Forum of Mathematics, Sigma (2026), **Vol. 14**:e73 1–23
doi:10.1017/fms.2026.10216

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Elliptic arrangements of complex multiplication type

Luca Moci[1], Roberto Pagaria[1], Maddalena Pismataro[1] and Alejandro Vargas[2]

[1]Dipartimento di Matematica, University of Bologna, Bologna, Italy; E-mail: roberto.pagaria@unibo.it (Corresponding author).
[2]Warwick Mathematics Institute (WMI), University of Warwick, Coventry, UK; E-mail: alejandro@vargas.page.

Received: 8 July 2025; **Revised:** 8 January 2026; **Accepted:** 31 March 2026

2020 Mathematical Subject Classification: *Primary* – 52C35; *Secondary* – 14N20

Abstract

We provide a natural definition of an elliptic arrangement, extending the classical framework to an elliptic curve $\mathcal{E}$ with complex multiplication. We analyze the intersections of elements of the arrangement and their connected components as $\mathrm{End}(\mathcal{E})$-modules. Furthermore, we prove that the combinatorial data of elliptic arrangements define both an arithmetic matroid and a matroid over the ring $\mathrm{End}(\mathcal{E})$. In this way, we obtain a class of arithmetic matroids that is different from the class of arithmetic matroids realizable via toric arrangements. Finally, we show that the Euler characteristic of the complement is an evaluation of the arithmetic Tutte polynomial.

Contents

1. Introduction

The theory of *hyperplane arrangements* is a bridge between geometry, algebra and combinatorics, with countless applications across these fields and other areas of mathematics. The leitmotif of arrangement theory consists in the explicit computation of geometric invariants – such as cohomology and homotopy type – from combinatorial data associated with the arrangement, such as the poset of layers or the poset of faces. Foundational results of this kind have been discovered for several decades: the Orlik-Solomon algebra [1, 28], the Salvetti complex [34], the cohomology of wonderful models of hyperplane arrangements [11, 19], among others.

In the last two decades, the researchers' attention has shifted toward more general types of arrangements, where the ambient vector space is replaced with richer and more interesting topological spaces. Motivated by applications to integral polytopes and box-splines, De Concini and Procesi [12, 13] initiated the study of *toric arrangements*, defined as collections of hypertori in a complex torus. More recently, other generalizations have emerged: *elliptic arrangements*, which are collections of a certain type of divisors in a product of elliptic curves $\mathcal{E}^n$, and *abelian arrangements*, defined as collections of certain subgroups in an ambient space G^n where G is an abelian Lie group.

The connection between topological invariants and combinatorics persists for toric arrangements, although in a subtler way [26, 9, 10, 8, 7, 27], [30], while it becomes weaker in the elliptic case [24, 15] [31] and in the abelian case [3, 37, 25, 4, 2].

The standard definition of elliptic arrangements, used in the literature, is independent of the specific elliptic curve $\mathcal{E}$. In fact, one only considers divisors in $\mathcal{E}^n$ of the form $\ker \phi$ where

$$\begin{array}{rccc} \phi\colon & \mathcal{E}^n & \longrightarrow & \mathcal{E} \\ & (p_1, \dots, p_n) & \mapsto & m_1 p_1 + \cdots + m_n p_n \end{array}$$

with $m_i \in \mathbb{Z}$. In fact, in this framework the elliptic curve is treated as a topological group $\mathcal{E} \simeq S^1 \times S^1$ that admits an Hodge structure.

In this work, we provide a more general and natural definition of an elliptic arrangement, which is also sensitive to the choice of the curve $\mathcal{E}$, thereby introducing number-theoretic aspects into the picture.

More precisely, let us recall that every endomorphism of an elliptic curve $\mathcal{E}$ arises from multiplication by a complex number lying in a subring $R_{\mathcal{E}}$ of a quadratic number field. The ring of integers $\mathbb{Z}$ is always contained in $R_{\mathcal{E}}$. Generically $R_{\mathcal{E}} = \mathbb{Z}$, but some elliptic curves, called of *complex multiplication type (CM)*, admit more endomorphisms. While all the cited papers assume that $m_i \in \mathbb{Z} \subseteq \mathrm{End}(\mathcal{E})$, in the present paper we drop such assumption, building a more general family of hypersurfaces. In other words, we define elliptic arrangements as collections of kernels of *arbitrary* morphisms of abelian varieties $\phi\colon \mathcal{E}^n \to \mathcal{E}$. We extend all previous results to this setting, which allows us to address new phenomena that arise when $\mathcal{E}$ has complex multiplication.

In Section 2, we recall some basic facts about the endomorphism ring $R_{\mathcal{E}} := \mathrm{End}(\mathcal{E})$ of an elliptic curve $\mathcal{E}$ of complex multiplication type. In particular $R_{\mathcal{E}}$ is an order in a quadratic imaginary number field. In Section 3 we introduce hyperplane, toric, and elliptic arrangements. While the description of the intersections is trivial in the hyperplane case, simple in the toric one, it becomes significantly more complex for the elliptic arrangements. A matrix $A \in \mathrm{Mat}_{k,n}(R_{\mathcal{E}})$ defines an elliptic arrangement of k

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

divisors in $\mathcal{E}^n$. For any $S \subseteq [k]$, we show that the intersection $\mathcal{A}_S$ of the corresponding divisors fits into a short exact sequence:

$$0 \to {}^{\ker \pi_S \circ A_{\mathbb{C}}}\!/_{\ker \pi_S \circ A} \to \mathcal{A}_S \to \operatorname{tor\,coker} \pi_S \circ A \to 0. \tag{1.1}$$

Here, π_S denotes the selection of rows of the matrix A indexed by S. In particular, each intersection $\mathcal{A}_S$ is an extension between an abelian variety and a finite group. Moreover, the number of connected components (or *layers*) in $\mathcal{A}_S$ equals $\#\operatorname{tor\,coker} \pi_S \circ A$ (see Lemma 3.1). We also characterize the abelian varieties that can appear as a layer of an elliptic arrangement in $\mathcal{E}^n$: they are products of elliptic curves $\mathcal{E}_i$ that are isogenous to $\mathcal{E}$, with the conductor of $\operatorname{End}(\mathcal{E}_i)$ dividing that of $\operatorname{End}(\mathcal{E})$ (see Lemma 3.3).

In Section 4, we study $\mathcal{A}_S$ as a $R_{\mathcal{E}}$-module, illustrating through examples how the situation is more subtle than considering $\mathcal{A}_S$ only as $\mathbb{Z}$-module. In particular, we present Example 4.2 where the short exact sequence (1.1) does not split as $R_{\mathcal{E}}$-modules, and we completely characterize when it splits (see Proposition 4.6). Furthermore, the matrix A induces maps $A_\Lambda \colon \Lambda^n \to \Lambda^k$ and $A_R \colon R^n \to R^k$ and we prove in Lemma 4.11 that

$$\Lambda^k / A_\Lambda(\Lambda^n) \cong R^k / A_R(R^n)$$

as $\mathbb{Z}$-modules, thereby filling a gap in the preprint [5] (c.f. Remark 4.12).

In Section 5, we recall the notion of *arithmetic matroid*, introduced in [9], and show that every elliptic arrangement $\mathcal{A}$ gives rise to such a structure. Specifically, the ground set $[k]$, the rank function $\operatorname{rk}_{\mathcal{A}}(S) = \dim_{\mathbb{C}} \ker \pi_S \circ A_{\mathbb{C}}$, and the multiplicity function $m_{\mathcal{A}}(S) = \#\operatorname{tor\,coker} \pi_S \circ A$ define an arithmetic matroid (Theorem 5.11). The proof relies on a new duality construction for elliptic arrangements (Lemma 5.9). Interestingly, while arithmetic matroids were originally introduced to study toric arrangements, we show an elliptic arrangement that corresponds to a nonrealizable arithmetic matroid (Example 5.13). We also observe that the so-called GCD property holds when $R_{\mathcal{E}}$ is Dedekind (Lemma 5.14), but not in general. We briefly relate elliptic arrangements to (poly-)matroids over $R_{\mathcal{E}}$, introduced in [20]. Finally, we compute the Euler characteristic of the complement of an elliptic arrangement (Theorem 5.16), extending Bibby's work [3] to the complex multiplication case.

2. Background on elliptic curves

2.1. Notation

We write $[k]$ for $\{1, 2, \ldots, k\}$; $\#X$ for cardinality of a set; $X \cup i$ for $X \cup \{i\}$; SES for Short Exact Sequence; $\mathrm{CC}(X)$ for connected components of X.

2.2. Elliptic curves with complex multiplication

Let $\mathcal{E}$ be a smooth complex Riemann surface of genus one. All such $\mathcal{E}$ are isomorphic to ${}^{\mathbb{C}}\!/_{\Lambda}$, with Λ a lattice generated by 1 and $\tau \in \mathbb{C} \setminus \mathbb{R}$, and group structure induced by $(\mathbb{C}, +)$. If $\operatorname{End}(\mathcal{E}) \neq \mathbb{Z}$ we say that $\mathcal{E}$ has complex multiplication type. We recall some properties of $\operatorname{End}(\mathcal{E})$ and Λ; for details and introductory references, see [35] or [36].

2.2.1. The endomorphism ring

We describe a ring $R_{\mathcal{E}}$ isomorphic to $\operatorname{End}(\mathcal{E})$. Denote by A_α the linear map given by multiplication by α, that is, $z \mapsto \alpha z$.

Lemma 2.1. *For an elliptic curve $\mathcal{E} = \mathbb{C}/\Lambda$, the ring $\operatorname{End}(\mathcal{E})$ is isomorphic to*

$$R_{\mathcal{E}} = \{\alpha \in \Lambda \colon \alpha\Lambda \subset \Lambda\}$$

via the map $R_{\mathcal{E}} \to \operatorname{End}(\mathcal{E})$ defined by $\alpha \mapsto A_\alpha$.

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Sketch of proof. One shows that if $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is an endomorphism, then an analytic continuation $\hat{f}\colon \mathbb{C} \to \mathbb{C}$ of f around 0 is of the form $\hat{f}(z) = \alpha z$. Moreover, α must have the property that $\alpha\Lambda \subset \Lambda$ for f being well defined when descending to $\mathbb{C}/\Lambda$. See Proposition 6.17 of [36] for details.

We write R instead of $R_{\mathcal{E}}$ when $\mathcal{E}$ is clear from the context.

2.2.2. A quadratic relation for τ

Assume that $\mathrm{End}(\mathcal{E}) \ncong \mathbb{Z}$ and consider α in R. By Lemma 2.1 we have $\alpha\Lambda \subset \Lambda$. Equivalently, $\alpha \cdot 1 \in \Lambda$ and $\alpha \cdot \tau \in \Lambda$. Having $\alpha \cdot 1 \in \Lambda$ gives $R \subset \Lambda$, so we can write $\alpha = x + y\tau$ with $x, y \in \mathbb{Z}$. Thus, $\alpha \cdot \tau = x\tau + y\tau^2$. Hence, the second condition $\alpha \cdot \tau \in \Lambda$ is true if and only if $y\tau^2$ is in Λ. That is, for some $h, k \in \mathbb{Z}$ we have

$$y\tau^2 = h\tau + k. \tag{2.1}$$

Lemma 2.2. *If $\mathcal{E} = \mathbb{C}/\langle 1, \tau\rangle$ has $\mathrm{End}(\mathcal{E}) \ncong \mathbb{Z}$, then $\mathbb{Q}[\tau]$ is a quadratic imaginary number field and $R_{\mathcal{E}}$ is an order, that is, a subring of the ring of integers $\mathcal{O}_{\mathbb{Q}[\tau]}$ of rank two.*

Proof. If $y = 0$ in Equation (2.1), then $\alpha = x$ is an integer. So $\mathrm{End}(\mathcal{E}) \ncong \mathbb{Z}$ is equivalent to the existence of a nonzero choice for y, which gives the quadratic relation $y\tau^2 - h\tau - k = 0$ for τ. Moreover, τ is nonreal because otherwise $\mathrm{span}_{\mathbb{Z}}\{1, \tau\}$ would not be a lattice in $\mathbb{C}$. This proves the statement about $\mathbb{Q}[\tau]$.

