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NECESSARY CONDITION IN A BREZIS-OSWALD-TYPE PROBLEM FOR MIXED LOCAL AND NONLOCAL OPERATORS

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ABSTRACT. In this note we complete the study started in [4] providing a full characterization of the existence of a unique positive weak solution of a p-sublinear Dirichlet boundary value problem driven by a mixed local-nonlocal operator.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. Moreover, let $1 < p < \infty$ and $s \in (0,1)$ be fixed. The aim of this short note is to complete the study started in [4] concerning the optimal solvability of the following p-sublinear Dirichlet problem

$$\begin{cases} \mathcal{L}_{p,s}u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here, $\mathcal{L}_{p,s}$ is the *mixed local and nonlocal* quasilinear operator

$$\mathcal{L}_{p,s} := -\Delta_p + (-\Delta)_p^s,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplace operator and $(-\Delta)_p^s$ denotes the so-called *fractional p-Laplacian in $\mathbb{R}^n$* which acts on sufficiently regular functions u and up to a suitable normalizing constant, as follows:

$$(-\Delta)_p^s u(x) := 2\,\text{P.V.}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,dx.$$

As usual, P.V. denotes the Cauchy principal value.

In order to clearly state the main theorem of this note and to review the results obtained in [4], it is worth introducing some assumptions and notation.

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The functional setting. To begin with, we fix once and for all the structural assumptions we require on the nonlinearity f:

(f1) $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function.
(f2) $f(\cdot, t) \in L^\infty(\Omega)$ for every $t \geq 0$.
(f3) There exists a constant $c_p > 0$ such that

$$|f(x,t)| \leq c_p(1 + t^{p-1}) \qquad \text{for a.e. } x \in \Omega \text{ and every } t \geq 0.$$

(f4) For a.e. $x \in \Omega$, the function $t \mapsto \dfrac{f(x,t)}{t^{p-1}}$ is strictly decreasing in $(0, \infty)$.
(f5) There exists $\rho_f > 0$ such that

$$f(x,t) > 0 \quad \text{for a.e. } x \in \Omega \text{ and every } 0 < t < \rho_f. \tag{1.2}$$

We observe that all the assumptions above are trivially satisfied in the particular case of power-type linearities $f(x, u) = u^\theta$, with $0 \leq \theta \leq p - 1$.

Remark 1.1. A few remarks on assumptions (f1)-(f5) are in order.

(1) If compared to the Brezis-Oswald's paper [7], our assumptions on the nonlinearity f are *more restrictive*: indeed, in [7] is required a *one-side* sublinear growth on f, and the sign assumption (f5) is not needed; thus, we can cover a *smaller class* of nonlinearities. For instance, the function

$$f(x,t) = f(t) := \begin{cases} \cos(t) & \text{if } 0 \leq t \leq \pi/2, \\ -(t - \pi/2)^2 & \text{if } t \geq \pi/2, \end{cases}$$

does not satisfy assumption (f3) (with $p = 2$), but it satisfies the one-side growth condition

$$f(t) \leq 1 + t \quad \text{for every } t \geq 0.$$

Hence, f is an 'admissible' nonlinearity in [7], but not for us.

(2) As pointed out in [4], assumption (f5) and the *two-side* growth condition in assumption (f3) are technical assumption which permit to overcome the lack of *boundary regularity* for $\mathcal{L}_{p,s}$, which is instead a crucial tool in [7, 11, 12]. Presently, the regularity for $\mathcal{L}_{p,s}$ is deeply investigated, see [1, 2, 3, 8, 10, 9, 13, 14] for the case of *weak solutions* and [5] for the case of *viscosity solutions*; however, the *optimal* boundary regularity for $\mathcal{L}_{p,s}$ in the context of weak solutions and a Hopf-type lemma seem lacking. As it will be clear from the proof of Theorem 1.3, assumptions (f3)-(f5) allows us to set up a suitable truncation/approximation argument which turns out to be a proper substitute of a Hopf-type lemma for $\mathcal{L}_{p,s}$.

