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An Elementary Approach to Euler's Reflection Formula and Its Role in the Infinite Product of the Sine Function and the Basel Problem

Antonio E. Bargellini † and **Daniele Ritelli** *,†

Department of Statistical Sciences, University of Bologna, 40126 Bologna, Italy; antonio.bargellini2@unibo.it
* Correspondence: daniele.ritelli@unibo.it
† These authors contributed equally to this work.

Abstract

We present a proof, using elementary methods, of the Euler reflection formula for the Gamma function, based on an integral computed by Laplace and on the Euler–Gauss infinite product representation of Gamma. This way, we reverse the classical path, and, using the reflection formula as a starting point, we obtain the representation of the sine as an infinite product and that of the cotangent in partial fractions, which, as is known, allows the explicit calculation of the zeta function with an even argument: all this without resorting to complex analysis or the Herglotz trick. We can present a teaching proposal that illustrates the complete proof of this fundamental formula using undergraduate-level mathematical analysis tools, such as the derivation of parametric integrals, the second Mean Value Theorem for Integrals (Bonnet formula), and the convergence criterion for Dirichlet oscillatory integrals.

Keywords: gamma function; Euler's reflection formula; Laplace integral; second Mean Value Theorem for Integrals; Dirichlet convergence criterion; oscillatory integrals; Basel problem

MSC: 33B15; 34A30; 40C99; 40A20

1. Introduction

The famous Euler reflection formula for Gamma states that, if $0 < \mathrm{Re}\,(x) < 1$, then

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{1}$$

where, for $\mathrm{Re}\,(x) > 0$ Gamma function is defined, following Legendre, [1] (p. 277) as

$$\Gamma(x) = \int_0^\infty t^{x-1}\,e^{-t}\,\mathrm{d}t.$$

As reported in [2], Euler obtained (1) in Section 43 of [3]. As noted by A. Ayckok in his commented translation of [3], available on the Euler Archive site [4], the proof relies on two earlier results obtained by Euler in [5,6]. In [5], Euler derived the integral representation of Gamma by interpolation, and in [6] he computed, as stated in the translation of [3], the infinite product for the sine, which we report here in modern notation:

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$$\prod_{n=1}^{\infty} \frac{n^2}{n^2-\lambda^2} = \frac{\lambda\,\pi}{\sin(\lambda\,\pi)}.$$

The usual modern approach to proving (1) is based on the Beta function, using the Euler identity, where $\mathrm{Re}\,(x)$, $\mathrm{Re}\,(y) > 0$

$$\mathrm{B}(x,y) = \int_0^{\infty} \frac{s^{x-1}}{(1+s)^{x+y}}\,\mathrm{d}s = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{2}$$

Then, taking $y = 1-x$ (2) yields

$$\Gamma(x)\,\Gamma(1-x) = \int_0^{\infty} \frac{s^{x-1}}{1+s}\,\mathrm{d}s. \tag{3}$$

The problem is solved by calculating the integral on the right-hand side of (3). Despite the ease of its formulation, the calculation of this integral requires the residue theorem to arrive, after the change of variable $x = \ln t$, at the equality, where $0 < a < 1$:

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}\,\mathrm{d}x = \frac{\pi}{\sin(a\,\pi)}. \tag{4}$$

See, for instance [7] (p. 133), [8] (p. 22) and [9] (p. 20). Alternatively, one can, following [10] (p. 254) and [8] (p. 73), use series integration, based on the following partial-fraction representation of the sine:

$$\frac{\pi}{\sin(\pi\,z)} = \frac{1}{z} - \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2-n^2}. \tag{5}$$

In turn, Formula (5) is obtained after having determined the Fourier series:

$$\cos(z\,x) = \frac{2\,z}{\pi}\,\sin(\pi\,z)\left(\frac{1}{2\,z^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\,\cos(n\,x)}{z^2-n^2}\right), \tag{6}$$

evaluated for $x = 0$. Then, (5) is used after breaking the integral on the right-hand side of (2) into

$$\int_0^1 \frac{s^{x-1}}{1+s}\,\mathrm{d}s + \int_0^1 \frac{s^{-x}}{1+s}\,\mathrm{d}s,$$

and eventually integrating term by term.

