curves will provide a structure theorem for birationally ruled varieties (the variety of negative kodaira dimension), as in the following cases:

Example 46. (Trivial, but useful remark):

0: If $S = \mathbb{P}^2$, then there exists a trivial morphism: $\pi : \mathbb{P}^2 \to \{pt.\}$ (the fiber of π is the whole $\mathbb{P}^2$).

1: If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then there exists a fibration $\pi : S \to \mathbb{P}^1$ (the projection on one factor), with fibers $\mathbb{P}^1$ (the other factor).

We still need an analogue of Castelnuovo's Theorem 35 which works in all dimensions. We will need the definitions of the following Section:

3.2.2. *The Mori cone; Ample, Nef, Pseudo-effective cones and Big divisors.*

More details on the definitions and results stated here can be found, for example, in [19], [15], [23].
A 1-*cycle* is a linear combination of subvarieties of dimension 1 in a variety X:

Definition 47. Let X be a smooth variety. Let $NE(X)$ be the cone generated by classes of effective 1-cycles, with *real coefficients* and let $\overline{NE(X)}$ denote its closure. $\overline{NE(X)}$ is the *Mori Cone* of X.

We consider the Mori cone modulo *numerical equivalence*, (which means: from the point of view of intersection with divisors, they behave the same).

Definition 48. A divisor D is called *ample* if mD is very ample (see Definition 16), for some $m \in \mathbf{N}$.

Theorem 49. *(Kleiman's criterion) D is ample if and only if:*
$D \cdot C > 0$, for every non-zero curve $C \in \overline{NE(X)}$.

We consider the cone of ample divisors and the cone of nef divisors, modulo *numerical equivalence*, (which means: from the point of view of intersection with curves, they behave the same).

Remark 50. The nef cone and the Mori cone are then dual cones with respect to the intersection pairing. The closure of the ample cone is the nef cone.

It is also useful to consider the cone of effective divisors, with real coefficients, modulo numerical equivalence. Its closure is called the *pseudo-effective cone*; a divisor D in the closure is called *pseudo-effective*. These divisors appear in more general versions of the statements of [3] than we discuss in Section 4, with the exception of Theorem 89. The following result is also used:

Proposition 51. *A big divisor can be written as the sum of an ample divisor and an effective divisor.*

Note that the following four inclusions hold:

$$\begin{array}{ccc} Ample & \hookrightarrow & Eff \\ \downarrow & & \downarrow \\ Nef & \hookrightarrow & Pseudoeff \end{array}$$

Definition 52. A ray R in $\overline{NE(X)}$ is defined as $R = \mathbb{R}^+[Z]$, for some $Z \in \overline{NE(X)}$. The ray R is *extremal* if $z_1 + z_2 \in R \Rightarrow z_1, z_2 \in R$ (in the sense of convex geometry) and $K_X \cdot R < 0$.

Example 53. (See for example [19]) Let S be a smooth surface and E a rational curve with $E^2 = -1$, then E generates an extremal ray.

Example 54. (See for example [19] and Example 46) Let $S = \mathbb{P}^2$, then the class of a line L is an extremal ray.

Example 55. (See for example [19] and Example 46) Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, then the class F of a factor $\mathbb{P}^1$ is an extremal ray.

Definition 56. Let R be an extremal ray; a Cartier divisor D is the *supporting divisors* of R if and only if $D \cdot Z \geq 0$, for all $Z \in \overline{NE(X)}$ with equality if and only if $Z \in R$.

Geometrically the hyperplane defined by the supporting divisor D intersects the cone $\overline{NE(S)}$ only on the ray R; this is a necessary condition for a morphism $\phi_{|mD|}$ (if it exists) to contract exactly the curves in ray R. Castenuovo's contraction Theorem shows that there exists a semiample divisor D (that is mD is *base point free* for sufficiently large integers m), such that $\phi_{|mD|} : X \to X_1$ contracts exactly the extremal ray generated by the rational exceptional curve E with $E^2 = -1$. The morphisms in Example 46 contract exactly the extremal rays L and F.

To generalize Castelnuovo's criterion we need to find a supporting divisor D for an extremal ray R such D is semiample. Note that this is what happens also for the morphisms in Example 46.

We will need to some Definitions and Properties.

3.3. **Higher dimensions: Minimal modes, Finite generation, Canonical models.** It is in higher dimension that minimal models, canonical models and finite generation get intertwined.

