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Asset Prices in Segmented and Integrated Markets

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Abstract This paper evaluates the effect of market integration on prices and welfare, in a model where two Lucas trees grow in separate regions with similar investors. We find equilibrium asset price dynamics and welfare both in segmentation, when each region holds its own asset and consumes its dividend, and in integration, when both regions trade both assets and consume both dividends. Integration always increases welfare. Asset prices may increase or decrease, depending on the time of integration, but decrease on average. Correlation in assets' returns is zero or negative before integration, but significantly positive afterwards, explaining some effects commonly associated with financialization.

Keywords asset pricing · integration · financialization, equilibrium

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1 Introduction

When investors gain access to a new asset class, diversification benefits loom large. Past price observations, from an era when the new asset class was unavailable to the new investors, suggest both significant risk premia and modest correlations with existing asset classes. Yet, as the new investors adjust their portfolios, they find that correlations have increased, largely reducing the putative diversification gains. International equities [41, 27], emerging markets [6], and commodity futures [46] offer recent examples of this phenomenon.

More broadly, the central problem is to understand how the integration of two markets affects the welfare of their participants, how such benefits – if any – are shared, and how integration changes the levels and dynamics of prices. To answer these questions, it is necessary to understand markets' equilibria both in the regime of segmentation – when in one market assets are available only to that market's investors – and in integration, when any investor can trade any asset.

This paper examines a general equilibrium model with two regions, each of them endowed with its own asset, and finds explicitly the two assets' dynamics and their implied welfare in both the segmentation and integration regimes. It also identifies the terms of sharing under which investors in both markets would agree to integration. Similar to the familiar Lucas Jr [32] tree, both regions have infinitely lived representative investors with the same isoelastic preferences. Outputs in the two regions have uncorrelated short-term fluctuations, but remain cointegrated, their shares oscillating around fractions that represent the markets' long-term relative sizes. The growth of total consumption has constant mean and volatility.

Prima facie, the model has contrasting implications for prices and welfare: integration increases welfare in both markets but, on average, it decreases asset prices – investors are happier, but poorer. Counterintuitive at first, this phenomenon stems from the change in consumption dynamics from segmentation to integration. In segmentation, consumption shocks are undiversifiable, and agents have no choice but to bid up the domestic asset in an effort to secure future consumption. In integration, however, agents can spread their wealth between domestic and foreign assets, thereby reducing consumption shocks, and in turn reducing the pressure to buy each of them. Thus, the loss of wealth from lower asset prices is more than compensated by the availability of a smoother consumption stream.

In the international asset pricing literature, the market segmentation hypothesis stipulates that integration increases the price of an asset by making it available to a larger pool of investors, and decreases its subsequent expected return by reducing its equilibrium risk premium.[1] In a similar vein, Merton's

[1] As Foerster and Karolyi [26] surmise, "International asset pricing models suggest that when investors realize that barriers to investments are to be removed, expected returns should decrease as prices are bid up on the expectation of the removal of these barriers." For partial equilibrium models of international asset pricing, see also Solnik [42, 43], Stulz [44, 45], Adler and Dumas [2].

[36] investor recognition hypothesis prescribes that an asset that is recognized (i.e., held or traded) by more investors has a lower expected return than another one, with the same exposures to risk factors, but recognized by fewer investors.

Our results are consistent with these hypotheses when a small market becomes integrated in a much larger one, but the model also uncovers new interactions that arise when the two markets have comparable size, potentially reversing these conclusions. While market integration brings new buyers for the domestic asset, i.e. capital inflows, it also leads domestic investors to buy foreign assets – capital outflows – which make the effect of flows ambiguous.

In segmentation, price-dividend ratios increase with the dividend share while decreasing in integration. In an integrated market, a smaller dividend share means a lower correlation between dividends and consumption growth, which implies a higher diversification value [16]. In segmentation, however, the dividend *is* the consumption, and a higher share increases demand for assets in the attempt to smooth consumption over time. As the supply of stock is fixed, in equilibrium such higher demand is reflected in increased stock prices.

Correlation between asset returns has also subtle properties. Only with logarithmic preferences it is true that uncorrelated dividend fluctuations imply uncorrelated returns.[2] With higher risk aversion, returns correlation is *negative* in segmentation and *positive* in integration. The latter effect has a natural interpretation as price-pressure from rebalancing:[3] when one asset gains value, its portfolio weight increases, spurring investors to sell it, to buy the other asset. In equilibrium, rebalancing is impossible, and prices adjust by moving together.

Negative correlation in segmentation arises from a more complex mechanism: Price correlation is the sum of correlation between dividends and price-dividend ratios and between the two price-dividend ratios (cf. (5.1) below). In both segmentation and integration, as the price-consumption ratio increases with that market's share and the shares add to one, the two assets' price-consumption ratios must be negatively correlated. In segmentation, where consumption and dividends coincide, this is the main determinant of return correlation, while in integration it is compensated by rebalancing effects. Because our model documents a shift – and even reversal – in correlation from the segmentation to the integration regime, it offers a rational explanation of this phenomenon as the equilibrium response to the removal of investment barriers.

Welfare implications are more forthright. Integration increases welfare for investors in both markets, regardless of their share. At the same time, aggregate wealth (i.e. the total value of both assets) also declines. The intuition is

[2] It is trivial to obtain uncorrelated returns with power utility by considering two segmented markets with independent Lucas trees. However, in such a model one market eventually overtakes the other, violating cointegration.

[3] With logarithmic preferences, see the integrated market of Cochrane et al. [16] and the partially integrated markets of Pavlova and Rigobon [38], Bhamra et al. [7]. For power utility, see Martin [33].

that, in segmentation, each asset has a domestic monopoly on consumption – in apple country, owning apple trees is the only way to eat. But in a country with apples and oranges (assumed to be perfect substitutes in this metaphor) one can buy both fruits, neither of them is a perfect consumption hedge, and consumption – a mixture of both – is overall less volatile over time.

Unanticipated, exogenous integration increases welfare for both markets, but it also changes the share of wealth that original investors own in the integrated market. Wealth shares implied by exogenous integration are weighted averages of the current and the long-term dividend shares, as prices reflect the reversion of future dividend shares to their means.

Endogenous integration, which would require the agreement of both investors while allowing for one-off transfers, needs to ensure that neither party is worse off. We determine the range of post-integration shares of wealth that are acceptable to both investors: In normal times (when consumption shares are close to their long-term means) the *integration range* is a narrow interval around the exogenous integration point. In times of stress, when one consumption share is abnormally low (hence the other is abnormally high), the interval widens for both investors, but especially for the one in distress, who is compelled to accept integration even for a wealth-share well below its exogenous level.

The rest of the article is organized as follows. Section 2 discusses in detail the relation of the paper with the literature. Section 3 describes the model, its main assumptions, and defines segmented and integrated equilibria. Section 4 states the main results: asset price dynamics, their correlation, and implied welfare in both regimes. Section 5 discusses the implications of market integration for price levels, excess returns, interest rates, correlation, and welfare, as well as the sharing ranges for endogenous integration. Section 6 discusses the effect of heterogenous preferences. Concluding remarks are in section 7. All proofs are in the Appendix.

2 Literature review

The results in this paper are related to several strands of literature. Our model of an integrated market is closest to the one of Menzly et al. [35]: assuming that consumption is a geometric Brownian motion and dividend shares follow multivariate mean-reverting process, they find linear approximations for asset prices in terms of dividend shares under logarithmic preferences with external habit. Santos and Veronesi [39, 40] obtain exact linear prices with power utility, at the cost of introducing predictability in consumption growth. Employing a novel volatility specification in the class of polynomial diffusions (Filipović and Larsson [23], Filipović et al. [25], Filipović and Larsson [24]), we combine exact linear prices for power utility with geometric Brownian motion for consumption.

Our results contribute to the literature on the financialization of commodity prices. Tang and Xiong [46] link the increase in both commodity prices and their correlation with equities to the entry of institutional investors in the market. Such effects, along with significant increase in volatility, alarmed regulators and policy makers in several countries [34, 10]. Yet, as the financial crisis has receded, so has some of this evidence [8], and the financialization debate remains mired in controversy (see Carmona [12] for a recent survey), as empirical observations attract competing interpretations – partly for the dearth of theoretical models that examine the implications of such explanations.

In an iterative equilibrium, where investors use recent price behavior to update their policies, Chan et al. [14] find that feedback effects on prices increase correlations. Likewise, Cont and Wagalath [17] observe an increase in correlations from price pressure in a model where institutional investors rebalance their portfolios subject to price impact. Basak and Pavlova [5] present a model in which the benchmarking incentives faced by institutional investors generate both higher prices and higher correlation. Yet, as they note, "The often-quoted intuition for this increase is that commodity futures markets were largely segmented before the inflow of institutional investors in the mid-2000s, and that institutions entering these markets have linked them together, as well as with the stock market, through the cross-holdings in their portfolios." This paper brings this oft-quoted intuition to life in a general equilibrium model, and examines its wider implications for asset dynamics and welfare. Consistent with intuition, our model shows that market integration does reproduce a substantial increase in correlation. Furthermore, it predicts that prices do not necessarily increase with integration, and in fact they may fall, depending on the circumstances of integration.

Our specification is related to the volatility-stabilized model of Banner et al. [3], Fernholz and Karatzas [22], Pal [37], and Cuchiero [18], who study models in which asset *prices* (without dividends) aggregate to a geometric Brownian motion, and characterize their no-arbitrage properties. By contrast, we posit similar diffusions as models of *dividends*, and exploit their aggregation properties to derive equilibrium prices that are arbitrage-free by construction, in view of the strictly positive stochastic discount factor implied by the equilibrium.

More broadly, our integrated market can be interpreted as a Lucas' orchard [33] with two trees, as in Cochrane et al. [16], Bhamra et al. [7], Buraschi et al. [11], and Chabakauri [13]. Hansen [29] solves numerically a model with two agents and different beliefs, showing that a large number of trees helps explaining empirical evidence on asset price dynamics. In contrast to these studies, our trees are cointegrated, in that the ratio of their outputs remains stationary. While Cochrane et al. [16] find that price-dividend ratios involve hypergeometric functions in the integrated market, in our model such functions arise in the segmentation regime, where consumption growth has a stationary component (with the exception of logarithmic preferences, for which price-dividend ratios remain constant).

In the international asset pricing literature, our model is closest to that of Pavlova and Rigobon [38], who also specify two countries with respective goods and representative investors. The main differences from their model are that, while we compare the segmentation and integration regimes, their agents trade through some endogenous exchange rate. Also, while they focus on a linear combination of logarithmic preferences across domestic and foreign goods, we assume that goods are perfect substitutes, but allow for power utility.

The representative investor of Dumas [19] shares our investors' preferences, but has a perfectly elastic asset supply through a linear technology, while trading incurs proportional costs. Bhamra et al. [7] represent imperfect integration as a tax on foreign dividends while keeping the logarithmic preferences and dividend dynamics of Cochrane et al. [16]. As in these previous studies, our integrated stock prices display "excess correlation" in the sense of Dumas et al. [20], i.e., higher than output correlation. In addition, we show that such correlation is a product of integration because we find that in the segmentation regime it is *negative*. As this latter effect arises from correlation in the price-dividend ratios, it is not visible in models with logarithmic preferences, which imply constant price-dividend ratios.

3 Model

3.1 Market

Two similar trees grow in separate islands with similar people, and each tree feeds its island. Their crops fluctuate independently, but have similar long-term growth. Crops are perishable – must be consumed immediately – and trees are the only property on the islands. With a bridge, people on both islands could share property (trees) and consumption (crops). Should they build such a bridge?

The island-tree metaphor has multiple interpretations: Certain countries (islands) compel their citizens to own only local securities (trees). Regulation prevents some industries (islands) from holding investments (trees) in other sectors. In each of these situations, a central policy question is how integration – enabling cross-ownership – affects asset prices and welfare.

Adopting a more conventional terminology, henceforth we refer to an island as a "region", to a tree as an "asset", and to its crop as a "dividend stream". The model begins with the dynamics of the dividend streams $(D_t^{(1)}, D_t^{(2)})_{t\geq 0}$, described by their sum $D_t = D_t^{(1)} + D_t^{(2)}$ and by the dividend share $X_t = D_t^{(1)}/D_t$, the proportion of dividends produced in the first region:

$$dD_t = \mu D_t dt + \sigma D_t dB_t^D \tag{3.1}$$

$$dX_t = \kappa(w_1 - X_t)dt + \sigma\sqrt{X_t(1 - X_t)}dB_t^X \tag{3.2}$$

where $\mu, \sigma > 0$, $w_1 \in (0,1)$, and B^D, B^X are independent Brownian motions supported on the probability space $(\Omega, \mathcal{F}, P)$ endowed with the augmentation $(\mathcal{F}_t)_{t\geq 0}$ of their natural filtration. Set $w_2 = 1 - w_1$.

Equation (3.1) means that aggregate dividends' growth has constant mean μ and volatility σ. This specification stems from the observation that dividend growth, while fluctuating over time, has remained positive on average over the past century, while the independence of increments reflects the unpredictability of dividend growth [15]. Equation (3.2) stipulates that the first region's share of total dividends fluctuates around its long-term mean w_1. The independence of the Brownian motions $(B_t^D)_{t\geq 0}$, driving aggregate dividends, and $(B_t^X)_{t\geq 0}$, driving the dividend share in the first market, is equivalent to the independence of dividend shocks, as shown by Proposition 3.2 below. This assumption, in turn, is made to isolate the price dependence that arises endogenously from agents' strategic behavior from any exogenous dependence among dividend shocks. Put differently, we can safely ascribe any price dependence to the endogenous demand for risky assets rather than to some exogenous correlation among dividend shocks, which is indeed absent.[4]

Neither dividend ever vanishes under the following[5]

Assumption 3.1 The parameters κ, σ, and w_1 satisfy

$$\frac{\sigma^2}{2\kappa} \leq w_1 \leq 1 - \frac{\sigma^2}{2\kappa}.$$

The above restriction is typically mild. For example, $\sigma = 1.5\%$ and $\kappa = 6\%$, used in the discussion below (cf. Figure 5.1), imply that $w_1 \in (4.5\%, 95.5\%)$, whereby, on average, one region should not overwhelm the other by more than twenty-two to one. (In particular, the above restriction implies that κ near zero is not a meaningful limit for the model.)

The specification in (3.1)-(3.2) implies the joint dynamics of separate dividend streams.

Proposition 3.2 *The dividend $D^{(1)}, D^{(2)}$ follow the bivariate diffusion:*

$$\begin{aligned} dD_t^{(1)} &= \left((\mu - \kappa w_2) D_t^{(1)} + \kappa w_1 D_t^{(2)}\right) dt + \sigma \sqrt{D_t^{(1)} \left(D_t^{(1)} + D_t^{(2)}\right)} dB_t^{(1)} \\ dD_t^{(2)} &= \left(\kappa w_2 D_t^{(1)} + (\mu - \kappa w_1) D_t^{(2)}\right) dt + \sigma \sqrt{D_t^{(2)} \left(D_t^{(1)} + D_t^{(2)}\right)} dB_t^{(2)} \end{aligned} \tag{3.3}$$

where the Brownian motions $B^{(1)}, B^{(2)}$ are defined as

$$dB_t^{(1)} = \sqrt{\frac{D_t^{(1)}}{D_t}} dB_t^D + \sqrt{\frac{D_t^{(2)}}{D_t}} dB_t^X, \qquad dB_t^{(2)} = \sqrt{\frac{D_t^{(2)}}{D_t}} dB_t^D - \sqrt{\frac{D_t^{(1)}}{D_t}} dB_t^X,$$

and therefore are independent.

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4 This assumption is also a reasonable approximation to the statistical properties of dividends across large economic areas. For example, in the period 1991-2018, the correlation among dividend growth between large-capitalization stocks in the United States, Europe, and Japan, were 0.25 (US-EU), -0.07 (US-JP), and -0.11 (EU-JP), as estimated from the annual dividends paid by the index funds VFINX, VEUREX, and VPACX, which track their respective markets.