The teaching problem that arises when one wishes to treat the derivation in a self-contained manner is significant when students lack the advanced background in complex analysis needed to compute the integral in (4) or to obtain the Fourier series for (5). This situation arises when students at the beginning of their careers encounter Eulerian functions.

Our contribution solves this problem by basing the proof on a couple of definite integrals. The first dates back to Laplace [11] (Livre premiere, p. 99):

$$\int_0^{\infty} \frac{\cos(z\,t)}{1+t^2}\,\mathrm{d}t = \frac{\pi}{2}\,e^{-z}, \tag{7}$$

where it is assumed that $z > 0$. The second integral is the Mellin transform of the cosine: recall that the Mellin transform of a locally integrable function $f : [0, \infty) \to \mathbb{C}$ is defined as, for a complex number s

$$\mathcal{M}[f](s) = \int_0^{\infty} x^{s-1}\,f(x)\,\mathrm{d}x\,, \tag{8}$$

and in our case we consider Formula (21) below. We will examine the integral (8) when $f(x) = \cos(b\,x)$ with $b > 0$, and for our purposes we will take $0 < s < 1$. Combining identities (7) and (21), we will demonstrate, in Theorem 6, the identity (28) from which the reflection Formula (1) follows.

In our exposition, we will assume that Fubini's theorem on nested integrals is known, citing the historical work [12], where the proof was presented, and the monograph [13], which treats the problem for Riemann integrals on hyperrectangles and normal domains. We limit citations to this theorem here since it is reported in numerous treatises on Mathematical Analysis and Measure Theory.

The original proof of (1) is not the sole result presented in the paper. We will also demonstrate how (1), in conjunction with the Euler–Gauss representation of the Gamma function, which was initially introduced by Euler in [14], and subsequently presented in the contemporary notation by C.F. Gauss in [15],

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\cdots(x+n)}, \quad x > 0, \tag{9}$$

provides, in a rigorous way, the representation of the sine in terms of an infinite product:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{10}$$

Formula (10) provides one of the possible keys to solving the Basel problem; for completeness, we will end the article by presenting this well-known approach briefly.

The significance of our contribution lies in two key aspects: first, the reverse logical sequence is rigorously defined; second, the proof of the reflection formula, the foundational element of our reasoning, is achieved using elementary tools. Notably, among papers employing elementary methods, D. Salwinski's paper [16], which concerns the formula for the infinite product of the sine function, stands out. While the cost of employing elementary methods may be the complexity of the demonstration, it is worth noting that this approach provides a rigorous foundation for our reasoning.

Finally, to emphasize the universality and significance of the reflection formula, we observe that its demonstration, presented in [17] using quantum mechanics, yields (1) as an approximate quantum result as a quantum system approaches the classical model.

The article is structured to provide a self-contained learning path. In the next two sections, we will present, respectively, the Mellin transform of the cosine function. For this purpose, we will recall Bonnet's theorem and the concept of the oscillatory integral. Additionally, we will compute the Laplace integral using a second-order differential equation with constant coefficients. In Section 4, we will prove the reflection formula. In Section 5, we will utilize the reflection formula to express the sine function as an infinite product. From this expression, we will derive the partial-fraction representation of the cotangent function, which ultimately leads to the solution of the Basel problem, Section 6. This article aims to provide undergraduate-level insights into the concepts presented in chapter 2 of [8].

2. Mellin Transform of Cosine

We present the explicit calculation of the Mellin transform of the cosine function in terms of the Gamma function. This procedure is reported in the encyclopedic monograph [18] on the Eulerian function, where the contributions by Legendre [1], Poisson [19], Cauchy [20], and Boncompagni [21] are highlighted. We believe it is appropriate to present the detailed proof of the definite integration Formula (21) below, based on elementary methods, in which the knowledge of complex analysis necessary for its understanding is limited to the exponential, in accordance with the aim of our contribution, which

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is to provide a proof of the reflection formula based only on undergraduate-level notions. We need three preliminary results, for the sake of completeness, in the spirit of presenting a logically consistent path. Therefore, we briefly present the proofs: the "Weighted Mean Value Theorem for Integrals", as seen in for instance [22] (p. 154), the Bonnet, "Second Mean Value Theorem for Integrals", as stated in Theorem 2, in [23], and more recently reported in [22] (p. 219), and Dirichlet's convergence criterion for improper integrals, Theorem 3 below, whose proof is outlined in [22,24]. We present here the detailed proof from [25]. Finally, we wish to highlight an interesting and recent publication [26], which is dedicated to the Mellin transforms of trigonometric functions.