In dimension 3 or higher one needs to allow certain mild singularities:

3.3.1. *Singularities.* In fact Mori [25] showed that if $dim(X) \geq 3$, contracting certain curves C on extremal rays on *smooth* varieties produces *singularities* on the image variety. The nice thing is that the program also works with these singularities, and these are called ($\mathbb{Q}$-factorial)-terminal singularities.

Definition 57. A Weil divisor D on a normal variety is called $\mathbb{Q}$-*Cartier* (resp. $\mathbb{R}$-*Cartier*) if mD is Cartier for some $m > 0$, $m \in \mathbb{Q}$ (resp. $m \in \mathbb{R}$). X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier. Let D and E be $\mathbb{Q}$ (resp. $\mathbb{R}$)-Cartier divisors. D and E are equivalent if there exists a m such that mD and mE are equivalent divisors. If C is a curve, we also define:

$$D \cdot C \stackrel{def}{=} \frac{1}{m}\,(mD \cdot C).$$

Note that the definitions in Section 3.2.2 can be applied also to $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors. In fact in the proofs in [3] the use of $\mathbb{R}$-divisors is essential.

Example 58. Let $X = \mathbb{C}^2/\tau$, where $\tau =< e^{\pi i/3}, e^{2\pi i/3}, 1 >$ and $\tau : (z_1, z_2) \mapsto (e^{\pi i/3} z_1, e^{\pi i/3} z_2)$. X is singular at $(0,0)$, but the canonical differential on $\mathbb{C}^2$ is not preserved by the group action. However, the third tensor power of the canonical differential is, and induces a Cartier divisor $3K_X$ on X. K_X is then a $\mathbb{Q}$-*Cartier divisor.*

Theorem-Definition 59. X has at most <u>*terminal*</u> singularities if K_X is $\mathbb{Q}$-Cartier and for every (equivalently, for any) resolution $h : \widetilde{X} \to X$, we have $mK_{\widetilde{X}} = h^*(mK_X) + F$, for some effective divisor F supported on the exceptional locus and every irreducible component of the exceptional locus appears in F with strictly positive multiplicity.

It is not hard to see that a surface S has at most terminal singularities if it is smooth; the singularities of the surface canonical models (see Remark 39) satisfy this similar property:

Proposition 60. *Let X^{can} be the canonical model of a surface of general type and $\phi : S \to X^{can}$ the minimal resolution as in Theorem 37. Then $K_S = \phi^*(K_{X^{can}})$.*

The definitions can be combined as follows:

Theorem-Definition 61. X has at most <u>*canonical*</u> singularities if K_X is $\mathbb{Q}$-Cartier and for every (equivalently, for any) resolution $h : \widetilde{X} \to X$, we have $mK_{\widetilde{X}} = h^*(mK_X) + F$, for some effective divisor F supported on the exceptional locus. If every irreducible component of the exceptional locus appears in F with strictly positive multiplicity, then X is said to have at most <u>*terminal*</u> singularities.

It can be shown that canonical and terminal singularities do not affect the plurigenera, hence the canonical ring; this property was in fact a motivation for considering this class of singularities (see [28], [29]).

Now we can state the higher dimensional definition of minimal (and canonical model)

Definition 62. X is a <u>*canonical*</u> (resp. <u>*minimal*</u>) model if X has at worst canonical (resp. terminal) singularities and K_X is an ample (resp. nef) $\mathbb{Q}$-Cartier divisor.

and the generalization of Castelnuovo's criterion:

Theorem 63. *(The contraction Theorem) Let X be a projective variety with at worst canonical singularities. Then every extremal ray R such that $K_X \cdot R < 0$ has a supporting semiample divisor.*

Key ingredients in the proof of the above theorem are the Vanishing Theorems, the Non-vanishing Theorem, the *Cone Theorem*, the Rationality Theorem, the Base point free Theorem (see for example, [19] or [23]).

Theorem-Definition 64. [Kawamata-Shokurov] If $\bar{X}$ is a minimal model of a projective variety X, and K_X is also big, then $K_{\bar{X}}$ is semiample, and the canonical ring is finitely generated. Then $X^{can} = \mathrm{Proj}(R(X, K_X))$ is called the <u>*canonical model*</u> of X, and the singularities are canonical.

Recall that the canonical model is unique. Now we can see the minimal Model Algorithm at work:

Theorem 65. [26] *Let X be a projective variety with only $\mathbb{Q}$-factorial terminal singularities. If X is not a minimal model, then there exists a surjective morphism $f : X \to Y$ to a normal projective variety Y with connected fibers and one of the following holds:*

(1) *$dim(X) > dim(Y)$ (Mori fiber space).*
(2) *f is birational and contracts a divisor (divisorial contraction)*
(3) *f is birational and contracts no divisor (small contraction).*

Furthermore:

Case (1) holds if and only if $\kappa(X) < 0$.
In Case (2) Y has at most terminal singularities.
In Case (3) Y is not $\mathbb{Q}$-factorial.