5 Under Assumption 3.1, the process X exists globally and never reaches zero or one, i.e., $P(X_t \in (0,1)\ \forall t \geq 0) = 1$. See, for example, Borodin and Salminen [9, No. 6, Chapter II.1]

The independence of $B^{(1)}$ and $B^{(2)}$ means that the dividend streams have uncorrelated fluctuations, thereby offering a natural model of two regions affected by unrelated risks. (Note that such independence is not an additional assumption, but stems from – and is equivalent to – the independence of fluctuations in their sum and share, posited in (3.1)-(3.2).) To see how dividend growth depends on assets' relative sizes, rewrite (3.3) as

$$\frac{dD_t^{(1)}}{D_t^{(1)}} = \left(\mu - \kappa w_2 + \kappa w_1 \frac{D_t^{(2)}}{D_t^{(1)}}\right) dt + \sigma\sqrt{1 + \frac{D_t^{(2)}}{D_t^{(1)}}} dB_t^{(1)}, \tag{3.4}$$

which shows that both growth rate and volatility are higher for the asset with smaller dividends.

3.2 Preferences and equilibrium

In both regions, asset owners are infinitely-lived representative agents with equal preferences. Each of them seeks to maximize time-additive utility from consumption, with a time-preference rate $\beta > 0$ and a constant relative risk aversion $\gamma > 0$.[6]

The asset pricing and welfare implications follow from the familiar premises of optimality and market clearing for consumption and investment, which underpin the definition of equilibrium, both in segmentation and in integration.

Definition 3.3 Equilibria in segmentation and integration are defined as:

(i) **Segmentation**
A segmented equilibrium for region $i = 1, 2$ is a pair $(r_t^{(i)}, P_t^{(i)})_{t\geq 0}$, where $(r_t^{(i)})_{t\geq 0}$ is adapted with $\int_0^T r_t^{(i)} dt < \infty$ a.s. for all $T > 0$, $(P_t^{(i)})_{t\geq 0}$ is a continuous semimartingale, and the optimal consumption-investment problem (the set of admissible strategies $\mathcal{A}$ is defined in Appendix A.1)

$$\max_{(c,\phi)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta s}\frac{c_s^{1-\gamma}}{1-\gamma} ds\right], \tag{3.5}$$

with safe rate $r^{(i)}$ and asset price $P^{(i)}$, hence with wealth $(Y_t)_{t\geq 0}$ satisfying the budget equation

$$dY_t = r_t^{(i)}(Y_t - \phi_t P_t^{(i)})dt + \phi_t\left(dP_t^{(i)} + D_t^{(i)}dt\right) - c_t dt \quad Y_0 = P_0^{(i)}, \tag{3.6}$$

is well-posed and has solution $c_t = D_t^{(i)}$ and $\phi_t = 1$ for all $t \geq 0$ (the market-clearing conditions for assets and consumption).

[6] The case $\gamma = 1$ corresponds to logarithmic utility, which leads to simpler but slightly different calculations, for brevity not included here.

(ii) **Integration**
An integrated equilibrium is a triplet of processes $(\bar{r}_t, \bar{P}_t^{(1)}, \bar{P}_t^{(2)})_{t\geq 0}$, where $(\bar{r}_t^{(i)})_{t\geq 0}$ is adapted with $\int_0^T r_t^{(i)} dt < \infty$ a.s. for all $T > 0$, $(\bar{P}_t^{(i)})_{t\geq 0}$ are continuous semimartingales, and the optimal consumption-investment problem[7] (3.5) with safe rate $\bar{r}$ and asset prices $\bar{P}^{(1)}, \bar{P}^{(2)}$, hence with wealth process $(Y_t)_{t\geq 0}$ satisfying

$$\begin{aligned} dY_t =& \bar{r}_t \left(Y_t - \phi_t^{(1)} \bar{P}_t^{(1)} - \phi_t^{(2)} \bar{P}_t^{(2)} \right) dt + \\ & \phi_t^{(1)} \left(d\bar{P}_t^{(1)} + D_t^{(1)} dt \right) + \phi_t^{(2)} \left(d\bar{P}_t^{(2)} + D_t^{(2)} dt \right) - c_t dt \\ Y_0 =& P_0^{(1)} + P_0^{(2)}, \end{aligned} \tag{3.7}$$

is well-posed and solved by $c_t = D_t^{(1)} + D_t^{(2)}$, $\phi_t^{(1)} = \phi_t^{(2)} = 1$ for all $t \geq 0$.

In particular, the above definition treats the goods of both markets as perfect substitutes in the integration regime, which implies a unit exchange rate. By contrast, in the segmentation regime the exchange rate is nontrivial, because a unit of the foreign good cannot be consumed domestically. In this case, the exchange rate reflects precisely the shadow price of the constraint that each market consume its own good. Accordingly, the foreign good becomes expensive precisely when it would be most needed, as discussed in Section 5.5 below.

3.3 Lucas' Tree

To set the stage for the main results, it is helpful to recall the continuous-time version of the familiar asset pricing model of Lucas Jr [32], with a representative agent maximizing (3.5) and a dividend stream following the geometric Brownian motion (3.1). The corresponding price-dividend ratio P_t/D_t and interest rate r_0 are constant:

$$\frac{P_t}{D_t} = \frac{1}{r_0 - \mu + \gamma\sigma^2} \tag{3.8}$$

$$r_0 = \beta + \gamma\mu - \frac{\sigma^2}{2}\gamma(\gamma + 1). \tag{3.9}$$

By the second equation, the interest rate increases with time preference and dividend growth, which reduces savings, and decreases with dividend volatility, which encourages precautionary savings. The first equation adjusts for risk the familiar dividend-discount model of Gordon and Shapiro [28].

The present model retains two attractive features of Lucas' model: stationary price-dividend ratios and interest rates, and stationary dividend growth rates. In fact, market integration leads to the same interest rate (3.9) as Lucas',

[7] Here the maximization takes place on $c \in \mathcal{C}$ and $\phi = (\phi^{(1)}, \phi^{(2)}) \in \overline{\mathcal{P}}$, as the integrated market includes two assets. See Appendix A.1 for details.

and the price-dividend ratio of the aggregate asset with price $\bar{P}_t = \bar{P}_t^{(1)} + \bar{P}_t^{(2)}$, paying the total dividend $D_t = D_t^{(1)} + D_t^{(2)}$, coincides with the value in the right-hands side of (3.8). The problem is to identify separately the two asset prices $\bar{P}_t^{(1)}, \bar{P}_t^{(2)}$ in integration, and to compare them with their counterparts $P_t^{(1)}, P_t^{(2)}$ in segmentation.

In addition, note that Lucas' model is well posed if and only if the asset price in (3.8) is finite, which corresponds to the following:

Assumption 3.4 The parameters $\mu, \sigma, \gamma, \beta > 0$ satisfy the condition:

$$\theta := r_0 - \mu + \gamma\sigma^2 = \beta - (1-\gamma)\mu + \gamma(1-\gamma)\frac{\sigma^2}{2} > 0.$$

This condition prescribes that preference and market parameters are such that, as the horizon increases, the discounted value of dividends declines sufficiently quickly, so that their accumulated value converges. It excludes combinations of preference and market parameters that would make it optimal to postpone consumption indefinitely, in the anticipation of ever higher future satisfaction.

4 Main results

This section summarizes the main results: first in segmentation and then in integration, we find explicit formulae for (i) asset prices, their expected excess returns, and safe rates, (ii) their corresponding welfares, and (iii) the instantaneous correlation of asset prices. Such quantities are most complex in segmentation, as the corresponding consumption processes and hence the stochastic discount factors have stochastic growth and volatility. In integration, instead, asset prices are *linear* in dividends, while welfare is homogeneous.

4.1 Segmentation

When the domestic investor owns and consumes only domestic dividends, each region has its own asset price, safe rate, and welfare. The next theorem expresses all these quantities in terms of two scalar functions $f^{(1)}, f^{(2)}$, which are found explicitly in Proposition A.3 in the appendix, in terms of (generalized) hypergeometric functions.

Theorem 4.1 (Segmentation) *Let dividends $(D^{(i)})_{i=1,2}$ be as in Proposition 3.2 and let Assumptions 3.1 and 3.4 hold. Define*

$$f^{(1)}(x) := \mathbb{E}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds \,\middle|\, X_0 = x\right], \tag{4.1}$$

$$f^{(2)}(x) := \mathbb{E}\left[\int_0^\infty e^{-\theta s} (1-X_s)^{1-\gamma} ds \,\middle|\, X_0 = x\right],$$

and assume that $f^{(1)}(x), f^{(2)}(x) < \infty$ for all $x \in (0,1)$. (Assumption 4.2 below clarifies when this condition is satisfied.) Then,

(i) segmentation equilibrium asset prices and safe rates $(P^{(i)}, r^{(i)})_{i=1,2}$ are

$$\begin{aligned} P_t^{(1)} &= D_t^{(1)} X_t^{\gamma-1} f^{(1)}(X_t), \\ P_t^{(2)} &= D_t^{(2)} (1-X_t)^{\gamma-1} f^{(2)}(X_t), \end{aligned} \tag{4.2}$$

$$r_t^{(1)} = \beta + \gamma(\mu-\kappa) + \frac{1}{X_t}\left(\gamma\kappa w_1 - \frac{\gamma(\gamma+1)\sigma^2}{2}\right), \tag{4.3}$$

$$r_t^{(2)} = \beta + \gamma(\mu-\kappa) + \frac{1}{1-X_t}\left(\gamma\kappa w_2 - \frac{\gamma(\gamma+1)\sigma^2}{2}\right). \tag{4.4}$$

(ii) segmentation welfares are

$$\mathcal{U}_t^{(i)} = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)} \frac{\left(D_s^{(i)}\right)^{1-\gamma}}{1-\gamma} ds\right] = \frac{D_t^{1-\gamma}}{1-\gamma} f^{(i)}(X_t), \qquad i = 1,2.$$

where $\mathbb{E}_t[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_t$.

(iii) segmentation correlation, defined as

$$\rho_t := \frac{\frac{d\langle P^{(1)}, P^{(2)}\rangle_t}{dt}}{\sqrt{\frac{d\langle P^{(1)}\rangle_t}{dt}}\sqrt{\frac{d\langle P^{(2)}\rangle_t}{dt}}}, \tag{4.5}$$

equals

$$\rho_t = \frac{\sqrt{X_t(1-X_t)}}{\sqrt{X_t\left(f^{(1)}(X_t)\right)^2 + \left(h^{(1)}(X_t)\right)^2 (1-X_t)}} \cdot \frac{f^{(1)}(X_t)f^{(2)}(X_t) + h^{(1)}(X_t)h^{(2)}(X_t)}{\sqrt{(1-X_t)\left(f^{(2)}(X_t)\right)^2 + \left(h^{(2)}(X_t)\right)^2 X_t}}, \text{ where}$$

$h^{(1)}(x) := \gamma f^{(1)}(x) + x f^{(1)\prime}(x)$ and $h^{(2)}(x) := -\gamma f^{(2)}(x) + (1-x) f^{(2)\prime}(x)$.

(iv) the assets' expected excess returns are

$$\begin{aligned} \mu_t^{(1)} - r_t^{(1)} &= \mu - \bar{r} + \gamma^2\sigma^2 \frac{1-X_t}{X_t} + \kappa(w - X_t)\frac{\left(f^{(1)}\right)'(X_t)}{f^{(1)}(X_t)} \\ &+ \frac{\sigma^2}{2}(1-X_t)\left(2\gamma\frac{\left(f^{(1)}\right)'(X_t)}{f^{(1)}(X_t)} + X_t\frac{\left(f^{(1)}\right)''(X_t)}{f^{(1)}(X_t)}\right) + \frac{1}{X_t^{\gamma-1} f^{(1)}(X_t)}, \end{aligned}$$

$$\begin{aligned} \mu_t^{(2)} - r_t^{(2)} &= \mu - \bar{r} + \gamma^2\sigma^2 \frac{X_t}{1-X_t} + \kappa(w - X_t)\frac{\left(f^{(2)}\right)'(X_t)}{f^{(2)}(X_t)} \\ &+ \frac{\sigma^2}{2}X_t\left(-2\gamma\frac{\left(f^{(2)}\right)'(X_t)}{f^{(2)}(X_t)} + (1-X_t)\frac{\left(f^{(2)}\right)''(X_t)}{f^{(2)}(X_t)}\right) + \frac{1}{(1-X_t)^{\gamma-1} f^{(2)}(X_t)}. \end{aligned}$$

Finding the functions $f^{(1)}, f^{(2)}$ that identify the segmentation equilibria is important both for computation and to understand when such functions are finite, i.e. the problem is well posed. In fact, such a condition holds under a parameter restriction that has no analogue in the Lucas' model (and in the integrated market).

Assumption 4.2 (Well-posedness) The values $\frac{2\kappa w_1}{\sigma^2}, \frac{2\kappa w_2}{\sigma^2}$ are not integers. In addition,

$$\gamma < 1 + \frac{2\kappa}{\sigma^2} \min\{w_1, w_2\}. \tag{4.6}$$

The exclusion of integer values for $\frac{2\kappa w_1}{\sigma^2}$ and $\frac{2\kappa w_2}{\sigma^2}$ is merely technical: as it is satisfied up to arbitrarily small perturbations of any parameter, it does not have a distinct economic meaning. The inequality in (4.6) specifically requires that risk aversion is not too high relative to growth, volatility, and market size, and holds for typical risk aversions. For example, with the combination $\mu = 1.5\%$, $\sigma = 6\%$, $w_1 = 2/3$ in Figure 5.1, the restriction becomes $\gamma < 8.4$.

To interpret this condition, note first that it is always satisfied for $\gamma \leq 1$, regardless of dividend dynamics and relative size, as for such investors the utility $U(x) = x^{1-\gamma}/(1-\gamma)$ of future dividends is always finite, and Assumption 3.4 guarantees that also its discounted value is finite. By contrast, for $\gamma > 1$ the marginal utility from states in which dividends are very low (X_t near 0 for region 1 or X_t near 1 for region 2) can be so large that the resulting asset price is infinite (an extra unit of consumption would lead to an unbounded increase in expected utility) and the resulting expected utility is infinitely negative – the specified dividend stream is unacceptable.

4.2 Integration

With market integration, each investor can own and consume both the domestic and the foreign asset. As both investors have the same preferences, they make the same choices in proportion to their respective wealth. How such wealth is shared at the time of integration depends on whether the integration occurs exogenously or endogenously. Exogenous integration takes place suddenly, and agents exchange their holdings according to their post-integration prices to form the optimal portfolios in both assets. Thus, the share of post-integration wealth of each agent equals the post-integration value of the domestic asset relative to total market capitalization. Vice versa, endogenous integration entails the consent of both participants, hence the negotiation of respective shares of post-integration wealth within a range that is compatible with a welfare gain for both parties. After relative shares are determined at time of integration, both agents effectively represent different fractions of a grand representative agent who consumes both dividend streams. Accordingly, the unique stochastic discount factor depends on overall consumption, i.e., the sum of dividends.

