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On a class of stochastic hyperbolic equations with double characteristics

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Abstract

We study the effect of Gaussian perturbations on a hyperbolic partial differential equation with double characteristics in two spatial dimensions. The coefficients of our partial differential operator depend polynomially on the space variables, while the noise is additive, white in time and coloured in space. We provide a sufficient condition on the spectral measure of the covariance functional describing the noise that allows for the existence of a random field solution for the resulting stochastic partial differential equation. Our approach is based on explicit computations for the fundamental solution of the partial differential operator and its Fourier transform.

Key words and phrases: hyperbolic equations with double characteristics, Gaussian noise, random field solution.

AMS 2000 classification: 60H15, 60H05, 35R60.

1 Introduction and statement of the main result

The aim of this note is to investigate the stochastic linear hyperbolic equation

$$\begin{cases} \left(\partial_t^2 - 2\partial_t\partial_{x_1} - x_1^2\partial_{x_2}^2\right) u(t,x_1,x_2) = \dot{F}(t,x_1,x_2), & t>0, (x_1,x_2)\in\mathbb{R}^2; \\ u(0,x_1,x_2)=0, & (x_1,x_2)\in\mathbb{R}^2; \\ \partial_t u(0,x_1,x_2)=0, & (x_1,x_2)\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

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where formally

$$F(\varphi) := \int_{\mathbb{R}^3} \varphi(t,x_1,x_2)\dot{F}(t,x_1,x_2)dtdx_1dx_2, \quad \varphi \in C_0^\infty(\mathbb{R}^3)$$

is a family of Gaussian random variables, defined on a common complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with mean zero and covariance

$$\mathbb{E}[F(\varphi)F(\psi)] = \int_0^{+\infty} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(t,x)f(x-y)\psi(t,y), \quad \varphi,\psi \in C_0^\infty(\mathbb{R}^3). \quad (1.2)$$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is assumed to be continuous in $\mathbb{R}^2 \setminus \{0\}$ with $f(-x) = f(x)$, for all $x \in \mathbb{R}^2$.

The most relevant feature of the differential operator appearing in (1.1) is the fact that its principal symbol, i.e.

$$p(x,\xi) = -\xi_0^2 - 2\xi_0\xi_1 + x_1^2\xi_2^2,$$

is hyperbolic with respect to ξ_0 and has double characteristics on the manifold

$$\Sigma = \{(x,\xi) \in \dot{T}^\star\mathbb{R}^3 : x_1 = 0, \xi_0 = 0, \xi_1 = 0\},$$

where $\dot{T}^\star\mathbb{R}^3$ denotes the phase-space cotangent bundle of $\mathbb{R}^3$ minus the 0-section (see e.g. [11],[12]). The fundamental matrix associated to p at a double point $\rho \in \Sigma$ is then computed as $F_p(\rho) = \frac{1}{2}dH_p(\rho)$, where H_p is the Hamilton vector field of p and it is readily seen that F_p has a Jordan block of order 4 at the eigenvalue 0; this is one of the (well-known) three possible non-effectively hyperbolic cases, if one excludes the case of a pair of non zero-real eigenvalues (the so called effectively hyperbolic case).
As it is the case with hyperbolic operators with multiple characteristics, lower order terms may modify the behavior of the well-posedness of the Cauchy problem. A large number of papers has been devoted to that question in the deterministic setting and in our case when studying the problem

$$(\partial_t^2 - 2\partial_t\partial_{x_1} - x_1^2\partial_{x_2}^2 + \mathtt{s}\partial_{x_2})u(t,x_1,x_2) = g(t,x_1,x_2)$$

the behavior changes whether $\mathtt{s} = 0$ or not. This is a special case of the Ivrii-Petkov conditions (see [13]) and we will deal with that and its effects on random perturbations in a following paper.

We recall that the analogue of (1.1) in the case of strictly hyperbolic operators, or the wave equation for that matter, has been thoroughly studied in the literature for different spatial dimensions. We mention for instance [14], [8], [9], [15], [16] and the reference quoted there. The framework adopted in this paper is the one proposed in [7] which extends the classical set up of [17]. We also mention [10] for a comparison between the abstract framework of [6] and the one discussed in [7].

Lastly, we mention that the analysis of some non strictly hyperbolic stochastic partial differential equations associated with linear operators with polynomially bounded coefficients has been performed in the recent papers [1], [2], and [3] (see also [4], for the investigation of random field solutions to linear strictly hyperbolic stochastic partial differential equations with uniformly bounded coefficients). In [1] and [3] the authors adopt the stochastic framework proposed in [7] and prove existence of random field solutions. The hyperbolic operators considered in [3] have multiple characteristics of constant multiplicities(in particular, they are strictly hyperbolic, when the maximum multiplicity equals one). Those considered in [1] have variable multiplicities and involutive characteristics. Finally, semilinear hyperbolic partial differential equations, associated with linear operators of the types studied in [1] and [3], are investigated in [2], adopting the setting given in [6], and existence and uniqueness of function-valued solutions is proved for them. However, the classes of operators studied in [1], [2], [3],and [4] do not cover the one treated here.

To state our main theorem, we now shortly describe the framework, referring the reader to [17] and [7] for further details.
We denote by $\mathcal{D}(\mathbb{R}^3)$ the space of functions $\varphi \in C_0^\infty(\mathbb{R}^3)$ endowed with the topology induced by the following notion of convergence: $\varphi_n \to \varphi$ if

1. there exists a compact set K of $\mathbb{R}^3$ such that the support of $\varphi_n - \varphi$ is contained in K, for all $n \geq 1$;

2. $\lim_{n\to+\infty} D^\alpha \varphi_n = D^\alpha \varphi$, uniformly on K for each multiindex α.