Recall that the ring of integers $\mathcal{O}$ is the subring of algebraic numbers $\alpha \in \mathbb{Q}[\tau]$ whose minimal polynomial f_α over $\mathbb{Z}$ is monic. To see that $\alpha \in R_{\mathcal{E}}$ is an algebraic integer, multiply Equation (2.1) by y to get the monic relation $(y\tau)^2 - h(y\tau) - yk = 0$, thus $y\tau \in \mathcal{O}$. Since $x \in \mathbb{Z}$, we have that $\alpha = x + y\tau$ is in $\mathcal{O}$ as desired. □

2.2.3. Quadratic number fields

In view of Lemma 2.2, we recall that every imaginary quadratic number field can be obtained via a unique choice of a square-free positive integer m: let $K_m = \mathbb{Q}(\sqrt{-m})$ be the field containing τ. Consider

$$\omega = \begin{cases} \frac{1+\sqrt{-m}}{2} & \text{if } m \equiv 3 \mod 4, \\ \sqrt{-m} & \text{otherwise,} \end{cases} \tag{2.2}$$

and recall that $\mathcal{O}_m = \mathbb{Z}[\omega]$ is the ring of integers of K_m. In the first case we have the minimal polynomial $f_\omega^{\mathbb{Z}}(\omega) = \omega^2 - \omega + m' = 0$ where $4m' - 1 = m$, and in the second case $f_\omega^{\mathbb{Z}}(\omega) = \omega^2 + m = 0$. Since τ is in K_m, we can choose integers a, b, c with $\gcd(a, b, c) = 1$ such that $\tau = \frac{a+b\omega}{c} \in K_m$. Calculate

$$\mathrm{tr}\, A_\tau = \begin{cases} (2a + b)/c & \text{if } m \equiv 3 \mod 4, \\ 2a/c & \text{else,} \end{cases} \tag{2.3}$$

$$\det A_\tau = \begin{cases} (a^2 + ab + b^2m')/c^2 & \text{if } m \equiv 3 \mod 4, \\ (a^2 + b^2m)/c^2 & \text{else;} \end{cases} \tag{2.4}$$

where $\mathrm{tr}\, A_\tau$ and $\det A_\tau$ are the trace and determinant of the linear map $z \mapsto \tau z$. Thus, the minimal polynomial over $\mathbb{Q}$ of τ is the characteristic polynomial of A_τ:

$$f_\tau^{\mathbb{Q}}(\tau) = \tau^2 - \mathrm{tr}\, A_\tau \tau + \det A_\tau. \tag{2.5}$$

2.2.4. Generators for $R_{\mathcal{E}}$

Consider the isomorphism $(x, y) \mapsto x \cdot 1 + y \cdot \tau$ between $\mathbb{Z}^2$ and Λ. Since R is a subgroup of $\Lambda \cong \mathbb{Z}^2$ with $1 \in R$, we conclude that R is equal to $\langle 1, N\tau\rangle$ for some nonzero integer N. As a consequence of Equation (2.5) and the fact that $N\tau^2$ must have integral coordinates in the basis $\{1, \tau\}$ we get that:

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Lemma 2.3. *We have that $R = \langle 1, N\tau \rangle$ if and only if $N f_\tau^{\mathbb{Q}}$ equals the primitive minimal polynomial $f_\tau^{\mathbb{Z}}$ of τ over $\mathbb{Z}$.*

We assume that N is positive, and calculate:

Lemma 2.4. *We have that*

$$N = c^2/\gcd(c, c^2 \det A_\tau).$$

Proof. This follows from clearing up denominators from $\operatorname{tr} A_\tau$ and $\det A_\tau$ in Equation (2.5). The equalities

$$\begin{aligned} \gcd(c^2, (2a+b)c, a^2+ab+b^2m') &= \gcd(c, a^2+ab+b^2m') && \text{if } m \equiv 3 \mod 4, \\ \gcd(c^2, 2ac, a^2+b^2m) &= \gcd(c, a^2+b^2m) && \text{otherwise;} \end{aligned}$$

hold by the hypothesis $\gcd(a, b, c) = 1$ and the claimed result follows. □

Remark 2.5. An order in a number field K is a subring whose additive group is finitely generated of rank equal to $[K : \mathbb{Q}]$. Thus, $R_{\mathcal{E}}$ is an order in K_m.

3. Arrangements

A central *abelian arrangement* $\mathcal{A}$ is a collection of codimension-1 Lie subgroups $H_1, \ldots, H_k$ in G^n where G is an abelian Lie group. The subgroups H_i are of the form $\ker \phi_i$ for some $\phi_i \in \operatorname{Hom}(G^n, G)$. Let $\operatorname{CM}(\mathcal{A}) = G^n \setminus \bigcup_{i \in [k]} H_i$ be the complement; the study of the topology of $\operatorname{CM}(\mathcal{A})$ is one of the motivations to study $\mathcal{A}$. We recall the affine and toric cases, to motivate our elliptic setting.

3.1. Abelian arrangements

3.1.1. Hyperplane arrangements

Here $G = (K, +)$ for some field K and ϕ_i is some functional in $\operatorname{Hom}(K^n, K)$, thus H_i is a hyperplane. A result by Orlik and Solomon establishes that an important combinatorial invariant of $\mathcal{A}$ is the *lattice of intersections* $\mathcal{P}(\mathcal{A})$, that is, affine spaces $\{\bigcap_{i \in S} H_i \colon S \subset [k]\}$ partially ordered by reverse inclusion. From $\mathcal{P}(\mathcal{A})$ one can reconstruct some topological data of the complement $\operatorname{CM}(\mathcal{A})$: cohomology when $K = \mathbb{C}$, number of connected components when $K = \mathbb{R}$, number of points when K is a finite field $\mathbb{F}_q$. See [17, Theorem 3.5] for an introduction.

It turns out that $\mathcal{P}(\mathcal{A})$ as a lattice is semimodular and atomic, thus cryptomorphically is a matroid, realizable over K, on the groundset $[k]$; see [29, Section 1.7]. Our philosophy is to generalize the matroid approach.

3.1.2. Toric arrangements

Here $G = (\mathbb{C}^*, *)$ is the multiplicative group and ϕ_i is some character in $\operatorname{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) \simeq \mathbb{Z}^n$, so H_i is a union of subtori of codimension 1. We will not consider here the case of subtori of arbitrary codimension, which has been studied in [27]. Write $\mathcal{A}_S$ for the intersection $\bigcap_{i \in S} H_i$ for some $S \subset [k]$. In the toric and elliptic cases $\mathcal{A}_S$ is generally not connected. We call an element of $\operatorname{CC}(\mathcal{A}_S)$ a *layer*, that is, a connected component of $\mathcal{A}_S$. All layers of $\operatorname{CC}(\mathcal{A}_S)$ have the same dimension. If ℓ_1 and ℓ_2 are layers such that $\#\operatorname{CC}(\ell_1 \cap \ell_2) \geq 2$, there is no unique minimal upper bound of ℓ_1 and ℓ_2 in the poset $\{\bigcap_{i \in S} H_i \colon S \subset [k]\}$, so we do not have a lattice. Thus, we call $\mathcal{C}(\mathcal{A})$ the *poset of layers*.

This also means that $\mathcal{C}(\mathcal{A})$ cannot correspond to a matroid. This is partially fixed in [26] by considering the triple $([k], \operatorname{rk}, m)$ of the functions $\operatorname{rk} S = \operatorname{codim} A_S$ and $m(S) = \#\operatorname{CC}(\mathcal{A}_S)$. These satisfy some axioms that are christened an *arithmetic matroid* in [9, 6]. The arithmetic matroid contains enough information to define an arithmetic Tutte polynomial that, like in the hyperplane case, yields important invariants as evaluations [26]. For example: the Poincare polynomial, whose coefficients encode the

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Betti numbers of the complement $\mathrm{CM}(\mathcal{A})$; the characteristic polynomial of $\mathcal{C}(\mathcal{A})$, associated with any poset and containing homological information of its order complex.

3.1.3. Elliptic arrangements

Here $G = \mathcal{E}$ for some elliptic curve $\mathcal{E} = \mathbb{C}/\Lambda$ and ϕ_i is in $\mathrm{Hom}(\mathcal{E}^n, \mathcal{E})$. By Section 2.2 we regard ϕ_i as a scalar product $\phi_i(p) = \langle \alpha_i, p\rangle$ with $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in R^n$. Let A be the matrix (α_{ij}). It gives rise to maps $A_R\colon R^n \to R^k$, $A_\Lambda\colon \Lambda^n \to \Lambda^k$, $A_\mathbb{C}\colon \mathbb{C}^n \to \mathbb{C}^k$ and $A_\mathcal{E}\colon \mathcal{E}^n \to \mathcal{E}^k$, and for convenience if we omit the subscript then we mean A_Λ.

Like in the toric case, we are interested in the number of layers in $\mathcal{A}_S$ for $S \subset [k]$. Note that $\ker \alpha_i = \ker \pi_i \circ A_\mathcal{E}$ is the i-th subvariety in $\mathcal{A}$. Thus, $\mathcal{A}_S = \bigcap_{i\in S} H_i = \ker \pi_S \circ A_\mathcal{E}$. We claim the following identity:

Lemma 3.1. *Let $\mathcal{A}$ be an elliptic arrangement in $\mathcal{E}^n$. For all $S \subset [k]$ the number of layers in $\mathcal{A}_S$ is:*

$$m(S) := \#\,\mathrm{CC}(\mathcal{A}_S) = \#\,\mathrm{tor\,coker}\,\pi_S \circ A_\Lambda.$$

For the proof, we find a short exact sequence with the middle term $\mathcal{A}_S$ such that the sequence splits, giving us a decomposition of $\mathcal{A}_S$ from which we derive the result. Consider the following diagram, where the second and third rows describe the elliptic arrangement and the first and fourth make an exact sequence, with ∂ the map obtained via the snake lemma:

$$\begin{tikzcd}
0 \arrow[r] & \ker A \arrow[r] \arrow[d] & \ker A_\mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}_S \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
0 \arrow[r] & \Lambda^n \arrow[r] \arrow[d, "\pi_S \circ A"] & \mathbb{C}^n \arrow[r] \arrow[d, "\pi_S \circ A_\mathbb{C}"] & \mathcal{E}^n \arrow[r] \arrow[d, "\pi_S \circ A_\mathcal{E}"] & 0 \\
0 \arrow[r] & \Lambda^S \arrow[r, "\iota"] \arrow[d] & \mathbb{C}^S \arrow[r] \arrow[d] & \mathcal{E}^S \arrow[r] \arrow[d] & 0 \\
 & \mathrm{coker}\,A \arrow[r, "\bar{\iota}"] & \mathrm{coker}\,A_\mathbb{C} \arrow[r] & \mathrm{coker}\,A_\mathcal{E} \arrow[r] & 0.
\end{tikzcd}$$

Diagram 1.

To simplify notation, for the remainder of this subsection we write A instead of $\pi_S \circ A$.

We write $\mathrm{rad\,Im}\,A_\Lambda$ for the *radical* of $\mathrm{Im}\,A_\Lambda$, that is, the elements in Λ^S such that they have a nonzero multiple in $\mathrm{Im}\,A$. We obtain our desired short exact sequence:

Lemma 3.2. *Let $\mathcal{A}$ be an elliptic arrangement in $\mathcal{E}^n$. For all $S \subset [k]$ we have the following short exact sequence:*

$$0 \to {}^{\ker A_\mathbb{C}}\!/_{\ker A_\Lambda} \to \mathcal{A}_S \to \mathrm{Im}\,\partial \to 0. \tag{3.1}$$

Moreover, this sequence splits as $\mathbb{Z}$-modules.

Proof. From the first and fourth row of Diagram 1 and the snake lemma we readily get the SES (3.1). Note that ${}^{\ker A_\mathbb{C}}\!/_{\ker A_\Lambda}$ is a divisible abelian group, thus as a $\mathbb{Z}$-module it is injective and the sequence splits as $\mathbb{Z}$-modules. □

Proof of Lemma 3.1. The vector space $\ker A_\mathbb{C} \cong \mathbb{C}^{n-r}$ is connected and so the quotient ${}^{\ker A_\mathbb{C}}\!/_{\ker A}$ is too. From Lemma 3.2 the number of connected components of $\mathcal{A}_S$ is equal to the one of $\mathrm{Im}\,\partial$.