Theorem 4.3 (Integration) *Let dividends $(D^{(i)})_{i=1,2}$ be as in Proposition 3.2 and let Assumptions 3.1 and 3.4 hold. Then,*

(i) integration asset prices and their safe rate are

$$\bar{P}_t^{(1)} = \frac{1}{\theta}\left(\frac{\theta+\kappa w_1}{\theta+\kappa}D_t^{(1)} + \frac{\kappa w_1}{\theta+\kappa}D_t^{(2)}\right) \tag{4.7}$$

$$\bar{P}_t^{(2)} = \frac{1}{\theta}\left(\frac{\kappa w_2}{\theta+\kappa}D_t^{(1)} + \frac{\theta+\kappa w_2}{\theta+\kappa}D_t^{(2)}\right) \tag{4.8}$$

$$\overline{r_t} = \bar{r} := \beta + \gamma\mu - \gamma(\gamma+1)\frac{\sigma^2}{2}.$$

(ii) denoting by $k_1, k_2 > 0$ the initial shares of ownership of the (risky) asset of the two agents (so that $k_1 + k_2 = 1$), integration welfares are

$$\bar{\mathcal{U}}_t^{(i)} := \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{(k_i D_s)^{1-\gamma}}{1-\gamma}ds\right] = k_i^{1-\gamma}\frac{D_t^{1-\gamma}}{(1-\gamma)}\frac{1}{\theta}, \qquad i=1,2.$$

If integration is exogenous, the shares $k_i, i = 1,2$ are

$$k_1 = \frac{\bar{P}_t^{(1)}}{\bar{P}_t^{(1)} + \bar{P}_t^{(2)}} = \frac{\theta}{\theta+\kappa}X_t + \frac{\kappa}{\theta+\kappa}w_1 \text{ and } k_2 = \frac{\theta}{\theta+\kappa}(1-X_t) + \frac{\kappa}{\theta+\kappa}w_2$$

(iii) integration correlation, defined as $\bar{\rho}_t := \dfrac{\frac{d\left\langle \bar{P}^{(1)}, \bar{P}^{(2)}\right\rangle_t}{dt}}{\sqrt{\frac{d\left\langle \bar{P}^{(1)}\right\rangle_t}{dt}}\sqrt{\frac{d\left\langle \bar{P}^{(2)}\right\rangle_t}{dt}}}$, *equals*

$$\bar{\rho}_t = \frac{b_1 b_2 + a_0\left(b_1(1-X_t) + b_2 X_t\right)}{\sqrt{b_1^2 + a_0\left(a_0+b_1\right)X_t}\sqrt{b_2^2 + a_0\left(a_0+b_2\right)\left(1-X_t\right)}},$$

where $a_0 := \frac{1}{\theta+\kappa}$, $b_i := \frac{\kappa w_i}{\theta(\theta+\kappa)}$, for $i = 1,2$.

(iv) the expected excess return of both assets is $\bar{\mu}_t^{(i)} - \bar{r} = \gamma\sigma^2$.

Note that in integration the safe rate is the same in both regions, as a common stochastic discount factor governs both markets. In fact, the price of the basket comprising both assets follows the usual risk-adjusted formula of Gordon and Shapiro [28]

$$P_t^{(1)} + P_t^{(2)} = \frac{1}{\theta}\left(D_t^{(1)} + D_t^{(2)}\right) = \frac{1}{\bar{r} - \mu + \gamma\sigma^2}\left(D_t^{(1)} + D_t^{(2)}\right), \tag{4.9}$$

confirming that the total consumption claim is priced as in the usual Lucas' model.

5 Implications

This section brings to life the theoretical results of the previous sections by discussing their significance for asset prices, their dynamics, and welfare.

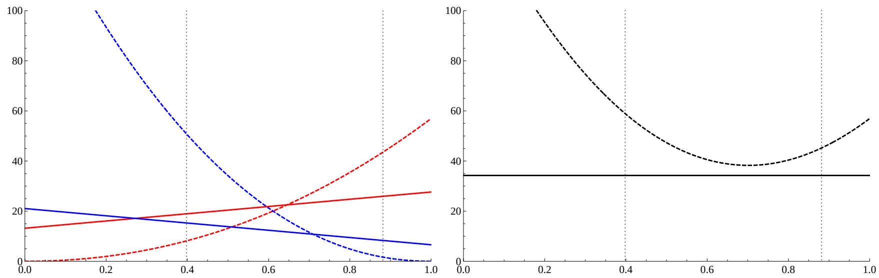

Fig. 5.1 Left: Price to aggregate consumption ratios (vertical) of the larger (red) and smaller (blue) asset, against dividend share of the larger asset X_t (horizontal), in segmentation (dashed) and integration (solid). ($P_t^{(1)}/D_t$ (dashed red), $P_t^{(2)}/D_t$ (dashed blue), $\bar{P}_t^{(1)}/D_t$ (solid red), $\bar{P}_t^{(2)}/D_t$ (solid blue).) Right: total market capitalization, i.e., sum of asset prices (vertical) against dividend proportion X_t (horizontal) in segmentation ($P_t^{(1)}+P_t^{(2)}$, dashed) and integration ($\bar{P}_t^{(1)}+\bar{P}_t^{(2)}$, solid). Vertical dotted lines delimit the 99% confidence interval of X_t for large t. Parameters are $\mu = 1.5\%$, $\sigma = 6\%$, $\beta = 1\%$, $w_1 = 2/3$, $\gamma = 3$, $\kappa = 4\%$, whence Assumptions 3.1, 3.4, and 4.2 hold.

5.1 Price levels

The intuitive effect of integration on price *levels* is ambivalent: On one hand, integration allows foreign investors to buy the domestic asset, increasing its demand and hence its price. On the other hand, integration also allows domestic investors to buy the foreign asset, thereby selling the domestic asset, hence decreasing its price. The overall effect is unclear.

Figure 5.1 plots the prices of both assets (per unit of aggregate consumption), as a function of the dividend share of the first asset, in segmentation and integration. Note that, in integration, the sum of both prices is constant (prices are symmetric with respect to the horizontal line through their intersection), confirming the validity of Lucas' formula for the aggregate asset.

Three features are apparent: First, price-consumption ratios increase with an asset's dividend share, reflecting the increased value of dividends from that asset. Second, price sensitivity to the dividend share is higher in segmentation and lower in integration for typical dividend shares (i.e., in the long-term confidence interval). Third, for a given current dividend share, the region with smaller long-term share has higher sensitivity. These three features share a common theme: the price of an asset increases as its alternatives dwindle.

This theme is apparent also in the effect of integration on price levels: As each solid curve is neither consistently below nor above the dashed curve with the same color, integration may either increase or decrease asset prices. An asset price increases only if integration takes place when its weight is relatively low, and therefore additional foreign demand overwhelms diminished domestic demand. Otherwise, integration typically *decreases* the prices of both assets – an effect that cannot be explained by the zero-sum logic of flows. In fact, the right panel in Figure 5.1 shows that integration reduces total market capitalization, regardless of the dividend share.

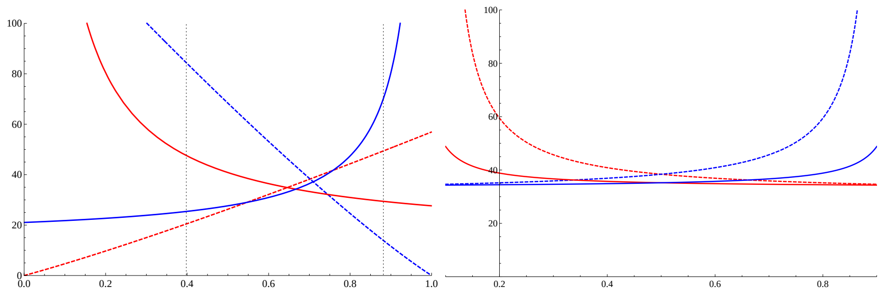

Fig. 5.2 Left: Price-dividend ratios (vertical) of the larger (red) and smaller (blue) asset, against dividend share of the larger asset X_t (horizontal), in segmentation (dashed) and integration (solid). ($P_t^{(1)}/D_t^{(1)}$ (dashed red), $P_t^{(2)}/D_t^{(2)}$ (dashed blue), $\bar{P}_t^{(1)}/D_t^{(1)}$ (solid red), $\bar{P}_t^{(2)}/D_t^{(2)}$ (solid blue).) Right: Average price-dividend ratios (vertical) against average dividend share w_1 of the first asset. Parameters as in Figure 5.1.

Integration depresses prices because it breaks the link between domestic dividends and future consumption growth, making each asset both less effective and less important as a consumption hedge. To see this mechanism, note that, in segmentation, consumption growth is mean-reverting, thus a high current consumption share is likely to be followed by lower consumption, which in turn generates demand for stocks in the attempt to smooth consumption over time, leading to higher stock prices. In integration, on the other hand, shocks to consumption growth are smaller and unforecastable, two characteristics that reduce the demand for stocks from the consumption-smoothing motive.

5.1.1 Price-dividend ratios

While the ratio of prices to total consumption helps to compare assets' prices, the price-dividend ratio gauges an asset's value relative to its own cash-flow. The left panel in Figure 5.2 displays price-dividend ratios, conditional on the dividend share of the first asset.

In segmentation, an asset becomes more expensive as its dividend share increases: this is an intertemporal-hedging effect, as a high current share forecasts a lower future share (by mean-reversion), which in turn spurs hedging against lower dividends through stock purchases, thereby increasing prices. In integration, an asset becomes cheaper as its dividend share increases because mean-reversion triggers a different response: as investors anticipate lower dividend growth, they shift their portfolios away from the asset, depressing its price while raising the other asset's prices.

An alternative explanation is based on the stochastic discount factor. In segmentation, each market has its separate discount factor, which determines prices through the ratio of future to present marginal utility. When the current dividend share is high, its marginal utility is low, hence future dividends are discounted at a low rate, leading to high prices. Crucially, for the typical values of risk aversion above one (which generate positive intertemporal hedging demand), this effect is so strong that not only prices, but even price-dividend ratios increase.

In integration, the discount factor depends on total dividends, which follow a geometric Brownian motion and lead to a constant aggregate price-dividend ratio (i.e., the sum of both asset prices divided by the sum of both dividends) as in the Lucas' model. Thus, the price-dividend ratio depends on the dividend share only through expected future dividends, which themselves reflect the mean-reversion in the dividend share, thereby leading, in integration, to a price-dividend ratio that decreases as the dividend share increases.

Consider now the *average* price-dividend ratios, defined as

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{P_T^{(i)}}{D_T^{(i)}}\right] \qquad \text{and} \qquad \lim_{T\to\infty} \mathbb{E}\left[\frac{\bar{P}_T^{(i)}}{D_T^{(i)}}\right] \qquad i=1,2$$

as functions of the *average* dividend share w_1. To compute such long-term averages, by the ergodicity of the divided share X_t it suffices to compute the average of the conditional price-dividend ratio with respect to the long-term (stationary) distribution of X_t, described by the Beta density

$$m(x) = \frac{1}{B(a+b-c_1+1,c_1)} x^{c_1-1}(1-x)^{a+b-c_1},$$

where a, b and c_1 are as in Proposition A.3 below and B is the beta function. The explicit formulae for conditional asset prices in Theorem 4.3 yield the average price-dividend ratios:

Proposition 5.1 *Let the dividends $(D^{(i)})_{i=1,2}$ be as in Proposition 3.2 and let Assumptions 3.1, 3.4, and 4.2 hold. In segmentation, the average price-dividend ratios are:*

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{P_t^{(i)}}{D_t^{(i)}}\right] = \frac{2}{\sigma^2\omega^{(i)}B(a+b-c_i+1,c_i)}\left(\Gamma_1^{(i)}\frac{G_1^{(i)}(1)}{c_i+1-\gamma}\frac{G_3^{(i)}(1)}{c_i+\gamma-1}\right.$$
$$\left. +\Gamma_2^{(i)}\int_0^1 x^{c_i}F_3^{(i)}(x)\left(\frac{G_1^{(i)}(x)}{c_i+1-\gamma}+\frac{G_3^{(i)}(x)}{c_i+\gamma-1}\right)dx\right),$$

where

$$F_3^{(i)}(x) := {}_2F_1(1-b,1-a;2-c_i;x),$$
$$G_1^{(i)}(x) := {}_3F_2(c_i-a,c_i-b,1+c_i-\gamma;c_i,c_i+2-\gamma;x),$$
$$G_3^{(i)}(x) := {}_3F_2(c_i-a,c_i-b,-1+c_i+\gamma;c_i,c_i+\gamma;x),$$

while the generalized hypergeometric functions ${}_2F_1$ and ${}_3F_2$ and constants a, b, c_i, $\omega^{(i)}$, and $\Gamma_j^{(i)}$ are as in Proposition A.3.

In integration, the average price-dividend ratios are

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{\bar{P}_t^{(i)}}{D_t^{(i)}}\right] = \frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_i}{\theta}\left(\frac{2\frac{\kappa}{\sigma^2}-1}{2\frac{\kappa w_i}{\sigma^2}-1}\right)\right).$$

These quantities are plotted in the right panel of Figure 5.2: consistent with Figure 5.1, the plot shows how integration on average decreases asset prices, especially in the smaller market. In addition, the plot shows that in integration average price-dividend ratios are rather insensitive to relative market size, while such sensitivity is significant in segmentation.

By comparison, with logarithmic investors ($\gamma = 1$), who are myopic and hence do not hedge investment opportunities, Equations (4.9) and (4.1)–(4.2) imply that price-dividend ratios are $1/\beta$ (hence independent of w_1) in both integration and segmentation, confirming that the sensitivity to w_1 stems from the motive of non-myopic investors to hedge against intertemporal changes in consumption growth.

5.2 Correlation

Although dividend streams in the two regions are uncorrelated by assumption, the resulting price correlation in segmentation and integration is far from granted, because the stochastic discount factor also depends on such dividends. To understand price correlations, note first that they are the same as return correlations:

$$\rho_t := \frac{\frac{d\langle P^{(1)},P^{(2)}\rangle_t}{dt}}{\sqrt{\frac{d\langle P^{(1)}\rangle_t}{dt}}\sqrt{\frac{d\langle P^{(2)}\rangle_t}{dt}}} = \frac{\frac{d\langle P^{(1)},P^{(2)}\rangle_t}{P_t^{(1)}P_t^{(2)}dt}}{\sqrt{\frac{d\langle P^{(1)}\rangle_t}{(P_t^{(1)})^2 dt}}\sqrt{\frac{d\langle P^{(2)}\rangle_t}{(P_t^{(1)}P_t^{(2)})^2 dt}}} = \frac{\frac{d\langle R^{(1)},R^{(2)}\rangle_t}{dt}}{\sqrt{\frac{d\langle R^{(1)}\rangle_t}{dt}}\sqrt{\frac{d\langle R^{(2)}\rangle_t}{dt}}},$$

where the total-return processes $R_t^{(i)}$ are defined by $dR_t^{(i)} = (D_t^{(i)}dt + dP_t^{(i)})/P_t^{(i)}$ and $R_0^{(i)} = 0$.

Next, note that

$$dR_t^{(i)} = \frac{D_t^{(i)}dt + dP_t^{(i)}}{P_t^{(i)}} = \frac{D_t^{(i)}}{P_t^{(i)}}dt + \frac{dD_t^{(i)}}{D_t^{(i)}} + \frac{d(P_t^{(i)}/D_t^{(i)})}{P_t^{(i)}/D_t^{(i)}} + \frac{d\langle D^{(i)}, P^{(i)}/D^{(i)}\rangle_t}{D_t^{(i)}(P_t^{(i)}/D_t^{(i)})}.$$

Now, as each price equals the dividend times the price-dividend ratio, and dividends are uncorrelated, price covariation in the model stems from covariation between (i) dividends and price-dividend ratios, and (ii) the two price-dividend ratios:

$$d\langle R^{(1)}, R^{(2)}\rangle_t = \frac{d\langle P^{(1)}/D^{(1)}, D^{(2)}\rangle_t}{\left(P_t^{(1)}/D_t^{(1)}\right) D_t^{(2)}} + \frac{d\langle P^{(2)}/D^{(2)}, D^{(1)}\rangle_t}{\left(P_t^{(2)}/D_t^{(2)}\right) D_t^{(1)}} + \frac{d\langle P^{(1)}/D^{(1)}, P^{(2)}/D^{(2)}\rangle_t}{\left(P_t^{(1)}/D^{(1)}\right)\left(P_t^{(2)}/D_t^{(2)}\right)} \tag{5.1}$$

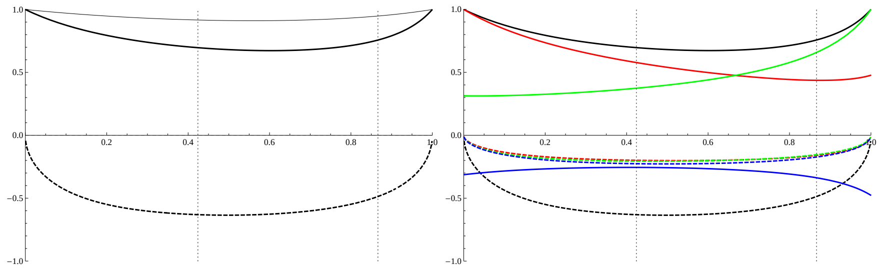

Fig. 5.3 Left: Price correlation in segmentation (ρ_t, dashed) and integration ($\overline{\rho_t}$, solid) against dividend share X_t. The thin upper line shows correlation in integration for logarithmic utility ($\gamma = 1$) while correlation is identically zero in segmentation for logarithmic utility. Right: decomposition of correlation in segmentation (dashed) and integration (solid), against dividend share X_t, as in Equation (5.1). The curves represent correlations between growths in one region and price-dividend ratio in the other (red and green), and between the two price-dividend ratios The vertical dotted lines delimit the 99% confidence interval of X_t for large t. Parameters as in Figure 5.1.