It is useful to see the algorithm at work in the case of surfaces; we will not discuss this here (see for example [23]). Case (2) is Castelnuovo's Theorem 35; $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ as in Example 46 are *Mori fiber spaces*, as in Case (1).

If the exceptional locus of a contraction morphism is not a divisor, but say a curve C on a threefold X, then the singularities on the image variety are such that the algorithm cannot be applied again[5]. To avoid the problem of these "small" contractions one would like, on a threefold, to *flip* the curve C, that is to construct another threefold (with mild singularities) X^+, isomorphic to X outside C, but such that K_{X^+} intersects positively the "transform" of C ($K_{X^+} \cdot C^+ > 0$). Then one would need to show that a sequence of flips terminates.

3.3.2. *The case of threefolds.*

Mori's result on existence of flips in 1988 [26] completed the algorithm for constructing minimal models for threefolds.

Theorem 66. [26] *Let $f : X \to Y$ be a birational morphism between normal projective threefolds as in Case (3) of the previous Theorem. Then there is a birational morphism $f^+ : X^+ \to Y$*

[5]It is easy to see that one would get the contradiction: $0 = K_X \cdot [R] < 0$.

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where X^+ is a projective threefold with only $\mathbb{Q}$-factorial terminal singularities, f^+ contracts no divisors and $K_{X^+} \cdot C > 0$, for any curve C contracted by f^+. The birational map between X and X^+ is an isomorphism in codimension 1.

Definition 67. A divisor D is f-ample if $D \cdot C > 0$, for any curve C contracted by f. The birational map between X and X^+ in the above Theorem is called a *flip*.

This gives the *minimal model algorithm* for threefolds:

Let X be a threefold with $\mathbb{Q}$-factorial terminal singularities: If K_X is nef, STOP: X is a minimal model. If K_X is not nef: contract any extremal ray R (as in Theorem 65). In case (1) X is a Mori fiber space: STOP. In case (2): replace X by Y, In case (3): replace X by X^+. The process is then repeated until the canonical bundle is nef or Case (1) holds.

It turns out that neither (2) nor (3) can occur an infinite number of times. The termination of Step (2) follows from the fact that at every step the second betti number decreases. The termination of (3) is subtle; it was shown by Shokurov, see for example [23].

Remark 68. The choice of a particular extremal ray to contract is arbitrary and typically far from unique and efficient (it is easy to construct surfaces with infinitely many extremal rays). The *Minimal Model Program with scaling* in [3] (see Section 4) provides a "guided", efficient, contraction algorithm.

The varieties of general type are somewhat special in this regard. It turns out (in the hypothesis of the cone theorem) that if a variety is of general type then the number of extremal rays is always finite. This principle is also used in the proof of termination of a certain kind of flips in [3]; see also Section 4.

3.3.3. *Log minimal pairs.* There exists also a *log version* of the minimal model program:
Let (X, Δ) be a pair consisting of a variety X and an effective $\mathbb{Q}(\mathbb{R})$-Cartier divisor in X. We might want to consider such a pair, for example, when we

have an open variety M and a compactification $X = \bar{M}$, in this case Δ is the boundary divisor (see Theorem 88). Another case is given by the elliptic fibrations, which we discuss here in the case of smooth surfaces.

Example 69. An elliptic surface is a morphism: $\pi : X \to C$, to a curve C where all the fibers, except a finite number, are elliptic curves; let also assume that X is minimal, for simplicity's sake. It is then a classical result [20] that $K_X = \pi^*(K_C + \Delta)$, where Δ is a $\mathbb{Q}$-divisor supported on the image of the fibers of π which are not smooth elliptic curves. The singular fibers determine the coefficients of $\Delta = \sum_i a_i \Delta_i$; it turns out that $0 \leq a_i < 1$. Here the log pair (C, Δ) in dimension 1 constitues as a step towards dimension 2.

Example 70. If $V \subset X$ is a smooth hypersurface, in a smooth variety X, then the *adjunction formula* (see Property 25) computes the canonical divisors of V in term of the log pair (X, V): $K_V = (K_X + V)_{|V}$.