The result follows from $\mathrm{rad\,Im}(A_\Lambda) = (\mathrm{Im}\,A_\mathbb{C}) \cap \Lambda^S$ because

$$\mathrm{Im}\,\partial = {}^{(\mathrm{Im}\,A_\mathbb{C}) \cap \Lambda^S}\!/_{\mathrm{Im}\,A_\Lambda} \text{ and } \mathrm{tor\,coker}\,A_\Lambda = {}^{\mathrm{rad\,Im}(A_\Lambda)}\!/_{\mathrm{Im}(A_\Lambda)}.$$

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Consider the case of CM elliptic curve, let $K = \mathrm{frac}(R)$ and observe that $K^S \cap \mathrm{Im}\, A_{\mathbb{C}} = \mathrm{Im}\, A_K$, moreover every element in K has a multiple in Λ, hence $\Lambda^S \cap \mathrm{Im}\, A_K = \mathrm{rad}\,\mathrm{Im}\, A_\Lambda$. The proof for a non-CM elliptic curve is similar, and we omit it.

Therefore $\mathrm{Im}\,\partial = \mathrm{tor\,coker}\, A$, from which the result follows. □

The behavior of the SES (3.1) is more intricate when regarded as R-modules. This is explored in Section 4.3, including an example where the sequence does not split as R-modules.

3.2. Description of connected components

Now we focus on the description of the connected component of the identity of $\mathcal{A}_S$ as an abelian variety.

The conductor of $R_{\mathcal{E}}$ is $f_{\mathcal{E}} := [\mathcal{O} : R_{\mathcal{E}}] = \frac{bc}{\gcd(c, c^2 \det A_\tau)}$.

Lemma 3.3. *Let us fix a complex multiplication elliptic curve $\mathcal{E}$.*

1. *Every connected component* ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A_\Lambda}$ *of some elliptic arrangement in $\mathcal{E}^n$ is a product of elliptic curves $\mathcal{E}_i$ isogenous to $\mathcal{E}$ such that $f_{\mathcal{E}_i} \mid f_{\mathcal{E}}$;*
2. *Vice versa, every product of elliptic curves $\mathcal{E}_i$ isogenous to $\mathcal{E}$ such that $f_{\mathcal{E}_i} \mid f_{\mathcal{E}}$ is a connected component of some elliptic arrangement in $\mathcal{E}^n$,*

Proof.

1. Consider a connected component of $A_{\mathcal{E}} \colon \mathcal{E}^n \to \mathcal{E}^k$, by the previous discussion or by [22, Theorem 4.4 (c)] every such morphism arises as a morphism of $R_{\mathcal{E}}$-module $A_R^T \colon R^k \to R^n$. Choosing a set of generators of $\mathrm{tor\,coker}\, A_R^T$, we construct another morphism $B_{\mathcal{E}} \colon \mathcal{E}^n \to \mathcal{E}^h$ such that $B_R^T \colon R^h \to R^n$ satisfies $\mathrm{Im}\, B_R^T = \mathrm{rad}\,\mathrm{Im}\, A_R^T$. This implies ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A_\Lambda} = \ker B_{\mathcal{E}}$ by [22, Theorem 4.4 (b)]. The result follows immediately from [22, Theorem 7.5] applied to $B_{\mathcal{E}}$.
2. Let X be an abelian variety satisfying the hypothesis, by [22, Theorem 7.5] it arises from a torsion-free R-module M. Choose a free presentation of M

$$R^k \to R^n \to M \to 0$$

by [22, Theorem 4.4 (b)] it corresponds to a sequence

$$0 \to X \to \mathcal{E}^n \to \mathcal{E}^k$$

hence X is a layer of an arrangement of k divisors in $\mathcal{E}^n$. □

Remark 3.4. The techniques of [22, 23] cannot be applied to the intersection of $\mathcal{A}_S$ but only to a connected component (layer). Indeed the functor $\mathscr{H}om_R(-, \mathcal{E})$ that they consider is not fully faithful on torsion modules.

4. Modules over $R_{\mathcal{E}}$

Our central aim is to describe $\mathcal{A}_S$. A good deal of our efforts are dedicated to studying the behaviour of the sequence

$$\zeta \colon 0 \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \to \mathcal{A}_S \to \mathrm{tor\,coker}\, A \to 0$$

as modules over $R_{\mathcal{E}}$. We give some conditions under which ζ splits, and an example in which it does not.

4.1. An example of nonsplitting

If either $\mathrm{tor\,coker}\, A$ is projective or ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ is injective, we get that ζ splits. The former only happens when $\mathrm{tor\,coker}\, A$ is trivial, because projective modules are torsion free. The latter offers more hope, as ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ is divisible, so we are done if we regard it over a principal ideal domain, for example, as

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

$\mathbb{Z}$-module. Unfortunately, R is not necessarily a PID, and quotients of $\mathbb{C}^n$ by a lattice are not necessarily injective as R-modules; we illustrate this now.

Example 4.1. Let $R = \mathbb{Z}[\sqrt{-3}]$ and $\Gamma = \left\langle 1, (1+\sqrt{-3})/2 \right\rangle$. We claim that $\mathbb{C}/\Gamma$ is not an injective R-module. Consider the map $A_R\colon R^2 \to R$ given by $(x, y) \mapsto 2x + (1+\sqrt{-3})y$. The kernel is generated by

$$\ker A_R = \left\langle v = \begin{matrix} 2 \\ -1+\sqrt{-3} \end{matrix}, w = \begin{matrix} 1+\sqrt{-3} \\ -2 \end{matrix} \right\rangle.$$

Note that $w = (1+\sqrt{-3})/2v$, so the map $v \mapsto 1$ shows that $\ker A_R$ is isomorphic to Γ, thus $\mathbb{C}/\Gamma$ is isomorphic to $\ker A_{\mathbb{C}}/\ker A_R$.

Let $\iota\colon \operatorname{Im} A_R \to R$ and $\pi\colon \mathbb{C} \to \mathbb{C}/\Gamma$ be the canonical injection and surjection, respectively. We construct an $f \in \operatorname{Hom}_R(\operatorname{Im} A_R, \mathbb{C}/\Gamma)$ that cannot be extended to $R \to \mathbb{C}/\Gamma$. By the first isomorphism theorem, a map f in $\operatorname{Hom}_R(\operatorname{Im} A_R, \mathbb{C}/\Gamma)$ is induced by a map $R^2 \to \mathbb{C}/\Gamma$ that vanishes on $\ker A_R$. This gives the following conditions:

$$\begin{aligned} 2f(2) &\equiv (1-\sqrt{-3})f(1+\sqrt{-3}) \mod \Gamma, \\ (1+\sqrt{-3})f(2) &\equiv 2f(1+\sqrt{-3}) \mod \Gamma. \end{aligned} \tag{4.1}$$

We take the following values:

$$f(2) = 0 \qquad\qquad f(1+\sqrt{-3}) = \frac{1}{1-\sqrt{-3}}.$$

Suppose $g\colon R \to \mathbb{C}/\Gamma$ lifts f. For some $\gamma \in \Gamma$ we have that

$$(1+\sqrt{-3})g(1) = g(1+\sqrt{-3}) = f(1+\sqrt{-3}) = \frac{1}{1-\sqrt{-3}} + \gamma.$$

Thus,

$$2g(1) = 2\left(\frac{1}{(1+\sqrt{-3})(1-\sqrt{-3})} + \frac{\gamma}{1+\sqrt{-3}}\right) = \frac{1}{2} + \frac{2\gamma}{1+\sqrt{-3}}.$$

Finally, a quick calculation on the generators verifies that $\frac{2}{1+\sqrt{-3}}\Gamma = \Gamma$, hence $g(2) \equiv 1/2 \mod \Gamma$, a contradiction. Therefore $\mathbb{C}/\Gamma \cong {}^{\ker A_{\mathbb{C}}}/_{\ker A_R}$ is not injective.

The failure of injectivity of ${}^{\ker A_{\mathbb{C}}}/_{\ker A}$ sets the stage for the failure of the sequence to split, which we verify by considering A in the ambient space $(\mathbb{C}/_{\mathbb{Z}[\sqrt{-3}]})^2$.

Example 4.2. Let A be the matrix from Example 4.1. We study the arrangement defined by A in the ambient space $(\mathbb{C}/\Lambda)^2$ with $\Lambda = \mathbb{Z}[\sqrt{-3}]$. That is, the parameters are $m = 3$ and $\tau = \sqrt{-3}$. Set $S = \{1, 2\}$, so $\mathcal{A}_S = \ker A_{\mathcal{E}} = \left\{z \in \mathbb{C}\colon\ A_{\mathbb{C}}(z) \in \mathbb{Z}[\sqrt{-3}]\right\}/(\mathbb{Z}[\sqrt{-3}])^2$. By Lemma 2.4 we have $N = 1$, so R equals $\mathbb{Z}[\sqrt{-3}]$ as well. Thus, $\ker A = \ker A_R$ and Example 4.1 tells us that ${}^{\ker A_{\mathbb{C}}}/_{\ker A}$ is not injective, so there is a chance that ζ does not split. Over $\mathbb{Z}$ we have

$$A_{\mathbb{Z}} = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 1 & 1 \end{pmatrix}. \tag{4.2}$$

Hence, $\#\operatorname{tor\,coker} A = \gcd(2\times 2 \text{ minors of } A) = 2$ and so $\operatorname{tor\,coker} A \cong \mathbb{Z}/2\mathbb{Z}$. This suggests to take ζ and look at elements of order 2 in each of the modules. Given a group X, write $X[2]$ for its

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

2-torsion, that is, elements x such that $2x = 0$. First, we have $(\operatorname{tor}\operatorname{coker} A)[2] = \operatorname{tor}\operatorname{coker} A \simeq R/I$ where $I = (2, 1+\sqrt{-3})$. Since the short exact sequence ζ splits as $\mathbb{Z}$-module, we have

$$\zeta[2]\colon 0 \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}[2] \to \mathcal{A}_S[2] \to \operatorname{tor}\operatorname{coker} A[2] \to 0$$

as R-modules. Since ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ is an elliptic curve, its order-2 points are generated by $v/2$ and $w/2$, with v and w as in Example 4.1. Namely,

$$({}^{\ker A_{\mathbb{C}}}\!/_{\ker A})[2] = \left\{ \begin{matrix} 0 & 1 & (1+\sqrt{-3})/2 & (3+\sqrt{-3})/2 \\ 0, & (-1+\sqrt{-3})/2, & -1 & (-3+\sqrt{-3})/2 \end{matrix} \right\} \simeq \left(R/I\right)^2.$$

We have that $\#\mathcal{A}_S[2] = (\#{}^{\ker A_{\mathbb{C}}}\!/_{\ker A}[2]) \cdot (\#\operatorname{tor}\operatorname{coker} A[2]) = 8$ since $\zeta[2]$ is an exact sequence over $\mathbb{Z}$. We already have four elements coming from the injection of ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$, plus the element $(1/2, 0)$ we had found before, so we compute:

$$(\mathcal{A}_S)[2] = \left\{ \begin{matrix} 0 & 0 & (1+\sqrt{-3})/2 & (1+\sqrt{-3})/2 \\ 0, & (-1+\sqrt{-3})/2, & 0 & (-1+\sqrt{-3})/2, \\ 1/2 & 1/2 & \sqrt{-3}/2 & \sqrt{-3}/2 \\ 0, & (-1+\sqrt{-3})/2, & 0, & (-1+\sqrt{-3})/2 \end{matrix} \right\} \simeq R/I \times R/(2).$$

In particular, $\zeta[2]$ does not split. Therefore ζ does not split, as a splitting of ζ would give a splitting of $\zeta[2]$.