A direct verification using identity (1.2) shows that the map $\varphi \mapsto F(\varphi)$ is linear and continuous in $\mathbb{L}^2(\Omega)$; this implies that F has a version with values in $\mathcal{D}'(\mathbb{R}^3)$ which in turn allows for a distributional ω-wise interpretation of the partial differential equation (1.1). For the distributional solution to be a real valued measurable stochastic process, we need to extend F to a worthy martingale (see for instance [17]) and interpret the distributional solution

$$u(t, x_1, x_2) = \int_0^t \int_{\mathbb{R}^2} \Gamma(t-s, x_1, y_1, x_2 - y_2) \dot{F}(s, y_1, y_2) dy_1 dy_2,$$

as a stochastic integral. Here, Γ denotes the fundamental solution of the differential operator $\partial_t^2 - 2\partial_t\partial_{x_1} - x_1^2\partial_{x_2}^2$ from (1.1) (see Section 2 below). To this aim, by suitably approximating indicator functions of bounded Borel subsets of $[0,+\infty[\times\mathbb{R}^2$ with elements from $\mathcal{D}(\mathbb{R}^3)$ and employing the $\mathbb{L}^2(\Omega)$-continuity mentioned above, we first define $F(A) := F(\mathbf{1}_A)$, $A \in \mathcal{B}_b([0,+\infty[\times\mathbb{R}^2)$ and

$$M_t(B) := F([0,t] \times B), \quad t \geq 0, B \in \mathcal{B}_b(\mathbb{R}^2).$$

Then, if we let

$$\mathcal{F}_t^0 := \sigma(M_s(B), 0 \leq s \leq t, B \in \mathcal{B}_b(\mathbb{R}^2)), \quad \mathcal{F}_t := \mathcal{F}_t^0 \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the σ-algebra generated by $\mathbb{P}$-null sets, we get that

$$\big(\{M_t(B)\}_{t\geq 0, B\in\mathcal{B}_b(\mathbb{R}^2)}, \{\mathcal{F}_t\}_{t\geq 0}\big)$$

is a worthy martingale measure. By construction, for all $B \in \mathcal{B}_b(\mathbb{R}^2)$ the stochastic process $\{M_t(B)\}_{t\geq 0}$ is a continuous martingale and we have

$$F(\varphi) = \int_0^{+\infty}\int_{\mathbb{R}^2} \varphi(t,x)M(dt,dx). \tag{1.3}$$

On the other hand, using elementary properties of the Fourier transform, we can rewrite identity (1.2) as

$$\mathbb{E}[F(\varphi)F(\psi)] = \int_0^{+\infty}\int_{\mathbb{R}^2} \mathcal{F}\varphi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)}d\mu(\xi)dt, \tag{1.4}$$

where $\mathcal{F}\eta$ denotes the Fourier transform of η, i.e.

$$\mathcal{F}\eta(\xi) := \int_{\mathbb{R}^2} e^{-i\xi\cdot x}\eta(x)dx, \quad \xi\in\mathbb{R}^2,$$

and μ, the *spectral measure of* f, is a non-negative tempered measure μ on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} f(x)\eta(x)dx = \int_{\mathbb{R}^2}\mathcal{F}\eta(\xi)d\mu(\xi), \qquad \text{for all } \eta\in S(\mathbb{R}^2).$$

Combining identity (1.4) with (1.3) we get

$$\mathbb{E}\left[\left|\int_0^{+\infty}\int_{\mathbb{R}^2}\varphi(t,x)M(dt,dx)\right|^2\right] = \int_0^{+\infty}\int_{\mathbb{R}^2}|\mathcal{F}\varphi(t,\xi)|^2 d\mu(\xi)dt. \tag{1.5}$$

The last isometry determines the class of admissible deterministic integrands for the stochastic integral in (1.3). We will say that $\{u(t,x_1,x_2)\}_{t\geq 0,(x_1,x_2)\in\mathbb{R}^2}$ is a *random field solution* to (1.1) if

$$u(t,x_1,x_2) := \int_0^t\int_{\mathbb{R}^2}\Gamma(t-s,x_1,y_1,x_2-y_2)M(ds,dy_1,dy_2), \tag{1.6}$$

is a well defined stochastic integral (i.e. the right hand side in (1.5) is finite) and the map

$$[0,+\infty[\times\mathbb{R}^2 \ni (t,x_1,x_2)\mapsto u(t,x_1,x_2)$$

is measurable. We are now ready to state the main theorem of the present paper; the proof can be found in Sections 2 and 3 (closed-form expression for the fundamental solution and existence for the random field solution, respectively).

Theorem 1.1. *Assume that the spectral measure μ satisfies*

$$\int_{\mathbb{R}^2} \frac{1}{1+|\xi|^{2/3}} d\mu(\xi) < +\infty. \tag{1.7}$$

Then the stochastic partial differential equation (1.1) *admits a random field solution* $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^2}$ *with representation* (1.6) *where*

$$\Gamma(t,x_1,y_1,x_2) = \begin{cases} \frac{\sqrt{3}}{2\pi}\frac{1}{\sqrt{(y_1^3-x_1^3)(2t+x_1-y_1)-3x_2^2}}, & \text{if } (t,x_1,x_2) \in A_{t,x_1,x_2}, \\ 0, & \text{if } (t,x_1,x_2) \notin A_{t,x_1,x_2}, \end{cases}$$

and

$$A_{t,x_1,x_2} := \left\{(t,x_1,x_2) \in \mathbb{R}^3 : t>0, y_1>x_1, (y_1^3-x_1^3)(2t+x_1-y_1)-3x_2^2>0\right\}.$$

Remark 1.2. *It is proved in [7] that for the stochastic wave equation*

$$\begin{cases} \left(\partial_t^2 - \Delta\right) u(t,x) = \dot{F}(t,x), & t>0, x\in\mathbb{R}^d; \\ u(0,x)=0, & x\in\mathbb{R}^d; \\ \partial_t u(0,x) = 0, & x\in\mathbb{R}^d, \end{cases}$$

with $\Delta := \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$, a sufficient condition for the existence of random field solutions is

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} d\mu(\xi) < +\infty,$$

for any spatial dimension $d \geq 1$. A comparison with (1.7) shows that the existence of a random filed solution for (1.1) requires more stringent assumptions on the spectral measure μ, and hence on the Gaussian noise F, than its simplest strictly hyperbolic counterpart.

The paper is organized as follows: in Section 2 we derive an explicit representation for the fundamental solution of the partial differential operator in (1.1), while in Section 3 we prove the existence for a random field solution under the integrability condition (1.7).