Definition 71. A *log resolution* of (X, Δ) is any resolution of X $f : \widetilde{X} \to X$ such that the union of the strict transform of Δ and the exceptional locus of f are supported on divisors with simple normal crossings. Let (X, Δ) be a log pair, $\Delta = \sum_i a_i \Delta_i$ and $f : \widetilde{X} \to X$ be a log resolution of (X, Δ); we write:

$$K_{\widetilde{X}} + \widetilde{\Delta} = f^*(K_X + \Delta) \text{ and } \widetilde{\Delta} = \sum d_j \widetilde{\Delta}_j.$$

Definition 72. We have the following:

- *klt:* The pair is kawamata log terminal if for any (equivalently for every) log resolution f, $d_j < 1, \forall j$.
- *lc:* The pair is log canonical if for any (equivalently for every) log resolution f, $d_j \leq 1, \forall j$.
- *dlt:* The pair (X, Δ) is divisorially log terminal if $0 < a_i \leq 1$ and there is a log resolution f such that $d_j < 1$ for every exceptional divisors D_j.
- *plt:* The pair is purely log terminal if for any log resolution f, $d_j < 1$, for every coefficient of an exceptional divisor D_j.

The pair (C, Δ) in the base of the elliptic fibration described in the Example 69 is *ktl*: C is taken to be smooth, and Δ consists of a finite set of points, so the only condition to verify is the one on the coefficients. The pair (X, V) in Example 70 is ptl; in fact, the notion of *plt* pair was introduced in the context of induction arguments (see Definition 83). *Lc* pairs are important also in the study of moduli spaces (see Theorem 88).

Remark 73. *Klt* pairs are *dlt* pairs; *dlt* pairs are *lc*; see also [19]. There are other variations on the definitions above, which are essential for the proofs of the theorems stated in Section 4; we do not state them here and we refer the readers to [3].

Definition 74. A curve C on X is *log-extremal* if $(K_X + \Delta) \cdot C < 0$ and C is extremal in the usual sense.

We are looking for suitable models (Y, Γ) with $\mathbb{Q}$-factorial singularities and $K_Y + \Gamma$ nef:

Definition 75. Let (X, Δ) be log canonical and let $\phi : X \dashrightarrow Y$ be a birational map such that ϕ^{-1} does not contract any divisors. Let $\Gamma = \phi_*(\Delta)$: Y is a *log terminal model* for $K_X + \Delta$ if ϕ is $(K_X + \Delta)$-negative, $K_Y + \Gamma$ is *dlt* and nef and Y is $\mathbb{Q}$-factorial.

In the next Section we will see the log model at work, in the proof of Theorem 81.

3.3.4. *Relative minimal model program; Finite generation and flips.* There is a "relative" version of the above algorithm, for a morphism $f : X \to Y$. This relative version is crucial in the proofs of [3]. In these notes we only mention the following result, which connects finite generation and flips (see for example [23], Remark 30 and Theorem 64):

Theorem 76. *Let $f : X \to Y$ be a small contraction.*
The sheaf $R_f = \bigoplus_{m \in \mathbf{N}} f_*(mK_X)$ is a sheaf of finitely generated algebras if an only if the flip of f exists. Then the variety $X^+ \stackrel{def}{=} \mathrm{Proj}(R)$ and the morphism f^+ are unique.

4. New results

Several groups of people have been working on extending the (log) minimal model program to arbitrary algebraic varieties and many important results have been obtained (see the Introduction of [3]).

In the summer of 2005 Siu [30] announced the proof of the finite generation of the canonical ring for a smooth variety of general type. Siu's proof is based on analysis and we will not discuss it here. In the Fall of 2006, Birkar, Cascini, Hacon and McKernan [3] posted a paper on ArXiv on the existence of minimal models in all dimensions for varieties of general type. A corollary is the finite generation of the canonical ring and the existence of flips. Their proof of the existence of minimal models for a n-dimensional variety of general type does not rely on the existence or termination of flips, but it is a clever round-about induction on n. In this section we present an outline of the combined inductions arguments in [10], [11] and [3].

The arguments of the proofs work in the relative log set up and the use of the adjunction formula 25 is crucial.

It should be emphasized that we do not state the results in the most general form derived in the original papers. Most notably, the theorems are not stated here in the relative setting. The relative setting is however crucial in the induction steps; see for example the comments on the proof of Theorem 85.