The left panel of Figure 5.3 displays price correlation in segmentation and integration, and the right-panel decomposes such correlation into its three components. The last term (blue in the right panel) represents correlation between price-dividend ratios, which is negative in segmentation, and even more so in integration, consistently with the observation that, in Figure 5.2, price-dividend ratios of the two assets move in opposite directions.

By contrast, the first two terms – the covariation between an asset's dividend and the other asset's price-dividend ratio – have a sharply different behavior in the two regimes. In segmentation, these terms are also negative, as an asset's dividend growth shrinks the other asset's price-dividend ratio (as in Figure 5.2).

Integration turns the tables completely, leading to a dramatic increase in correlation, driven by the positive covariation between dividends in one region and price-dividend ratio in another. Such a positive dependence is clear from the pricing formulae (4.7)-(4.8), but has also a clear interpretation: in an integrated market, as an asset's dividend increases, also its price increases, and hence its proportion in the portfolio. The investor responds by increasing the demand for the other asset, attempting to rebalance the portfolio. But in equilibrium rebalancing cannot take place, and instead the price of the other asset rises.

5.3 Interest rates and excess returns

To understand the properties of interest rates and excess returns in the model, note first that in integration the safe rate is constant because aggregate consumption is a geometric Brownian motion, and therefore the stochastic discount factor has the same dynamics as in the Lucas' model.

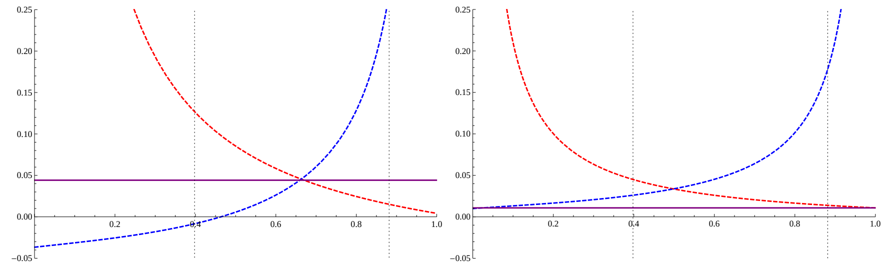

Fig. 5.4 Safe rate (left, vertical), and expected excess return (right, vertical) of the first (red) and second (blue) market, against dividend share X_t (horizontal), in segmentation (dashed) and integration (solid). Vertical dotted lines delimit the 99% confidence interval of X_t for large t. Parameters as in Figure 5.1. (Left: $r_t^{(1)}$ (dashed red), $r_t^{(2)}$ (dashed blue), $\bar{r}_t$ (solid purple); Right: $\mu_t^{(1)} - r_t^{(1)}$ (dashed red), $\mu_t^{(2)} - r_t^{(1)}$ (dashed blue), $\bar{\mu}_t^{(1)} - \bar{r}_t$ (solid red), $\bar{\mu}_t^{(2)} - \bar{r}_t$ (solid blue).)

The left panel of Figure 5.4 compares the common safe rate in integration with the two safe rates in segmentation. In each segmented market, the safe rate decreases as the market's dividend share increases. This effect is consistent with the increase in asset prices noted earlier, because a decrease in the interest rate is equivalent to an increase in bond prices.

Yet, the interest rate formulae in Theorem 4.1 show that this directional effect in general depends on the parameter values, and results from the tradeoff between two countervailing forces: on one hand, a higher dividend share implies lower future growth, thereby lowering the interest rate through the intertemporal substitution channel (Equations (4.3) and (4.4)). On the other hand, a higher dividend share also implies a lower consumption volatility (Equation (3.4)), which increases the interest rate by reducing the precautionary motive for saving. In the example considered, intertemporal substitution is stronger than precautionary savings, hence the interest rate declines.

Expected excess returns (right panel in Figure 5.4) have a qualitatively similar behavior, as they are constant (and equal to each other) in integration and decreasing with the dividend share in segmentation. It is noteworthy that in integration both assets have the same expected excess return – a fact that can be seen analytically from the explicit formulae for prices. As in Santos and Veronesi [39, Equation 24], an asset has an expected return above the market only if its dividend *share* covaries with the total consumption, while here it is uncorrelated. Note, however, that equal expected excess returns do not correspond to equal asset volatilities, which are higher when the dividend share is lower by Equation (3.4), therefore the Sharpe ratios are stochastic and different from each other.

The inverse relation between the dividend share and expected excess returns in segmentation can also be seen from the stochastic discount factor viewpoint. A low dividend share increases current marginal utility, thereby depressing prices. As the dividend share rises, the price increases for two rea-

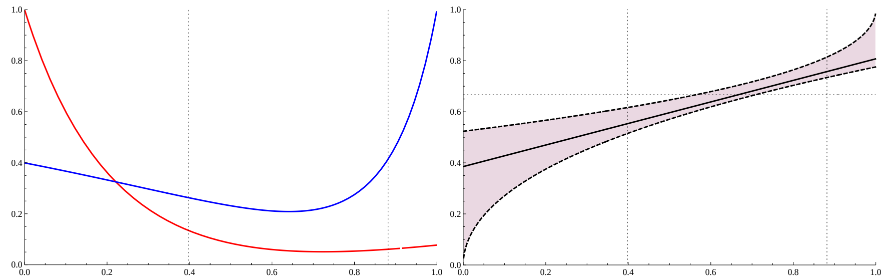

Fig. 5.5 Left: Fraction of wealth that each agent would forego in segmentation in exchange for integration (vertical), for the first (red) and second (blue) agent, against dividend share X_t (horizontal) in segmentation (dashed) and integration (solid). As integration always increases welfare, both curves are always positive. Right: Range of wealth shares k_1 (vertical) against the dividend share X_t (horizontal) compatible with voluntary integration. The solid line represents the wealth share corresponding to exogenous integration. The horizontal dotted line represents the long-term dividend share w_1. Vertical dotted lines delimit the 99% confidence interval of X_t for large t. Parameters as in Figure 5.1 and $D_t = 1$. $\mathcal{U}_t^{(1)}$ (dashed red), $\mathcal{U}_t^{(2)}$ (dashed blue), $\bar{\mathcal{U}}_t^{(1)}$ (solid red), $\bar{\mathcal{U}}_t^{(2)}$ (solid blue).

sons: first, the dividend has increased; second marginal utility has decreased, boosting prices further.

5.4 Welfare and endogenous integration

The left panel in Figure 5.5 displays welfare in segmentation and integration in terms of the fraction of wealth that each agent would forego in segmentation in exchange for exogenous integration. As such fraction is always positive for both agents, integration always increases welfare. The agent with a smaller dividend share has more to gain from integration through diversification with a larger, more stable consumption stream, therefore is also more willing to integrate (the fraction of wealth foregone is higher, cf. Section 4.1).

Importantly, integration increases welfare even when it reduces wealth for one, or even both, investors. The intuition is that the integrated market offers a smoother consumption stream or, equivalently, superior diversification, which more than offset any decrease in welfare resulting from lower wealth. Note also that the welfare increase is more significant for the smaller (blue) market, which benefits more than the larger market from increased diversification opportunities.

Although this analysis finds that market integration is beneficial for both investors, it assumes that it happens exogenously, with the investors surprised by a sudden change in wealth, after which they reallocate their holdings by exchanging them according to the new prices. By contrast, if each region can independently choose to remain segmented, integration requires mutual consent, which in turn may lead to a negotiation over the respective shares of post-integration wealth.

The right panel in Figure 5.5 shows the range of post-integration wealth shares that are Pareto improving, and hence compatible with both regions agreeing to integrate. The straight line, denoting the wealth shares implied by exogenous integration, lies inside the integration region, as exogenous integration increases the welfare of both investors.

In normal times, when dividend shares are close to their long-term averages, the integration region is actually a rather narrow interval around the exogenous integration share, wider for the smaller market than for the larger one. The intuition is that, because the smaller market has the most to gain from integration, it is also the one that is more willing to pay a higher price to see that integration takes place.

In times of distress, when dividend shares are away from their means, the integration range widens – in both directions. In other words, the range of shares under which both markets are better off integrated is larger both for the market that is abnormally small, and for the one that is abnormally large. Deviations from the mean make both markets more willing to integrate for different reasons: the abnormally small market faces the double whammy of low and volatile consumption – both mitigated by integration. The abnormally large market, on the other hand, has the incentive to integrate because its future consumption is likely to worsen (by mean-reversion), potentially leading to a similar predicament as the one in which the other market is now.

In summary, while in reality the exact shares of post-integration wealth may depend on the bargaining power of the two regions, on the irreversibility of integration, and on other aspects that are not in the present model, the Pareto optimality discussed here suggests that integration may be easier to achieve in times of dislocation from the long-term averages.

5.5 Exchange rates

While in the integration regime each unit of dividend can be consumed interchangeably by both agents, the segmentation regime can be described, equivalently, either by a ban on transfers between regions, or by an equilibrium real exchange rate [19, 21] that represents the shadow price of such transfers (and hence prevents them from occurring).

The next proposition calculates the equilibrium exchange rate $(p_t)_{t\geq 0}$, defined as the price of one unit of region-two dividend in terms of region-one dividends.

Proposition 5.2 *In segmentation, the equilibrium real exchange rate* $(p_t)_{t\geq 0}$ *is*

$$p_t = \left(\frac{1-X_t}{X_t}\right)^{\gamma}.$$

(In integration, the equilibrium real exchange rate is $\bar{p}_t := 1$*.)*

Thus, the equilibrium exchange rate in segmentation is a power of the ratio of the two dividends, and decreases in the dividend share X_t (Figure 5.6). As

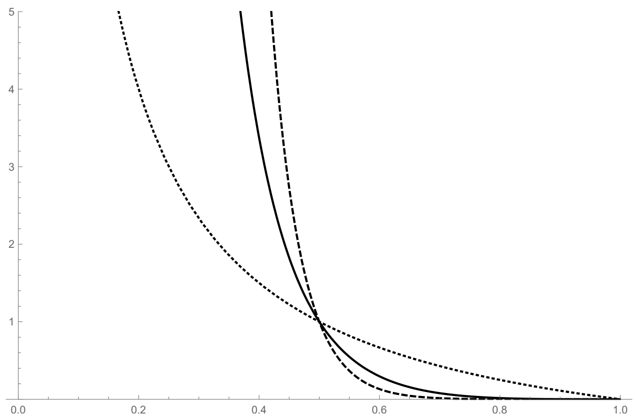

Fig. 5.6 Equilibrium real exchange rate (vertical axis), in segmentation, against dividend share (horizontal) under risk aversion $\gamma = 1$ (dotted), $\gamma = 3$ (solid), and $\gamma = 5$ (dashed).

the dividend share declines, the agent in the first region would be increasingly keen to buy foreign dividends, which are both uncorrelated and less volatile than domestic ones. Of course, the variation in the equilibrium exchange rate perfectly offsets such a shift, deterring agents from buying or selling.

6 Heterogeneous preferences

The previous analysis assumes that preferences in both regions are the same. Such an assumption allows to study the dependence or prices and welfare on market factors, such as relative size, but it also raises the question of whether it is possible to construct, in the integration regime, an equilibrium in which regions with different preferences coexist in the long run (in segmentation, the regions do not interact, therefore can certainly coexist with arbitrary preferences). The result in this section indicates that different preferences typically lead one region to overtake the other one in the integration regime, thereby suggesting that long-term coexistence would have to require additional self-regulating mechanisms to the ones described in the model.

Let the representative agent in the i-th market have time-preference rate $\beta_i > 0$ and relative risk aversion $\gamma_i > 0$. In segmentation, the main results of the paper remain valid without change, as the equilibrium is obtained separately for each market.

In integration, the market-clearing conditions are

$$\begin{aligned} D_t = D_t^{(1)} + D_t^{(2)} &= c_t^{(1)} + c_t^{(2)} \\ Y_t^{(1)} + Y_t^{(2)} &= \bar{P}_t^{(1)} + \bar{P}_t^{(2)} \\ \phi_t^{(1j)} Y_t^{(1)} + \phi_t^{(2j)} Y_t^{(2)} &= \bar{P}_t^{(j)} \text{ for } j = 1, 2, \end{aligned}$$

where $c^{(i)}, Y^{(i)}, \phi^{(ij)}$ are the consumption, wealth, and the number of shares held in the the j-th asset by the i-th agent.

With similar arguments as in Proposition A.2, the optimal consumptions of both agents are respectively

$$c_t^{(1)} = \left(y^{(1)} e^{\beta_1 t} \bar{M}_t\right)^{-\frac{1}{\gamma_1}} \qquad \text{and} \qquad c_t^{(2)} = \left(y^{(2)} e^{\beta_2 t} \bar{M}_t\right)^{-\frac{1}{\gamma_2}}, \tag{6.2}$$

where $\bar{M} > 0$ is the stochastic discount factor and $y^{(i)} > 0$ is the Lagrange multiplier that fulfills the budget constraint $\mathbb{E}[\int_0^\infty \bar{M}_t c_t^{(i)} dt] = Y_0^{(i)}$ for $i = 1, 2$; $Y_0^{(i)} > 0$ is the initial wealth of agent i.

The next result shows that, in the tractable case of risk aversion in one market being twice the risk aversion in the other, one market overtakes the entire economy.

Proposition 6.1 *Let $\gamma_1 = \gamma \geq 1$, $\gamma_2 = 2\gamma$, and let Assumption 3.4 hold for β_i and γ_i, $i = 1, 2$. Assuming further that*

$$\beta_1 \left(1 - \frac{1}{2\gamma}\right) + \frac{\beta_2}{2\gamma} > \left(\gamma - \frac{1}{2}\right) \left(\left(\frac{1}{2} + \gamma\right) \frac{\sigma^2}{2} - \mu\right), \tag{6.3}$$

it follows that:

$$\lim_{t\to\infty} \frac{Y_t^{(2)}}{Y_t^{(1)}} = \begin{cases} 0 & \mu - \frac{\sigma^2}{2} + \frac{\beta_2 - \beta_1}{\gamma} > 0 \\ +\infty & \mu - \frac{\sigma^2}{2} + \frac{\beta_2 - \beta_1}{\gamma} < 0 \end{cases} \qquad a.s.$$

Note that condition (6.3) ensures well-posedness by requiring that agents' overall time-preference is large enough to prevent indefinite deferral of consumption to achieve arbitrarily high utility.

To interpret this result, consider first the case where the time-preference parameters are the same ($\beta_1 = \beta_2$). Then, the above proposition implies that, in the typical case of $\mu - \sigma^2/2$ (which corresponds to a dividend stream that increases indefinitely), the second market (which has higher risk aversion) asymptotically disappears, eventually leaving the first market, which has lower risk aversion, to take over all of the wealth in the economy. Note that, in the present setting of time-additive preferences, lower risk aversion is equivalent to higher elasticity of intertemporal substitution, therefore it is not possible to determine which of these is more important for long-term dominance.

Lower risk aversion is not enough to guarantee dominance when time-preference is heterogeneous: indeed, if low risk aversion is combined with a sufficiently high time-preference, so that $\mu - \frac{\sigma^2}{2} + \frac{\beta_2 - \beta_1}{\gamma} < 0$, then it is possible for the high risk aversion market to overtake the economy, as the other market's proclivity for consumption depletes its share of wealth over time.

In both cases, the result suggests that a substantial difference in preferences between the two markets is poised to lead one market to dominate the overall economy in the long run. The dominating market may have lower risk aversion, lower time-preference, or both.