2 The fundamental solution

In this section we describe a derivation of the fundamental solution for the partial differential operator in (1.1). Following [12], we set $D_t := \frac{1}{i}\partial_t$, $D_{x_j} := \frac{1}{i}\partial_{x_j}$, for $j = 1,2$, and consider the problem

$$\left(-D_t^2 + 2D_tD_{x_1} + x_1^2 D_{x_2}^2\right)\Gamma(t,x_1,y_1,x_2) = \delta_{(0,y_1,0)}(t,x_1,x_2), \tag{2.1}$$

for $t>0$ and $(x_1,x_2)\in\mathbb{R}^2$; here δ_z stands for the Dirac's delta distribution with mass at $z\in\mathbb{R}^3$ and $y_1\in\mathbb{R}$ is a fixed parameter. We observe that the operator under investigation is not invariant by translation in the variable x_1; hence, the parameter y_1 serves to keep trace of this fact.
We now denote by $\hat\Gamma(t,x_1,y_1,\xi_2)$ the Fourier transform w.r.t. x_2 of $x_2\mapsto\Gamma(t,x_1,y_1,x_2)$, i.e.

$$\hat\Gamma(t,x_1,y_1,\xi_2):=\int_{\mathbb{R}} e^{-i\xi_2x_2}\Gamma(t,x_1,y_1,x_2)dx_2,\quad \xi_2\in\mathbb{R},$$

and by $\Gamma^\dagger(\xi_0,x_1,y_1,x_2)$ the Fourier-Laplace transform w.r.t. t of $t\mapsto\Gamma(t,x_1,y_1,x_2)$, i.e.

$$\Gamma^\dagger(\xi_0,x_1,y_1,x_2):=\int_0^{+\infty} e^{-i\xi_0t}\Gamma(t,x_1,y_1,x_2)dt,\quad \mathtt{Im}(\xi_0)<0.$$

Transforming equation (2.1) we get

$$\left(-\xi_0^2+2\xi_0D_{x_1}+x_1^2\xi_2^2\right)\hat\Gamma^\dagger(\xi_0,x_1,y_1,\xi_2)=\delta(x_1-y_1), \tag{2.2}$$

with δ being now the Dirac's delta distribution with mass at $0\in\mathbb{R}$. Recalling that $D_{x_1}=\frac{1}{i}\partial_{x_1}$ and setting

$$q(x_1;\xi_0,\xi_2):=x_1^2\xi_2^2-\xi_0^2,$$

we can rewrite equation (2.2) as

$$\left(\partial_{x_1}+\frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\right)\hat\Gamma^\dagger(\xi_0,x_1,y_1,\xi_2)=\frac{i}{2\xi_0}\delta(x_1-y_1). \tag{2.3}$$

The homogeneous part of the last equation can be solved as

$$\begin{aligned}v(\xi_0,x_1,\xi_2)&=C(\xi_0,\xi_2)e^{-\frac{i}{2\xi_0}\int_0^{x_1}q(t;\xi_0,\xi_2)dt}\\&=C(\xi_0,\xi_2)e^{-\frac{i}{2\xi_0}\int_0^{x_1}t^2\xi_2^2-\xi_0^2dt}\\&=C(\xi_0,\xi_2)e^{-\frac{i}{2\xi_0}\left(\frac{\xi_2^2}{3}x_1^3-\xi_0^2x_1\right)}\\&=C(\xi_0,\xi_2)e^{i\frac{\xi_0}{2}x_1-i\frac{\xi_2^2}{6\xi_0}x_1^3}.\end{aligned}$$

On the other hand, since it is natural to expect that $\hat\Gamma^\dagger(\xi_0,x_1,y_1,\xi_2)$ vanishes when $x_1>y_1$, we choose

$$\hat\Gamma^\dagger(\xi_0,x_1,y_1,\xi_2)=v(\xi_0,x_1,\xi_2)H(y_1-x_1),$$

as a particular solution of (2.3) (here $x\mapsto H(x)$ denotes the Heaviside function). We now have to find the constant $C(\xi_0,\xi_2)$ that makes identity (2.3) true:

$$\left(\partial_{x_1}+\frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\right)\hat\Gamma^\dagger(\xi_0,x_1,y_1,\xi_2)$$

$$
\begin{aligned}
&= \partial_{x_1}\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2) + \frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2)\\
&= \partial_{x_1}(v(\xi_0,x_1,\xi_2)H(y_1-x_1)) + \frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2)\\
&= (\partial_{x_1}v(\xi_0,x_1,\xi_2))H(y_1-x_1) + v(\xi_0,x_1,\xi_2)\partial_{x_1}H(y_1-x_1)\\
&\quad + \frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2)\\
&= -\frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)v(\xi_0,x_1,\xi_2)H(y_1-x_1) - v(\xi_0,x_1,\xi_2)\delta(y_1-x_1)\\
&\quad + \frac{i}{2\xi_0}q(x_1;\xi_0,\xi_2)\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2)\\
&= -v(\xi_0,x_1,\xi_2)\delta(y_1-x_1)\\
&= -C(\xi_0,\xi_2)e^{i\frac{\xi_0}{2}x_1 - i\frac{\xi_2^2}{6\xi_0}x_1^3}\delta(y_1-x_1).
\end{aligned}
$$

This gives

$$
C(\xi_0,\xi_2) = -\frac{i}{2\xi_0}e^{-i\frac{\xi_0}{2}y_1 + i\frac{\xi_2^2}{6\xi_0}y_1^3},
$$

and hence

$$
\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2) = -\frac{i}{2\xi_0}e^{i\frac{\xi_0}{2}(x_1-y_1) - i\frac{\xi_2^2}{6\xi_0}(x_1^3-y_1^3)}H(y_1-x_1).
$$

We now proceed inverting the transforms. To this aim we write the last expression as

$$
\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2) = G(\xi_0,x_1)e^{-\frac{A(\xi_0)}{2}\xi_2^2},
$$

where

$$
G(\xi_0,x_1,y_1) := -\frac{i}{2\xi_0}e^{i\frac{\xi_0}{2}(x_1-y_1)}H(y_1-x_1), \tag{2.4}
$$

and

$$
A(\xi_0,x_1,y_1) := i\frac{x_1^3-y_1^3}{3\xi_0}. \tag{2.5}
$$

Then,

$$
\begin{aligned}
\Gamma^\dagger(\xi_0,x_1,y_1,x_2) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}\hat{\Gamma}^\dagger(\xi_0,x_1,y_1,\xi_2)d\xi_2\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}G(\xi_0,x_1,y_1)e^{-\frac{A(\xi_0,x_1,y_1)}{2}\xi_2^2}d\xi_2\\
&= \frac{1}{2\pi}G(\xi_0,x_1,y_1)\int_{\mathbb{R}} e^{ix_2\xi_2-\frac{A(\xi_0,x_1,y_1)}{2}\xi_2^2}d\xi_2
\end{aligned}
$$

$$= \frac{1}{\sqrt{2\pi}} G(\xi_0, x_1, y_1) A(\xi_0, x_1, y_1)^{-\frac{1}{2}} e^{-\frac{x_2^2}{2} A(\xi_0, x_1, y_1)^{-1}}. \tag{2.6}$$