To prove Theorem 2, we use the following lemma.

Lemma 1. *Given $f, g : [a,b] \to \mathbb{R}$ assume that*

(i) f *is continuous in* $[a,b]$;

(ii) g *is decreasing,* $\mathcal{C}^1$ *and such that* $g(b) = 0$.

Then there exists $\xi \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x\,. \tag{11}$$

Proof. For any $x \in [a,b]$ define

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Integrating by parts, we get

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \Big[F(x)g(x)\Big]_{x=a}^{x=b} - \int_a^b F(x)g'(x)\,\mathrm{d}x = -\int_a^b F(x)g'(x)\,\mathrm{d}x.$$

The last step follows from the fact that, by construction, $F(a) = 0$ and by hypothesis $g(b) = 0$. Now, by the hypothesis of the decrease in g, we have that $g'(x) \le 0$ for every $x \in [a,b]$: therefore the hypotheses of the generalized mean theorem are satisfied, so we can conclude that there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = -\int_a^b F(x)g'(x)\mathrm{d}x = -F(\xi)\int_a^b g'(x)\mathrm{d}x = -F(\xi)(g(b)-g(a)) = F(\xi)g(a),$$

showing (11). □

Theorem 1. *Assume $f : [a,\, b] \to \mathbb{R}$ continuous and $g : [a,\, b] \to \mathbb{R}$ Riemann integrable, and it does not change sign. Then there exists $\xi \in [a,b]$ such that*

$$\int_a^b f(x)\,g(x)\,\mathrm{d}x = f(\xi)\int_a^b g(x)\,\mathrm{d}x. \tag{12}$$

Proof. Without loss of generality we can assume $g(x) \ge 0$ in $[a,\, b]$. If m and M are the maximum and minimum of f, we have

$$m\int_a^b g(x)\,\mathrm{d}x \le \int_a^b f(x)g(x)\,\mathrm{d}x \le M\int_a^b g(x)\,\mathrm{d}x.$$

Notice that if

$$\int_a^b g(x)\mathrm{d}x = 0,$$

then (12) follows trivially, and hence we can assume

$$\int_a^b g(x)\,\mathrm{d}x > 0.$$

In this case, we find

$$m \leq \frac{\displaystyle\int_a^b f(x)g(x)\mathrm{d}x}{\displaystyle\int_a^b g(x)\mathrm{d}x} \leq M.$$

The thesis is a consequence at this point of the Intermediate Value Theorem (Bolzano) for continuous functions. □

Now we can prove Theorem 2.

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$ such that the following hold:*

(i) f is continuous in $[a, b]$;
(ii) g is monotonic and $\mathcal{C}^1$ in $[a, b]$.

Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x\,. \tag{13}$$

Proof. First, let us note that if the thesis (13) holds for an increasing g, then changing g to $-g$ ensures it also holds for a decreasing function, and vice versa; therefore, we do not lose generality by restricting to the case of decreasing g. Then, crucially, we can limit ourselves to proving (13) under the hypothesis that $g(b) = 0$. In fact, if $g(b) \neq 0$, we set $g_1(x) = g(x) - g(b)$. If we suppose that g_1 satisfies (13), then, observing that the left-hand side becomes

$$\int_a^b f(x)g_1(x)\mathrm{d}x = \int_a^b f(x)(g(x) - g(b))\mathrm{d}x = \int_a^b f(x)g(x)\mathrm{d}x - g(b)\int_a^b f(x)\mathrm{d}x. \tag{14}$$

The right side is

$$(g(a) - g(b))\int_a^\xi f(x)\,\mathrm{d}x + \big(g(b) - g(b)\big)\int_\xi^b f(x)\,\mathrm{d}x = \big(g(a) - g(b)\big)\int_a^\xi f(x)\,\mathrm{d}x. \tag{15}$$

If the thesis (13) is satisfied, expressions (29) and (30) must be equal, and then

$$\int_a^b f(x)g(x)\,\mathrm{d}x - g(b)\int_a^b f(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x - g(b)\int_a^\xi f(x)\,\mathrm{d}x.$$