7 Conclusion

This paper examines the asset pricing implications of a two-region model in which shocks to dividend growth are independent across regions, and shocks to aggregate dividends are independent over time – a cointegrated, two-way split of the usual Lucas' tree.

In segmentation, when each region owns and consumes its dividends, correlation among asset prices is null with logarithmic preferences and negative with higher risk aversion, reflecting intertemporal hedging demand. Correlation increases significantly as markets integrate, even as the shocks to assets' cash flows remain independent.

Integration has an ambiguous effect on price levels, but on average it decreases prices, even as welfare always increases for both regions. Price-dividend ratios increase with the dividend share in segmentation, while in integration they decline. Expected excess returns decline with a region's dividend share in segmentation, while they are equalized by integration.

If integration requires the consent of both regions, they agree to integrate only for wealth shares within a narrow range around the values implied by exogenous integration. Such a range widens in unusual times, when the dividend share significantly departs from its long-term mean.

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A Proofs

A.1 Preliminaries

The next proof relies on the definition of consumption and investment policies $\mathcal{C}$ and $\mathcal{P}$ respectively, which combine into admissible strategies.

Definition A.1 In all markets (segmented or integrated), the set of consumption policies $\mathcal{C}$ denotes all adapted processes $(c_t)_{t\geq 0}$ such that $\mathbb{E}[\int_0^T c_t dt] < +\infty$ for all $T > 0$.

In each segmented market, the set of investment strategies $(\mathcal{P}^i)_{i=1,2}$ consists of all adapted processes $(\phi_t)_{t\geq 0}$ such that $\mathbb{E}[\int_0^T \phi_t^2 d\left\langle P^{(i)}\right\rangle_t] < +\infty$, where $P^{(i)}$ is the respective asset price. In the integrated market the set of investment policies $\overline{\mathcal{P}}$ consist of $\mathbb{R}^2$-valued, adapted processes $(\phi_t)_{t\geq 0}$ such that $\mathbb{E}[\langle \int_0^\cdot \phi \cdot dP\rangle_T] < +\infty$.

The set of admissible strategies $\mathcal{A}$ is consists of all pairs (c, ϕ), such that $c \in \mathcal{C}$, $\phi \in \mathcal{P}^{(i)}$ for each segmented market and $\phi \in \bar{\mathcal{P}}$ for the integrated market, and the corresponding wealth in Equations (3.6) or (3.7) satisfies $Y_t \geq 0$ a.s. for all $t \geq 0$.

The equilibrium Definition 3.3 leads to the familiar representation of prices as discounted cash flows and the safe rate as opposite of the growth rate of the stochastic discount factor. Although the result is often taken for granted and can be informally derived from a perturbation argument, it is not guaranteed to hold in general, and the literature does offer counterexamples (See Remark 4 and Corollary 3 in Basak and Cuoco [4] and the discussion in Karatzas et al. [31]). Thus, we offer a proof that applies to the model considered here.

Proposition A.2 *In the segmented markets, equilibrium asset prices are:*

$$P_t^{(i)} = \mathbb{E}_t\left[\int_t^\infty \frac{M_s^{(i)}}{M_t^{(i)}} D_s^{(i)} ds\right] \qquad \text{where} \qquad M_t^{(i)} = e^{-\beta t}(D_t^{(i)})^{-\gamma},$$

while equilibrium safe rates $r_t^{(i)}$ are identified by the local-martingale condition for $(M_t^{(i)} e^{\int_0^t r_s^{(i)} ds})_{t\geq 0}$. Likewise, in the integrated market the equilibrium asset prices are

$$\bar{P}_t^{(i)} = \mathbb{E}_t\left[\int_t^\infty \frac{\bar{M}_s}{\bar{M}_t} D_s^{(i)} ds\right] \qquad \text{where} \qquad \bar{M}_t = e^{-\beta t}(D_t^{(1)} + D_t^{(2)})^{-\gamma}.$$

Proof. Note that, if $Y_0 = P_0^{(i)}$, $c_t = D_t^{(i)}$ and $\phi_t = 1$, then (3.6) implies that $Y_t = P_t^{(i)}$ for all $t \geq 0$. Fix $i = 1, 2$, $t_0 > 0$, $\vartheta > 0$ and $K > 1$. Define the stopping time

$$\tau_0 := (t_0 + \vartheta) \wedge \inf\left\{t > t_0 \,:\, P_t^{(i)} \leq \frac{1}{K} P_{t_0}^{(i)}, D_t^{(i)} \geq K D_{t_0}^{(i)}\right\}.$$

For any $\delta \in (-\vartheta, \vartheta)$, consider a $\mathcal{F}_{t_0}$-measurable event A on which we adopt an alternative strategy, whereby from time t_0 to τ_0 the consumption changes from $D_t^{(i)}$ to $c_t^\delta = (1 - \frac{\delta}{\vartheta}) D_t^{(i)}$, with the difference in consumption invested in the risky asset, so that the number of shares changes from 1 to $\phi_t^\delta := 1 + \frac{\delta}{\vartheta}\Delta_t$ for some process $(\Delta_t)_{t\geq 0}$. Thus, the corresponding wealth becomes

$$Y_t^\delta = \left(1 + \frac{\delta}{\vartheta}\Delta_t\right) P_t^{(i)}.$$

To satisfy the budget constraints (3.6), Δ_t satisfies

$$d\left(1 + \frac{\delta}{\vartheta}\Delta_t\right) P_t^{(i)} = r_t^{(i)}\left(\left(1 + \frac{\delta}{\vartheta}\Delta_t\right) P_t^{(i)} - \left(1 + \frac{\delta}{\vartheta}\Delta_t\right) P_t^{(i)}\right) dt$$
$$+ \left(1 + \frac{\delta}{\vartheta}\Delta_t\right)\left(dP_t^{(i)} + D_t^{(i)} dt\right) - \left(1 - \frac{\delta}{\vartheta}\right) D_t^{(i)} dt$$

$$\frac{\delta}{\vartheta} P_t^{(i)} d\Delta_t + \frac{\delta}{\vartheta} d\left\langle \Delta, P^{(i)}\right\rangle_t = \left(\frac{\delta}{\vartheta} + \frac{\delta}{\vartheta}\Delta_t\right) D_t^{(i)} dt$$

$$d\Delta_t = (1 + \Delta_t)\frac{D_t^{(i)}}{P_t^{(i)}} dt.$$

Hence,

$$\Delta_t = \int_{t_0}^{t} e^{\int_s^t \frac{D_u^{(i)}}{P_u^{(i)}}du} \frac{D_s^{(i)}}{P_s^{(i)}} ds \leq \vartheta K^2 e^{\vartheta K^2 \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}}} \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}} \qquad \text{for all} \qquad t \in [t_0, \tau_0].$$

After τ_0, the investor holds $\phi_t^\delta = 1 + \frac{\delta}{\vartheta}\Delta_{\tau_0}$ unit of risky asset and consumes $c_t^\delta = (1 + \frac{\delta}{\vartheta}\Delta_{\tau_0})D_t^{(i)}$ for $t \geq \tau_0$. To ensure that consumption remains positive, assume also that

$$\delta > -\frac{1}{K^2} e^{-\vartheta K^2 \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}}} \frac{P_{t_0}^{(i)}}{D_{t_0}^{(i)}}.$$

The change in expected utility from $(c_t, \phi_t)_{t\geq 0}$ to $(c_t^\delta, \phi_t^\delta)_{t\geq 0}$ is thus

$$\Delta^\delta J = \mathbb{E}\left[1_A \left(\int_{t_0}^{\tau_0} \frac{e^{-\beta t}\left(\left(1 - \frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt \right.\right.$$
$$\left.\left. + \int_{\tau_0}^{+\infty} \frac{e^{-\beta t}\left(\left(1 + \frac{\delta}{\vartheta}\Delta_{\tau_0}\right) D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt - \int_{t_0}^{+\infty} \frac{e^{-\beta t}\left(D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt\right)\right].$$

Because $\frac{x^{1-\gamma}}{1-\gamma}$ is concave, $y^{-\gamma}(y-x) \leq \frac{y^{1-\gamma}}{1-\gamma} - \frac{x^{1-\gamma}}{1-\gamma}$ for any $x, y > 0$, whence

$$\Delta^\delta J \geq \mathbb{E}\left[1_A \left(-\frac{\delta}{\vartheta}\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(\left(1 - \frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{-\gamma} dt \right.\right.$$
$$\left.\left. + \frac{\delta}{\vartheta}\Delta_{\tau_0} \int_{\tau_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(\left(1 + \frac{\delta}{\vartheta}\Delta_{\tau_0}\right) D_t^{(i)}\right)^{-\gamma} dt\right)\right].$$

As $(c_t, \phi_t)_{t\geq 0}$ is optimal, it follows that $\lim_{\delta\downarrow 0} \frac{\Delta^\delta J}{\delta} \leq 0$ and $\lim_{\delta\uparrow 0} \frac{\Delta^\delta J}{\delta} \geq 0$, whence

$$\lim_{\delta\to 0} \frac{1}{\vartheta}\mathbb{E}\left[1_A \left(-\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(\left(1 - \frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{-\gamma} dt \right.\right.$$
$$\left.\left. + \Delta_{\tau_0} \int_{\tau_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(\left(1 + \frac{\delta}{\vartheta}\Delta_{\tau_0}\right) D_t^{(i)}\right)^{-\gamma} dt\right)\right] = 0$$

and therefore

$$\frac{\mathbb{E}\left[1_A \left(-\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(D_t^{(i)}\right)^{-\gamma} dt + \Delta_{\tau_0} \int_{\tau_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(D_t^{(i)}\right)^{-\gamma} dt\right)\right]}{\vartheta} = 0. \tag{A.1}$$

Because (A.1) holds for all $\vartheta \neq 0$,

$$\lim_{\vartheta \downarrow 0} \frac{\mathbb{E}\left[1_A \left(-\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(D_t^{(i)}\right)^{-\gamma} dt + \Delta_{\tau_0} \int_{\tau_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(D_t^{(i)}\right)^{-\gamma} dt\right)\right]}{\vartheta} = 0. \quad \text{(A.2)}$$

Note that

$$\lim_{\vartheta \downarrow 0} \frac{\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(D_t^{(i)}\right)^{-\gamma} dt}{\vartheta} = e^{-\beta t_0} \left(D_{t_0}^{(i)}\right)^{1-\gamma},$$

$$\lim_{\vartheta \downarrow 0} \int_{\tau_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(D_t^{(i)}\right)^{-\gamma} dt = \int_{t_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(D_t^{(i)}\right)^{-\gamma} dt$$

and

$$\lim_{\vartheta \downarrow 0} \frac{\Delta_{\tau_0}}{\vartheta} = \lim_{\vartheta \downarrow 0} \frac{1}{\vartheta} \int_{t_0}^{\tau_0} e^{\int_s^{\tau_0} \frac{D_u^{(i)}}{P_u^{(i)}} du} \frac{D_s^{(i)}}{P_s^{(i)}} ds = \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}}.$$

Thus, an application of the dominated convergence theorem to (A.2) yields

$$\mathbb{E}\left[1_A \left(-e^{-\beta t_0} \left(D_{t_0}^{(i)}\right)^{1-\gamma} + \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}} \mathbb{E}_{t_0}\left[\int_{t_0}^{+\infty} D_t^{(i)} e^{-\beta t} \left(D_t^{(i)}\right)^{-\gamma} dt\right]\right)\right] = 0.$$

As A is an arbitrary $\mathcal{F}_{t_0}$ event, it follows that

$$P_{t_0}^{(i)} = \mathbb{E}_{t_0}\left[\int_{t_0}^{+\infty} D_t^{(i)} e^{-\beta(t-t_0)} \left(\frac{D_t^{(i)}}{D_{t_0}^{(i)}}\right)^{-\gamma} dt\right]. \quad \text{(A.3)}$$

To identify the safe rate $r^{(i)}$, fix $i = 1, 2$, $t_1 > t_0 > 0$, $\vartheta \in (0, t_1 - t_0)$, and $K > 1$, and define stopping times

$$\eta_K := \inf\left\{t > t_0 \,:\, e^{\int_{t_0}^t |r_u^{(i)}| du} \geq K, P_t^{(i)} \leq \frac{1}{K} P_{t_0}^{(i)}\right\}$$

$$\tau_0 := (t_0 + \vartheta) \wedge \eta_K \wedge \inf\left\{t > t_0 : D_t^{(i)} \geq K D_{t_0}^{(i)}, \frac{1}{P_t^{(i)}} \int_{t_0}^t e^{\int_s^t |r_u^{(i)}| du} D_s^{(i)} ds \geq \frac{K}{2}\right\}.$$

For any $0 < |\delta| < \frac{\vartheta}{K}$, further define another stopping time

$$\tau_1 := t_1 \wedge \eta_K \wedge \inf\left\{t > \tau_0 \,:\, \left|\frac{\delta}{\vartheta} \frac{1}{P_t^{(i)}} \int_{t_0}^{\tau_0} e^{\int_s^t r_u^{(i)} du} D_s^{(i)} ds\right| \geq \frac{1}{2}\right\}.$$

Consider a $\mathcal{F}_{t_0}$-measurable event A on which to adopt an alternative strategy in which, as before, from t_0 to τ_0 the consumption changes from $D_t^{(i)}$ to $c_t^\delta = (1 - \frac{\delta}{\vartheta}) D_t^{(i)}$, while the difference now is invested in the safe asset. Thus, the corresponding wealth is $Y_t^\delta = P_t^{(i)} + \frac{\delta}{\vartheta}\Delta_t$ where $\Delta_t = \int_{t_0}^t e^{\int_s^t r_u^{(i)} du} D_s^{(i)} ds$. From time τ_0 to τ_1, the strategy remains $(c_t^\delta, \phi_t^\delta) = (D_t^{(i)}, 1)$, thus wealth becomes $Y_t^\delta = P_t^{(i)} + \frac{\delta}{\vartheta}\Delta_t$ where $\Delta_t = \int_{t_0}^{\tau_0} e^{\int_s^t r_u^{(i)} du} D_s^{(i)} ds$. After τ_1, the lump-sum $\frac{\delta}{\vartheta}\Delta_{\tau_1}$ is used to buy $\frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}$ units of stock, so that the number of shares becomes $\phi_t^\delta = 1 + \frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}$ and the consumption rate $c_t^\delta = (1 + \frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}) D_t^{(i)}$ for $t \geq \tau_1$.