4.2. Splitting of $\mathcal{A}_S$

Choose $S \subseteq [k]$ and for brevity write A instead of A_S. We now relate the splitting of the sequence

$$\zeta\colon 0 \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \to \mathcal{A}_S \to \operatorname{tor}\operatorname{coker} A \to 0$$

with the splitting of the sequence

$$\eta\colon 0 \to \ker A \to \Lambda^n \to \operatorname{Im} A \to 0.$$

The left term of ζ is described by the sequence

$$\mu\colon 0 \to \ker A \to \ker A_{\mathbb{C}} \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \to 0,$$

and the right one by

$$\nu\colon 0 \to \operatorname{Im} A \to \operatorname{rad}\operatorname{Im} A \to \operatorname{tor}\operatorname{coker} A \to 0.$$

Recall that ζ and η correspond to classes in the ext groups $\operatorname{Ext}^1(,\operatorname{tor}\operatorname{coker} A)^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ and $\operatorname{Ext}^1(,\ker A)\operatorname{Im} A$, respectively. We relate these two groups:

Lemma 4.3. *The Diagram 2 of exact sequences commutes.*

Proof. Combine μ and ν using the bifunctoriality of $\operatorname{Hom}(-,-)$. Most of the zeros follow from the fact that $\ker A_{\mathbb{C}}$ is a $\mathbb{C}$-vector space, thus is an injective R-module, hence $\operatorname{Ext}^1(,-)\ker A_{\mathbb{C}}$ is zero. Moreover, $\operatorname{Hom}(\operatorname{tor}\operatorname{coker} A, \ker A_{\mathbb{C}})$ is zero because $\ker A_{\mathbb{C}}$ being a $\mathbb{C}$-vector space has trivial torsion. □

Looking at the middle term of the 4th row of Diagram 2, if there were an $f \in \operatorname{Hom}(\operatorname{Im} A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A})$ that lifts both ζ and η, we could perform diagram chasing to relate ζ and η. Since $A_{\mathbb{C}}$ is a map of vector

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

$$\begin{tikzcd}
 & & 0 \arrow[d] & \mathrm{Hom}(\mathrm{Im}\,A, \ker A) \arrow[d] & \\
0 \arrow[r] \arrow[d] & \mathrm{Hom}(\mathrm{tor\,coker}\,A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathrm{tor\,coker}\,A, \ker A) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}(\mathrm{rad\,Im}\,A, \ker A_{\mathbb{C}}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathrm{rad\,Im}\,A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathrm{rad\,Im}\,A, \ker A) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}(\mathrm{Im}\,A, \ker A_{\mathbb{C}}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathrm{Im}\,A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathrm{Im}\,A, \ker A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(\mathrm{tor\,coker}\,A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}) \arrow[r] \arrow[d] & \mathrm{Ext}^2(\mathrm{tor\,coker}\,A, \ker A) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Ext}^1(\mathrm{rad\,Im}\,A, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}) \arrow[r] & \mathrm{Ext}^2(\mathrm{rad\,Im}\,A, \ker A) &
\end{tikzcd}$$

Diagram 2.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\,A \arrow[r, "\iota"] \arrow[d, "f"] & \mathrm{rad\,Im}\,A \arrow[r] \arrow[d] & \mathrm{tor\,coker}\,A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \arrow[r] & Y \arrow[r] & \mathrm{tor\,coker}\,A \arrow[r] & 0.
\end{tikzcd}$$

Diagram 3.

spaces, there exists a section $s\colon \mathrm{Im}\,A_{\mathbb{C}} \to \mathbb{C}^n$ with $A_{\mathbb{C}} \circ s = \mathrm{id}_{\mathrm{Im}\,A_{\mathbb{C}}}$. Consider $f\colon \mathrm{Im}\,A \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ given by

$$A\lambda \mapsto s(A\lambda) - \lambda.$$

This is well defined because for another λ' such that $A\lambda = A\lambda'$ we have that $A(\lambda - \lambda') = 0$, so $\lambda - \lambda'$ is in $\ker A$ and $(s(A\lambda) - \lambda) - (s(A\lambda') - \lambda') \equiv 0 \bmod \ker A$. We show that f is mapped on the one hand to $[\zeta]$, and on the other to $-[\eta]$.

Lemma 4.4. *In Diagram 2 we have that $f \mapsto [\zeta]$.*

Proof. Let Y be the pushout of $f\colon \mathrm{Im}\,A \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ and $\iota\colon \mathrm{Im}\,A \to \mathrm{rad\,Im}\,A$. By [33, Lemma 7.28] the following diagram commutes Applying the functor $\mathrm{Hom}(-, {}^{\ker A_{\mathbb{C}}}\!/_{\ker A})$ gives that f maps to the class of the bottom row in the corresponding ext group. Thus, we are done if the bottom row is equivalent to ζ. We deal first with the square on the left. Consider the diagram:

$$\begin{tikzcd}
\mathrm{Im}\,A \arrow[r] \arrow[d] & \mathrm{rad\,Im}\,A \arrow[d] \arrow[ddr, bend left, "s"] & \\
{}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \arrow[r] \arrow[drr, bend right, "\iota"'] & Y \arrow[dr, "g"] & \\
 & & \mathcal{A}_S
\end{tikzcd}$$

The diagram commutes: take an arbitrary element $A\lambda$ in $\mathrm{Im}\,A$. Going right and then down we have $A\lambda \mapsto s(A\lambda)$; down and right gives $A\lambda \mapsto s(A\lambda) - \lambda$. The difference of both images is

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

$s(A\lambda) - (s(A\lambda) - \lambda) = \lambda \equiv 0 \bmod \mathcal{A}_S \subset \mathbb{C}^n/\Lambda^n$. Thus, the outer square commutes and by the universal property of the pushout the map $g\colon Y \mapsto \mathcal{A}_S$ exists. We argue that g is an isomorphism.

Recall that Y can be taken equal to ${}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \oplus \operatorname{rad}\operatorname{Im} A$ quotiented by the submodule $\langle(-f(\lambda), \lambda) \mid \lambda \in \operatorname{Im} A\rangle$. In this presentation g is given by $(z, \lambda) \mapsto z + s(\lambda)$.

Surjectivity of g: take an element in $\mathcal{A}_S$ with representative $z \in \mathbb{C}^n$. Thus, $A_{\mathbb{C}}(z) \in \Lambda^S$ and $\lambda := A_{\mathbb{C}}(z) \in \operatorname{rad}\operatorname{Im} A = \Lambda^S \cap \operatorname{Im} A_{\mathbb{C}}$. In particular, $z - s(A_{\mathbb{C}}(z)) \in \ker A_{\mathbb{C}}$ and

$$g((z - s(A_{\mathbb{C}}(z)), A_{\mathbb{C}}(z))) = z - s(A_{\mathbb{C}}(z)) + s(A_{\mathbb{C}}(z)) = z.$$

Injectivity of g: take $z \in \ker A_{\mathbb{C}}$ and $\lambda \in \operatorname{rad}\operatorname{Im} A$ and suppose that $(z, \lambda) \in \ker g$, that is, $z + s(\lambda) = \mu$ for some μ in Λ^n. Since $z \in \ker A_{\mathbb{C}}$, we have $\lambda = A_{\mathbb{C}} \circ s(\lambda) = A(\mu)$. So λ is in $\operatorname{Im} A$; also $f(\lambda) = s(\lambda) - \mu = -z$. Thus, $(z, \lambda) = (-f(\lambda), \lambda) \equiv 0$ in Y, as desired.

To prove the equivalence it remains to show that $Y \to \operatorname{tor}\operatorname{coker} A$ equals $\partial \circ g$. Given (z, λ) as before, the former map sends it to λ in $\operatorname{tor}\operatorname{coker} A$. The latter map first sends it to $z + s(\lambda)$, and then ∂ sends it to $A_{\mathbb{C}}(z + s(\lambda))$, which equals λ. □

The above lemma implies that the short exact sequence $\pi^*\zeta$ always splits, where $\pi\colon \operatorname{rad}\operatorname{Im} A \to \operatorname{tor}\operatorname{coker} A$. However, we already know this fact because it is equivalent to the existence of the section s.

Lemma 4.5. *In Diagram 2 we have that $f \mapsto -[\eta]$.*

Proof. Let X be the pullback of $-f : \operatorname{Im} A \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ and $\iota : \ker A \to \ker A_{\mathbb{C}}$. By [33, Lemma 7.29] the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \ker A \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \operatorname{Im} A \arrow[r] \arrow[d, "-f"] & 0 \\
0 \arrow[r] & \ker A \arrow[r] & \ker A_{\mathbb{C}} \arrow[r] & {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \arrow[r] & 0
\end{tikzcd}$$

Diagram 4.

The functor $\operatorname{Hom}(\operatorname{Im} A, -)$ maps $-f$ to the class of the bottom row in the corresponding Ext^1. Thus, we are done if the top row is equivalent to η. We deal first with the square on the right. Consider the diagram:

$$\begin{tikzcd}
 & \ker A_{\mathbb{C}} \arrow[r] & {}^{\ker A_{\mathbb{C}}}\!/_{\ker A} \\
 & X \arrow[u] \arrow[r] & \operatorname{Im} A \arrow[u, "-f"'] \\
\Lambda^n \arrow[uur, bend left, "\hat{f}"] \arrow[ur, "h"] \arrow[urr, bend right, "A"'] & &
\end{tikzcd}$$

where $\hat{f}$ sends $\lambda \in \Lambda^n$ to $\lambda - s(A\lambda)$; this $\hat{f}$ makes the diagram commute. Thus, by the universal property of the pullback we have a map $h : \Lambda^n \mapsto X$.

Recall that X can be taken to be the submodule of $\ker A_{\mathbb{C}} \oplus \operatorname{Im} A$ of pairs $(z, A\lambda)$ such that $z \equiv -f(\lambda) \bmod \ker A$, and so h maps λ to $(\hat{f}(\lambda), A\lambda)$. Suppose the latter pair is $(0, 0)$ in X, so $A\lambda = 0$, and $0 = \hat{f}(\lambda) = \lambda - s(A\lambda) = \lambda$, which proves injectivity. On the other hand, given an arbitrary element $(z, A\lambda)$ of X, we have $z \equiv -f(\lambda) \bmod \ker A$ and $z = \lambda - s(A\lambda) + \mu$ for some $\mu \in \ker A$. Therefore $\hat{f}(\lambda + \mu) = \lambda + \mu - f(A(\lambda + \mu)) = \lambda - s(A\lambda) + \mu = z$, thus $\lambda - \mu$ maps to $(z, A\lambda)$, proving surjectivity.

Lastly, $\lambda \in \ker A$ gets mapped to $(\hat{f}(z), A\lambda) = (\lambda, 0)$ in X, showing that the bottom row is equivalent to η, so $-f$ maps to $[\eta]$. Since Ext^1 is a group, we conclude that f maps to $-[\eta]$. □

In the following write $\pi\colon \ker A_{\mathbb{C}} \to {}^{\ker A_{\mathbb{C}}}\!/_{\ker A}$ and $\iota\colon \operatorname{Im} A \to \operatorname{rad}\operatorname{Im} A$ for the canonical projection and immersion, respectively. The previous three results give:

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Proposition 4.6. *The element* $[\eta]$ *is in* $\operatorname{Im}\operatorname{Ext}^1(,\iota)\ker A$ *if and only if the sequence* ζ *splits.*

Proof. Notice that by Lemma 4.4 and Lemma 4.5 the image of $[\eta]$ and the one of $[\zeta]$ in $\operatorname{Ext}^2(\operatorname{tor}\operatorname{coker} A, \ker A)$ coincide. The sequence ζ splits if and only if the image of $[\zeta]$ in $\operatorname{Ext}^2(\operatorname{tor}\operatorname{coker} A, \ker A)$ is zero. The latter is equivalent to $\eta \in \operatorname{Im}\operatorname{Ext}^1(,\iota)\ker A$. □

As a corollary we get sufficiently easy conditions for the splitting of ζ.

Corollary 4.7. *If R is Dedekind, then the sequence* ζ *splits.*

Proof. If R is Dedekind, all Ext^2 groups vanish, so $\operatorname{Ext}^1(\iota, \ker A)$ is surjective, thus $[\eta]$ lifts. Alternatively, if we regard the fifth row of Diagram 2 we see that $\operatorname{Ext}^1(\operatorname{tor}\operatorname{coker} A, {}^{\ker A_{\mathbb{C}}}/_{\ker A})$ vanishes when Ext^2 vanishes. □

If the map $A_S\colon \Lambda^n \to \operatorname{Im} A_S$ has a section then the extension ζ is trivial, indeed:

Corollary 4.8. *If* η *splits, then the sequence* ζ *splits.*

Proof. If η splits then $[\eta] = 0$ in $\operatorname{Ext}^1(\operatorname{Im} A, \ker A)$ and the zero class always lifts. □

4.3. The lattice Λ

As $\mathbb{Z}$-module we have that $\Lambda \cong \mathbb{Z}^2$, thus it is free. As R-module we have that Λ is not free because $R \cong \mathbb{Z}^2 \cong \Lambda$ as $\mathbb{Z}$-modules, $\Lambda \cong R$ would be the only option for freeness as R-module, but evidently this is not the case. Clearly Λ is not injective either, since it is not divisible. We show that Λ is projective.