Owing to assumption (f4), we then introduce the following functions:

$$a_0(x) := \lim_{t \downarrow 0} \frac{f(x,t)}{t^{p-1}} \qquad a_\infty(x) := \lim_{t \uparrow \infty} \frac{f(x,t)}{t^{p-1}} \qquad (\text{for } x \in \Omega).$$

We explicitly observe that, taking into account assumption (f5), the function a_0 is *non-negative but possibly unbounded from above* in Ω, and even *infinite in a non-null subset of* Ω; on the other hand, since the *two-side* growth condition on f in assumption (f3) gives

$$\left| \frac{f(x,t)}{t^{p-1}} \right| \leq c_p \frac{1 + t^{p-1}}{t^{p-1}} \leq 2c_p \quad \text{for a.e. } x \in \Omega \text{ and } t \geq 1,$$

we readily infer that $a_\infty \in L^\infty(\Omega)$. Summing up, recalling (f4), we have

(1) $\max\{0, a_\infty(x)\} \leq a_0(x) \leq \infty$ for a.e. $x \in \Omega$;
(2) $a_\infty \in L^\infty(\Omega)$.

We now introduce the function space

$$\mathbb{X}_p(\Omega) := \big\{u \in W^{1,p}(\mathbb{R}^n) : \, u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\big\}. \tag{1.3}$$

In view of the regularity assumption on $\partial\Omega$, we can identify $\mathbb{X}_p(\Omega)$ with the space $W^{1,p}_0(\Omega)$. Indeed, denoting with $\mathbf{1}_\Omega$ the indicator function of Ω, we have

$$u \in W^{1,p}_0(\Omega) \iff u \cdot \mathbf{1}_\Omega \in \mathbb{X}_p(\Omega). \tag{1.4}$$

From now on, we shall tacitly identify a function $u \in W^{1,p}_0(\Omega)$ with its 'zero-extension' $\hat{u} := u \cdot \mathbf{1}_\Omega \in \mathbb{X}_p(\Omega)$.

By the Poincaré inequality and (1.4), we get that the quantity

$$\|u\|_{\mathbb{X}_p} := \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}, \qquad u \in \mathbb{X}_p(\Omega),$$

endows $\mathbb{X}_p(\Omega)$ with a structure of real Banach space, which is actually isometric to $W^{1,p}_0(\Omega)$. Moreover, $\mathbb{X}_p(\Omega)$ is separable and reflexive and $C^\infty_0(\Omega)$ is dense in $\mathbb{X}_p(\Omega)$.

The space $\mathbb{X}_p(\Omega)$ is the right one where solutions can be found, according to the following definition.

Definition 1.2. Let the above assumptions and notations be in force. We say that a function $u \in \mathbb{X}_p(\Omega)$ is a *weak solution* of (1.1) if

(1) for every function $\varphi \in \mathbb{X}_p(\Omega)$ one has

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi\rangle \, dx \\ &\qquad + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}} \, dx \, dy \\ &= \int_\Omega f(x,u)\varphi \, dx; \end{aligned} \tag{1.5}$$

(2) $u \geq 0$ a.e. in Ω and $|\{x \in \Omega : \, u(x) > 0\}| > 0$,

where $|\cdot|$ denotes the n-dimensional Lebesgue measure of a measurable set.

The main result. Taking into account all the definitions and notations introduced so far, we are able to state the main result of this note.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. Moreover, assume that f satisfies* (f1)–(f5)*.*

Then, if a solution $u \in \mathbb{X}_p(\Omega)$ of (1.1) *exists, we have*

$$\lambda_1(\mathcal{L}_{p,s} - a_\infty) > 0. \tag{1.6}$$

Following the notation in [4], the number $\lambda_1(\mathcal{L}_{p,s} - a_\infty)$ in (1.6) is the *smallest eigenvalue* of the operator $\mathcal{L}_{p,s} - a_\infty$ with nonlocal Dirichlet boundary conditions.