Then the change in expected utility from $(c_t, \phi_t)_{t\geq 0}$ to $(c_t^\delta, \phi_t^\delta)_{t \geq 0}$ is

$$\Delta^\delta J = \mathbb{E}\left[1_A \left(\int_{t_0}^{\tau_0} \frac{e^{-\beta t}\left(\left(1-\frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt \right.\right.$$
$$\left.\left. + \int_{\tau_1}^{+\infty} \frac{e^{-\beta t}\left(\left(1+\frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}\right) D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt - \int_{t_0}^{+\infty} \frac{e^{-\beta t}\left(D_t^{(i)}\right)^{1-\gamma}}{1-\gamma} dt \right)\right].$$

Again, concavity implies that $y^{-\gamma}(y-x) \leq \frac{y^{1-\gamma}}{1-\gamma} - \frac{x^{1-\gamma}}{1-\gamma}$, for any $x, y > 0$, thus

$$\Delta^\delta J \geq \mathbb{E}\left[1_A \left(-\frac{\delta}{\vartheta}\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(\left(1-\frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{-\gamma} dt \right.\right.$$
$$\left.\left. + \frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}} \int_{\tau_1}^{+\infty} e^{-\beta t} D_t^{(i)} \left(\left(1+\frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}\right) D_t^{(i)}\right)^{-\gamma} dt\right)\right].$$

As $(c_t, \phi_t)_{t\geq 0}$ is optimal, it is necessary that $\lim_{\delta\downarrow 0} \frac{\Delta^\delta J}{\delta} \leq 0$ and $\lim_{\delta\uparrow 0} \frac{\Delta^\delta J}{\delta} \geq 0$, and thus

$$\lim_{\delta\to 0} \frac{1}{\vartheta}\mathbb{E}\left[1_A \left(-\int_{t_0}^{\tau_0} e^{-\beta t} D_t^{(i)} \left(\left(1-\frac{\delta}{\vartheta}\right) D_t^{(i)}\right)^{-\gamma} dt \right.\right.$$
$$\left.\left. + \frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}} \int_{\tau_1}^{+\infty} D_t^{(i)} e^{-\beta t} \left(\left(1+\frac{\delta}{\vartheta}\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}}\right) D_t^{(i)}\right)^{-\gamma} dt\right)\right] = 0;$$

Then, considering that $\lim_{\delta\to 0} \tau_1 = \tau_2 := \min\{t_1, \eta_K\}$ almost surely and $\frac{\Delta_{\tau_1}}{P_{\tau_1}^{(i)}} \leq (\tau_1 - t_0) K^3 \frac{D_{t_0}^{(i)}}{P_{t_0}^{(i)}}$, the dominated convergence theorem yields

$$\lim_{\vartheta\downarrow 0} \frac{\mathbb{E}\left[1_A\left(-\int_{t_0}^{\tau_0} e^{-\beta t}\left(D_t^{(i)}\right)^{1-\gamma} dt + \frac{\Delta_{\tau_2}}{P_{\tau_2}^{(i)}}\int_{\tau_2}^{+\infty} e^{-\beta t}\left(D_t^{(i)}\right)^{1-\gamma} dt\right)\right]}{\vartheta} = 0. \tag{A.4}$$

Recalling that $\Delta_{\tau_2} = \int_{t_0}^{\tau_0} e^{\int_s^{\tau_2} r_u^{(i)} du} D_s^{(i)} ds$, it follows that

$$\lim_{\vartheta \downarrow 0} \frac{\Delta_{\tau_2}}{\vartheta} = e^{\int_{t_0}^{\tau_2} r_u^{(i)} du} D_{t_0}^{(i)}$$

Applying the dominated convergence theorem to (A.4), (A.3) implies that

$$\mathbb{E}\left[1_A \left(-e^{-\beta t_0}\left(D_{t_0}^{(i)}\right)^{1-\gamma} + \frac{e^{\int_{t_0}^{\tau_2} r_u^{(i)} du} D_{t_0}^{(i)}}{P_{\tau_2}^{(i)}} \mathbb{E}_{\tau_2}\left[\int_{\tau_2}^{+\infty} D_t^{(i)} e^{-\beta t}\left(D_t^{(i)}\right)^{-\gamma} dt\right]\right)\right]$$
$$= e^{-\beta t_0}\mathbb{E}\left[1_A D_{t_0}^{(i)}\left(-\left(D_{t_0}^{(i)}\right)^{-\gamma} + e^{\int_{t_0}^{\tau_2} r_u^{(i)} du} e^{-\beta(\tau_2 - t_0)}\left(D_{\tau_2}^{(i)}\right)^{-\gamma}\right)\right] = 0.$$

As A is any arbitrary $\mathcal{F}_{t_0}$ event,

$$e^{\int_0^{t_0}\left(r_u^{(i)}-\beta\right)du}\left(D_{t_0}^{(i)}\right)^{-\gamma} = \mathbb{E}_{t_0}\left[e^{\int_0^{\min\{t_1,\eta_K\}}\left(r_u^{(i)}-\beta\right)du}\left(D_{\min\{t_1,\eta_K\}}^{(i)}\right)^{-\gamma}\right],$$

which implies that $(e^{\int_0^t (r_u^{(i)}-\beta)du}(D_t^{(i)})^{-\gamma})_{t\geq 0}$ is a local martingale.

A.2 Proof of Theorem 4.1

The proof focuses on region 1, as the argument for region 2 is analogous.

(i) Recalling that $e^{\int_0^t (r_s^{(1)})ds} M_t^{(1)} = e^{\int_0^t (r_s^{(1)}-\beta)ds}(c_t)^\gamma = e^{\int_0^t (r_s^{(1)}-\beta)ds}(D_t^{(1)})^\gamma$ and applying Itô's lemma, it follows that

$$d\left(e^{\int_0^t (r_s^{(1)}-\beta)ds}\left(D_t^{(1)}\right)^{-\gamma}\right) = \left(r_t^{(1)} - \beta\right) e^{\int_0^t (r_s^{(1)}-\beta)ds}\left(D_t^{(1)}\right)^{-\gamma} dt$$
$$- \gamma e^{\int_0^t (r_s^{(1)}-\beta)ds}\left(D_t^{(1)}\right)^{-\gamma-1}\left((\mu-\kappa)D_t^{(1)} + \kappa w_1 D_t\right) dt$$
$$+ \frac{\gamma(1+\gamma)\sigma^2}{2} e^{\int_0^t (r_s^{(1)}-\beta)ds}\left(D_t^{(1)}\right)^{-\gamma-1} D_t dt + L_t$$
$$= e^{\int_0^t (r_s^{(1)}-\beta)ds}\left(D_t^{(1)}\right)^{-\gamma} \times$$
$$\left(r_t^{(1)} - \beta - \gamma(\mu-\kappa) - \gamma\kappa w_1 \frac{D_t}{D_t^{(1)}} + \frac{\gamma(\gamma+1)\sigma^2}{2}\frac{D_t}{D_t^{(1)}}\right) dt + L_t$$

for some local martingale L (a stochastic integral with respect to Brownian motion, whose expression is inconsequential). Thus, the equilibrium safe rate is

$$r_t^{(1)} = \beta + \gamma(\mu-\kappa) + \gamma\mu w_1 \frac{D_t}{D_t^{(1)}} - \frac{\gamma(\gamma+1)\sigma^2}{2}\frac{D_t}{D_t^{(1)}}.$$

Write the price of asset 1 in terms of X_t and D_t, as

$$\begin{aligned}
P_t^{(1)} &= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{\left(D_s^{(1)}\right)^{1-\gamma}}{\left(D_t^{(1)}\right)^{-\gamma}}ds\right]\\
&= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{(D_s)^{1-\gamma}}{(D_t)^{-\gamma}}\frac{X_s^{1-\gamma}}{X_t^{-\gamma}}ds\right]\\
&= \int_t^\infty e^{-\beta(s-t)}\mathbb{E}_t\left[\frac{(D_s)^{1-\gamma}}{(D_t)^{-\gamma}}\frac{X_s^{1-\gamma}}{X_t^{-\gamma}}\right]ds\\
&= \int_t^\infty e^{-\beta(s-t)}\mathbb{E}_t\left[\frac{(D_s)^{1-\gamma}}{(D_t)^{-\gamma}}\right]\mathbb{E}_t\left[\frac{X_s^{1-\gamma}}{X_t^{-\gamma}}\right]ds\\
&= D_tX_t^\gamma\int_t^\infty e^{-\beta(s-t)}\mathbb{E}_t\left[\left(\frac{D_s}{D_t}\right)^{1-\gamma}\right]\mathbb{E}_t\left[X_s^{1-\gamma}\right]ds, \quad \text{(A.5)}
\end{aligned}$$

where the fourth equality follows by the independence of $(B_t^D)_{t\geq 0}$ and $(B_t^X)_{t\geq 0}$. As $(D_t)_{t\geq 0}$ is a geometric Brownian motion, it satisfies

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{D_s}{D_t}\right)^{1-\gamma}\right] &= \mathbb{E}\left[e^{(1-\gamma)\left(\mu-\frac{\sigma^2}{2}\right)(s-t)+(1-\gamma)\sigma\left(B_s^D-B_t^D\right)}\right]\\
&= e^{(1-\gamma)\left(\mu-\frac{\gamma\sigma^2}{2}\right)(s-t)}.
\end{aligned}$$

Then, (A.5) becomes

$$\begin{aligned}
P_t^{(1)} &= D_tX_t^\gamma\int_t^\infty e^{-\theta(s-t)}\mathbb{E}_t\left[X_s^{1-\gamma}\right]ds = D_tX_t^\gamma\mathbb{E}_t\left[\int_t^\infty e^{-\theta(s-t)}X_s^{1-\gamma}ds\right]\\
&= D_tX_t^\gamma\mathbb{E}\left[\int_t^\infty e^{-\theta(s-t)}X_s^{1-\gamma}ds\,\middle|\,X_t\right] = D_tX_t^\gamma f^{(1)}(X_t),
\end{aligned}$$

where the third equality follows by the Markov property of $(X_t)_{t\geq 0}$, and

$$f^{(1)}(x) := \mathbb{E}\left[\int_t^\infty e^{-\theta(s-t)}X_s^{1-\gamma}ds\,\middle|\,X_t = x\right] = \mathbb{E}\left[\int_0^\infty e^{-\theta s}X_s^{1-\gamma}ds\,\middle|\,X_0 = x\right]$$

(Note that the second equality holds true because of Markov property.)

(ii) For the welfare, recall that

$$\mathcal{U}_t^{(1)} = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{\left(D_s^{(1)}\right)^{1-\gamma}}{1-\gamma}ds\right]$$

Comparing this expression to the formula in $P_t^{(1)}$ in (A.5), note that

$$\mathcal{U}_t^{(1)} = \frac{1}{1-\gamma}\left(D_t^{(1)}\right)^{-\gamma} P_t^{(1)} = \frac{1}{1-\gamma}\left(D_t^{(1)}\right)^{1-\gamma} X_t^{\gamma-1} f^{(1)}(X_t)$$
$$= \frac{1}{1-\gamma}\left(D_t\right)^{1-\gamma} f^{(1)}(X_t)$$

(iii) As $P_t^{(1)} = D_t X_t^\gamma f^{(1)}(X_t)$ where $f^{(1)}$ is as in (4.1), Ito's formula yields

$$\begin{aligned}
dP_t^{(1)} &= \sigma D_t X_t^\gamma f^{(1)}(X_t)\, dB_t^D \\
&+ \sigma D_t X_t^{\gamma-1}\left(\gamma f^{(1)}(X_t) + X_t f^{(1)\prime}(X_t)\right)\sqrt{X_t(1-X_t)}dB_t^X + A_t^1 dt \\
&= \sigma D_t X_t^\gamma f^{(1)}(X_t)\, dB_t^D + \sigma D_t X_t^{\gamma-1} h^{(1)}(X_t)\sqrt{X_t(1-X_t)}dB_t^X \\
&+ A_t^1 dt \\
dP_t^{(2)} &= \sigma D_t(1-X_t)^\gamma f^{(2)}(X_t)\, dB_t^D + A_t^2 dt + \sigma D_t (1-X_t)^{\gamma-1}\times \\
&\quad \left(-\gamma f^{(2)}(X_t) + (1-X_t) f^{(2)\prime}(X_t)\right)\sqrt{X_t(1-X_t)}dB_t^X \\
&= \sigma D_t(1-X_t)^\gamma f^{(2)}(X_t)\, dB_t^D + \\
&\sigma D_t(1-X_t)^{\gamma-1} h^{(2)}(X_t)\sqrt{X_t(1-X_t)}dB_t^X + A_t^2 dt.
\end{aligned}$$

for some adapted processes A^1, A^2. Hence,

$$\begin{aligned}
\frac{d\left\langle P^{(1)}\right\rangle_t}{dt} &= \sigma^2 D_t^2 X_t^{2(\gamma-1)}\left(\left(X_t f^{(1)}(X_t)\right)^2 + \left(h^{(1)}(X_t)\right)^2 X_t(1-X_t)\right) \\
\frac{d\left\langle P^{(2)}\right\rangle_t}{dt} &= \\
\sigma^2 D_t^2 (1-X_t)^{2(\gamma-1)} &\left(\left((1-X_t) f^{(2)}(X_t)\right)^2 + \left(h^{(2)}(X_t)\right)^2 X_t(1-X_t)\right) \\
\frac{d\left\langle P^{(1)}, P^{(2)}\right\rangle_t}{dt} &= \\
\sigma^2 D_t^2 X_t^\gamma (1-X_t)^\gamma &\left(f^{(1)}(X_t) f^{(2)}(X_t) + h^{(1)}(X_t) h^{(2)}(X_t)\right).
\end{aligned}$$

and the claim follows by (4.5).

(iv) Again, the proof focuses on region 1, as the argument for region 2 is analogous. As $P_t^{(1)} = D_t X_t^\gamma f^{(1)}(X_t)$ where $f^{(1)}$ are defined in (4.1), Ito's formula yields

$$
\begin{aligned}
dP_t^{(1)} &= X_t^\gamma f^{(1)}(X_t)dD_t + D_t X_t^{\gamma-1}\left(\gamma f^{(1)}(X_t) + X_t\left(f^{(1)}\right)'(X_t)\right)dX_t \\
&+ X_t^{\gamma-1}\left(\gamma f^{(1)}(X_t) + X_t\left(f^{(1)}\right)'(X_t)\right)d\left\langle D, X\right\rangle_t \\
&+ \frac{1}{2}D_t X_t^{\gamma-2}\left(\gamma(\gamma-1)f^{(1)}(X_t) + 2\gamma X_t\left(f^{(1)}\right)'(X_t)\right. \\
&\qquad\qquad \left. + X_t^2\left(f^{(1)}\right)''(X_t)\right)d\left\langle X\right\rangle_t \\
&= D_t X_t^{\gamma-1}\left(\left(\mu X_t f^{(1)}(X_t) + \kappa(w - X_t)\left(\gamma f^{(1)}(X_t) + X_t\left(f^{(1)}\right)'(X_t)\right)\right.\right. \\
&\qquad + \frac{\sigma^2}{2}(1-X_t)\left(\gamma(\gamma-1)f^{(1)}(X_t) + 2\gamma X_t\left(f^{(1)}\right)'(X_t)\right. \\
&\qquad\qquad \left.\left. + X_t^2\left(f^{(1)}\right)''(X_t)\right)\right)dt + \sigma X_t f^{(1)}(X_t)dB_t^D \\
&\qquad \left. + \sigma\sqrt{X_t(1-X_t)}\left(\gamma f^{(1)}(X_t) + X_t\left(f^{(1)}\right)'(X_t)\right)dB_t^X\right)
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&\frac{dP_t^{(1)}}{P_t^{(1)}} - \sigma dB_t^D - \sigma\sqrt{\frac{1-X_t}{X_t}}\left(\gamma + X_t\frac{\left(f^{(1)}\right)'(X_t)}{f^{(1)}(X_t)}\right)dB_t^X = \\
&\qquad \left(\mu + \kappa\frac{w_1 - X_t}{X_t}\left(\gamma + X_t\frac{\left(f^{(1)}\right)'(X_t)}{f^{(1)}(X_t)}\right) + \right. \\
&\left. \frac{\sigma^2}{2}\frac{1-X_t}{X_t}\left(\gamma(\gamma-1) + 2\gamma X_t\frac{\left(f^{(1)}\right)'(X_t)}{f^{(1)}(X_t)} + X_t^2\frac{\left(f^{(1)}\right)''(X_t)}{f^{(1)}(X_t)}\right)\right) := \\
&\qquad\qquad \mu_t^{(1)} - \frac{D_t^{(1)}}{P_t^{(1)}}dt
\end{aligned}
$$

which proves the claim.

Next, under Assumption 4.2, closed-form expressions of $f^{(i)}$ in terms of hypergeometric functions follow (cf. the formulas derived by Hurd and Kuznetsov [30] for related functionals of the Jacobi process).