The latter, due to integral additivity, can be written as

$$\int_a^b f(x)g(x)\,\mathrm{d}x - g(b)\int_a^\xi f(x)\,\mathrm{d}x - g(b)\int_\xi^b f(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x - g(b)\int_a^\xi f(x)\,\mathrm{d}x,$$

which leads to (13). Therefore, it is sufficient to prove the theorem under the hypothesis $g(b) = 0$, and this is what is asserted in Lemma 1. □

Theorem 2 is necessary for the proof of the Dirichlet convergence criterion for oscillating generalized integrals. For simplicity and because our application falls under these hypotheses, we state and prove the result assuming the functions that form the integrand are continuous and differentiable.

Theorem 3. *Assume that the function f is continuous on $[a, +\infty)$ and such that*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \tag{16}$$

is bounded on $[a, +\infty)$ and the function g is differentiable on $[a, +\infty)$, decreasing, i.e., $g'(x) \leq 0$ for all $x \geq a$, and satisfies

$$\lim_{x \to +\infty} g(x) = 0. \tag{17}$$

Then the improper integral

$$\int_a^{+\infty} f(x)\,g(x)\,\mathrm{d}x, \tag{18}$$

converges.

Proof. Let $\varepsilon > 0$. For (16), there exists $M > 0$ such that for any $x \geq a$

$$|F(x)| \leq M, \tag{19}$$

and for (17) there exists $N \geq a$ such that for all $x \geq N$,

$$|g(x)| < \frac{\varepsilon}{4M}. \tag{20}$$

Fix $x_2 > x_1 > N$. By Theorem 2, applied to f and the monotone function g on $[x_1, x_2]$, there exists $\xi \in [x_1, x_2]$ such that

$$\int_{x_1}^{x_2} f(x)\,g(x)\,\mathrm{d}x = g(x_1) \int_{x_1}^{\xi} f(x)\,\mathrm{d}x + g(x_2) \int_{\xi}^{x_2} f(x)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned} \left| \int_{x_1}^{x_2} f(x)\,g(x)\,\mathrm{d}x \right| &\leq |g(x_1)| \left| \int_{x_1}^{\xi} f(x)\,\mathrm{d}x \right| + |g(x_2)| \left| \int_{\xi}^{x_2} f(x)\,\mathrm{d}x \right| \\ &= |g(x_1)|\,|F(\xi) - F(x_1)| + |g(x_2)|\,|F(x_2) - F(\xi)|. \end{aligned}$$

Recalling (19), we have

$$\begin{aligned} |F(\xi) - F(x_1)| &\leq |F(\xi)| + |F(x_1)| \leq 2M, \\ |F(x_2) - F(\xi)| &\leq |F(x_2)| + |F(\xi)| \leq 2M, \end{aligned}$$

so

$$\left| \int_{x_1}^{x_2} f(x)\,g(x)\,\mathrm{d}x \right| \leq 2M\big(|g(x_1)| + |g(x_2)|\big) < 2M\Big(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\Big) = \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there exists N such that for all $x_2 > x_1 > N$,

$$\left| \int_{x_1}^{x_2} f(x)\,g(x)\,\mathrm{d}x \right| < \varepsilon.$$

By the Cauchy criterion for improper integrals, (18) converges. □

Remark 1. *The most popular application of Theorem 3 is about the convergence of the Dirichlet integral*

$$\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x\,.$$

The Dirichlet convergence criterion allows us to proceed rigorously to the calculation of the Mellin transform of the cosine, which in turn is used in the proof of the reflection formula.

Theorem 4 and the following Remark below are a classical result due to Euler [27] and are present also in [18]. In keeping with the didactic spirit of the present paper, which aims to provide a unified exposition, we also include its proof.