Lemma 4.9. *The lattice* Λ *is a projective R-module.*

Proof. Recall that $R = \mathbb{Z} \oplus N\tau\mathbb{Z}$ and that $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$. Since Λ is closed under sums and $R\Lambda \subset \Lambda$, it is an R-module in $\operatorname{Quot}(R)$, that is, a fractional R-ideal. Thus, Λ is projective if and only if it is invertible; see, for example, [33, Proposition 4.21]. We claim that $N \cdot \Lambda \cdot \sigma(\Lambda) = R$, where σ is the automorphism of $\mathbb{Q}[\sqrt{-m}]$ that sends $\sqrt{-m} \mapsto -\sqrt{-m}$. Indeed,

$$I = N \cdot \Lambda \cdot \sigma(\Lambda) = N \cdot \langle 1, \tau\rangle \cdot \langle 1, \bar{\tau}\rangle = N \cdot \langle 1, \tau, \tau + \bar{\tau}, \tau\bar{\tau}\rangle.$$

From the expression on the right, we see that $I \subset R$. Moreover, I contains $N(\tau + \bar{\tau}) = N \operatorname{tr} A_\tau$, also $N\tau\bar{\tau} = N \det A_\tau$, and N. By Lemma 2.4 these are the coefficients of the minimal polynomial of τ over $\mathbb{Z}$, so $\gcd(N \operatorname{tr} A_\tau, N \det A_\tau, N) = 1$, which means $1 \in I$ as desired. □

4.4. Multiplication by $\alpha \in R$

Let $A \in \operatorname{Mat}_{k\times n}(R_{\mathcal{E}})$ encode an elliptic arrangement in $\mathcal{E}^n$. Consider the induced maps $A_R\colon R^n \to R^k$ and $A_\Lambda\colon \Lambda^n \to \Lambda^k$. As in Section 2.2.1, we have $\tau = (a + b\omega)/c$ with $\gcd(a, b, c) = 1$. We will show that $R^k/\operatorname{Im} A_R$ and $\Lambda^k/\operatorname{Im} A_\Lambda$ are isomorphic as abelian groups. First, a motivating example that corresponds to smallest nontrivial case.

Example 4.10. When $n = k = 1$ we have a map of the form A_α for some $\alpha = x + yN\tau \in R$. We claim that

$$\Lambda/\alpha\Lambda \cong R/\alpha R$$

as $\mathbb{Z}$-modules. For this we write A_α as a matrix A in the basis $\{1, \tau\}$ of Λ, and as a matrix $\widetilde{A}$ in the basis $\{1, N\tau\}$, and we compare their Smith normal form. Consider the integers $\gamma = c \operatorname{tr} A_\tau$, $\delta = c^2 \det A_\tau$, $g = \gcd(c, \delta)$, $c = gc'$ and $\delta = g\delta'$. So that $N = c^2/g = g(c')^2$, we get

$$A = \begin{pmatrix} x & -y\delta' \\ yg(c')^2 & x + y\gamma c' \end{pmatrix} \qquad \widetilde{A} = \begin{pmatrix} x & -y\delta' g(c')^2 \\ y & x + y\gamma c' \end{pmatrix}$$

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Clearly $\det A = \det \widetilde{A}$. We are done if we prove that the g.c.d. of the entries of A and of $\widetilde{A}$ coincide. By Euclidean algorithm this is equivalent to:

$$\gcd\Big(x, y \gcd\Big(g(c')^2, \delta', \gamma c'\Big)\Big) = \gcd\Big(x, y \gcd\Big(1, \delta' g(c')^2, \gamma c'\Big)\Big),$$

which is true if

$$\gcd\Big(g(c')^2, \delta', \gamma c'\Big) = 1. \tag{4.3}$$

By Lemma 2.3 and Theorem 2.4, these three numbers are the coefficients of the minimal polynomial $f_\tau^{\mathbb{Z}}$ of τ over the integers, which implies the coprimality.

Now we consider the general case. Consider the basis $\{1, \tau\}$ of Λ, each entry of A_Λ expands into a 2×2 matrix, as in Example 4.10, to get a matrix in $\mathrm{Mat}_{\mathbb{Z}}(2k, 2n)$ representing A_Λ. Likewise, the basis $\{1, N\tau\}$ gives a matrix in $\mathrm{Mat}_{\mathbb{Z}}(2k, 2n)$ representing A_R. By reordering the bases, we can write:

$$A_\Lambda = \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} \qquad A_R = \begin{pmatrix} X & -Y\delta' N \\ Y & X + Y\gamma c' \end{pmatrix},$$

with $X, Y \in \mathrm{Mat}_{\mathbb{Z}}(k, n)$. We claim that the cokernels of both matrices are isomorphic:

Lemma 4.11. *Given $A_R \colon R^n \to R^k$ and $A_\Lambda \colon \Lambda^n \to \Lambda^k$ we have that*

$$\Lambda^k / A_\Lambda(\Lambda^n) \cong R^k / A_R(R^n)$$

as additive groups.

Proof. We regard R and Λ as $\mathbb{Z}$-modules and we write A for A_Λ and $\widetilde{A}$ for A_R. By the structure theorem for finitely generated modules over PID, it is enough to prove that for all primes $p \in \mathbb{Z}$ the localizations $A_{(p)} \colon (\mathbb{Z}_{(p)})^{2n} \to (\mathbb{Z}_{(p)})^{2k}$ and $\widetilde{A}_{(p)} \colon (\mathbb{Z}_{(p)})^{2n} \to (\mathbb{Z}_{(p)})^{2k}$ have isomorphic cokernels. This property is preserved by applying elementary row and columns operations. We distinguish three cases:

- $p \nmid N$ since N is invertible in $\mathbb{Z}_{(p)}$ we can multiply the second column of $\tilde{A}_{(p)}$ by N^{-1} and the second row by N and obtain the matrix $A_{(p)}$.
- $p \nmid \delta'$ since δ' is invertible in $\mathbb{Z}_{(p)}$ we can multiply the second column of $\tilde{A}_{(p)}$ by $-\delta'^{-1}$ and the second row by $-\delta'$ and obtain the matrix

$$\begin{pmatrix} X & YN \\ -Y\delta' & X + Y\gamma c' \end{pmatrix} \sim \begin{pmatrix} X & YN + \frac{\gamma c'}{\delta'} X \\ -Y\delta' & X \end{pmatrix} \sim \begin{pmatrix} X + Y\gamma c' & YN \\ -Y\delta' & X \end{pmatrix}.$$

  where firstly we added to the second column $\frac{\gamma c'}{\delta'}$ times the first one and then to the first row $\frac{\gamma c'}{\delta'}$ times the second one. Finally, by exchanging both rows and columns we obtain the matrix $A_{(p)}$.
- $p \mid N$ **and** $p \mid \delta'$ Let I_k be the $k \times k$ identity, and I_n the $n \times n$ identity. We modify A and $\widetilde{A}$ so that the integer N is replaced by another integer $N(s)$ coprime with p and then the result follows from the first case. We have for some $s \in \mathbb{Z}$:

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

$$
\begin{aligned}
A_{(p)} &\sim \begin{pmatrix} I_k & 0 \\ -sI_k & I_k \end{pmatrix} \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ sI_n & I_n \end{pmatrix} \\
&= \begin{pmatrix} X - Ys\delta' & -Y\delta' \\ Y(N + s\gamma c' + s^2\delta') & X + Y(\gamma c' + s\delta') \end{pmatrix} \\
\widetilde{A}_{(p)} &\sim \begin{pmatrix} I_k & -\delta' sI_k \\ 0 & I_k \end{pmatrix} \begin{pmatrix} X & -Y\delta' N \\ Y & X + Y\gamma c' \end{pmatrix} \begin{pmatrix} I_n & \delta' sI_n \\ 0 & I_n \end{pmatrix} \\
&= \begin{pmatrix} X - Ys\delta' & -Y\delta'(N + s\gamma c' + s^2\delta') \\ Y & X + Y(\gamma c' + s\delta') \end{pmatrix}
\end{aligned}
$$

Write $N(s)$ for $N + s\gamma c' + s^2\delta'$. We are done if there is a choice of s such that $p \nmid N(s)$. As in Example 4.10, we have $\gcd(\delta', \gamma c', N) = 1$, so $p \nmid \gamma c'$, and we can choose any $s \not\equiv 0 \bmod p$. □

Remark 4.12. Lemma 4.11 is used implicitly in the proof of Theorem 6.1 and Theorem 6.2 of [5]. They only provide the justification for the 1-dimensional case, which corresponds to our Example 4.10.

5. Arithmetic matroid structure

5.1. Arithmetic matroids

Let E be a finite set. A *matroid* on E is given by a function $\mathrm{rk}\colon \mathcal{P}(E) \to \mathbb{N}$ that satisfies:

(r1) $\mathrm{rk}\,\varnothing = 0$,
(r2) $\mathrm{rk}\,X \leq \mathrm{rk}(X \cup i) \leq \mathrm{rk}\,X + 1$ for every $X \subset E$ and $i \in E$,
(r3) $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}\,X + \mathrm{rk}\,Y$ for every $X, Y \subset E$.

These axioms are an abstraction of the following example, called the *realizable case*: given a list $(v_e)_{e\in E}$ of vectors (indexed by E) in some finite dimensional vector space V over some field K, set $\mathrm{rk}\,S = \dim_K \langle v_e \colon e \in S\rangle$ for every subset $S \subset E$. The vectors $(v_e)_{e\in E}$ define an arrangement of hyperplanes on the dual vector space V^*; the cohomology of the complement of this arrangement can be described explicitly in terms of the matroid.

On the other hand, the cohomology of the complement of a *toric arrangement* is not determined by the matroid. Let us recall that a toric arrangement is a family of hypersurfaces in a complex torus $T = (\mathbb{C}^*)^n$, each hypersurface being defined by a character $\chi \in Hom(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$. We set $\mathrm{rk}\,S$ to be the rank of the submodule $\langle \chi_e \colon e \in S\rangle_{\mathbb{Z}}$. This satisfies axioms (r1), (r2), and (r3). Consider also the number $m(S)$ of connected components in the intersection $\bigcap_{i\in S} H_i$. The question of axiomatizing $m(S)$ and studying its properties was addressed in [9, 6, 14]; we now recall some basic facts.

Denote by $[X, Y]$ the interval $\{S \subset E : X \subseteq S \subseteq Y\}$ in $(\mathcal{P}(E), \subseteq)$. We say that $[X, Y]$ is a *molecule* if we can write Y as a disjoint union $Y = X \sqcup F \sqcup T$ such that for each $S \in [X, Y]$ we have

$$\mathrm{rk}(S) = \mathrm{rk}(X) + \#(S \cap F).$$

This amounts to saying that, after contracting the elements of X, the elements of T become loops and the elements of F become coloops. Now, an *arithmetic matroid* is a matroid (E, rk) endowed with a *multiplicity function* $m\colon \mathcal{P}(E) \to \mathbb{N}$ such that the following axioms are satisfied:

(A1) For all $S \subset E$ and $i \in E$: if $\mathrm{rk}(S \cup i) = \mathrm{rk}(S)$, then $m(S \cup i)$ divides $m(S)$; otherwise $m(S)$ divides $m(S \cup i)$.
(A2) If $[X, Y]$ is a molecule then

$$m(X)m(Y) = m(X \cup F)m(X \cup T).$$

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

(P) If $[X, Y]$ is a molecule, $Y = X \sqcup F \sqcup T$, then the number $\rho(X, Y)$ given by

$$\rho(X, Y) = (-1)^{|T|} \sum_{S \in [X,Y]} (-1)^{|Y|-|S|} m(S)$$

is greater or equal than 0.

In the realizable case, Axioms (A1) and (A2) hold by basic algebraic facts on injections, surjections and sums of modules, while Axiom (P) expresses a count of connected components in a toric arrangement, through inclusion-exclusion.

5.2. The arithmetic matroid of an elliptic arrangement

We now show that to an elliptic arrangement one can naturally associate an arithmetic matroid.

Construction 5.1. Let $\mathcal{E}$ be an elliptic curve and let $A \in \mathrm{Mat}_{k\times n}(R_{\mathcal{E}})$ encode an elliptic arrangement in $\mathcal{E}^n$. Given a subset $S \subset [k]$ define:

$$\mathrm{rk}_{\mathcal{A}}(S) = \operatorname{codim} \mathcal{A}_S$$
$$m_{\mathcal{A}}(S) = \#\mathrm{CC}(\mathcal{A}_S)$$

Our aim is to show that the triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ of Construction 5.1 is an arithmetic matroid, by proving that axioms (A1), (A2) and (P) hold in this case.

First, by Lemma 3.1 the multiplicity $m(S)$ equals $\operatorname{tor\,coker} A_S$. For convenience, let us write G_S for $\operatorname{tor\,coker} A_S$. Note that if $X \subset Y$, then the natural projection $\pi\colon \Lambda^Y \to \Lambda^X$ induces a map $\overline{\pi}\colon G_Y \to G_X$ with the following properties.