More explicitly, taking into account that $a_\infty \in L^\infty(\Omega)$, we have

$$\lambda_1(\mathcal{L}_{p,s} - a_\infty) := \inf_{\substack{u\in\mathbb{X}_p(\Omega)\\ \|u\|_{L^p(\Omega)}=1}} \Big\{\mathcal{Q}_{p,s}(u) - \int_\Omega a_\infty\,|u|^p\,dx\Big\}, \tag{1.7}$$

where we have introduced the shorthand notation

$$\mathcal{Q}_{p,s}(u) := \int_\Omega |\nabla u|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy, \qquad u \in \mathbb{X}_p(\Omega).$$

Remark 1.4. Since $a_\infty \in L^\infty(\Omega)$, we know from [4, Prop. 5.1] that the infimum in (1.7) is actually achieved, so that $\lambda_1(\mathcal{L}_{p,s} - a_\infty) \in \mathbb{R}$. Moreover, there exists a *unique non-negative* function $u_0 \in \mathbb{X}_p(\Omega)$ such that $\|u\|_{L^p(\Omega)} = 1$ and

$$\lambda_1(\mathcal{L}_{p,s} - a_\infty) = \mathcal{Q}_{p,s}(u_0) - \int_\Omega a_\infty\, u_0^p\,dx.$$

The relevance of Theorem 1.3 becomes clear if we combine this theorem with the main result obtained in [4], which is the following.

Theorem 1.5 ([4, Thm. 1.2])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. Assume that f satisfies* (f1)–(f5)*.*

Then, the following assertions hold.

(1) *If a solution to* (1.1) *exists, it is* unique, bounded and positive *in Ω.*
(2) *There exists a solution to* (1.1) *if*
$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0 < \lambda_1(\mathcal{L}_{p,s} - a_\infty).$$
Moreover, if a solution to (1.1) *exists, then*
$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0.$$
(3) *In the linear case $p = 2$, there exists a solution to* (1.1) if and only if *the following condition is satisfied*
$$\lambda_1(\mathcal{L}_{2,s} - a_0) < 0 < \lambda_1(\mathcal{L}_{2,s} - a_\infty).$$

Clearly, Theorem 1.5 provides a *complete characterization* for the unique solvability of (1.1) in the linear case $p = 2$. By combining Theorem 1.5-(1) with our Theorem 1.3, we are able to *close the gap*: indeed, we derive that the condition

$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0 < \lambda_1(\mathcal{L}_{p,s} - a_\infty),$$

is both *necessary and sufficient* for the (unique) solvability of (1.1). This gives an extension of the classical result by Brezis-Oswald [7] and in particular of its extension to the quasilinear case [11].

Remark 1.6. Some remarks concerning Theorem 1.5 are in order.

(1) The positivity property in assertion (1) is a consequence of the *Strong Maximum Principle* for the equation
$$\mathcal{L}_{p,s}u = f(x,u)$$
proved in [4, Thm. 3.1]. As pointed out in [4, Rem. 3.4], this result holds for *any nonlinearity* f satisfying the following properties:
  (a) $f(x,0) \geq 0$ for a.e. $x \in \Omega$;
  (b) $f(x,t) \geq -c_f t^{p-1}$ for a.e. $x \in \Omega$ and every $0 < t < 1$;
  (c) $|f(x,t)| \leq c_p(1 + t^{p-1})$ for a.e. $x \in \Omega$ and every $t \geq 1$.

In particular, the sign assumption (f5) is not necessary for the strong maximum principle.

(2) As for the case of a_∞, the number $\lambda_1(\mathcal{L}_{p,s} - a_0)$ appearing in Theorem 1.5 indicates the *smallest eigenvalue* of the operator $\mathcal{L}_{p,s} - a_0$ with nonlocal Dirichlet boundary conditions. However, since the map a_0 is non-negative but *possibly unbounded from above or infinite*, we define (see [4, 7])

$$\lambda_1(\mathcal{L}_{p,s} - a_0) := \inf_{\substack{u\in\mathbb{X}_p(\Omega)\\ \|u\|_{L^p(\Omega)}=1}} \Big\{\mathcal{Q}_{p,s}(u) - \int_{\{u\neq 0\}} a_0\,|u|^p\,dx\Big\}.$$

We point out that, in this case, we can have $\lambda_1(\mathcal{L}_{p,s} - a_0) = -\infty$.

2. Proof of Theorem 1.3

We now turn to prove Theorem 1.3.