Theorem 4. *If* $0 < s < 1$, $b > 0$ *then*

$$\int_0^\infty z^{s-1}\cos(b\,z)\,\mathrm{d}z = \frac{\Gamma(s)}{b^s}\cos\left(\frac{\pi}{2}s\right). \tag{21}$$

Proof. We start studying the convergence of the integral (21). Near $z = 0$, since $\cos(bz) \sim 1$, the integrand behaves like z^{s-1}. Hence, integrability at the origin requires, as previously assumed $s > 0$. As $z \to \infty$ the function $\cos(bz)$ oscillates, and the integral is not absolutely convergent. However, by Dirichlet's test, Theorem 3, convergence holds if $z^{s-1} \to 0$ monotonically, which requires $s - 1 < 0$ that is $s < 1$. Thus the natural strip of convergence is, exactly as supposed,

$$0 < s < 1.$$

Now we introduce a convergence factor $e^{-\varepsilon z}$ with $\varepsilon > 0$:

$$I_\varepsilon(s) = \int_0^\infty z^{s-1}e^{-\varepsilon z}\cos(b\,z)\,\mathrm{d}z.$$

For every $\varepsilon > 0$ and $s > 0$, this integral is absolutely convergent. Using

$$\cos(b\,z) = \mathrm{Re}\left(e^{i\,b\,z}\right),$$

we write

$$I_\varepsilon(s) = \mathrm{Re}\left(\int_0^\infty z^{s-1}e^{-(\varepsilon-i\,b)z}\,\mathrm{d}z\right).$$

Now, recall the classical Gamma–Laplace formula: for $s > 0$ and $\mathrm{Re}\,(a) > 0$,

$$\int_0^\infty z^{s-1}e^{-a\,z}\,dz = \frac{\Gamma(s)}{a^s}\,.$$

Here we set

$$a = \varepsilon - i\,b, \qquad \mathrm{Re}\,(a) = \varepsilon > 0,$$

so that

$$I_\varepsilon(s) = \mathrm{Re}\left(\frac{\Gamma(s)}{(\varepsilon - i\,b)^s}\right).$$

Write $\varepsilon - i\,b$ in polar form:

$$\varepsilon - ib = r_\varepsilon e^{-i\theta_\varepsilon}, \qquad r_\varepsilon = \sqrt{\varepsilon^2 + b^2}, \quad \theta_\varepsilon = \arctan\left(\frac{b}{\varepsilon}\right).$$

As $\varepsilon \to 0^+$, we have

$$r_\varepsilon \to b, \qquad \theta_\varepsilon \to \frac{\pi}{2}.$$

Using the principal branch of the complex power,

$$(\varepsilon - ib)^{-s} = r_\varepsilon^{-s}e^{is\theta_\varepsilon} \longrightarrow b^{-s}e^{i\pi s/2}.$$

Therefore

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(s) = \mathrm{Re}\Big(\Gamma(s) b^{-s} e^{i\pi s/2}\Big) = \frac{\Gamma(s)}{b^s}\cos\Big(\frac{\pi s}{2}\Big).$$

For $0 < s < 1$, one checks that

$$\lim_{\varepsilon\to 0^+}\int_0^\infty z^{s-1} e^{-\varepsilon z}\cos(b\,z)\,\mathrm{d}z = \int_0^\infty z^{s-1}\cos(b\,z)\,\mathrm{d}z\,.$$

Formula (21) follows by combining dominated convergence on finite intervals with Dirichlet-type estimates on the tail $[R,\infty)$. □

Remark 2. *By the same method, one may derive the companion identity involving the sine function. Indeed, using*

$$\sin(bz) = \mathrm{Im}(e^{i\,b\,z}),$$

repeating the regularization argument with the exponential factor $e^{-\varepsilon z}$, one finds, for $0 < s < 1$,

$$\int_0^\infty z^{s-1}\sin(b\,z)\,\mathrm{d}z = \mathrm{Im}\left(\frac{\Gamma(s)}{(-i\,b)^s}\right) = \frac{\Gamma(s)}{b^s}\sin\Big(\frac{\pi\,s}{2}\Big).$$

3. Laplace Integral

Although the Laplace integral (7) can be computed using the residue theorem, as demonstrated in [28,29] (pp. 107–110), and [30] (pp. 255–256), it can also be computed, as indicated in these references, by differentiation under the integral sign, effectively circumventing complex analysis. We provide a concise summary of the proof: by denoting $L(z)$ as the left-hand side of (7), it can be established that $L(z)$ satisfies the linear second-order differential equation $L''(z) = L(z)$, thereby yielding the existence of c_1, $c_2 \in \mathbb{R}$ such that $L(z) = c_1\,e^z + c_2\,e^{-z}$. Subsequently, we delve into the intricate integration process that culminates in (7).