Theorem 77. [8], [3, 11], [30] *Let X be a smooth variety. Then the canonical ring $R(X, K_X)$ is finitely generated.*

Theorem 78. *(Corollary of Theorem 77) Let X be a smooth variety of general type, then* $\mathrm{Proj}(R(X, K_X)) = X^{can}$ *is the canonical model of X (see Remark 30).*

Recall that the canonical model is then unique. The *log* statement of the finite generation is:

Theorem 79. *Let X be a projective variety. Suppose $K_X + \Delta$ is is* klt*, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbf{N}} H^0((\llcorner m(K_X + \Delta) \lrcorner)$$

is finitely generated. Here $\llcorner \sum a_i D_i \lrcorner = \sum \llcorner a_i \lrcorner D_i$ and $\lfloor a_i \rfloor$ means the round down to an integer of a_i.

Theorem 80. [3] *Let X be a smooth projective variety of general type of any dimension. Then the minimal model of X exists.*

The *log* statement of the above theorem is:

Theorem 81. *Let X be a projective variety. If $K_X + \Delta$ is big and $K_X + \Delta$ is* klt*, then there exists a log terminal model of $K_X + \Delta$.*

The proofs of the above theorems are intertwined and form a circular ascending induction on the dimension of the (log) variety (X, Δ). The ascending induction on dimension [3, 11] goes roughly as follows (but there are other inductions within the induction, and many delicate technical points, not mentioned here):

$$\boxed{\text{Th. 79 in } \dim n \;\Leftarrow\; \text{Th. 81 in } \dim n \;\Leftarrow\; \text{Th. 79 in } \dim n - 1.} \tag{4.1}$$

The proofs of 81 and 79 are different from the one outlined in the previous sections for the minimal model algorithm; in particular the termination and existence of flips is not a building block of the proof.

In the Sections 4.1.1 and 4.1.2 below we discuss some ideas involved in the proof of the arrows in (4.1):

4.1. **Some ideas in the proofs.**

4.1.1. *Existence of log minimal models for log varieties of general types implies finite generation for the log canonical ring:*

$$\boxed{\text{Theorem 79 in } \dim n \Leftarrow \text{Theorem 81 in } \dim n}$$

(Theorem 77 in $\dim n$ follows from Theorem 79).

Case 1. $\kappa(X, \Delta) \leq 0$. Then $R(X, K_X + \Delta)$ is obviously finitely generated, as in Examples 27 and 28.

Case 2. $\kappa(X,\Delta) = \dim(X)$, that is (X,Δ) is of general type. Then by Theorem 81 (X,Δ) has a log terminal model $(\bar{X},\bar{\Delta})$. The divisor $K_{\bar{X}} + \bar{\Delta}$ is semiample, since it is nef and big, by the base point free theorem of Kawamata and Shokurov (Theorem 64). Then $R(\bar{X}, K_{\bar{X}} + \bar{\Delta})$ is finitely generated (as in Example 29); since the log canonical ring is a birational invariant we have $R(X, K_X + \Delta) = R(\bar{X}, K_{\bar{X}} + \bar{\Delta})$.

Case 3. $0 \leq \kappa(X,\Delta) < \dim(X)$. This case follows from the previous one, as the following holds:

Theorem 82 (Fujino-Mori)**.** [8] *Let (X,Δ) be a klt pair with $\kappa(X,\Delta) \geq 0$, then there exists a variety Y and a* klt *log pair (K_Y, Δ_1) of general type such that $\oplus_r H^0(X, \ell r(K_X + \Delta)) = \oplus_r H^0(Y, \ell r(K_Y + \Delta_1))$, $\ell >> 0$.*

A particular case, and a "model" for the above Theorem 82 is given by the minimal elliptic surfaces X of $\kappa(X) = 1$, where $K_C + \Delta$ is big on C (see Section 3.3.3). (In general birational Iitaka's fibrations are used, see Theorem 90.)

4.1.2. *Finite generation and existence of certain flips imply the existence of log minimal models for varieties of general type:*

$$\boxed{\text{Theorem 81 in } \dim n \Leftarrow \text{Theorem 79 in } \dim n-1.} \tag{4.2}$$

The argument uses several log versions of the minimal model program, and it relies on induction on n. Some of the properties of the varieties of general types which have been discussed in Example 29 and Remark 68 become crucial here.

We have seen in the previous Sections that the crucial points in completing the minimal model algorithm are the <u>existence</u> and <u>termination</u> of flips. The argument of [3] cleverly avoids both issues: First the problem is modified by considering a suitable *log pair* $K_X + D$ and showing that flips exist for these particular pairs, following ideas of Shokurov; these are called pre-limiting flips, *pl flips* [10], [11].

