Useful elements in common between gH-based extended-interval and probabilistic arithmetic

Maria Letizia Guerra[1], Laerte Sorini[2], and Luciano Stefanini[2]

[1] University of Bologna, Italy - mletizia.guerra@unibo.it
[2] University of Urbino, Italy - {laerte.sorini;luciano.stefanini}@uniurb.it

Abstract. Building on new properties of generalized Hukuhara (gH) operations - in particular, the fact that gH-based arithmetic identifies the exact core (i.e., the points of maximum PDF value) of variable $Z = XoY$ when X, Y are uniform random variables on compact intervals and o is one of the four arithmetic operations - we show that, when combined with the well-known property that Minkowski-based arithmetic provides the exact support of Z, the PDF of Z, in the family of generalized trapezoidal distributions, can be accurately approximated by linearly connecting the extremes of its support and core. This result enables a new, efficient implementation of histogram-based arithmetic for general random variables X, Y whether independent or dependent.

1 Extended gH-based arithmetic for intervals.

The arithmetic operations $C = A \circ B$ in standard interval analysis (SIA), with A, B compact intervals and $\circ \in \{+, -, \times, /\}$, primarily introduced in Moore's work ([1]), are strongly related to probabilistic arithmetic (PA), say $Z = X \circ Y$; in particular, if A, B, C are the supports of X, Y, Z, respectively and X, Y are uniform and independent, then the resulting operations in SIA coincide with the support of Z in PA ([2]).

Starting with [5], where the interval SIA operations are extended with the generalized Hukuhara arithmetic (gH-type $\{+, -, \times, /\}$), we highlight that all these operations are related to probabilistic arithmetic; in particular, not only they offer a natural (exact) computation of the support (assuming it is bounded), but also of the *core* (corresponding to the points at which the probability density assumes its maximum value). Their knowledge allows to obtain an approximation of the PDF (or the CDF) of a probabilistic arithmetic operation in terms of trapezoidal distributions, for uniform independent X, Y variables. In addition, the obtained approximation can be embedded into a general histogram-based arithmetic procedure to approximate the arithmetic of uniform or non uniform variables for the independent (using marginal distributions) or dependent (using joint distributions) cases.

As conventionally assumed, $\mathbb{R}$ denotes the set of real numbers and $\mathcal{K}_C$ is the space of all real compact intervals $[a^-, a^+]$ where $a^-, a^+ \in \mathbb{R}$ and $a^- \leq a^+$; equivalently, an interval is denoted through *midpoint* representation $(\widehat{a}; \widetilde{a})$, where

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$\widehat{a}, \widetilde{a} \in \mathbb{R}$ with $\widetilde{a} \geq 0$ such that $\widehat{a} = \frac{a^+ + a^-}{2}$ and $\widetilde{a} = \frac{a^+ - a^-}{2}$, i.e., $a^- = \widehat{a} - \widetilde{a}$, $a^+ = \widehat{a} + \widetilde{a}$.

We then write an interval $A \in \mathcal{K}_C$, either as $A = [a^-, a^+]$ or, equivalently, $A = (\widehat{a}; \widetilde{a})$.

Given $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}_C$ and $\tau \in \mathbb{R}$, we have the following classical Moore-Minkowski interval scalar multiplication and arithmetic operations:

- $\tau A = \{\tau a : a \in A\} = \begin{cases} [\tau a^-, \tau a^+], \text{ if } \tau \geq 0, \\ [\tau a^+, \tau a^-], \text{ if } \tau \leq 0 \end{cases};$
- $A \oplus_M B = [a^- + b^-, a^+ + b^+]$,
- $A \ominus_M B = A \oplus_M (-1)B = [a^- - b^+, a^+ - b^-]$,
- $A \otimes_M B = [p^-, p^+]$, where $p^- = min\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}$ and $p^+ = max\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}$,
- $A \oslash_M B = A \otimes_M B^{-1}$, assuming $0 \notin B$, where $B^{-1} = [1/b^+, 1/b^-]$.

Remark that, if $0 \notin A$ and $0 \notin B$, then $A \otimes_M B = A \oslash_M B^{-1} = B \oslash_M A^{-1}$.

We denote the generalized Hukuhara operations (gH operations, see [5]) of two intervals A and B as $\oplus_{gH}$, $\ominus_{gH}$, $\otimes_{gH}$, $\oslash_{gH}$; they are obtained as follows:

- the gH-difference is the interval $C \in \mathcal{K}_C, C = A \ominus_{gH} B$ such that (i) $A = B \oplus_M C$ or (ii) $B = A \ominus_M C$ and we have $A \ominus_{gH} B = (\widehat{a} - \widehat{b}; \left|\widetilde{a} - \widetilde{b}\right|)$;
- the gH-addition is $A \oplus_{gH} B = A \ominus_{gH} (-B) = (\widehat{a} + \widehat{b}; \left|\widetilde{a} - \widetilde{b}\right|)$;
- the gH-division is defined to be the interval $C \in \mathcal{K}_C$, $C = A \oslash_{gH} B$ such that (i) $A = B \otimes_M C$ or (ii) $B = A \otimes_M C^{-1}$; the details on computation of interval C, assuming $0 \notin B$, are given in [5];
- the gH-multiplication $A \otimes_{gH} B$ is defined in terms of the gH-division by

$$A \otimes_{gH} B = \begin{cases} \{0\} & \text{if } 0 \in A \text{ and } 0 \in B, \\ A \oslash_{gH} B^{-1} & \text{if } 0 \notin B, \\ B \oslash_{gH} A^{-1} & \text{if } 0 \notin A; \end{cases}$$

(if $0 \notin A$ and $0 \notin B$ we have $A \oslash_{gH} B^{-1} = B \oslash_{gH} A^{-1}$).