In the last equality we utilized the identity

$$\int_{\mathbb{R}} e^{-ixy - \frac{1}{2}Ax^2} dx = \sqrt{2\pi} A^{-\frac{1}{2}} e^{-\frac{1}{2}A^{-1}y^2},$$

with $A^{\frac{1}{2}}$ chosen in a such a way that $A^{\frac{1}{2}} > 0$, if $\mathtt{Re}(A) > 0$. In our case, $A(\xi_0, x_1, y_1) = i\frac{x_1^3 - y_1^3}{3\xi_0}$, $\mathtt{Im}(\xi_0) < 0$ and $x_1 < y_1$; this means that $\xi_0 = |\xi_0|e^{i\theta}$, for some $\theta \in]-\pi, 0[$, that $x_1^3 - y_1^3 = |x_1^3 - y_1^3|e^{-i\pi}$ and hence

$$A(\xi_0, x_1, y_1) = e^{i\frac{\pi}{2}} \frac{|x_1^3 - y_1^3|e^{-i\pi}}{3|\xi_0|e^{i\theta}} = \frac{|x_1^3 - y_1^3|}{3|\xi_0|} e^{i(-\frac{\pi}{2} - \theta)}.$$

Since $-\frac{\pi}{2} - \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, we conclude that $\mathtt{Re}(A(\xi_0, x_1, y_1)) > 0$. Replacing in (2.6) the definitions of $G(\xi_0, x_1, y_1)$ and $A(\xi_0, x_1, y_1)$ from (2.4) and (2.5), respectively, we get

$$\begin{aligned}
\Gamma^\dagger(\xi_0, x_1, y_1, x_2) &= -\frac{1}{\sqrt{2\pi}} \frac{i}{2\xi_0} e^{i\frac{\xi_0}{2}(x_1 - y_1)} \left(\frac{y_1^3 - x_1^3}{3i\xi_0} \right)^{-\frac{1}{2}} e^{\frac{i}{2}\frac{3\xi_0}{x_1^3 - y_1^3} x_2^2} H(y_1 - x_1) \\
&= -\frac{1}{\sqrt{2\pi}} \frac{i}{2\xi_0} \left(\frac{3i\xi_0}{y_1^3 - x_1^3} \right)^{\frac{1}{2}} e^{i\frac{\xi_0}{2}\left(x_1 - y_1 + \frac{3}{x_1^3 - y_1^3} x_2^2\right)} H(y_1 - x_1) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \frac{1}{i\xi_0} \left(\frac{3}{y_1^3 - x_1^3} \right)^{\frac{1}{2}} (i\xi_0)^{\frac{1}{2}} e^{i\frac{\xi_0}{2}\left(x_1 - y_1 + \frac{3}{x_1^3 - y_1^3} x_2^2\right)} H(y_1 - x_1) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{3}{y_1^3 - x_1^3} \right)^{\frac{1}{2}} (i\xi_0)^{-\frac{1}{2}} e^{i\frac{\xi_0}{2}\left(x_1 - y_1 + \frac{3}{x_1^3 - y_1^3} x_2^2\right)} H(y_1 - x_1).
\end{aligned} \tag{2.7}$$

We now observe that the function $\xi_0 \mapsto (i\xi_0)^{-\frac{1}{2}}$ is the Fourier-Laplace tranform of $t \mapsto (\pi t)^{-\frac{1}{2}} H(t)$. In fact, writing $\xi_0 = \alpha_0 + i\beta_0$ and recalling that $\beta_0 < 0$, we get

$$\begin{aligned}
\int_0^{+\infty} e^{-i\xi_0 t} (\pi t)^{-\frac{1}{2}} dt &= \int_0^{+\infty} e^{-i\alpha_0 t + \beta_0 t} (\pi t)^{-\frac{1}{2}} dt \\
&= \int_0^{+\infty} e^{-i\alpha_0 t} \frac{1}{\sqrt{\pi}} t^{1/2 - 1} e^{-(-\beta_0)t} dt \\
&= (-\beta_0)^{-1/2} \int_0^{+\infty} e^{-i\alpha_0 t} \frac{(-\beta_0)^{1/2}}{\sqrt{\pi}} t^{1/2 - 1} e^{-(-\beta_0)t} dt \\
&= (-\beta_0)^{-1/2} \left(1 - i\frac{\alpha_0}{\beta_0} \right)^{-1/2} \\
&= (i\xi_0)^{-\frac{1}{2}}.
\end{aligned}$$

In the fourth equality above we recognize the characteristic function of a Gamma distribution with parameters $(\frac{1}{2}, -\beta_0)$. Lastly, denoting the function $t \mapsto (\pi t)^{-\frac{1}{2}} H(t)$ with

$\chi(t)$ and $\left(x_1 - y_1 + \frac{3}{x_1^3 - y_1^3} x_2^2\right)/2$ with $\mathcal{T}$, we can continue in (2.7) as

$$\begin{aligned}\Gamma^\dagger(\xi_0, x_1, y_1, x_2) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{3}{y_1^3 - x_1^3}\right)^{\frac{1}{2}} \chi^\dagger(\xi_0) e^{i\xi_0 \mathcal{T}} H(y_1 - x_1) \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{3}{y_1^3 - x_1^3}\right)^{\frac{1}{2}} (\chi(\cdot + \mathcal{T}))^\dagger(\xi_0) H(y_1 - x_1).\end{aligned}$$

This entails

$$\begin{aligned}\Gamma(t, x_1, y_1, x_2) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{3}{y_1^3 - x_1^3}\right)^{\frac{1}{2}} (\pi(t + \mathcal{T}))^{-\frac{1}{2}} H(t) H(t + \mathcal{T}) H(y_1 - x_1) \\ &= \frac{\sqrt{3}}{2\pi} \frac{1}{\sqrt{(y_1^3 - x_1^3)(2t + x_1 - y_1) - 3x_2^2}} H(t) H(t + \mathcal{T}) H(y_1 - x_1).\end{aligned}$$

This completes the proof of the following result.