The arrow of (4.2) consists then of two other arrows, which are the bulk of the inductive arguments (remember that this is only a rough outlines of the proofs!):

$$\boxed{\begin{array}{l} \textit{Min. mod. in } \dim\ n \text{ (Th. 81)} \Leftarrow \textit{ Pl-flips } \exists \textit{ in } \dim\ n \ \ ([11]) \\ \Leftarrow \textit{ Fin. gen. in } \dim\ n-1 \text{ (Th. 79)}. \end{array}} \tag{4.3}$$

Here we assume the existence of *pl*-flips, as in [3]; note that in [10], the existence of pl-flips is proved by induction, following ideas of Kawamata, Shokurov, Siu.

Definition 83. Let (X,Δ) a *plt* pair and $f : X \to Z$ be a projective morphism between normal projective varieties. Then f is a pl-flipping contraction if Δ is a $\mathbb{Q}$-divisor and

(1) f is small, of relative Picard number one,
(2) $-(K_X + \Delta)$ is f-ample[6],
(3) X is $\mathbb{Q}$-factorial,
(4) $S = \llcorner \Delta \lrcorner$ is irreducible and $-S$ is f-ample.

The flip of a pl-flipping contraction $f : X \to Z$ is a small projective morphism $g : Y \to Z$ of relative Picard number one, with $\Gamma = \mu_*(\Delta)$,

$$\begin{tikzcd} (X,\Delta) \arrow[rr, dashed, "\mu"] \arrow[dr, "f"'] & & (Y,\Gamma) \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

such that $K_Y + \Gamma$ is g-ample.

As in Theorem 76, the sheaf $R_f = \bigoplus_{m \in \mathbf{N}} f_*(m(K_X + \Delta))$ is a sheaf of finitely generated algebras if an only if the flip of f exists. The variety Y and the morphism g are then unique (see Remark 30). The main results of [11] implies:

Theorem 84. *Pl flips exist in dimension n.*

The strategy is to consider a *klt* pair (X, Δ) with Δ big, $\dim X = n - 1$, *assume* that the canonical ring $R(X, \Delta)$ is finitely generated and that other technical hypothesis are satisfied *(in* $\dim n - 1$*)*. In the notation of [11] and [3], Theorem 84 (in dimension n) is Theorem A_n, while the assumptions are Theorem F_{n-1}.

We now discuss some ideas in the following implication:

$$\boxed{\text{Theorem 81 in } \dim n \Leftarrow \text{ Pl-flips exist in dimension } n.} \tag{4.4}$$

To give a flavor of the argument used in [3] we follow the reasoning, presented in [3] for a very particular case (recall that Theorem 80 is a particular case of Theorem 81). This will allow us to introduce some of the ideas used in the arguments: *minimal model program with scaling* and *finiteness of models* combined with the *induction step(s)*.

A very particular case of Theorem 80.

Assume that X is a smooth variety (there is no log divisor), K_X is *big* (see Section 3.2.2) and that there exists a divisor D linearly equivalent to mK_X smooth and irreducible.

Consider the pair $K_X + D$. It is clear that K_X is nef if and only if $K_X + D$ is nef; note that R is an extremal ray for K_X if and only if it is an extremal ray for $K_X + D$. If K_X is not nef, there exists an extremal ray R and a contraction $\phi : X \to Z$, as in Theorem 63.

Case 1: ϕ is a divisorial contraction: then K_Z is trivial and Z is a minimal model.

[6]see Definition 67

Case 2: ϕ is a small contraction: ϕ is a small contraction for the pair $K_X + D$ and we are in the situation of a *pl*-flip of Definition 83; we assume that such flips exists by Theorem 84. Let $\varphi : X \dashrightarrow X'$ be the flip. One still should show that any sequence of such flips terminates, but this results is not available; instead the the authors of [3] run the minimal model program with scaling together with induction:

The minimal model program with scaling: Consider an ample divisor H on X and a positive number $t_0 \in \mathbb{R}_{\geq 0}$ such that $K_X + D + t_0 H$ is still ample. Then there exists a number $0 \leq t_1 \leq t_0$, a ray R such that $K_X + D + t_1 H$ is *nef* and $(K_X + D + t_1 H) \cdot R = 0$. Note that if $t_1 = 0$, $K_X + D$ is nef and we are done. We can then assume that $0 < t_1$, so that $(K_X + D) \cdot R < 0$; it can be shown that we can also assume R extremal. So there exists a divisorial extremal contraction or a flip, as above, and we can reiterate the algorithm. The novelty here is that the strict transforms of the divisor $(K_X + D + t_1 H)$ remain *nef* at each step, as the algorithm is run, and this is enough to guarantee the termination of these flips. Assume in fact that the contraction of R gives a flip: we restrict to D and use induction. Then $(K_X + D)_{|D} = K_D$ (by adjunction, see 25); let $G = H_{|_D}$. We have then the pair $K_D + t_1 G$ on D, with $\dim(D) = n - 1$. Termination is proved by showing the *finiteness* of the possible nef pairs $\{K_D + \tau G\}$ that one would get by running this algorithm; the hypothesis of K_X of general type is used (see Remark 68). A delicate point is that these minimal models are not necessarily log terminal, so a result for broader singularities is needed.