Lemma 5.2. *Let $S \subseteq [k]$ a set and $i \in [k] \setminus S$ an element. Consider the map $\overline{\pi}\colon G_{S\cup i} \to G_S$.*

1. *If $\mathrm{rk}(S \cup i) = \mathrm{rk}(S)$, then $\overline{\pi}$ is injective.*
2. *If $\mathrm{rk}(S \cup i) \neq \mathrm{rk}(S)$, then $\overline{\pi}$ is surjective.*

Proof. Let e_i be the standard basis vector with 1 in the i-th coordinate and zeros everywhere else. We observe that there exists a nonzero integer k with $ke_i \in \operatorname{Im} A_{S\cup i}$ if and only if $\mathrm{rk}(S \cup i) > \mathrm{rk}(S)$. This is because $\mathrm{rk}(S \cup i) = \mathrm{rk}(S)$ if and only if the i-th coordinate is linearly dependent on those indexed by S.

1. let $\overline{v} \in G_{S\cup i}$ be nonzero, and $v \in \Lambda^{S\cup i}$ a representative. So $mv \in \operatorname{Im} A_{S\cup i}$ for a nonzero m. If $\overline{v} \in \ker \overline{\pi}$, there is $x \in \Lambda^n$ such that $\pi(v) = A_S(x)$. Thus, $v + \lambda e_i \in \operatorname{Im} A_{S\cup i}$ for $\lambda = A_{S\cup i}(x)_i - v_i$; also $m\lambda e_i = m(v + \lambda e_i) - mv$ is in $\operatorname{Im} A_{S\cup i}$. By the observation $m\lambda = 0$, so $\lambda = 0$ and $\ker \overline{\pi}$ is trivial.
2. Let $\overline{v} \in G_S$, and $v \in \Lambda^S$ a representative. So $mv \in \operatorname{Im} A_S$ for a nonzero m, thus by a similar argument as before there is λ such that $(mv, 0) + \lambda e_i$ is in $\operatorname{Im} A_{S\cup i}$. Moreover, by the observation let $k \in \mathbb{Z} \setminus 0$ such that $ke_i \in \operatorname{Im} A_{S\cup i}$. Thus, $km(v, 0) = k((mv, 0) + \lambda e_i) - \lambda(ke_i)$ is in $\operatorname{Im} A_{S\cup i}$, so $\overline{(v, 0)}$ is a torsion element in $G_{S\cup i}$ and also a lift of $\overline{v}$, proving that $\overline{\pi}$ is surjective. □

Corollary 5.3. *The triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (A1).*

Proof. For any $S \subset [k]$ and $i \in [k] \setminus S$, if $\mathrm{rk}_{\mathcal{A}}(S \cup i) = \mathrm{rk}_{\mathcal{A}}(S)$ then $G_{S\cup i} \hookrightarrow G_S$ by Lemma 5.2 and so $m_{\mathcal{A}}(S \cup i) \mid m(S)$ by Lemma 3.1. Otherwise, $\mathrm{rk}_{\mathcal{A}}(S \cup i) = \mathrm{rk}_{\mathcal{A}}(S) + 1$ and the surjection $G_{S\cup i} \twoheadrightarrow G_S$ implies $m_{\mathcal{A}}(S) \mid m_{\mathcal{A}}(S \cup i)$. □

Next, if $[X, Y]$ is a molecule with $Y = X \sqcup F \sqcup T$ we can chain the maps from the previous lemma to get a commutative square.

Lemma 5.4. *The triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (A2). That is, if $[X, Y]$ is a molecule with $Y = X \sqcup F \sqcup T$, we have that*

$$m_{\mathcal{A}}(X) \cdot m_{\mathcal{A}}(X \sqcup F \sqcup T) = m_{\mathcal{A}}(X \sqcup F) \cdot m_{\mathcal{A}}(X \sqcup T).$$

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

$$\begin{tikzcd}
0 \arrow[r] & G_{X\sqcup F\sqcup T} \arrow[r] \arrow[d] & G_{X\sqcup F} \arrow[d] \\
0 \arrow[r] & G_{X\sqcup T} \arrow[r] \arrow[d] & G_X \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Diagram 5.

$$\begin{tikzcd}
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[d] & \ker\psi \arrow[r] \arrow[d] & \ker\overline{\psi} \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
0 \arrow[r] & G_{X\sqcup F\sqcup T} \arrow[r] \arrow[d, "\varphi"] & G_{X\sqcup F} \arrow[r] \arrow[d, "\psi"] & G_{X\sqcup F}/G_{X\sqcup F\sqcup T} \arrow[r] \arrow[d, "\overline{\psi}"] & 0 \\
0 \arrow[r] & G_{X\sqcup T} \arrow[r] \arrow[d] & G_X \arrow[r] \arrow[d] & G_X/G_{X\sqcup F} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Diagram 6.

Proof. We complete Diagram 5 with cokernels and apply the snake lemma to get Diagram 6.

The result follows from the third column if we prove that $\ker\overline{\psi}$ is trivial, that is $\ker\varphi$ and $\ker\psi$ are isomorphic. So we show that $\ker\varphi \to \ker\psi$ is surjective. If $\overline{y}$ is in $\ker\psi$, there is a representative in $\Lambda^{X\sqcup F}$ of the form $(0, v)$, where the zeros are for the coordinates indexed by X. There is a nonzero m such that $(0, mv) = A_{X\sqcup F}(x)$ for some $x \in \Lambda^n$. Since $\mathrm{rk}(X \sqcup T) = \mathrm{rk}(X)$, the coordinates indexed by T are dependent on those indexed by X. Thus, $(0, mv) = A_{X\sqcup F}(x)$ implies $(0, mv, 0) = A_{X\sqcup F\sqcup T}(x)$. Hence $\overline{(0, v, 0)}$ is a torsion element in $G_{X\sqcup F\sqcup T}$ and also the desired lift for $\overline{y}$. □

The remaining Axiom (P) is proved in the next subsection by using duality.

5.3. *Arithmetic matroid duality*

Given a triple $M = (E, \mathrm{rk}, m)$ we define the *dual rank function* $\mathrm{rk}^*(S)$ and the *dual multiplicity* $m^*(S)$ as

$$\mathrm{rk}^*(S) = \#S - (\mathrm{rk}\, E - \mathrm{rk}(E \setminus S)) \qquad m^*(S) = m(E \setminus S). \tag{5.1}$$

Note that $(M^*)^*$ equals M again. By [9, Lemma 2.2], if M is an arithmetic matroid, then so is $M^* = (E, \mathrm{rk}^*, m^*)$ and we call it the *dual arithmetic matroid*. By [6, Section 2] Axiom (P) is equivalent to Axiom (A2) plus:

(P1) For all $X \subset E$, if $Y \in [X, E]$ and $\mathrm{rk}\, X = \mathrm{rk}\, Y$, then $\rho(X, Y) \geq 0$.
(P2) For all $X \subset E$, if $Y \in [X, E]$ and $\mathrm{rk}^*\, X = \mathrm{rk}^*\, Y$, then $\rho^*(X, Y) \geq 0$.

Here ρ^* is the analogous expression for the dual matroid. Looking at Axiom (P) in this setting, we must prove for an elliptic arrangement $\mathcal{A}$ that

$$\rho(X, Y) = \sum_{S \in [X,Y]} (-1)^{|S|-|X|} m(S)$$

is non-negative. We abuse notation by writing $\mathrm{CC}(S)$ instead of $\mathrm{CC}(\mathcal{A}_S)$, that is, the set of connected components of $\mathcal{A}_S = \bigcap_{i \in S} H_i$.

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Lemma 5.5. *Let $\mathcal{A}$ be an elliptic arrangement, for all $X \subset Y \subseteq [k]$ with* $\operatorname{rk} X = \operatorname{rk} Y$*, we have*

$$\rho(X,Y) = \#\Big(\mathrm{CC}(X) \setminus \bigcup_{i \in Y \setminus X} \mathrm{CC}(X \cup i)\Big).$$

Proof. The expression on the right considers all the layers ℓ such that ℓ is in $\mathcal{A}_X$ and for any $i \in Y \setminus X$ we have that ℓ is not a subset of $H_i = \ker A_i$. Note that since $\operatorname{rk} X = \operatorname{rk} Y$, taking connected components $\mathrm{CC}(-)$ is an inclusion-reversing operation on the interval $[X,Y]$, that is, for all $S,T \in [X,Y]$ we have that $S \subset T$ implies that $\mathrm{CC}(T) \supset \mathrm{CC}(S)$. For similar reason we have $\mathrm{CC}(S) \cap \mathrm{CC}(T) = \mathrm{CC}(S \cup T)$ for all $S,T \in [X,Y]$. Recall that $m(S) = \#\,\mathrm{CC}(S)$. The result follows from a straightforward inclusion-exclusion count. □

Corollary 5.6. *In the context of Construction 5.1, the triple* $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ *satisfies Axiom (P1).*

Proof. From Lemma 5.5 follows that $\rho(X,Y)$ is non-negative, as it counts the cardinality of a set. □

Finally, Axiom (P2) would follow from duality if we were able to build an elliptic arrangement realizing the dual arithmetic matroid. We believe this construction to be possible, by developing an analogue of the *generalized toric arrangements* developed in [9, Section 4], but we deem it excessively technical for our aims. Thus, in the next section we approach Axiom (P2) with a weaker construction.

5.4. The dual matroid as a minor

We describe an elliptic arrangement that gives an arithmetic matroid $M_{\mathcal{B}}$ with a minor isomorphic to the dual of M. Thus, $M_{\mathcal{B}}$ satisfies Axiom (P1). Since Axiom (P1) is inherited to minors, M^* satisfies Axiom (P1), and $(M^*)^* = M$ satisfies Axiom (P2) as desired.

The main ingredient is the elliptic analogue of a toric construction from [9, Section 3.4]. Firstly, let us recall that k elements $\mathcal{P} = \{p_1, \dots, p_k\} \subset \mathbb{Z}^n$ give an arithmetic matroid $M_{\mathcal{P}}$ as follows: for $S \subset \mathcal{P}$ let $G_S = \mathrm{rad}_{\mathbb{Z}^n}\langle p\colon\ p \in S\rangle$. Consider then $\mathrm{rk}_{\mathcal{P}}(S) = \mathrm{rk}\langle p\colon\ p \in S\rangle$ and

$$m_{\mathcal{P}}(S) = \#\,\mathrm{tor}(\mathbb{Z}^n/\langle p\colon\ p \in S\rangle) = \#G_S/\langle p\colon\ p \in S\rangle.$$

The triple $M_{\mathcal{P}} = ([k], \mathrm{rk}_{\mathcal{P}}, m_{\mathcal{P}})$ is an arithmetic matroid; see [9, Section 2.4].

Secondly, we recall the contraction of arithmetic matroids $M = (E, \mathrm{rk}, m)$: the contraction M/T by a set $T \subset E$ is an arithmetic matroid on $E \setminus T$ with rank $r_{M/T}(A) = r(A \cup T) - r(T)$ and multiplicity $m_{M/T}(A) = m(A \cup T)$.

Finally, let $\mathcal{Q} = \{q_1, \dots, q_n\} \subset \mathbb{Z}^k$ be the columns of the $(k \times n)$-matrix A whose i-th row is the vector p_i of $\mathcal{P}$. Also let e_i be the i-th standard vector of $\mathbb{Z}^k$ and $\mathcal{B} = \{e_1, \dots, e_k\} \subset \mathbb{Z}^k$ the collection of standard vectors. We consider the matroid $M_{\mathcal{B}\cup\mathcal{Q}}$ associated to the matrix $\mathcal{B} \cup \mathcal{Q}$ with $n + k$ columns:

Lemma 5.7 (Theorem 3.8 from [9]). *Let $\mathcal{P}$ be a list of elements in $\mathbb{Z}^n$, the dual $(M_{\mathcal{P}})^*$ is isomorphic to the contraction $M_{\mathcal{B}\cup\mathcal{Q}}/\mathcal{Q}$.*

We perform a similar construction for elliptic arrangements. Let $\mathcal{A}$ be an elliptic arrangement defined by a matrix $A\colon \mathcal{E}^n \to \mathcal{E}^k$ and consider the dual homomorphism of abelian varieties $A^H\colon (\mathcal{E}^\vee)^k \to (\mathcal{E}^\vee)^n$, where $\mathcal{E}^\vee$ is the dual elliptic curve and A^H is the conjugate transpose of A. Consider $\mathcal{B}$ the arrangement given by the matrix

$$(\mathcal{E}^\vee)^k \xrightarrow[\begin{pmatrix} I_k \\ A^H \end{pmatrix}]{} (\mathcal{E}^\vee)^{k+n}$$

Let $T = \{k+1, \dots, n\}$ be the indexes of the rows of A^H.