Proposition 2.1. *The fundamental solution of the operator* $\partial_t^2 - 2\partial_t \partial_{x_1} - x_1^2 \partial_{x_2}^2$ *on* $]0, +\infty[\times \mathbb{R}^2$ *is given by*

$$\Gamma(t, x_1, y_1, x_2) = \begin{cases} \frac{\sqrt{3}}{2\pi} \frac{1}{\sqrt{(y_1^3 - x_1^3)(2t + x_1 - y_1) - 3x_2^2}}, & \text{if } (t, x_1, x_2) \in A_{t,x_1,x_2}, \\ 0, & \text{if } (t, x_1, x_2) \notin A_{t,x_1,x_2}, \end{cases}$$

where

$$A_{t,x_1,x_2} := \left\{(t, x_1, x_2) \in \mathbb{R}^3 : t > 0, y_1 > x_1, (y_1^3 - x_1^3)(2t + x_1 - y_1) - 3x_2^2 > 0\right\}.$$

3 Existence of the random field solution

We now prove that

$$u(t, x_1, x_2) := \int_0^t \int_{\mathbb{R}^2} \Gamma(t - s, x_1, y_1, x_2 - y_2) M(ds, dy_1, dy_2), \tag{3.1}$$

where M denotes the worthy martingale measure associated with F and Γ the function defined in Proposition 2.1, is a random field solution to the Cauchy problem (1.1). To do so we first need to verify the bound

$$\int_0^t \int_{\mathbb{R}^2} |\mathcal{F}\Gamma(t - s, x_1, \cdot, x_2 - \cdot)(\xi_1, \xi_2)|^2 \mu(d\xi_1 d\xi_2) ds < +\infty, \tag{3.2}$$

for the Fourier transform of the function

$$(y_1, y_1) \mapsto \Gamma(t - s, x_1, y_1, x_2 - y_2);$$

this will ensure that $u(t,x_1,x_2)$ is a well defined stochastic integral. To ease the notation we set

$$h := h(t-s,x_1,y_1) := (2(t-s)+x_1-y_1)(y_1^3-x_1^3)/3;$$

then,

$$\begin{aligned}
&\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot)(\xi)\\
&\quad=\int_{\mathbb{R}} dy_1 e^{-iy_1\xi_1}\int_{\mathbb{R}} dy_2 e^{-iy_2\xi_2}\Gamma(t-s,x_1,\xi,x_2-\xi_2)\\
&\quad=\frac{1}{2\pi}\int dy_1 e^{-iy_1\xi_1}\int dy_2 e^{-iy_2\xi_2}\frac{1}{\sqrt{h-(x_2-y_2)^2}}H(h-(x_2-y_2)^2)H(y_1-x_1).
\end{aligned}$$

Observe that condition $h(t-s,x_1,y_1)-(x_2-y_2)^2>0$ implies $h(t-s,x_1,y_1)>0$ which, in combination with $x_1<y_1$, gives $x_1<y_1<x_1+2(t-s)$. Therefore, making the change of variable $\sigma := \frac{y_2-x_2}{\sqrt{h(t-s,x_1,y_1)}}$ and observing that

$$h(t-s,x_1,y_1)-(x_2-y_2)^2>0 \quad\Longleftrightarrow\quad \sigma\in]-1,1[,$$

we get

$$\begin{aligned}
\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot)(\xi) &= \frac{1}{2\pi}\int_{x_1}^{x_1+2(t-s)} dy_1 e^{-iy_1\xi_1}\int_{-1}^{1} e^{-i(x_2+\sigma\sqrt{h})\xi_2}\frac{1}{\sqrt{1-\sigma^2}}d\sigma\\
&= \frac{e^{-ix_2\xi_2}}{2\pi}\int_{x_1}^{x_1+2(t-s)} dy_1 e^{-iy_1\xi_1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\sqrt{h}\sin(\theta)\xi_2}d\theta. \qquad (3.3)
\end{aligned}$$

Here, we performed the further change of variable $\sigma := \sin(\theta)$, $\theta\in]-\pi/2,\pi/2[$. Note that the last integral above can be written as

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\sqrt{h}\sin(\theta)\xi_2}d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\left(\sqrt{h}\sin(\theta)\xi_2\right)d\theta - i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\left(\sqrt{h}\sin(\theta)\xi_2\right)d\theta\\
&= 2\int_0^{\frac{\pi}{2}}\cos\left(\sqrt{h}\sin(\theta)\xi_2\right)d\theta\\
&= \pi\mathrm{J}_0(\sqrt{h}\xi_2),
\end{aligned}$$

where J_0 denotes the first Bessel function of order zero. Therefore, equation (3.3) reads

$$\begin{aligned}
&\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot)(\xi)\\
&\quad=\frac{e^{-ix_2\xi_2}}{2}\int_{x_1}^{x_1+2(t-s)} e^{-iy_1\xi_1}\mathrm{J}_0(\sqrt{h}\xi_2)dy_1\\
&\quad=\frac{e^{-ix_2\xi_2}}{2}\int_{x_1}^{x_1+2(t-s)} e^{-iy_1\xi_1}\mathrm{J}_0\left(\xi_2\sqrt{(2(t-s)+x_1-y_1)(y_1^3-x_1^3)/3}\right)dy_1.
\end{aligned}$$