Theorem 5. *Assume $z \in \mathbb{R}$ positive, then identity (7) holds.*

Proof. Define

$$L(z) = \int_0^\infty \frac{\cos(z\,t)}{1+t^2}\,\mathrm{d}t\,. \tag{22}$$

Integrating (22) by parts, we obtain

$$z\,L(z) = 2\int_0^\infty \frac{t\,\sin(z\,t)}{(1+t^2)^2}\,\mathrm{d}t. \tag{23}$$

Indeed, let

$$u = \frac{1}{1+t^2}, \qquad dv = \cos(z\,t)\,dt,$$

then

$$\mathrm{d}u = -\frac{2t}{(1+t^2)^2}\,\mathrm{d}t, \qquad v = \frac{\sin(z\,t)}{z}.$$

Therefore, integration by parts gives

$$L(z) = \left[\frac{\sin(z\,t)}{z(1+t^2)}\right]_0^\infty + \frac{2}{z}\int_0^\infty \frac{t\sin(z\,t)}{(1+t^2)^2}\,\mathrm{d}t.$$

Since

$$\left[\frac{\sin(z\,t)}{z(1+t^2)}\right]_0^\infty = 0,$$

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it follows that

$$L(z) = \frac{2}{z} \int_0^\infty \frac{t \sin(z\,t)}{(1+t^2)^2}\,\mathrm{d}t.$$

Then, we differentiate (23) with respect to z, which is justified by the dominated convergence of the integrand:

$$L(z) + z\,L'(z) = 2 \int_0^\infty \frac{t^2 \cos(z\,t)}{(1+t^2)^2}\,\mathrm{d}t\,. \tag{24}$$

Now, we use the partial decomposition:

$$\frac{t^2}{(1+t^2)^2} = \frac{1}{1+t^2} - \frac{1}{(1+t^2)^2},$$

and plugging it into (24), we get

$$L(z) + z\,L'(z) = 2\,L(z) - 2 \int_0^\infty \frac{\cos(z\,t)}{(1+t^2)^2}\,\mathrm{d}t\,,$$

that is

$$z\,L'(z) - L(z) = -2 \int_0^\infty \frac{\cos(z\,t)}{(1+t^2)^2}\,\mathrm{d}t\,. \tag{25}$$

Differentiating (25), we obtain

$$z\,L''(z) = 2 \int_0^\infty \frac{t\,\sin(z\,t)}{(1+t^2)^2}\,\mathrm{d}t.$$

So that is recalling (23), we obtain the differential equation $L''(z) = L(z)$, hence

$$L(z) = c_1\,e^z + c_2\,e^{-z}.$$

To compute the integration constants c_1 and c_2, we start from (7), evaluated at $z = 0$ yielding immediately

$$L(0) = \int_0^\infty \frac{\mathrm{d}t}{1+t^2} = \frac{\pi}{2}.$$

Thus $c_1 + c_2 = \pi/2$. Moreover, from (23), we see that

$$L(z) = \frac{2}{z} \int_0^\infty \frac{t\,\sin(z\,t)}{(1+t^2)^2}\,\mathrm{d}t.$$

Hence

$$\lim_{z\to+\infty} L(z) = 0\,.$$

So we infer that $c_1 = 0$, and this finally implies (7). $\square$

The role of the integral (7) in the proof of (1) is to provide the following integral representation of the exponential:

Corollary 1.

$$e^{-z} = \frac{1}{\pi} \int_0^\infty \frac{\cos(z\sqrt{u})}{\sqrt{u}\,(1+u)}\,\mathrm{d}u. \tag{26}$$

Proof. Formula (26) follows immediately using the change of variable $t^2 = u$ in (7). Indeed, by (7)

$$e^{-z} = \frac{2}{\pi}\int_0^\infty \frac{\cos(z\,t)}{1+t^2}\,\mathrm{d}t \overset{t^2=u}{=} \frac{2}{\pi}\int_0^\infty \frac{\cos(z\sqrt{u})}{2\sqrt{u}\,(1+u)}\,\mathrm{d}u = \frac{1}{\pi}\int_0^\infty \frac{\cos(z\sqrt{u})}{\sqrt{u}\,(1+u)}\,\mathrm{d}u. \tag{27}$$

□

4. The Proof of the Reflection Formula

Having all the necessary tools at our disposal, we can finally demonstrate our main result. We begin with a preliminary identity that is the key to proving (1).