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Lemma 5.8. *Let $\mathcal{A}$ be an elliptic arrangement, the dual matroid $M^*_{\mathcal{A}}$ is isomorphic to the contraction $M_{\mathcal{B}}/T$*

Before the proof of Lemma 5.8 we need a technical result.

Lemma 5.9. *For any matrix $A \in \mathrm{Mat}_{k,n}(R)$ we have $\operatorname{tor\,coker}(A\colon \Lambda^n \to \Lambda^k) \simeq \operatorname{tor\,coker}(A^H\colon (\Lambda^\vee)^k \to (\Lambda^\vee)^n)$ as abelian groups.*

Proof. Observe that $\operatorname{tor\,coker} A \simeq \operatorname{tor\,coker} A^T$ because their elementary divisors coincides. It remains to show that $\operatorname{tor\,coker} A \simeq \operatorname{tor\,coker} \overline{A}$ where $\overline{A}$ is the complex conjugate of the matrix A. As in Section 4.4, consider $\mathbb{Z}$-bases of Λ^n and Λ^k given by $\{e_i, \tau e_i\}_i$ for $i \leq n$, respectively $i \leq k$. The matrices that represent A and $\overline{A}$ in the $\mathbb{Z}$-basis are of the form

$$A_{\mathbb{Z}} = \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} \text{ and } \overline{A}_{\mathbb{Z}} = \begin{pmatrix} X + Y\gamma c' & Y\delta' \\ -YN & X \end{pmatrix}$$

We show that they have the same Smith normal form, indeed by exchanging rows and columns

$$A_{\mathbb{Z}} \sim \begin{pmatrix} X + Y\gamma c' & YN \\ -Y\delta' & X \end{pmatrix} \sim \begin{pmatrix} X + Y\gamma c' & Y \\ -Y\delta' N & X \end{pmatrix} \sim \begin{pmatrix} X + Y\gamma c' & -Y\delta' \\ YN & X \end{pmatrix} \sim \overline{A}_{\mathbb{Z}},$$

where the two middle steps follow from the proof of Lemma 4.11. □

Proof of Lemma 5.8. The statement about underlying matroid is classical, we check only that the two multiplicity functions coincide. Let $S \subseteq [k]$, we want to compute

$$m_{\mathcal{B}/T}(S) = m_{\mathcal{B}}(S \cup T) = \#\operatorname{tor\,coker}\begin{pmatrix} (I_k)_S \\ A^H \end{pmatrix}$$

Using row operation we simplify the matrix and obtain

$$\#\operatorname{tor\,coker}\begin{pmatrix} (I_k)_S \\ A^H \end{pmatrix} = \#\operatorname{tor\,coker}\begin{pmatrix} I_s & 0 \\ 0 & (A_{[k]\setminus S})^H \end{pmatrix} = \#\operatorname{tor\,coker}\big((A_{[k]\setminus S})^H\big)$$

where for simplicity in the middle step we have assumed that S are the first s indices. Finally, Lemma 5.9 implies $m_{\mathcal{B}/T}(S) = m_{\mathcal{A}}([k] \setminus S)$ and this completes the proof. □

Corollary 5.10. *In the context of Construction 5.1, the triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (P2).*

Proof. By Corollary 5.3 the triple $M_{\mathcal{B}} = ([k+n], \mathrm{rk}_{\mathcal{B}}, m_{\mathcal{B}})$ satisfies Axiom (P1). This implies that the minor $M_{\mathcal{B}}/T$ satisfies Axiom (P1). To prove the relevant equality, given a molecule $[X, Y]$ in $M_{\mathcal{B}}/T$ just consider the molecule $[X \cup T, Y \cup T]$ in $M_{\mathcal{B}}$. Now by Lemma 5.8, Axiom (P1) holds for $M^*_{\mathcal{A}}$, which as observed before proves Axiom (P2) for $M_{\mathcal{A}}$ by duality. □

Theorem 5.11. *The data of ground set $[k]$, of rank function $\mathrm{rk}_{\mathcal{A}}$, and of number of connected components $m_{\mathcal{A}}(S) = \#\operatorname{tor\,coker} A_S$ gives rise to an arithmetic matroid.*

Proof. Note that $\operatorname{codim} \mathcal{A}_S = \operatorname{rk} A_S$, which implies that $([k], \mathrm{rk}_{\mathcal{A}})$ is an underlying matroid. We are left with: corollary 5.3 proving Axiom (A1); Lemma 5.4 proving Axiom (A2); Corollaries 5.6 and 5.10 proving Axiom (P). □

Remark 5.12. Another proof of Theorem 5.11 is suggested by Emanuele Delucchi. It consists in checking that the action of Λ^n on the periodic hyperplane arrangement $\mathcal{A}^\uparrow$ in $\mathbb{C}^n$ by translations is *arithmetic*, in the sense of [16, Definition 3.15]. Then Theorem 5.11 follows from [16, Theorem C].

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

5.5. Examples

We provide two interesting examples: the first is an arithmetic matroid realizable via elliptic arrangements, but not via toric arrangements. The second is a variation in which we change the elliptic curve, but the defining matrix is the same. This second arrangement defines an arithmetic matroid realizable as toric arrangement.

Recall from [9, Section 3] that all arithmetic matroids realizable in the usual sense (that is, via toric arrangements) have the so-called *GCD property*, that is, the multiplicity function is defined by the values on the independent sets:

$$m(A) = \gcd(\{m(I) \colon\ I \subseteq A \text{ and } |I| = \mathrm{rk}(I) = \mathrm{rk}(A)\}).$$

The other implication and the realizability space were studied in [32]. We now show a generalized elliptic arrangement whose arithmetic matroid does not satisfy the GCD property.

Example 5.13. Let $\Lambda = \mathbb{Z}[\sqrt{-3}]$, $\mathcal{E} = \mathbb{C}/\Lambda$, so $\mathrm{End}(\mathcal{E}) \cong \mathbb{Z}[\sqrt{-3}]$ as in Section 4.3. Consider the arrangement $\mathcal{A}$ associated with the matrix

$$A = \begin{pmatrix} 2 \\ 1+\sqrt{-3} \end{pmatrix}.$$

We get:

$$m_{\mathcal{A}}(1,2) = \#\mathbb{Z}[\sqrt{-3}]/(2, 1+\sqrt{-3}) = 2,$$
$$m_{\mathcal{A}}(1) = \#\mathbb{Z}[\sqrt{-3}]/2R = 4, \qquad m_{\mathcal{A}}(2) = \#\mathbb{Z}[\sqrt{-3}]/(1+\sqrt{-3}) = 4,$$
$$m_{\mathcal{A}}(\varnothing) = 1.$$

This arithmetic matroid does not satisfy the GCD property, since $m(1,2) \neq 4$.

On the other hand, if we consider $\Gamma = \mathbb{Z}[\omega]$ where ω is a third root of unity, the arrangement $\mathcal{B}$ in $\mathbb{C}/\Gamma$ defined by the matrix

$$B = \begin{pmatrix} 2 \\ 1+\sqrt{-3} \end{pmatrix}.$$

has multiplicities $m_{\mathcal{B}}(\varnothing) = 1$ and $m_{\mathcal{B}}(1) = m_{\mathcal{B}}(2) = m_{\mathcal{B}}(1,2) = 4$, which is realizable via a toric arrangement associated to the matrix $B' = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

5.6. The GCD property

The proof of the GCD property for toric arrangements uses the fact that over a PID every matrix has a Smith normal form. This is no longer necessarily true for matrices over R. Recall that if R is Dedekind, all localizations $R_{\mathfrak{p}}$ over all maximal ideals $\mathfrak{p} \subset R$ are discrete valuation rings, therefore PID. This fact, together with a local-global principle argument, allows us to prove:

Lemma 5.14. *Let $\mathcal{A}$ be an elliptic arrangement in $\mathcal{E}^n$. If $R = \mathrm{End}(\mathcal{E})$ is Dedekind, then the arithmetic matroid $M_{\mathcal{A}}$ satisfies the GCD property.*

Proof. Let $\mathfrak{p} \subset R$ be a maximal prime. As usual, denote by $R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}R$ the localization of R and $M_{\mathfrak{p}}$ the $R_{\mathfrak{p}}$-module obtained by localizing M. We use the following facts about the localization of R over a maximal prime ideal $\mathfrak{p} \subset R$:

- $\mathrm{tor}\, M_{\mathfrak{p}} = (\mathrm{tor}\, M)_{\mathfrak{p}}$, see, for example, [20, Proposition 3.3];
- $\mathrm{coker}\, A_{\mathfrak{p}} = (\mathrm{coker}\, A)_{\mathfrak{p}}$, because localization is an exact functor.

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

- If M is a finitely generated torsion module over R, then $M \simeq \bigoplus_{\mathfrak{p}} M_{\mathfrak{p}}$. The isomorphism follows by the structure theorem for modules over a Dedekind domain [18, Chapter 16, Theorem 22] and by the isomorphism $R/\mathfrak{p}^e \simeq R_{\mathfrak{p}}/\mathfrak{p}^e R_{\mathfrak{p}}$.

Applying these facts we get

$$\begin{aligned} m_{\mathcal{A}}(S) &= \#(\operatorname{tor}\operatorname{coker} A_S) \\ &= \#\left(\bigoplus_{\mathfrak{p}} (\operatorname{tor}\operatorname{coker} A_S)_{\mathfrak{p}}\right) = \prod_{\mathfrak{p}} \#\operatorname{tor}\operatorname{coker}(A_S)_{\mathfrak{p}} \end{aligned} \tag{5.2}$$

Recall that R is Dedekind if and only if R is Noetherian and all localizations $R_{\mathfrak{p}}$ at maximal primes are discrete valuation rings. Thus, $m_{\mathfrak{p}}(S) = \#\operatorname{tor}\operatorname{coker}(A_S)_{\mathfrak{p}}$ satisfies the GCD property. Since Equation (5.2) expresses $m_{\mathcal{A}}$ as the product of $m_{\mathfrak{p}}$ over all maximal primes $\mathfrak{p} \subset R$, we have that $m_{\mathcal{A}}$ satisfies the GCD property. □

5.7. *Matroids over rings*

Elliptic arrangements with complex multiplication define naturally matroids over a ring, see [20, 21] for the definition.

Let $\mathcal{A}$ be the arrangement defined by a matrix $A \in \operatorname{Mat}_{k,n}(R)$. Consider the R-module $(\Lambda^\vee)^n$ and the submodules

$$\operatorname{Im}\left(A_S^H : (\Lambda^\vee)^S \to (\Lambda^\vee)^n\right)$$

for each $S \subseteq [k]$. This data defines a *realizable polymatroid over R* with the desirable properties

$$\begin{aligned} \operatorname{codim} \mathcal{A}_S &= \operatorname{rk}_R(\operatorname{coker}(A_S^H)), \\ \#CC(\mathcal{A}_S) &= \#\operatorname{tor}(\operatorname{coker}(A_S^H)). \end{aligned}$$

These properties follow from Lemmas 3.1 and 5.9. This is not a matroid over R because Λ could be a nonfree R-module: take $k = n = 1$ and $A = (1)$, the surjection $\Lambda^\vee \to 0$ does not have a R-cyclic kernel.

The construction presented above is the most natural one; however – if one is more comfortable working with matroids instead of polymatroids – an alternative construction is available. Consider the module R^n and the elements $A_R^H e_i$ for each $i \in [k]$, where e_i is the standard base element of R^k. This data defines a *realizable matroid over R* with the properties

$$\begin{aligned} \operatorname{codim} \mathcal{A}_S &= \operatorname{rk}_R(R^n/\langle A_R^H e_i \mid i \in S\rangle), \\ \#CC(\mathcal{A}_S) &= \#\operatorname{tor}(R^n/\langle A_R^H e_i \mid i \in S\rangle). \end{aligned}$$

The equalities follow from Lemmas 3.1, 5.9 and 4.11.

5.8. *Calculating the Euler characteristic*

The result [3, Remark 4.4] computes the Euler characteristic of the complement $\mathrm{CM}(\mathcal{A})$ for abelian arrangements $\mathcal{A}$ that are defined by a $(k \times n)$-matrix A with integer coefficients. That work can be adapted to our setting where A has coefficients in R.