Proof of Theorem 1.3. Let $u \in \mathbb{X}_p(\Omega)$ be a (weak) solution of problem (1.1), according to Definition 1.2. On account of [4, Thm. 4.1], we know that u is globally bounded in Ω; thus, setting $M := \|u\|_{L^\infty(\Omega)} + 1 > 1$, we can define

$$\overline{a} : \Omega \to \mathbb{R}, \qquad \overline{a}(x) := \frac{f(x,M)}{M^{p-1}}.$$

Owing to assumption (f4), it is readily seen that $\overline{a} \in L^\infty(\Omega)$; as a consequence, we know from [4, Prop. 5.1] that the *eigenvalue problem*

$$\begin{cases} \mathcal{L}_{p,s}v - \overline{a}(x)|v|^{p-2}v = \lambda|v|^{p-2}v & \text{in } \Omega, \\ v \not\equiv 0 & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.1}$$

admits a smallest eigenvalue, say $\mu \in \mathbb{R}$, whose associated eigenfunctions are globally bounded and do not change sign in Ω. We then choose an eigenfunction

$$\psi_0 \in \mathbb{X}_p(\Omega) \cap L^\infty(\Omega), \quad \psi_0 \gneqq 0$$

for (2.1) relative to μ, and we claim that the following inequality holds.

$$\int_\Omega \Big[\Big(\frac{f(x,u)}{u^{p-1}} - \overline{a}(x)\Big) - \mu\Big](u^p - \psi_0^p)\,dx \geq 0. \tag{2.2}$$

Taking this claim for granted for a moment, we can complete the proof of the theorem. In fact, since also the function $\psi_k = k\psi_0$ (with $k > 0$) is a non-negative and bounded eigenfunction for (2.1) relative to μ, from (2.2) we infer that

$$\int_\Omega \Big[\Big(\frac{f(x,u)}{u^{p-1}} - \overline{a}(x)\Big) - \mu\Big](u^p - k^p\psi_0^p)\,dx \geq 0 \qquad \forall\, k > 0. \tag{2.3}$$

On the other hand, by assumption (f4) and the very definition of $\overline{a}$, we have

$$\frac{f(x,u)}{u^{p-1}} - \overline{a}(x) > 0 \quad \text{a.e. in } \Omega.$$

Thus, by combining this last inequality with (2.3) (and taking into account that $\psi_0 > 0$ in Ω by the Strong Maximum Principle [4, Thm. 3.1]), we infer that

$$\mu > 0. \tag{2.4}$$

With (2.4) at hand, it now suffices to proceed as in [7]: using again (f4), we readily see that $\overline{a} > a_\infty$ a.e. in Ω; this, together with the definition of $\lambda_1(\mathcal{L}_{p,s} - a_\infty)$ and the variational characterization of μ (see [4, Eq. (5.2)]), implies that

$$\lambda_1(\mathcal{L}_{p,s} - a_\infty) \geq \inf_{\substack{u\in \mathbb{X}_p(\Omega) \\ \|u\|_{L^p(\Omega)}=1}} \left\{ \mathcal{Q}_{p,s}(u) - \int_\Omega \overline{a}(x)\,|u|^p\,dx \right\} = \mu > 0,$$

which is exactly what we wanted to prove. Hence, we are left to prove (2.2). To this end, we exploit an *approximation argument* already used in [4] and originally introduced in [6] to study *purely nonlocal problems* at critical growth.

First of all, we arbitrarily fix $\varepsilon > 0$ and we define

$$\varphi_{1,\varepsilon} := r_{1,\varepsilon} - u, \qquad \varphi_{2,\varepsilon} := r_{2,\varepsilon} - \psi_0,$$

where

$$r_{1,\varepsilon} := \frac{\psi_0^p}{(u+\varepsilon)^{p-1}}, \qquad r_{2,\varepsilon} := \frac{u^p}{(\psi_0+\varepsilon)^{p-1}}.$$