We remark that the convergence of the integral would require $0 < \mathrm{Re}\,(s) < 1$. However, in order to apply formula (21), we further restrict the parameter and assume $0 < s < 1$.

Theorem 6. *Let* $0 < s < 1$, *then*

$$\frac{\pi}{\cos\left(\frac{\pi}{2}\,s\right)} = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right). \tag{28}$$

Proof. We begin with the basic definition of Gamma, where we assume $0 < s < 1$: at the end, we list all the steps, which we tag to illustrate them step by step at the end of the procedure .

$$\begin{aligned}
1 &= \frac{1}{\Gamma(s)}\int_0^\infty z^{s-1}e^{-z}\,\mathrm{d}z \\
&= \frac{1}{\pi\,\Gamma(s)}\int_0^\infty z^{s-1}\left(\int_0^\infty \frac{\cos\left(z\sqrt{u}\right)}{\sqrt{u}\,(1+u)}\,\mathrm{d}u\right)\mathrm{d}z && (29)\\
&= \frac{1}{\pi\,\Gamma(s)}\int_0^\infty \frac{1}{\sqrt{u}\,(1+u)}\left(\int_0^\infty z^{s-1}\cos\left(z\sqrt{u}\right)\mathrm{d}z\right)\mathrm{d}u && (30)\\
&= \frac{1}{\pi\,\Gamma(s)}\int_0^\infty \frac{1}{\sqrt{u}\,(1+u)}\,\frac{\Gamma(s)}{u^{s/2}}\cos\left(\frac{\pi}{2}\,s\right)\mathrm{d}u && (31)\\
&= \frac{\cos\left(\frac{\pi}{2}\,s\right)}{\pi}\int_0^\infty \frac{u^{-\frac{s+1}{2}}}{1+u}\,\mathrm{d}u && (32)\\
&= \frac{\cos\left(\frac{\pi}{2}\,s\right)}{\pi}\,\mathrm{B}\left(\frac{1-s}{2},\frac{1+s}{2}\right) && (33)\\
&= \frac{\cos\left(\frac{\pi}{2}\,s\right)}{\pi}\,\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right). && (34)
\end{aligned}$$

The proof steps are explained below:

(29) follows from (26);
(30) follows from Fubini's theorem;
(31) follows from (21);
(32) follows from standard simplifications;
(33) follows from the Beta representation (2);
(34) follows from the representation of Beta in terms of Gamma.

Observe that

$$0 < s < 1 \implies 0 < \frac{1 \pm s}{2} < 1$$

and in conclusion, we have demonstrated (28). □

The reflection formula at this point is obtained through the following simple observation.

Remark 3. *We observe that* (1) *follows from* (28) *introducing the parameter σ defined by*

$$\frac{1+s}{2} = 1-\sigma \iff s = 2\sigma - 1.$$

5. Sine Infinite Product and Cotangent Partial Fractions Representations and Basel Problem

In this section, we will demonstrate how, if we assume the Euler–Gauss formula that expresses Gamma as an infinite product:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)}, \quad x > 0$$

Euler's formula for the representation of the sine in terms of the infinite product follows.

Theorem 7. *Assume* $\mathrm{Re}\,(x) > 0$ *then*

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Proof. Choosing $0 < x < 1$ and observing that, form (9) follows

$$\Gamma(1-x) = \lim_{n\to\infty} \frac{n!\, n^{1-x}}{(1-x)(2-x)\cdots(n-x)(n+1-x)}. \tag{35}$$