The hyperplane arrangement induced by $\mathcal{A}$ on the tangent space at a general point of a layer W is denoted by $\mathcal{A}_W^{\uparrow}$. Part of the article [3] holds for any arrangement of divisors, in particular:

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Theorem 5.15 [3, Theorem 3.3]. *Let $\mathcal{A}$ be an arrangement of smooth connected divisors that intersect like hyperplanes in a smooth complex projective variety $\mathcal{X}$. The rational cohomology of the complement* $\mathrm{CM}(\mathcal{A})$ *is isomorphic as a graded algebra to the cohomology of*

$$E_2^{*,*}(\mathcal{A}) = \bigoplus_{W \in \mathcal{P}(\mathcal{A})} H^{top}(M(\mathcal{A}_W^{\restriction})) \otimes H^*(W;\mathbb{Q}) \tag{5.3}$$

with respect to its differential.

We omit the definition of the differential, because we only need that it is homogeneous of bidegree $(-1,2)$. We are going to apply the above theorem to the projective variety $\mathcal{X} = \mathcal{E}^n$ and $\mathcal{A}$ an abelian arrangement in $\mathcal{X}$, possibly defined by non integer coefficients.

Recall that the arithmetic Tutte polynomial of an arithmetic matroid M is

$$T_M(x,y) = \sum_{S \subseteq [k]} m(S)(x-1)^{\mathrm{rk}[k]-\mathrm{rk}\,S}(y-1)^{|S|-\mathrm{rk}\,S}$$

and the characteristic polynomial of a finite graded poset $\mathcal{L}$ is

$$\chi_{\mathcal{P}}(t) = \sum_{W \in \mathcal{P}} \mu_{\mathcal{P}}(\hat{0},W)t^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}\,W}.$$

We get the following result.

Theorem 5.16. *Let $\mathcal{A}$ be an essential elliptic arrangement of rank n. The Euler characteristic of the complement is*

$$e_{\mathrm{CM}(\mathcal{A})} = (-1)^n T_{M_{\mathcal{A}}}(1,0) = \chi_{\mathcal{P}(\mathcal{A})}(0) = (-1)^n \sum_{P \in \mathcal{P}_n(\mathcal{A})} \mathrm{nbc}(P).$$

where $\mathrm{nbc}(P)$ *is the number of no broken circuits with support equals to P.*

Proof. Observe that the poset of layers of $\mathcal{A}_W^{\restriction}$ can be identified with the interval $[\hat{0},W]$ in $\mathcal{L}(\mathcal{A})$ and by the Orlik-Solomon result on the cohomology of hyperplane arrangement we have $\dim_{\mathbb{Q}}(H^{\mathrm{top}}(M(\mathcal{A}_W^{\restriction}))) = (-1)^{\mathrm{rk}\,W}\mu_{\mathcal{L}(\mathcal{A}_W^{\restriction})}(\hat{0},\hat{1}) = (-1)^{\mathrm{rk}\,W}\mu_{\mathcal{L}(\mathcal{A})}(\hat{0},W)$, that is, the number of no-broken circuits with support equals to W. Let us start by considering the bigraded Poincaré polynomial of $E_2^{*,*}$, described in Equation (5.3):

$$P_{E_2}(t,s) = \sum_{p,q} \dim(E_2^{p,q})t^p s^q = \sum_{W \in \mathcal{P}(\mathcal{A})} (-1)^{\mathrm{rk}\,W}\mu_{\mathcal{P}(\mathcal{A})}(\hat{0},W)(1+t)^{2(n-\mathrm{rk}\,W)}s^{\mathrm{rk}\,W}.$$

Recall that

$$\chi_{\mathcal{P}(\mathcal{A})}(t) = (-1)^n T_{M_{\mathcal{A}}}(1-t,0)$$

by combining the proof of [26, Theorem 5.6] with Theorem 5.11 or by applying [16, Theorem F]. Therefore,

$$P_{E_2}(t,s) = (-s)^n \chi_{\mathcal{P}(\mathcal{A})}\left(\frac{(1+t)^2}{-s}\right) = s^n T_{M_{\mathcal{A}}}\left(1+\frac{(1+t)^2}{s},0\right).$$

Finally, setting $s = t = -1$ we have

$$e_{\mathrm{CM}(\mathcal{A})} = (-1)^n T_{M_{\mathcal{A}}}(1,0) = \chi_{\mathcal{P}(\mathcal{A})}(0) = (-1)^n \sum_{P \in \mathcal{P}_n(\mathcal{A})} \mathrm{nbc}(P).$$

□

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press

Proposition 5.17. *The E-polynomial of an essential elliptic arrangement $\mathcal{A}$ is*

$$E_{\mathrm{CM}(\mathcal{A})}(x,y) = (xy)^{\mathrm{rk}(\mathcal{A})}\chi_{\mathcal{P}(\mathcal{A})}\left(\frac{(1-x)(1-y)}{xy}\right).$$

Proof. The mixed Hodge polynomial of $E_2^{*,*}$ with the natural Hodge structure defined in [3] is

$$H(x,y,t) = \sum_{W\in\mathcal{P}(\mathcal{A})} (-1)^{\mathrm{rk}\,W}\mu_{\mathcal{P}(\mathcal{A})}(\hat{0},W)((1+xt)(1+yt))^{(\mathrm{rk}(\mathcal{A})-\mathrm{rk}\,W)}(xyt)^{\mathrm{rk}\,W}$$

because the cohomology of the complement of hyperplane arrangement in degree k is pure of type (k,k). The E-polynomial is the specialization at $t=-1$ and the claimed equality follows. □

Competing interests. The authors have no competing interest to declare.

Funding statement. The authors are members of the project PRIN 2022 "ALgebraic and TOPological combinatorics (ALTOP)" CUP J53D23003660006. The second author is partially supported by INdAM - GNSAGA Project CUP E53C23001670001. The fourth author has received support from EPSRC grant EP/X02752X/1. First and second authors are members of the INdAM - GNSAGA group.

References

[1] V. I. Arnol' d, 'The cohomology ring of the group of dyed braids', *Mat. Zametki* **5** (1969), 227–231. MR 242196
[2] E. Bazzocchi, R. Pagaria and M. Pismataro, 'Cohomology ring of non-compact abelian arrangements', *Proceedings of the London Mathematical Society* **131**(6) (2025), e70113.
[3] C. Bibby, 'Cohomology of abelian arrangements', *Proceedings of the American Mathematical Society* **144**(7) (2015), 3093–3104.
[4] C. Bibby and E. Delucchi, 'Supersolvable posets and fiber-type abelian arrangements', *Selecta Math. (N.S.)* **30**(5) (2024), Paper No. 89, 39. MR 4808965
[5] A. Borzì and I. Martino, 'Set of independencies and tutte polynomial of matroids over a domain', arXiv preprint arXiv:1909.00332 (2019).
[6] P. Brändén and L. Moci, 'The multivariate arithmetic Tutte polynomial', *Trans. Amer. Math. Soc.* **366**(10) (2014), 5523–5540. MR 3240933
[7] F. Callegaro, M. D'Adderio, E. Delucchi, L. Migliorini and R. Pagaria, 'Orlik-Solomon type presentations for the cohomology algebra of toric arrangements', *Trans. Amer. Math. Soc.* **373**(3) (2020), 1909–1940. MR4068285
[8] F. Callegaro and E. Delucchi, 'The integer cohomology algebra of toric arrangements', *Adv. Math.* **313** (2017), 746–802. MR 3649237
[9] M. D'Adderio and L. Moci, 'Arithmetic matroids, the Tutte polynomial and toric arrangements', *Adv. Math.* **232** (2013), 335–367. MR 2989987
[10] G. d'Antonio and E. Delucchi, 'A Salvetti complex for toric arrangements and its fundamental group', *Int. Math. Res. Not. IMRN* **15** (2012), 3535–3566. MR 2959041
[11] C. Concini and C. Procesi, 'Wonderful models of subspace arrangements', *Selecta Math. (N.S.)* **1**(3) (1995), 459–494. MR 1366622
[12] C. Concini and C. Procesi, 'On the geometry of toric arrangements', *Transform. Groups* **10**(3–4) (2005), 387–422. MR 2183118
[13] C. Concini and C. Procesi, *Topics in Hyperplane Arrangements, Polytopes and Box-splines*, Universitext (Springer, New York, 2011). MR 2722776
[14] E. Delucchi and L. Moci, 'Products of arithmetic matroids and quasipolynomial invariants of CW-complexes', *J. Combin. Theory Ser. A* **157** (2018), 28–40. MR 3780405
[15] E. Delucchi and R. Pagaria, 'The homotopy type of elliptic arrangements', *Algebr. Geom. Topol.* **21**(4) (2021), 2037–2063. MR 4302492
[16] E. Delucchi and S. Riedel, 'Group actions on semimatroids', *Adv. in Appl. Math.* **95** (2018), 199–270. MR 3759217
[17] A. Dimca, *Hyperplane Arrangements*, Universitext (Springer, Cham, 2017), An introduction. MR 3618796
[18] D. S. Dummit and R. M. Foote, *Abstract Algebra*, third ed. (John Wiley & Sons, Inc., Hoboken, NJ, 2004). MR 2286236
[19] E. Feichtner and S. Yuzvinsky, 'Chow rings of toric varieties defined by atomic lattices', *Invent. Math.* **155**(3) (2004), 515–536. MR 2038195
[20] A. Fink and L. Moci, 'Matroids over a ring', *J. Eur. Math. Soc.* **18**(4) (2016), 681–731.
[21] A. Fink and L. Moci, 'Polyhedra and parameter spaces for matroids over valuation rings', *Adv. Math.* **343** (2019), 448–494.
[22] B. W. Jordan, A. G. Keeton, B. Poonen, E. M. Rains, N. Shepherd-Barron and J. T. Tate, 'Abelian varieties isogenous to a power of an elliptic curve', *Compos. Math.* **154**(5) (2018), 934–959. MR 3798590

[23] E. Kani, 'Products of CM elliptic curves', *Collect. Math.* **62**(3) (2011), 297–339. MR 2825715
[24] A. Levin and A. Varchenko, *Cohomology of the Complement to an Elliptic Arrangement*, Configuration Spaces (A. Bjorner, F. Cohen, C. De Concini, C. Procesi, and M. Salvetti, eds.), Scuola Normale Superiore, 2012, pp. 373–388.
[25] Y. Liu, T. Nhat Tran and M. Yoshinaga, 'G-Tutte polynomials and abelian Lie group arrangements' , *Int. Math. Res. Not. IMRN* **1** (2021), 152–190. MR4198494
[26] L. Moci, 'A Tutte polynomial for toric arrangements', *Trans. Am. Math. Soc.* **364**(2) (2012), 1067–1088.
[27] L. Moci and R. Pagaria, 'On the cohomology of arrangements of subtori', *J. London Math. Soc.* **106**(3) (2022), 1999–2029.
[28] P. Orlik and L. Solomon, 'Combinatorics and topology of complements of hyperplanes', *Invent. Math.* **56**(2) (1980), 167–189. MR 558866
[29] J. Oxley, *Matroid Theory*, vol. 3 (Oxford University Press, Oxford, 2011), xiv+684 pp.
[30] R. Pagaria, 'Two examples of toric arrangements', *J. Combin. Theory Ser. A* **167** (2019), 389–402. MR 3954076
[31] R. Pagaria, 'Poincaré polynomial of elliptic arrangements is not determined by the Tutte polynomial', *Discrete Math.* **343**(6) (2020), 111710, 4. MR 4087210
[32] R. Pagaria and G. Paolini, 'Representations of torsion-free arithmetic matroids', *European J. Combin.* **93** (2021), Paper No. 103272, 17. MR 4186616
[33] J. J. Rotman, *An Introduction to Homological Algebra* (Springer Science & Business Media, New York, 2009), xiv+709 pp.
[34] M. Salvetti, 'Topology of the complement of real hyperplanes in $\mathbb{C}^N$', *Invent. Math.* **88**(3) (1987), 603–618. MR 884802
[35] J. H. Silverman, *The Arithmetic of Elliptic Curves*, second ed., Graduate Texts in Mathematics, vol. 106 (Springer, Dordrecht, 2009). MR 2514094
[36] J. H. Silverman and J. T. Tate, *Rational Points on Elliptic Curves* (Springer, Cham, 2015), xxii+332 pp.
[37] T. Tran and M. Yoshinaga, 'Combinatorics of certain abelian Lie group arrangements and chromatic quasi-polynomials', *J. Combin. Theory Ser. A* **165** (2019), 258–272. MR 3916005

https://doi.org/10.1017/fms.2026.10216 Published online by Cambridge University Press