We now set $\tau := t - s > 0$ and for $\lambda \in]0,1[$ we use the change of variable $y_1 = x_1 + 2\lambda\tau$; this gives

$$\begin{aligned}
&\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi) \\
&\quad = \tau e^{-ix_2\xi_2} \int_0^1 e^{-i(x_1+2\lambda\tau)\xi_1} \mathtt{J}_0 \left(\xi_2 \sqrt{4\tau^2\lambda(1-\lambda)(4\tau^2\lambda^2 + 6\tau x_1\lambda + 3x_1^2)/3} \right) d\lambda \\
&\quad = \tau e^{-ix_2\xi_2 - ix_1\xi_1} \int_0^1 e^{-i2\lambda\tau\xi_1} \mathtt{J}_0 \left(\xi_2 \sqrt{4\tau^2\lambda(1-\lambda)(4\tau^2\lambda^2 + 6\tau x_1\lambda + 3x_1^2)/3} \right) d\lambda \\
&\quad = \tau e^{-ix_2\xi_2 - ix_1\xi_1} \int_0^1 e^{-i2\lambda\tau\xi_1} \mathtt{J}_0 \left(\xi_2 \tilde{h}(\tau, x_1, \lambda) \right) d\lambda, \qquad (3.4)
\end{aligned}$$

where we introduced the shorthand notation

$$\begin{aligned}
\tilde{h}(\tau, x_1, \lambda) :=& \sqrt{4\tau^2\lambda(1-\lambda)(4\tau^2\lambda^2 + 6\tau x_1\lambda + 3x_1^2)/3} \\
=& 2\tau\sqrt{\lambda(1-\lambda)}\sqrt{(4\tau^2\lambda^2 + 6\tau x_1\lambda + 3x_1^2)/3}. \qquad (3.5)
\end{aligned}$$

Taking the modulus of the first and last members in (3.4) we see that

$$|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)| \le \tau \left| \int_0^1 e^{-i2\lambda\tau\xi_1} \mathtt{J}_0 \left(\xi_2 \tilde{h}(\tau, x_1, \lambda) \right) d\lambda \right|. \qquad (3.6)$$

To establish the bound (3.2), we can focus on the behaviour of $\xi \mapsto |\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)|$ for large values of $|\xi|$ only; in fact, such function is smooth and bounded on any compact set containing the origin. According to formula (1) page 206 in [18], the Bessel function $\mathtt{J}_0$ can be represented for $z > 0$ as

$$\mathtt{J}_0(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\pi}{4}\right) \mathtt{P}_+(z) - \sin\left(z - \frac{\pi}{4}\right) \mathtt{P}_-(z) \right] \qquad (3.7)$$

with

$$\mathtt{P}_+(z) = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-u} \frac{1}{\sqrt{u}} \left\{ \left(1 + \frac{iu}{2z}\right)^{-\frac{1}{2}} + \left(1 - \frac{iu}{2z}\right)^{-\frac{1}{2}} \right\} du$$

and

$$\mathtt{P}_-(z) = \frac{1}{2i\sqrt{\pi}} \int_0^{+\infty} e^{-u} \frac{1}{\sqrt{u}} \left\{ \left(1 + \frac{iu}{2z}\right)^{-\frac{1}{2}} - \left(1 - \frac{iu}{2z}\right)^{-\frac{1}{2}} \right\} du.$$

Since $\left|1 + \frac{iu}{2z}\right| = \sqrt{1 + \frac{u^2}{4z^2}} \ge 1$, we see that $|\mathtt{P}_\pm(z)| \le 1$; this together with (3.7) implies $|\mathtt{J}_0(z)| \le \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{z}}$ and

$$\left| \int_0^1 e^{-i2\lambda\tau\xi_1} \mathtt{J}_0 \left(\xi_2 \tilde{h}(\tau, x_1, \lambda) \right) d\lambda \right| \le C \int_0^1 \frac{1}{|\xi_2|^{\frac{1}{2}} |\tilde{h}(\tau, x_1, \lambda)|^{\frac{1}{2}}} d\lambda$$

$$=\frac{C}{|\xi_2|^{\frac{1}{2}}}\int_0^1 \frac{1}{|\tilde{h}(\tau,x_1,\lambda)|^{\frac{1}{2}}}d\lambda. \tag{3.8}$$

Here and in the sequel we denote by C a positive constant whose value may change from line to line. We now evaluate the last integral above; recalling the definition of $\tilde{h}$ in (3.5) we can write

$$\int_0^1 \frac{1}{|\tilde{h}(\tau,x_1,\lambda)|^{\frac{1}{2}}}d\lambda = \int_0^1 \frac{1}{\sqrt{2\tau}}\frac{1}{(\lambda(1-\lambda))^{\frac{1}{4}}}\frac{1}{L^{\frac{1}{4}}}d\lambda, \tag{3.9}$$

where $L := (4\tau^2\lambda^2 + 6\tau x_1\lambda + 3x_1^2)/3 \geq \tau^2\lambda^2/3$ and hence $L^{\frac{1}{4}} \geq \sqrt{\tau}\sqrt{\lambda}/3^{\frac{1}{4}}$. This gives

$$\int_0^1 \frac{1}{|\tilde{h}(\tau,x_1,\lambda)|^{\frac{1}{2}}}d\lambda \leq \frac{C}{\tau}\int_0^1 \frac{1}{\lambda^{3/4}(1-\lambda)^{1/4}}d\lambda = \frac{C}{\tau}.$$

A combination of this estimate with (3.8) and (3.6) yields

$$|\mathcal{F}\Gamma(\tau,x_1,\cdot,x_2-\cdot)(\xi)| \leq \frac{C}{|\xi_2|^{\frac{1}{2}}}. \tag{3.10}$$

To get a bound involving also $|\xi_1|$, we go back to the integral in (3.6) and perform an integration by parts (recall from (3.5) that $\tilde{h}(\tau,x_1,0) = \tilde{h}(\tau,x_1,1) = 0$ and $\mathtt{J}_0(0) = 1$); this yields

$$\begin{aligned}
&\int_0^1 e^{-i2\lambda\tau\xi_1}\mathtt{J}_0\left(\xi_2\tilde{h}(\tau,x_1,\lambda)\right)d\lambda \\
&= \frac{1}{i2\tau\xi_1} - \frac{e^{-i2\tau\xi_1}}{i2\tau\xi_1} + \frac{1}{i2\tau\xi_1}\int_0^1 e^{-i2\lambda\tau\xi_1}\mathtt{J}_0'\left(\xi_2\tilde{h}(\tau,x_1,\lambda)\right)\xi_2\partial_\lambda\tilde{h}(\tau,x_1,\lambda)d\lambda.
\end{aligned}$$