Proposition A.3 *Let Assumptions 3.4 and 4.2 hold. Then*

$$f^{(1)}(x) = \frac{2}{\sigma^2\omega^{(1)}}\left(\frac{\Gamma_1^{(1)}}{1+c_1-\gamma}F_1^{(1)}(x)G_1^{(1)}(1) + \frac{\Gamma_2^{(1)}}{1+c_1-\gamma}x^{2-\gamma}F_2^{(1)}(x)G_1^{(1)}(x) + \frac{\Gamma_2^{(1)}}{2-\gamma}F_1^{(1)}(x)\left(G_2^{(1)}(1) - x^{2-\gamma}G_2^{(1)}(x)\right)\right)$$

$$f^{(2)}(x) = \frac{2}{\sigma^2\omega^{(2)}}\left(\frac{\Gamma_1^{(2)}}{1+c_2-\gamma}F_1^{(2)}(1-x)G_1^{(2)}(1) + \frac{\Gamma_2^{(2)}(1-x)^{2-\gamma}}{1+c_2-\gamma}F_2^{(2)}(1-x)G_1^{(2)}(1-x) + \frac{\Gamma_2^{(2)}}{2-\gamma}F_1^{(2)}(1-x)\left(G_2^{(2)}(1) - (1-x)^{2-\gamma}G_2^{(2)}(1-x)\right)\right),$$

where

$$F_1^{(i)}(x) := {}_2F_1(a,b;c_i;x),$$
$$G_1^{(i)}(x) := {}_3F_2(c_i-a, c_i-b, 1+c_i-\gamma; c_i, 2+c_i-\gamma; x),$$
$$F_2^{(i)}(x) := {}_2F_1(b+1-c_i, a+1-c_i; 2-c_i; x),$$
$$G_2^{(i)}(x) := {}_3F_2(1-b, 1-a, 2-\gamma; 2-c_i, 3-\gamma; x),$$
$$a := \frac{-\left(1-\frac{2\kappa}{\sigma^2}\right) + \sqrt{\left(1-\frac{2\kappa}{\sigma^2}\right)^2 - \frac{8\theta}{\sigma^2}}}{2},\ b := \frac{-\left(1-\frac{2\kappa}{\sigma^2}\right) - \sqrt{\left(1-\frac{2\kappa}{\sigma^2}\right)^2 - \frac{8\theta}{\sigma^2}}}{2},$$
$$c_i := \frac{2\kappa w_i}{\sigma^2},$$

$\Gamma_1^{(i)} := \frac{\Gamma(a+b+1-c_i)\Gamma(1-c_i)}{\Gamma(a+1-c_i)\Gamma(b+1-c_i)}$, $\Gamma_2^{(i)} := \frac{\Gamma(a+b+1-c_i)\Gamma(c_i-1)}{\Gamma(a)\Gamma(b)}$, *and the Wronskian constant* $\omega^{(i)}$ *is*

$$\omega^{(i)} := \left(\varphi^{(i)}(y)\left(F_1^{(i)}\right)'(y) - \left(\varphi^{(i)}\right)'(y)F_1^{(i)}(y)\right)(1-y)^{\frac{2\kappa(1-w_i)}{\sigma^2}}y^{\frac{2\kappa w_i}{\sigma^2}}$$

with $\varphi^{(i)}(x) = {}_2F_1(a,b;a+b+1-c_i;1-x)$ *and hypergeometric functions* ${}_2F_1$ *and* ${}_3F_2$

$$ {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$
$$ {}_3F_2(a,b,p;c,q;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(p)_n}{(c)_n(q)_n}\frac{x^n}{n!},$$

where $(p)_n := p\cdot(p+1)\cdots(p+n-2)\cdot(p+n-1)$.

Proof. The proof focuses on region 1, as the formulae for region 2 follow by replacing x with $1-x$ and w_1 with w_2. First, characterize the function $f^{(1)}$ in terms of the density function and speed measure of the process X:

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds \middle| X_0 = x\right] &= \int_0^\infty e^{-\theta s}\left(\int_0^1 y^{1-\gamma} p(s;x,y) m(y) dy\right) ds \\ &= \int_0^1 y^{1-\gamma}\left(\int_0^\infty e^{-\theta s} p(s;x,y) ds\right) m(y) dy\end{aligned} \tag{A.6}$$

where $p(s;x,y)$ is the density function with respect to the speed measure $m(dy) = m(y)dy$. The representation of the Green function for scalar diffusions (e.g. Borodin and Salminen [9, II.10-11]) yields $m(x) = \frac{2}{\sigma^2} x^{c_1-1}(1-x)^{a+b-c_1}$ and

$$\int_0^\infty e^{-\theta s} p(s;x,y) ds = \begin{cases} \frac{1}{\omega^{(1)}} F_1^{(1)}(x)\varphi^{(1)}(y), & x \leq y, \\ \frac{1}{\omega^{(1)}} F_1^{(1)}(y)\varphi^{(1)}(x), & x \geq y, \end{cases}$$

where $F_1^{(1)}$ and $\varphi^{(1)}$ are the fundamental solutions of the ordinary differential equation

$$\left(x - x^2\right) g''(x) + \frac{2\mu}{\sigma^2}(w - x) g'(x) = \frac{2\theta}{\sigma^2} g(x),$$

with the respective boundary conditions

$$\begin{aligned} F_1^{(1)}(0+) > 0, &\quad \left(F_1^{(1)}\right)'(0+) \cdot s(0+) = 0, \\ \varphi^{(1)}(0+) = +\infty, &\quad \left(\varphi^{(1)}\right)'(0+) \cdot s(0+) > -\infty \\ \varphi^{(1)}(1-) > 0, &\quad \left(\varphi^{(1)}\right)'(1-) \cdot s(1-) = 0, \\ F_1^{(1)}(1-) = +\infty, &\quad \left(F_1^{(1)}\right)'(1-) \cdot s(1-) > -\infty \end{aligned}$$

where $s(x) := \frac{2}{\sigma^2} x^{c_1-1}(1-x)^{a+b-c_1} = x(1-x)m(x)$. Thus, (A.6) further simplifies to

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds \middle| X_0 = x\right] = \frac{2}{\sigma^2 \omega^{(1)}} \times \\ &\Bigg({}_2F_1(a,b;a+b+1-c_1;1-x) \int_0^x {}_2F_1(a,b;c_1;y)(1-y)^{a+b-c_1} y^{c_1-\gamma} dy \\ &\quad + {}_2F_1(a,b;c_1;x) \int_x^1 {}_2F_1(a,b;a+b+1-c_1;1-y)(1-y)^{a+b-c_1} y^{c_1-\gamma} dy\Bigg).\end{aligned}$$

Note that Assumption 4.2 ensures that the first integral in the right-hand side converges. Applying the identities for hypergeometric functions in Abramowitz and Stegun [1, Chapter 15.3], the expression for $f^{(1)}$ simplifies as follows

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds \middle| X_0 = x\right] \\
&= \frac{2}{\sigma^2\omega^{(1)}} \left(\Gamma_1^{(1)} \cdot {}_2F_1(a,b;c_1;x) \int_0^x {}_2F_1(a,b;c_1;y)(1-y)^{a+b-c_1} y^{c_1-\gamma} dy \right.\\
&+ \Gamma_2^{(1)} \cdot x^{1-c_1} {}_2F_1(b+1-c_1, a+1-c_1; 2-c_1; x) \times \\
&\int_0^x {}_2F_1(a,b;c_1;y)(1-y)^{a+b-c_1} y^{c_1-\gamma} dy \\
&+ \Gamma_1^{(1)} \cdot {}_2F_1(a,b;c_1;x) \int_x^1 {}_2F_1(a,b;c_1;y)(1-y)^{a+b-c_1} y^{c_1-\gamma} dy + \Gamma_2^{(1)} \times \\
&\left. {}_2F_1(a,b;c_1;x) \int_x^1 {}_2F_1(b+1-c_1, a+1-c_1; 2-c_1; y)(1-y)^{a+b-c_1} y^{1-\gamma} dy \right) \\
&= \frac{2}{\sigma^2\omega^{(1)}} \left(\Gamma_1^{(1)} \cdot {}_2F_1(a,b;c_1;x) \int_0^1 {}_2F_1(c_1-a, c_1-b; c_1; y) y^{c_1-\gamma} dy + \Gamma_2^{(1)} \times \right.\\
&x^{1-c_1} {}_2F_1(b+1-c_1, a+1-c_1; 2-c_1; x) \int_0^x {}_2F_1(c_1-a, c_1-b; c_1; y) y^{c-\gamma} dy \\
&\left. + \Gamma_2^{(1)} \cdot {}_2F_1(a,b;c_1;x) \int_x^1 {}_2F_1(1-b, 1-a; 2-c_1; y) y^{1-\gamma} dy \right) \\
&= \frac{2}{\sigma^2\omega^{(1)}} \left(\frac{\Gamma_1^{(1)} {}_2F_1(a,b;c_1;x)}{1+c_1-\gamma} {}_3F_2(c_1-a, c_1-b, 1+c_1-\gamma; c_1, 2+c_1-\gamma; 1) \right.\\
&+ \Gamma_2^{(1)} \frac{x^{2-\gamma}}{1+c_1-\gamma} {}_2F_1(b+1-c_1, a+1-c_1; 2-c_1; x) \\
&\times {}_3F_2(c_1-a, c_1-b, 1+c_1-\gamma; c_1, 2+c_1-\gamma; x) \\
&+ \Gamma_2^{(1)} \frac{1}{2-\gamma} {}_2F_1(a,b;c_1;x) \Big({}_3F_2(1-b, 1-a, 2-\gamma; 2-c_1, 3-\gamma; 1) \\
&\qquad\qquad \left. - x^{2-\gamma} {}_3F_2(1-b, 1-a, 2-\gamma; 2-c_1, 3-\gamma; x) \Big) \right)
\end{aligned}
$$

which yields the claim for $f^{(1)}$.

A.3 Proof of Theorem 4.3

(i) Note that

$$\begin{aligned}
\bar{P}_t^{(1)} &= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\left(\frac{D_s}{D_t}\right)^{1-\gamma}\frac{D_t}{D_s}D_s^{(1)}ds\right]\\
&= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\left(\frac{D_s}{D_t}\right)^{1-\gamma}X_s ds\right]D_t\\
&= D_t\int_t^\infty e^{-\beta(s-t)}\mathbb{E}_t\left[\left(\frac{D_s}{D_t}\right)^{1-\gamma}\right]\mathbb{E}_t\left[X_s\right]ds\\
&= D_t\int_t^\infty e^{-\theta(s-t)}\mathbb{E}_t\left[X_s\right]ds = D_t\int_t^\infty e^{-\theta(s-t)}\mathbb{E}\left[X_s|X_t\right]ds\\
&= D_t f(X_t).
\end{aligned}$$

where the last equality follows by the Markov property of $(X_t)_{t\geq 0}$, and f is defined as

$$f(x) := \int_t^\infty e^{-\theta(s-t)}\mathbb{E}\left[X_s|X_t = x\right]ds = \int_0^\infty e^{-\theta s}\mathbb{E}\left[X_s|X_0 = x\right]ds. \tag{A.7}$$

where the second equality again uses the Markov property. Integrating (3.2) and taking the expectation yields $\mathbb{E}[X_t|X_0 = x] = e^{-\kappa t}x + w_1(1 - e^{-\kappa t})$. Therefore, (A.7) further simplifies to

$$f(x) = \int_0^\infty e^{-\theta s}\left(e^{-\kappa s}x + w_1\left(1 - e^{-\kappa s}\right)\right)ds = \frac{1}{\theta+\kappa}x + \frac{\kappa}{(\theta+\kappa)\theta}w_1,$$

whence the price of asset 1 is

$$\begin{aligned}
&\bar{P}_t^{(1)} = f(X_t)D_t\\
&= \frac{1}{\theta+\kappa}X_tD_t + \frac{\kappa}{(\theta+\kappa)\theta}w_1D_t = \frac{1}{\theta+\kappa}D_t^{(1)} + \frac{\kappa w_1}{(\theta+\kappa)\theta}\left(D_t^{(1)} + D_t^{(2)}\right)\\
&= \frac{1}{\theta}\left(\frac{\theta+\kappa w_1}{\theta+\kappa}D_t^{(1)} + \frac{\kappa w_1}{\theta+\kappa}D_t^{(2)}\right)
\end{aligned}$$

and the price of asset 2 follows analogously.

As $e^{\int_0^t \overline{r_s}ds}\bar{M}_t = e^{\int_0^t(\overline{r_s}-\beta)ds}(D_t)^{-\gamma}$ is a local martingale and

$$\begin{aligned}
&d\left(e^{\int_0^t(\overline{r_s}-\beta)ds}\left(D_t\right)^{-\gamma}\right)\\
&= \left(\overline{r_t}-\beta\right)e^{\int_0^t(\overline{r_s}-\beta)ds}\left(D_t\right)^{-\gamma}dt\\
&\quad -\gamma e^{\int_0^t(\overline{r_s}-\beta)ds}\left(D_t\right)^{-\gamma}\left(\mu + \frac{(-\gamma-1)\sigma^2}{2}\right)dt + L_t\\
&= e^{\int_0^t(\overline{r_s}-\beta)ds}\left(D_t\right)^{-\gamma}\left(\overline{r_t}-\beta-\gamma\left(\mu - \frac{(\gamma+1)\sigma^2}{2}\right)\right)dt + L_t,
\end{aligned}$$

for some local martingale L, it follows that $\overline{r_t} = \beta + \gamma\mu - \frac{\gamma(\gamma+1)\sigma^2}{2}$.

(ii) For the welfare, note that

$$
\begin{aligned}
\overline{\mathcal{U}}_t &= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{(D_s)^{1-\gamma}}{1-\gamma}ds\right]\\
&= \frac{D_t^{1-\gamma}}{1-\gamma}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\left(\frac{D_s}{D_t}\right)^{1-\gamma}ds\right]\\
&= \frac{D_t^{1-\gamma}}{1-\gamma}\int_t^\infty e^{-\beta(s-t)}\mathbb{E}_t\left[\left(\frac{D_s}{D_t}\right)^{1-\gamma}\right]ds = \frac{D_t^{1-\gamma}}{1-\gamma}\int_t^\infty e^{-\theta(s-t)}ds\\
&= \frac{D_t^{1-\gamma}}{1-\gamma}\frac{1}{\theta}.
\end{aligned}
$$

(iii) In the integrated market, recalling that a_0 and b_i are defined in Theorem 4.3, $\bar{P}_t^{(1)} = a_0 D_t X_t + b_1 D_t$ and $\bar{P}_t^{(2)} = a_0 D_t(1-X_t) + b_2 D_t$. Ito's formula yields

$$
\begin{aligned}
d\bar{P}_t^{(1)} &= (\mu(a_0X_t+b_1)D_t + a_0\kappa(w_1-X_t)D_t)\,dt\\
&\quad + \sigma(a_0X_t+b_1)D_t dB_t^D + \sigma a_0 D_t\sqrt{X_t(1-X_t)}dB_t^X \qquad \text{(A.8)}\\
d\bar{P}_t^{(2)} &= (\mu(a_0(1-X_t)+b_2)D_t - a_0\kappa(w_1-X_t)D_t)\,dt\\
&\quad + \sigma(a_0(1-X_t)+b_2)D_t dB_t^D - \sigma a_0 D_t\sqrt{X_t(1-X_t)}dB_t^X \qquad \text{(A.9)}
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{d\left\langle\bar{P}^{(1)}\right\rangle_t}{dt} &= \sigma^2 D_t^2\left((a_0X_t+b_1)^2 + a_0^2X_t(1-X_t)\right)\\
\frac{d\left\langle\bar{P}^{(2)}\right\rangle_t}{dt} &= \sigma^2 D_t^2\left((a_0(1-X_t)+b_2)^2 + a_0^2X_t(1-X_t)\right)\\
\frac{d\left\langle\bar{P}^{(1)},\bar{P}^{(2)}\right\rangle_t}{dt} &= \sigma^2 D_t^2\Big((a_0X_t+b_1)(a_0(1-X_t)+b_2) - a_0^2X_t(1-X_t)\Big).
\end{aligned}
$$

and the claim follows by (4.5).