In the notation of [3], the *"finiteness of models"* is Theorem E_{n-1}; Theorem 81 is Theorem C_n. Theorem E_{n-1} is also proved by induction, assuming Theorem C_{n-1}.

Quite a few subtleties and difficulties occur in the general case and the relative case of the minimal program needs to be considered.

5. Remarks

5.1. **Other results and applications.** The results stated in the previous Section have generated a cascade of other new results; we mention here only a few.

Theorem 85. Klt *flips exist.*

The existence of flips in fact follows from a relative version of the finite generation, see also Theorem 76.

Another result is the following Theorem 87, which describes how two different minimal models are related; in fact the minimal models in $\dim \geq 3$ are not necessarily unique (see Example 86). Note that if X is a minimal model and K_X is ample (then X is of general type), then the minimal model is then unique (we know also that $X = \mathrm{Proj}(R(X, K_X))$).

Example 86. Let $\bar{X} = \{xy + zw = 0\} \subset \mathbb{C}^4$; $\bar{X}$ is singular at the origin. Now consider the threefolds $X \subset \mathbb{C}^4 \times \mathbb{P}^1$ and $X' \subset \mathbb{C}^4 \times \mathbb{P}^1$ defined by the equations: $\{y_0w+y_1y = 0, y_0x+y_1z = 0\}$ and $\{y_0w+y_1x = 0, y_0y+y_1z = 0\}$ respectively, with $[y_0, y_1] \in \mathbb{P}^1$. X and X' are smooth and the natural projection morphisms $\phi : X \longrightarrow \mathbb{C}^4$ and $\phi' : X' \longrightarrow \mathbb{C}^4$ are a isomorphism outside the origin of $\mathbb{C}^4$. The composition $\varphi = \phi \cdot \phi'^{-1} : X \dashrightarrow X'$ is an isomorphism outside $L = \phi^{-1}(0, 0, 0, 0)$ and $L' = \phi'^{-1}(0, 0, 0, 0)$. It can be shown that $K_X \cdot L = K_{X'} \cdot L' = 0$. [Exercise.] This transformation is called a <u>*flop*</u>.

Theorem 87. [3, 13]*(2008) Let X and X' two minimal models and $\phi : X \dashrightarrow X'$ a birational map. Then ϕ is a composition of flops.*

This result was first proved in [3] for varieties of general type. The proof in [13] uses the existence of *klt* flips (Theorem 85), the terminations of a certain kind of flips, proved in [3] and the algorithm of the minimal model program with scaling. The statement is obtained by the considering a *klt* log pair (X, D), where D is a suitable effective divisor in X; then $\phi : X \dashrightarrow X'$ is shown to be a composition of *klt flips* for the pair. Each *klt* flip for the pair (X_i, D_i) is also a flop for the terminal variety X.

The following is an application to the moduli space and involves *lc* pairs:

Theorem 88. [3] *Let $X = M_{g,n}$ the moduli space of stable curves of genus g with n marked points and let Δ_i, $1 \leq i \leq k$ denote the boundary divisors. Let $\Delta = \sum a_i\Delta_i$ be a boundary. Then the pair (X, Δ) is* lc *and if $K_X + \Delta$ is big then there is a log canonical model.*

5.2. **Classification.**

Finally a few words on how the Minimal Model Program provides a powerful tool to understand varieties in the sense of classification; majors players are also the Mori fibrations, the Iitaka fibrations and canonical models. We have already seen this in special case of surfaces.

The Minimal Model Program with Scaling provides the following:

Theorem 89. [3] *Let (X, Δ) be a* klt *$\mathbb{Q}$-factorial pair. If $K + \Delta$ is not pseudo-effective, then there exists a birational map $X \dashrightarrow \bar{X}$ and a Mori fiber space $\pi : \bar{X} \to B$.*

Here we do not discuss Mori fiber spaces. We have seen that if C is any curve contracted by π then $K_{\bar{X}} \cdot C < 0$, equivalently, since π is a fibration, $-K_F$ is ample, for a fiber F of π (this follows from the adjunction formula and Kleiman's criterion); F is a called *Fano variety.* In the case of surfaces we have seen (see Theorem 34) that the fibers are birationally ruled, in particular they are covered by rational curves. It is not known if the same is true in all dimensions. We refer to [9], [24], [18] for an introduction to Mori fiber spaces and several results.