Recalling that for all $z \in \mathbb{C}$ we have $\mathtt{J}_0'(z) = -\mathtt{J}_1(z)$, the Bessel function of order one, and that $|\mathtt{J}_1(x)| \leq M$, for all $x \in \mathbb{R}$ and a suitable positive constant M, we can write

$$\begin{aligned}
&\left|\int_0^1 e^{-i2\lambda\tau\xi_1}\mathtt{J}_0\left(\xi_2\tilde{h}(\tau,x_1,\lambda)\right)d\lambda\right| \\
&\leq \frac{C}{\tau|\xi_1|} + \frac{C|\xi_2|}{\tau|\xi_1|}\int_0^1 \left|\mathtt{J}_0'\left(\xi_2\tilde{h}(\tau,x_1,\lambda)\right)\right||\partial_\lambda\tilde{h}(\tau,x_1,\lambda)|d\lambda \\
&\leq \frac{C}{\tau|\xi_1|} + \frac{MC|\xi_2|}{\tau|\xi_1|}\int_0^1 |\partial_\lambda\tilde{h}(\tau,x_1,\lambda)|d\lambda.
\end{aligned} \tag{3.11}$$

Now,

$$\begin{aligned}
\int_0^1 |\partial_\lambda\tilde{h}(\tau,x_1,\lambda)|d\lambda =& \tau\int_0^1 \frac{|1-2\lambda|}{\sqrt{\lambda(1-\lambda)}}L^{1/2}d\lambda \\
&+ \tau\int_0^1 \frac{\sqrt{\lambda(1-\lambda)}}{L^{1/2}}|\partial_\lambda L|d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sqrt{\tau^2 + x_1^2} + c_2(\tau + |x_1|) \\
&\leq C(\tau + |x_1|);
\end{aligned}
$$

here, we utilized in the first integral the bound $L \leq C(\tau^2 + x_1^2)$ while in the second $L^{1/2} \geq \frac{1}{\sqrt{3}}\tau\lambda$ (recall the inequalities following identity (3.9)). Therefore, using this estimate with (3.11) in (3.6) we get

$$
\begin{aligned}
|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)| &\leq \frac{C}{|\xi_1|} + \frac{MC|\xi_2|}{|\xi_1|} c_3(\tau + |x_1|) \\
&= \frac{C}{|\xi_1|} + K(\tau, x_1)\frac{|\xi_2|}{|\xi_1|}
\end{aligned} \tag{3.12}
$$

Combining (3.10) and (3.12) we can now complete the estimate of $|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)|$ for large value of $|\xi|$. Let $\theta \in]0, 1[$ (to be fixed later):

- if ξ is such that $|\xi_2| \leq M|\xi_1|^\theta$, for some positive constant M, then from inequality (3.12) we get

$$
\begin{aligned}
|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)| &\leq \frac{C}{|\xi_1|} + K(\tau, x_1)M\frac{|\xi_1|^\theta}{|\xi_1|} \\
&= \frac{C}{|\xi_1|} + K(\tau, x_1)M\frac{1}{|\xi_1|^{1-\theta}} \\
&\leq \tilde{K}(\tau, x_1)\frac{1}{|\xi_1|^{1-\theta}} \\
&\leq \tilde{K}(\tau, x_1)\frac{1}{|\xi|^{1-\theta}};
\end{aligned} \tag{3.13}
$$

  here, the last inequality is due to $|\xi| \lesssim |\xi_1| + |\xi_2| \lesssim |\xi_1| + |\xi_1|^\theta \lesssim |\xi_1|$ and hence $\frac{1}{|\xi_1|} \lesssim \frac{1}{|\xi|}$;

- otherwise, if ξ is such that $|\xi_2| \geq M|\xi_1|^\theta$, then $|\xi| \lesssim |\xi_1| + |\xi_2| \lesssim |\xi_2|^{1/\theta} + |\xi_2| \lesssim |\xi_2|^{1/\theta}$ or equivalently $|\xi|^\theta \lesssim |\xi_2|$. This condition, combined with (3.10), yields

$$
|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)| \leq \frac{C}{|\xi|^{\theta/2}}. \tag{3.14}
$$

To match the exponents in (3.13) and (3.14) we need to impose $1 - \theta = \theta/2$, that means $\theta = 2/3$. Hence, for large values of $|\xi|$ we get the estimate

$$
|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)|^2 \leq \frac{\kappa(\tau, x_1)^2}{|\xi|^{2/3}}, \tag{3.15}
$$

with $\kappa(\tau, x_1)$ being with linear growth in τ and $|x_1|$. For a global (in ξ) estimate we can simply set

$$
|\mathcal{F}\Gamma(\tau, x_1, \cdot, x_2 - \cdot)(\xi)|^2 \leq \frac{\tilde{\kappa}(\tau, x_1)}{1 + |\xi|^{2/3}}, \quad \xi \in \mathbb{R}^2, \tag{3.16}
$$

and conclude that

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^2} |\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot)(\xi_1,\xi_2)|^2 \mu(d\xi_1 d\xi_2)ds\\ &\quad \le \int_0^t \tilde{\kappa}(t-s,x_1)\int_{\mathbb{R}^2}\frac{1}{1+|\xi|^{2/3}}\mu(d\xi_1 d\xi_2)ds,\end{aligned}$$

which turns out to be finite by virtue of (1.7) (and the nice behaviour of $\tilde{\kappa}(t-s,x_1)$ with respect to its first argument).

We now prove that the map $(t,x_1,x_2)\mapsto u(t,x_1,x_2)$ is measurable by showing the $\mathbb{L}^2(\Omega)$-continuity of $\{u(t,x)\}_{t\ge 0, x\in\mathbb{R}^2}$. Starting with the time increment we can write for $t\in[0,T]$, $x\in\mathbb{R}^2$ and $h>0$ that

$$\begin{aligned}&\mathbb{E}\left[|u(t,x_1,x_2)-u(t+h,x_1,x_2)|^2\right]\\ &\quad\le 2\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^2}\Gamma(t-s,x_1,y_1,x_2-y_2)-\Gamma(t+h-s,x_1,y_1,x_2-y_2)M(ds,dy_1,dy_2)\right|^2\right]\\ &\qquad + 2\mathbb{E}\left[\left|\int_t^{t+h}\int_{\mathbb{R}^2}\Gamma(t+h-s,x_1,y_1,x_2-y_2)M(ds,dy_1,dy_2)\right|^2\right]\\ &\quad = 2\int_0^t\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot))(\xi)-(\mathcal{F}\Gamma(t+h-s,x_1,\cdot,x_2-\cdot))(\xi)\right|^2 d\mu(\xi)ds\\ &\qquad + 2\int_t^{t+h}\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t+h-s,x_1,\cdot,x_2-\cdot))(\xi)\right|^2 d\mu(\xi)ds\\ &\quad = 2\int_0^t\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,\cdot))(\xi)-(\mathcal{F}\Gamma(t+h-s,x_1,\cdot,\cdot))(\xi)\right|^2 d\mu(\xi)ds\\ &\qquad + 2\int_t^{t+h}\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t+h-s,x_1,\cdot,\cdot))(\xi)\right|^2 d\mu(\xi)ds.\end{aligned}$$