By multiplying the terms of the two sequences on the right-hand side of (9) and (35), we get

$$\begin{aligned}
&\frac{n!\, n^x}{x(1+x)(2+x)\ldots(n+x)(n+1-x)} \, \frac{n!\, n^{1-x}}{(1-x)(2-x)\cdots(n-x)(n+1-x)} = \\
&\frac{(n!)^2\, n}{x(1+x)(1-x)(2+x)(2-x)\ldots(n+x)(n-x)(n+1-x)} = \\
&\frac{(n!)^2\, n}{x(1-x^2)(2^2-x^2)\ldots(n^2-x^2)(n+1-x)} = \\
&\frac{n}{x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\left(1-\frac{x^2}{n^2}\right)(n+1-x)} = \\
&\frac{1}{x\displaystyle\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)\left(1+\frac{1-x}{n}\right)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Gamma(x)\Gamma(1-x) &= \lim_{n\to\infty} \frac{1}{x\displaystyle\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)(1+\frac{1-x}{n})} \\
&= \frac{1}{x\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)}.
\end{aligned} \tag{36}$$

Comparing (36) with reflection Formula (1) leads to our statement. In fact, we have

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = \frac{\sin(\pi x)}{\pi x}. \tag{37}$$

which agrees with (10). □

Remark 4. *Rewriting* (37) *as*

$$\sin(\pi x) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), \tag{38}$$

we see that Formula (38) *is well defined for* $x \in \mathbb{C}$ *with zeros in the integers.*

We conclude with the representation of the cotangent.

Corollary 2. *If* $z \notin \mathbb{Z}$ *then*

$$\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}, \tag{39}$$

Proof. By logarithmically differentiating the left-hand side of (38), we get

$$\frac{d}{dz} \ln \sin(\pi z) = \pi \cot(\pi z),$$

and on the right side

$$\frac{d}{dz} \ln(\pi z) + \sum_{n=1}^{\infty} \frac{d}{dz} \ln\left(1 - \frac{z^2}{n^2}\right).$$

This completes the proof, after standard computation. □

6. Epilogue

In our final step, we use (39) to solve the Basel problem, which consists of determining the exact value of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{40}$$

We refrain here from presenting the history of this famous problem, which is discussed in several contributions; we limit ourselves here to addressing [31], the seminal paper by R. Ayoub [32,33].

To show (40), we analyze both sides of (39) in a neighborhood of $z = 0$.

We begin with the right-hand side of (39). For $|z| < 1$ and each fixed $n \geq 1$, we may write the term using the geometric series expansion

$$\frac{1}{z^2 - n^2} = -\frac{1}{n^2} \frac{1}{1 - \frac{z^2}{n^2}} = -\frac{1}{n^2} \sum_{k=0}^{\infty} \frac{z^{2k}}{n^{2k}} = -\frac{1}{n^2}\left(1 + \frac{z^2}{n^2} + \frac{z^2}{n^4} + \frac{z^4}{n^6} + \cdots\right).$$

Since the resulting series is absolutely convergent for $|z| < 1$, we may sum term by term to obtain

$$2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = -2z \sum_{n=1}^{\infty} \frac{1}{n^2} + O(z^3) \quad \text{as } z \to 0.$$

Therefore, the right-hand side of (39) admits the expansion

$$\frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{1}{n^2} + O(z^3). \tag{41}$$

Now, we work on the left-hand side of (39). From the Taylor expansions of $\sin x$ and $\cos x$, it follows that

$$\cot x = \frac{1}{x} - \frac{x}{3} + O(x^3) \quad \text{as } x \to 0.$$

Substituting $x = \pi z$ yields

$$\pi \cot(\pi z) = \frac{1}{z} - \frac{\pi^2}{3} z + O(z^3). \tag{42}$$

Comparing the coefficients of the linear term in z in (41) and (42), we obtain

$$-2 \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{3}.$$

Solving for the series gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This completes the derivation of the Basel problem solution.

7. Conclusions

The sole original contribution of the present work lies in the proof of Theorem 6, which is based on the Mellin transform of the cosine function and an integral originally computed by Laplace. The objective of the paper is to present a didactic pathway constructed around this result, complemented by the Euler–Gauss representation of the Gamma function and Fubini's theorem on the reduction of double integrals. This approach enables the derivation, employing solely elementary tools of analysis accessible to undergraduate students, of a proof of the reflection formula for the Gamma function, without resorting to advanced methods such as complex analysis or Fourier series.

This framework also facilitates an elementary derivation of the infinite product formula for the sine function, the partial fraction expansion of the cotangent, and consequently the solution of the Basel problem. This provides an intriguing didactic perspective that can be effectively presented in undergraduate analysis courses.

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