Taking into account that $u, \psi_0 \in \mathbb{X}_p(\Omega)$, $u, \psi_0 \geq 0$ a.e. in Ω and that u, ψ_0 are *globally bounded in* Ω, we readily infer that

$$\varphi_{i,\varepsilon} \in \mathbb{X}_p(\Omega) \text{ for every } \varepsilon > 0 \text{ and } i = 1, 2.$$

Hence, using $\varphi_{1,\varepsilon}$, $\varphi_{2,\varepsilon}$ as test functions in (1.5) for u and ψ_0, respectively, and adding the resulting integral identities, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega |\nabla \psi_0|^{p-2}\langle \nabla \psi_0, \nabla \varphi_{2,\varepsilon}\rangle\,dx \\
&\quad + \iint_{\mathbb{R}^{2n}} \frac{J_p(u(x)-u(y))(\varphi_{1,\varepsilon}(x)-\varphi_{1,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2n}} \frac{J_p(\psi_0(x)-\psi_0(y))(\varphi_{2,\varepsilon}(x)-\varphi_{2,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&= \int_\Omega \big(f(x,u)\varphi_{1,\varepsilon} + (\overline{a}(x)+\mu)\psi_0^{p-1}\varphi_{2,\varepsilon}\big)\,dx,
\end{aligned} \tag{2.5}$$

where we have used the notation $J_p(t) := |t|^{p-2}t$ (for $t \in \mathbb{R}$). Now, a direct computation based on the very definition of $\varphi_{i,\varepsilon}$, gives

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega |\nabla \psi_0|^{p-2}\langle \nabla \psi_0, \nabla \varphi_{2,\varepsilon}\rangle\,dx \\
&\quad = -\int_\Omega \mathcal{A}_p\Big(\nabla u, \frac{u}{\psi_0+\varepsilon}\nabla \psi_0\Big)\,dx - \int_\Omega \mathcal{A}_p\Big(\nabla \psi_0, \frac{\psi_0}{u+\varepsilon}\nabla u\Big)\,dx,
\end{aligned}$$

where we have set

$$\mathcal{A}_p(v,w) := |v|^p + (p-1)|w|^p - p|w|^{p-2}\langle v, w\rangle \qquad (\text{for } v, w \in \mathbb{R}^n).$$

As a consequence, since $\mathcal{A}_p(v,w) \geq 0$ for every $v,w \in \mathbb{R}^n$ (see, e.g., [4, Lem. 4.4]), identity (2.5) boils down to

$$
\begin{aligned}
&\int_\Omega \big(f(x,u)\varphi_{1,\varepsilon} + (\overline{a}(x)+\mu)\psi_0^{p-1}\varphi_{2,\varepsilon}\big)\,dx \\
&\quad\leq \iint_{\mathbb{R}^{2n}} \frac{J_p(u(x)-u(y))(r_{1,\varepsilon}(x)-r_{1,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad+ \iint_{\mathbb{R}^{2n}} \frac{J_p(\psi_0(x)-\psi_0(y))(r_{2,\varepsilon}(x)-r_{2,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad- \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy - \iint_{\mathbb{R}^{2n}} \frac{|\psi_0(x)-\psi_0(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad=: \mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2,
\end{aligned}
\tag{2.6}
$$

We now aim at passing to the limit as $\varepsilon \to 0^+$ in the above (2.6).

To this end, we first remind the following discrete Picone inequality: *for every fixed $p \in (1,+\infty)$ and every $a,b,c,d \in [0,+\infty)$ with $a,b>0$, one has*

$$J_p(a-b)\left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}}\right) \leq |c-d|^p$$

(for a proof see, e.g., [6, Prop. 2.2]). By using this inequality, we have

(i) $J_p(u(x)-u(y))(r_{1,\varepsilon}(x)-r_{1,\varepsilon}(y)) \leq |\psi_0(x)-\psi_0(y)|^p$;

(ii) $J_p((\psi_0(x)-\psi_0(y))(r_{2,\varepsilon}(x)-r_{2,\varepsilon}(y)) \leq |u(x)-u(y)|^p$.