The integrand in the first integral is bounded by a constant times $\frac{1}{1+|\xi|^{2/3}}$, which is by assumption integrable with respect to μ; therefore, by dominated convergence the first integral tends to zero as $h\to 0$. The second integral also converges to zero by virtue of (3.2).

The increment in the variable x_2 is treated similarly; in fact, for $t\in[0,T]$, $x_1,x_2,z_2\in\mathbb{R}$ we have

$$\begin{aligned}&\mathbb{E}\left[|u(t,x_1,x_2)-u(t,x_1,z_2)|^2\right]\\ &\quad=\int_0^t\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot))(\xi)-(\mathcal{F}\Gamma(t-s,x_1,\cdot,z_2-\cdot))(\xi)\right|^2 d\mu(\xi)ds\\ &\quad=\int_0^t\int_{\mathbb{R}^2}|e^{ix_2\xi_2}-e^{iz_2\xi_2}|^2\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,\cdot))(\xi)\right|^2 d\mu(\xi)ds,\end{aligned}$$

and by dominated convergence we conclude that the last integral tends to zero as $|x_2-z_2|\to 0$. Lastly, we consider the increment in the variable x_1: for $t\in[0,T]$, $x_1,z_1,x_2\in\mathbb{R}$

we get

$$\begin{aligned}
&\mathbb{E}\left[|u(t,x_1,x_2)-u(t,z_1,x_2)|^2\right]\\
&\quad=\int_0^t\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,x_2-\cdot))(\xi)-(\mathcal{F}\Gamma(t-s,z_1,\cdot,x_2-\cdot))(\xi)\right|^2 d\mu(\xi)ds\\
&\quad=\int_0^t\int_{\mathbb{R}^2}\left|(\mathcal{F}\Gamma(t-s,x_1,\cdot,\cdot))(\xi)-(\mathcal{F}\Gamma(t-s,z_1,\cdot,\cdot))(\xi)\right|^2 d\mu(\xi)ds.
\end{aligned}$$

Recalling the estimate (3.16), we can upper bound the integrand above by a constant of the form $\tilde{k}(t-s,x_1)+\tilde{k}(t-s,z_1)$ times $\frac{1}{1+|\xi|^{2/3}}$; this fact together with dominated convergence implies that the last integral tends to zero as $|x_1-z_1|\to 0$.

References

[1] A. Abdeljawad, A. Ascanelli and S. Coriasco, Deterministic and stochastic Cauchy problems for a class of weakly hyperbolic operators on $\mathbb{R}^n$, *Monatsh. Math.* **192** (2020) 1-38.

[2] A. Ascanelli, S. Coriasco, S. and A. Süß, Solution theory to semilinear hyperbolic stochastic partial differential equations with polynomially bounded coefficients, *Nonlinear Anal. Theory Methods Appl.* **189** (2019) 111–574.

[3] A. Ascanelli, S. Coriasco and A. Süß, Random-field solutions of weakly hyperbolic stochastic partial differential equations with polynomially bounded coefficients, *J. Pseudo-Differ. Oper. Appl.* **11** 1 (2020) 387-424.

[4] A. Ascanelli and A. Süß, Random-field solutions to linear hyperbolic stochastic partial differential equations with variable coefficients, *Stochast. Processes Appl.* **128** (2018) 2605–2641.

[5] E. Bernardi and T. Nishitani, On the Cauchy problem for noneffectively hyperbolic operators: The Gevrey 4 well-posedness, *Kyoto J. Math.*, **51** (2011) 767-810.

[6] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, II Edition, Encyclopedia of Mathematics and its Applications, **152** Cambridge University Press (2014).

[7] R. C. Dalang, Extending martingale measure stochastic integral with application to spatially homogenous S.P.D.E.'s, *Electr. J. Probab.* **4** (1999) 1-29.

[8] R. C. Dalang, R.C. and N. E. Frangos, The stochastic wave equation in two spatial dimensions, *Ann. Probab.*, **26** 1 (1998), 187-212.

[9] R. C. Dalang and C. Mueller, Some non-linear S.P.D.E.'s that are second order in time, *Electron. J. Probab.* **8** (2003) 1–21.

[10] R. C. Dalang and L. Quer-Sardanyons, Stochastic integral for spde's: a comparison, *Expositiones Mathematicae* **29** (2011) 67–109.

[11] L. Hörmander, The Cauchy problem for differential equations with double characteristics, *J. Anal. Math.* **32** (1977) 118–196.

[12] L. Hörmander, Quadratic hyperbolic operators, *Microlocal Analysis and Applications, L. Cattabriga and L. Rodino (eds)* , Lecture Notes in Mathematics, **1495**, Springer Verlag (1991).

[13] V. Ja. Ivrii and V. M. Petkov, Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations (in Russian), *Uspehi Mat. Nauk.* **29** (1974) 3–70.

[14] E. Orsingher, Randomly forced vibrations of a string, *Annales de l'I.H.P. Probabilités et Statistiques* **18** (1982) 367-394.

[15] S. Peszat, The Cauchy problem for a nonlinear stochastic wave equation in any dimension, *J. Evolut. Equ.* **2** (2002) 383–394.

[16] M. Sanz-Solé and A. Süß, The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity, *Electron. J. Probab.* **18** (2013) 1-28.

[17] J. B. Walsh, An Introduction to Stochastic Partial Differential Equations, *Ecole d'Ete de Prob. de St-Flour XIV 1984*, Lect. Notes in Math **1180**, Springer Verlag (1986).

[18] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge University Press (1944).