(iv) Equations (A.8)-(A.9) yield

$$
\begin{aligned}
\frac{d\bar{P}_t^{(1)}}{\bar{P}_t^{(1)}} &= \left(\mu + \frac{a_0\kappa(w_1-X_t)}{a_0X_t+b_1}\right)dt + \sigma dB_t^D + \sigma\frac{a_0\sqrt{X_t(1-X_t)}}{a_0X_t+b_1}dB_t^X\\
\frac{d\bar{P}_t^{(2)}}{\bar{P}_t^{(2)}} &= \left(\mu - \frac{a_0\kappa(w_1-X_t)}{a_0(1-X_t)+b_2}\right)dt + \sigma dB_t^D - \sigma\frac{a_0\sqrt{X_t(1-X_t)}}{a_0(1-X_t)+b_2}dB_t^X
\end{aligned}
$$

whence the expected returns are

$$
\begin{aligned}
\bar{\mu}_t^{(1)} &= \mu + \frac{a_0\kappa(w_1-X_t)}{a_0X_t+b_1} + \frac{X_t}{a_0X_t+b_1} = \mu+\theta\\
\bar{\mu}_t^{(2)} &= \mu - \frac{a_0\kappa(w_1-X_t)}{a_0(1-X_t)+b_2} + \frac{1-X_t}{a_0(1-X_t)+b_2} = \mu+\theta
\end{aligned}
$$

A.4 Proof of Proposition 5.1

The proof focuses on region 1, as region 2 is analogous.

In the segmented market the pricing formula in Proposition A.3 yields

$$
\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left[\frac{P_t^{(1)}}{D_t^{(1)}}\right] = \lim_{t\to\infty}\mathbb{E}\left[X_t^{\gamma-1}f^{(1)}(X_t)\right] = \int_0^1 x^{\gamma-1}f^{(1)}(x)m(x)dx \\
&= \int_0^1 \frac{2x^{c_1+\gamma-2}(1-x)^{a+b-c_1}}{\sigma^2\omega^{(1)}B(a+b-c_1+1,c_1)}\times \\
&\quad\left(\mathit{\Gamma}_1^{(1)}\cdot {}_2F_1(a,b;c_1;x)\int_0^1 {}_2F_1(c_1-a,c_1-b;c_1;y)y^{c_1-\gamma}dy + \mathit{\Gamma}_2^{(1)}x^{1-c_1}\times\right. \\
&\quad\quad {}_2F_1(b+1-c_1,a+1-c_1;2-c_1;x)\int_0^x {}_2F_1(c_1-a,c_1-b;c_1;y)y^{c_1-\gamma}dy \\
&\quad\quad\left. +\mathit{\Gamma}_2^{(1)}\cdot {}_2F_1(a,b;c_1;x)\int_x^1 {}_2F_1(1-b,1-a;2-c_1;y)y^{1-\gamma}dy\right)dx \\
&= \frac{2}{\sigma^2\omega^{(1)}B(a+b-c_1+1,c_1)}\times \\
&\quad\left(\mathit{\Gamma}_1^{(1)}\int_0^1 {}_2F_1(c_1-a,c_1-b;c_1;x)x^{c_1+\gamma-2}dx\times\right. \\
&\quad\quad\quad \int_0^1 {}_2F_1(c_1-a,c_1-b;c_1;y)y^{c_1-\gamma}dy \\
&\quad\quad + \mathit{\Gamma}_2^{(1)}\int_0^1 x^{\gamma-1}{}_2F_1(1-b,1-a;2-c_1;x)\times \\
&\quad\quad\quad \left(\int_0^x {}_2F_1(c_1-a,c_1-b;c_1;y)y^{c_1-\gamma}dy\right)dx \\
&\quad\quad + \mathit{\Gamma}_2^{(1)}\int_0^1 {}_2F_1(1-b,1-a;2-c_1;y)y^{1-\gamma}\times \\
&\quad\quad\quad \left.\left(\int_0^y x^{c_1+\gamma-2}{}_2F_1(a,b;c_1;x)dx\right)dy\right) \\
&= \frac{2}{\sigma^2\omega^{(1)}B(a+b-c_1+1,c_1)}\left(\mathit{\Gamma}_1^{(1)}\frac{G_1^{(1)}(1)}{c_1+1-\gamma}\frac{G_3^{(1)}(1)}{c_1+\gamma-1}\right. \\
&\quad\quad\left. +\mathit{\Gamma}_2^{(1)}\int_0^1 x^{c_1}F_3^{(1)}(x)\left(\frac{G_1^{(1)}(x)}{c_1+1-\gamma}+\frac{G_3^{(1)}(x)}{c_1+\gamma-1}\right)dx\right).
\end{aligned}
$$

Vice versa, in the integrated market, Theorem 4.3 implies that

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left[\frac{\bar{P}_t^{(1)}}{D_t^{(1)}}\right]\\
&=\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_1}{\theta}\frac{1}{X_t}\right)\right]=\frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_1}{\theta}\int_0^1\frac{1}{x}m(x)dx\right)\\
&=\frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_1}{\theta}\int_0^1\frac{x^{c_1-2}(1-x)^{a+b-c_1}}{B(a+b-c_1+1,c_1)}dx\right)=\frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_1}{\theta}\frac{a+b}{c_1-1}\right)\\
&=\frac{1}{\theta+\kappa}\left(1+\frac{\kappa w_1}{\theta}\left(\frac{2\frac{\kappa}{\sigma^2}-1}{2\frac{\kappa w_1}{\sigma^2}-1}\right)\right),
\end{aligned}$$

where $m(x) := \frac{x^{c_1-1}(1-x)^{a+b-c_1}}{B(a+b-c_1+1,c_1)}$.

A.5 Proof of Proposition 5.2

The real exchange rate is the ratios between the stochastic discount factors in the two countries, i.e., $p_t = \frac{M_t^{(1)}}{M_t^{(2)}}$. Under segmentation, Proposition A.2 yields that $p_t = \frac{e^{-\beta t}(D_t^{(1)})^{-\gamma}}{e^{-\beta t}(D_t^{(2)})^{-\gamma}}$, and Proposition 5.2 follows by recalling that $D_t^{(1)} = D_t X_t$ and $D_t^{(2)} = D_t(1-X_t)$.

A.6 Proof of Proposition 6.1

Because of the market clearing condition (6.1) and the obtained optimal consumption (6.2), the stochastic discount factor $\bar{M}_t$ solves

$$D_t = c_t^{(1)} + c_t^{(2)} = \left(y^{(1)}e^{\beta_1 t}\bar{M}_t\right)^{-\frac{1}{\gamma}} + \left(y^{(2)}e^{\beta_2 t}\bar{M}_t\right)^{-\frac{1}{2\gamma}}.$$

Solving the above quadratic equation gives

$$\begin{aligned}
\bar{M}_t^{-\frac{1}{2\gamma}} &= \frac{-\left(y^{(2)}e^{\beta_2 t}\right)^{-\frac{1}{2\gamma}}+\sqrt{\left(y^{(2)}e^{\beta_2 t}\right)^{-\frac{1}{\gamma}}+4D_t\left(y^{(1)}e^{\beta_1 t}\right)^{-\frac{1}{\gamma}}}}{2\left(y^{(1)}e^{\beta_1 t}\right)^{-\frac{1}{\gamma}}}\\
&=\frac{\left(y^{(2)}\right)^{-\frac{1}{2\gamma}}}{2\left(y^{(1)}\right)^{-\frac{1}{\gamma}}}e^{\left(\frac{\beta_1}{\gamma}-\frac{\beta_2}{2\gamma}\right)t}\left(-1+\sqrt{1+\frac{4\left(y^{(1)}\right)^{-\frac{1}{\gamma}}}{\left(y^{(2)}\right)^{-\frac{1}{\gamma}}}D_t e^{\frac{\beta_2-\beta_1}{\gamma}t}}\right)
\end{aligned}$$

With the stochastic discount factor $\bar{M}_t$, the wealth process $Y^{(i)}$ of agent i is

$$\bar{M}_t Y_t^{(i)} = \mathbb{E}_t\left[\int_t^\infty \bar{M}_u c_u^{(i)} du\right].$$

Thus, the ratio of wealth becomes

$$\frac{Y_t^{(2)}}{Y_t^{(1)}} = \frac{\mathbb{E}_t\left[\int_t^\infty \bar{M}_u c_u^{(2)} du\right]}{\mathbb{E}_t\left[\int_t^\infty \bar{M}_u c_u^{(1)} du\right]} = \frac{\left(y^{(2)}\right)^{-\frac{1}{2\gamma}}}{\left(y^{(1)}\right)^{-\frac{1}{\gamma}}} \frac{\mathbb{E}_t\left[\int_t^\infty \bar{M}_u^{1-\frac{1}{2\gamma}} e^{-\frac{\beta_2}{2\gamma}u} du\right]}{\mathbb{E}_t\left[\int_t^\infty \bar{M}_u^{1-\frac{1}{\gamma}} e^{-\frac{\beta_1}{\gamma}u} du\right]}$$
$$= 2\frac{\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{1-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right]}{\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{2-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right]}, \tag{A.10}$$

where K is a positive constant dependent on $y^{(i)}$ and γ only.

To establish an upper bound (respectively, lower bound) for the expectation in the numerator (respectively, denominator), consider first the case of $\mu - \frac{1}{2}\sigma^2 + \frac{\beta_2-\beta_1}{\gamma} > 0$. On one hand,

$$\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{1-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right]$$
$$\leq \mathbb{E}_t\left[\int_t^\infty \left(\frac{2+\sqrt{KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}}{KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{2\gamma-1} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right]$$
$$\leq 2^{2\gamma-2}\mathbb{E}_t\left[\int_t^\infty \left(2^{2\gamma-1}\left(KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{1-2\gamma} + \left(KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{\frac{1-2\gamma}{2}}\right) \times \right.$$
$$\left. e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right]$$
$$= \mathbb{E}_t\left[\int_t^\infty \left(K_1 e^{-\beta_2 u} D_u^{1-2\gamma} + K_2 e^{\left(\frac{\beta_1}{2\gamma}(1-2\gamma)-\frac{\beta_2}{2\gamma}\right)u} D_u^{\frac{1-2\gamma}{2}}\right) du\right]$$
$$= K_1 D_t^{1-2\gamma}\int_t^\infty e^{-\beta_2 u} e^{\left((1-2\gamma)\mu-\gamma(1-2\gamma)\sigma^2\right)(u-t)} du$$
$$+ K_2 D_t^{\frac{1-2\gamma}{2}}\int_t^\infty e^{\left(\frac{\beta_1}{2\gamma}(1-2\gamma)-\frac{\beta_2}{2\gamma}\right)u} e^{\left(\frac{1-2\gamma}{2}\mu-\frac{1}{2}\frac{1-2\gamma}{2}\frac{1+2\gamma}{2}\sigma^2\right)(u-t)} du$$
$$= K_1 e^{-\beta_2 t} D_t^{1-2\gamma} + K_2 e^{\left(\frac{\beta_1}{2\gamma}(1-2\gamma)-\frac{\beta_2}{2\gamma}\right)t} D_t^{\frac{1-2\gamma}{2}}, \tag{A.11}$$

where K_1 and K_2 are some positive constants independent of t, while the first inequality follows from $(\sqrt{1+x}-1)^{-\delta} = (\frac{\sqrt{1+x}+1}{x})^\delta \leq (\frac{2+\sqrt{x}}{x})^\delta$ for all $x > 0, \delta \geq 0$ and the second inequality from the version Jensen's inequality

$(x+y)^\delta \leq 2^{\delta-1}(x^\delta + y^\delta)$ for all $x, y \geq 0, \delta \geq 1$. On the other hand,

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{2-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&\geq \mathbb{E}_t\left[\int_t^\infty \left(KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{1-\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&= K^{1-\gamma}\mathbb{E}_t\left[\int_t^\infty e^{-\beta_1 u} D_u^{1-\gamma} du\right] = K_3 e^{-\beta_1 t} D_t^{1-\gamma}, \qquad \text{(A.12)}
\end{aligned}$$

where K_3 is a positive constant independent of t and the first inequality follows from $-1+\sqrt{1+x} \leq \sqrt{x}$. With (A.11) and (A.12), (A.10) becomes

$$\begin{aligned}
\frac{Y_t^{(2)}}{Y_t^{(1)}} &\leq 2\frac{K_1 e^{-\beta_2 t} D_t^{1-2\gamma} + K_2 e^{\left(\frac{\beta_1}{2\gamma}(1-2\gamma)-\frac{\beta_2}{2\gamma}\right)t} D_t^{\frac{1-2\gamma}{2}}}{K_3 e^{-\beta_1 t} D_t^{1-\gamma}} \\
&= K_1 e^{(\beta_1-\beta_2)t} D_t^{-\gamma} + K_2 e^{\frac{\beta_1-\beta_2}{2\gamma}t} D_t^{-\frac{1}{2}} \\
&= K_1 D_0^{-\gamma} e^{-\gamma\left(\mu-\frac{1}{2}\sigma^2+\frac{\beta_2-\beta_1}{\gamma}\right)t-\gamma\sigma W_t} + K_2 D_0^{-\frac{1}{2}} e^{-\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2+\frac{\beta_2-\beta_1}{\gamma}\right)t-\frac{1}{2}\sigma W_t}
\end{aligned}$$

which tends to zero as $t \to \infty$ because $\mu - \frac{1}{2}\sigma^2 + \frac{\beta_2-\beta_1}{\gamma} > 0$.

Now, consider the case of $\mu - \frac{1}{2}\sigma^2 + \frac{\beta_2-\beta_1}{\gamma} < 0$. On one hand,

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{2-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&\leq \mathbb{E}_t\left[\int_t^\infty \left(\frac{2+\sqrt{KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}}{KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{2\gamma-2} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&\leq \mathbb{E}_t\left[\int_t^\infty \left(K_1\left(D_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{2-2\gamma} + K_2\left(D_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{1-\gamma}\right) \times \right. \\
&\qquad\qquad \left. e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&= \mathbb{E}_t\left[\int_t^\infty \left(K_1 e^{\left(\frac{\beta_2}{\gamma}(1-\gamma)-\frac{\beta_1}{\gamma}\right)u} D_u^{2-2\gamma} + K_2 e^{-\beta_1 u} D_u^{1-\gamma}\right) du\right] \\
&= K_1 e^{\left(\frac{\beta_2}{\gamma}(1-\gamma)-\frac{\beta_1}{\gamma}\right)t} D_t^{2-2\gamma} + K_2 e^{-\beta_1 t} D_t^{1-\gamma},
\end{aligned}$$

where K_1 and K_2 are positive constants independent of t. On the other hand,

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^\infty \left(-1+\sqrt{1+KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}}\right)^{1-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&\geq \mathbb{E}_t\left[\int_t^\infty \left(KD_u e^{\frac{\beta_2-\beta_1}{\gamma}u}\right)^{1-2\gamma} e^{\left(\frac{\beta_1}{\gamma}(1-2\gamma)-\frac{\beta_2}{\gamma}(1-\gamma)\right)u} du\right] \\
&= K^{1-2\gamma}\mathbb{E}_t\left[\int_t^\infty e^{-\beta_2 u} D_u^{1-2\gamma} du\right] = K_3 e^{-\beta_2 t} D_t^{1-2\gamma},
\end{aligned}$$

where K_3 is a positive constant independent of t and the first inequality holds due to $-1+\sqrt{1+x} = \frac{x}{1+\sqrt{1+x}} \leq x$. Then, (A.10) becomes

$$\begin{aligned}\frac{Y_t^{(2)}}{Y_t^{(1)}} &\geq 2\frac{K_3 e^{-\beta_2 t} D_t^{1-2\gamma}}{K_1 e^{\left(\frac{\beta_2}{\gamma}(1-\gamma)-\frac{\beta_1}{\gamma}\right)t} D_t^{2-2\gamma} + K_2 e^{-\beta_1 t} D_t^{1-\gamma}} \\ &= \frac{1}{K_1 e^{\frac{\beta_2-\beta_1}{\gamma}t} D_t + K_2 e^{(\beta_2-\beta_1)t} D_t^{\gamma}} \\ &= \frac{1}{K_1 D_0 e^{\left(\mu-\frac{1}{2}\sigma^2+\frac{\beta_2-\beta_1}{\gamma}\right)t+\sigma W_t} + K_2 D_0^{\gamma} e^{\gamma\left(\mu-\frac{1}{2}\sigma^2+\frac{\beta_2-\beta_1}{\gamma}\right)t+\gamma\sigma W_t}},\end{aligned}$$

which tends to $+\infty$ as $t \to \infty$ because $\mu - \frac{1}{2}\sigma^2 + \frac{\beta_2-\beta_1}{\gamma} < 0$.

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