Hence, we can apply the Fatou lemma for the integrals $\mathrm{I}_{1,\varepsilon}, \mathrm{I}_{2,\varepsilon}$, obtaining

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \big(\mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2\big) \\
&\quad\leq \iint_{\mathbb{R}^{2n}} \frac{J_p(u(x)-u(y))}{|x-y|^{n+ps}}\left(\frac{\psi_0^p}{u^{p-1}}(x) - \frac{\psi_0^p}{u^{p-1}}(y)\right)dx\,dy \\
&\quad+ \iint_{\mathbb{R}^{2n}} \frac{J_p(\psi_0(x)-\psi_0(y))}{|x-y|^{n+ps}}\left(\frac{u^p}{\psi_0^{p-1}}(x) - \frac{u^p}{\psi_0^{p-1}}(y)\right)dx\,dy \\
&\quad- \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy - \iint_{\mathbb{R}^{2n}} \frac{|\psi_0(x)-\psi_0(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad=: \kappa(u_1,u_2,p),
\end{aligned}
\tag{2.7}
$$

where $\kappa(u_1,u_2,p) \in [-\infty,0]$ again by the discrete Picone inequality (here, to give a meaning to the integrals when x or y are not in Ω, we have tacitly set $0/0=0$).

We now turn our attention to the left hand side of (2.6). Taking into account the very definition of $\varphi_{i,\varepsilon}$, we first write

$$
\begin{aligned}
&\int_\Omega \big(f(x,u)\varphi_{1,\varepsilon} + (\overline{a}(x)+\mu)\psi_0^{p-1}\varphi_{2,\varepsilon}\big)\,dx \\
&= \int_\Omega f(x,u)\,r_{1,\varepsilon}\,dx + \int_\Omega (\overline{a}(x)+\mu)\psi_0^{p-1}\,r_{2,\varepsilon}\,dx \\
&- \int_\Omega f(x,u)u\,dx - \int_\Omega (\overline{a}(x)+\mu)\psi_0^p\,dx \\
&=: \mathrm{A}_{1,\varepsilon} + \mathrm{A}_{2,\varepsilon} - \mathrm{B}_1 - \mathrm{B}_2.
\end{aligned}
$$

Moreover, recalling the value $\rho_f > 0$ in (1.2), we further split $\mathrm{A}_{1,\varepsilon}$ as

$$\mathrm{A}_{1,\varepsilon} = \int_{\{u<\rho_f\}} f(x,u)\, r_{1,\varepsilon}\, dx + \int_{\{u\geq\rho_f\}} f(x,u)\, r_{1,\varepsilon}\, dx =: \mathrm{A}'_{1,\varepsilon} + \mathrm{A}''_{1,\varepsilon}.$$

Now, by assumption (f3), for every $\varepsilon > 0$ we have

$$|f(x,u)\, r_{1,\varepsilon}| \cdot \mathbf{1}_{\{u\geq\rho_f\}} \leq c_p\big(1+\rho_f^{1-p}\big)\, \psi_0^p \equiv c_{p,f}\, \psi_0^p;$$

on the other hand, since $\overline{a} \in L^\infty(\Omega)$, we have

$$\big|(\overline{a}(x)+\mu)\psi_0^{p-1}\, r_{2,\varepsilon}\big| \leq \big|\|a\|_{L^\infty(\Omega)} + \mu\big| u^p \equiv c\, u^p.$$

Thus, we can then apply the Dominated Convergence theorem, obtaining

$$\begin{aligned} &\mathrm{A}''_1 := \lim_{\varepsilon\to 0^+} \mathrm{A}''_{1,\varepsilon} = \int_{\{u\geq\rho_f\}} \frac{f(x,u)}{u^{p-1}}\, \psi_0^p\, dx \in \mathbb{R} \qquad \text{and} \\ &\mathrm{A}_2 := \lim_{\varepsilon\to 0^+} \mathrm{A}_{2,\varepsilon} = \int_\Omega (\overline{a}(x)+\mu)\, u^p\, dx \in \mathbb{R}. \end{aligned} \tag{2.8}$$

Hence, it remains to study the behavior of $A'_{1,\varepsilon}$ when $\varepsilon \to 0^+$.