For the variety of positive Kodaira dimension the following holds:

Theorem 90. (The Iitaka fibration, [34]) *If $\kappa(X) > 0$, there exists an integer $m >> 0$ and a fibre space $I_X : \tilde{X} \to X^{can}$ which is birationally equivalent to the rational maps $\phi_{|mK_X|} : X \dashrightarrow W_m$, with $\dim W_m = \dim X^{can} = \kappa(X)$ and the general fiber F of I_X has $\kappa(F) = 0$.*

Note that Theorem 77 implies $W_m = \mathrm{Proj}(R(X, K_X)) = X^{can}$, $\forall m >> 0$; note also that the all the relevant W_m are isomorphic, but given in different projective embeddings. The Theorem above states that a variety X of kodaira dimension $0 < \kappa(X) < \dim(X)$ is birationally fibered by varieties F of smaller dimension and $\kappa(F) = 0$.

The statement of the theorem is trivial for $\kappa(X) = 0$, but formally true. If $\kappa(X) = \dim(X)$, and X is of general type, then X^{can} is the canonical model of X.

It is then important to know under which condition the canonical divisor is semiample and find the lowest m such that mK_X is base point free; this an active area of current study. Recall that if X is minimal and of general type, then K_X is semiample by Theorem 64.

Conjecture 91. *(Abundance) Let X be a minimal model of a variety with $\kappa(X) \geq 0$; then the canonical divisor K_X is semiample, and it defines a morphism $\phi_{|mK_X|} : X \to B$, for $m >> 0$, where $B = X^{can}$ and $\kappa(F) = 0$, where F is the general fiber of ϕ.*

Conjecture 92. *Let X be a smooth projective variety. Then:*

- **a:** *If $\kappa(X) < 0$, then there exists a variety Y birational to X and a morphism $f : Y \twoheadrightarrow B$ which gives W the structure of Mori fiber space.*
- **b:** *If $\kappa(X) \geq 0$, then X is birationally equivalent to a minimal model M.*

Let us combine the above theorems in the case of $\dim X = 3$ (the abundance conjecture holds):

Example 93. Let X be a smooth projective variety, with $\dim X = 3$:

- **a:** If $\kappa(X) < 0$, then there exists a variety Y birational to X and a morphism $f : Y \twoheadrightarrow B$ which gives W the structure of Mori fiber space.
- **b:** If $\kappa(X) = 0$, there is an explicit birational classification of minimal models.
- **c:** If $\kappa(X) = 1, 2$, then there exists a morphism $f : M \twoheadrightarrow X^{can}$, where M is the minimal model of X; furthermore the fibers F have dim $2, 1$ respectively, with $\kappa(F) = 0$ (as in Theorem 90).
- **d:** If $\kappa(X) = \dim(X)$, then X is of general type, and the minimal model M is the terminal resolution $M \to X^{can}$ of the canonical model.

6. Summary/Timeline

- ∼ **1850s:** $\dim(X) = 1$. [Riemann et al.] Each smooth curve X has a unique smooth minimal model, the same curve X.
- ∼ **1900-1950:** $\dim(X) = 2$. [Castelnuovo-Enriques] If $\kappa(X) \geq 0$, then X has a unique minimal model. If $\kappa(X) < 0$, then X is birational to a Mori fiber space.
- ∼ **1960:** [Severi, Kodaira, Mumford, Bombieri et al.] A classification exists.
- ∼ **1980s:** $\dim(X) = 3$. [Kawamata, Kollár, Mori, Reid, Shokurov et al.]
  If $\kappa(X) \geq 0$, then X has a minimal model, *unique in codimension* 2.
  If $\kappa(X) < 0$, then X is birational to a Mori fiber space.
  For $\dim X \leq 3$ a finer structure exists.

1990s-2005: Many important results (see Introduction of [3].

2005-2008: [Birkar, Cascini, Hacon, McKernan, et al.] $\dim X$ any, X of general type, then Minimal model exists.

2005-2008: [Siu; Birkar, Cascini, Hacon, McKernan, et al.] Let X be a smooth variety, then the canonical ring $R(X, K_X)$ is finitely generated.

2005-2008: [Birkar, Cascini, Hacon, McKernan, et al.] $\dim X$ any: Flips exists.

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