First of all, using (1.2) and the fact that $r_{1,\varepsilon}$ is nonnegative and monotone increasing with respect to ε, we can apply the Beppo Levi theorem, obtaining

$$\mathrm{A}'_1 := \lim_{\varepsilon\to 0^+} \mathrm{A}'_{1,\varepsilon} = \int_{\{u_1<\rho_f\}} \frac{f(x,u)}{u^{p-1}}\, \psi_0^p\, dx \in [0,+\infty]. \tag{2.9}$$

On the other hand, going back to estimate (2.6) and taking into account the very definitions of the integrals $\mathrm{A}'_{1,\varepsilon}, \mathrm{A}''_{1,\varepsilon}, \mathrm{A}_{2,\varepsilon}, \mathrm{B}_i$, we get

$$0 \leq \mathrm{A}'_{1,\varepsilon} \leq \big(\mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2\big) + \mathrm{B}_1 + \mathrm{B}_2 - \mathrm{A}''_{1,\varepsilon} - \mathrm{A}_{2,\varepsilon}.$$

Then, by letting $\varepsilon \to 0^+$ with the aid of (2.7)–(2.8), we obtain

$$0 \leq \mathrm{A}'_1 \leq \kappa(u_1,u_2,p) + \mathrm{B}_1 + \mathrm{B}_2 - \mathrm{A}''_1 - \mathrm{A}_2,$$

from which we derive at once that

$$\kappa(u_1,u_2,p) > -\infty \qquad \text{and} \qquad \mathrm{A}'_1 < +\infty. \tag{2.10}$$

Gathering (2.8)–(2.9), and taking into account (2.10), we finally have

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\bigg(\int_\Omega \big(f(x,u)\varphi_{1,\varepsilon} + (\overline{a}(x)+\mu)\psi_0^{p-1}\varphi_{2,\varepsilon}\big)\, dx\bigg) \\ &\quad = \lim_{\varepsilon\to 0^+} \big(\mathrm{A}'_{1,\varepsilon} + \mathrm{A}''_{1,\varepsilon} + \mathrm{A}_{2,\varepsilon} - \mathrm{B}_1 - \mathrm{B}_2\big) \\ &\quad = \int_\Omega \Big(\frac{f(x,u)}{u^{p-1}}\psi_0^p + (\overline{a}(x)+\mu)u^p - f(x,u)u - (\overline{a}(x)+\mu)\psi_0^p\Big)\, dx \\ &\quad = -\int_\Omega \Big(\frac{f(x,u)}{u^{p-1}} - (\overline{a}(x)+\mu)\Big)(u^p - \psi_0^p)\, dx. \end{aligned} \tag{2.11}$$

With (2.7) and (2.11) at hand, we can easily conclude the proof of the theorem. Indeed, using these cited identities we can let $\varepsilon \to 0^+$ in (2.6), obtaining

$$-\int_\Omega \Big(\frac{f(x,u)}{u^{p-1}} - (\overline{a}(x)+\mu)\Big)(u^p - \psi_0^p)\, dx \leq \kappa(u_1,u_2,p) \leq 0.$$

This is exactly the claimed (2.2), and the proof is now complete. □

Remark 2.1. By carefully scrutinizing the proof of Theorem 1.3, it is clear that the regularity of $\partial\Omega$ plays an effective role only in (1.4). Following [3], it would be also possible to look for solutions in the function space

$$\mathcal{X}_0^{1,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\mathbb{R}^n)}} \subseteq W^{1,p}(\mathbb{R}^n),$$

On the other hand, since our techniques do not rely on the regularity up the boundary for $\mathcal{L}_{p,s}$ nor on an Hopf-type lemma (which are not available, as far as we know), they are also independent of the regularity of $\partial\Omega$; hence, Theorems 1.3-1.5 hold for *any bounded open set*, by replacing $\mathbb{X}_p(\Omega)$ with the space $\mathcal{X}_0^{1,p}(\Omega)$. In this perspective, our assumptions (f3)-(f5) can be viewed as the *price to pay* for considering general open sets (differently to case considered in [7]).

We also point out that a related approach could be used for the case of p-sublinear nonlocal problems with Robin nonlocal boundary conditions as considered in [15], once proved that solutions are bounded.

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