In this paper, we make use of the following probabilistic division $C = A \oslash_{pgH} B$, similar to $A \oslash_{gH} B$ in [5], defined as follows, according to position of A, B:

(1) case with $0 < a^- \leq a^+$ and $b^- \leq b^+ < 0$: (two sub-cases)
if $a^-b^- \geq a^+b^+$, then $C = [a^+/b^-, a^-/b^+]$, else $C = \{a^+/b^-\}$;
(2) case with $0 < a^- \leq a^+$ and $0 < b^- \leq b^+$: (two sub-cases)
if $a^-b^+ \leq a^+b^-$, then $C = [a^-/b^-, a^+/b^+]$, else $C = \{a^+/b^+\}$;
(3) case with $a^- \leq a^+ < 0$ and $b^- \leq b^+ < 0$: (two sub-cases)
if $a^+b^- \leq a^-b^+$, then $C = [a^+/b^+, a^-/b^-]$, else $C = \{a^-/b^-\}$;
(4) case with $a^- \leq a^+ < 0$ and $0 < b^- \leq b^+$: (two sub-cases)
if $a^-b^- \leq a^+b^+$, then $C = [a^-/b^+, a^+/b^-]$, else $C = \{a^-/b^+\}$;
(5) case with $a^- \leq 0 \leq a^+$ and $b^- \leq b^+ < 0$: $C = [a^+/b^-, a^-/b^-]$;
(6) case with $a^- \leq 0 \leq a^+$ and $0 < b^- \leq b^+$: $C = [a^-/b^+, a^+/b^+]$.

2 Between interval and probabilistic arithmetic.

The cumulative distribution function (CDF) gives the probability that a (real) random variable X takes value less than or equal to a particular number; it is a non-decreasing function $F_X : \mathbb{R} \longrightarrow [0,1]$ that provides a complete description of the distribution of X. In the continuous case, the CDF is the integral of the probability distribution (or density) function $p_X(x)$ (PDF).

For a given variable X, its support set $supp(X) \subseteq \mathbb{R} \bigcup \{-\infty, +\infty\}$ is defined to be the set of points $x \in \mathbb{R}$ where $p_X(x) = F'_X(x)$ exists and is nonzero; we define the *support interval* of X to be its convex hull, i.e., the interval

$$Isupp(X) = [\inf \{x|p_X(x) > 0\}, \sup \{x|p_X(x) > 0\}] \subseteq [-\infty, +\infty]. \quad (1)$$

Defining the *modal density value* of X to be $p^*_X = sup\{p_X(x), x \in \mathbb{R}\}$, the *core interval* of X, assuming $0 < p^*_X < +\infty$, is defined to be the interval

$$Icore(X) = [\inf \{x|p_X(x) \geq p^*_X\}, \sup \{x|p_X(x) \geq p^*_X\}] \subset \mathbb{R}. \quad (2)$$

In the case of a variable X with CDF F_X and bounded support interval $Isupp(X) = A = [a^-, a^+]$, we clearly have $Icore(X) \subseteq Isupp(X)$ with equality if and only if X is uniform on its support. It is also obvious that, if $Icore(X) = [a^-_*, a^+_*]$ is not a singleton, then, on $[a^-_*, a^+_*]$, its PDF has constant value p^*_X and the CDF $F_X(x)$ is linear.

Consider any two intervals $A, B \in \mathcal{K}_\mathcal{C}$ and operations $+, -, \times, /$ to the pair A, B; we associate A and B to independent uniform random variables X and Y, respectively, having support intervals $Isupp(X) = A$ and $Isupp(Y) = B$.

Our main result is the following

Theorem 1. *Let $A = [a^-, a^+], B = [b^-, b^+] \in \mathcal{K}_C$, $a^- < a^+, b^- < b^+$ be any two (non-singleton) compact intervals and let X, Y be uniform random variables having $Isupp(X) = A$, $Isupp(Y) = B$; then, assuming that X and Y are independent,*
(a) the support and core of the sum and difference of X and Y are:

$$Isupp(X + Y) = A \oplus_M B \text{ and } Icore(X + Y) = A \oplus_{gH} B, \quad (3)$$

$$Isupp(X - Y) = A \ominus_M B \text{ and } Icore(X - Y) = A \ominus_{gH} B; \quad (4)$$

(b) the support and core of the product $X \times Y$ are:

$$Isupp(X \times Y) = A \otimes_M B \text{ and } Icore(X \times Y) = A \otimes_{gH} B; \quad (5)$$

(c) assuming also $0 \notin B$, the support and core intervals of the X/Y is given by:

$$Isupp(X/Y) = A \oslash_M B \text{ and } Icore(X/Y) = A \oslash_{pgH} B. \quad (6)$$

Proof. We just show that the proof for all cases is simple but somehow tedious; indeed, it is well known (e.g., [2]) that computing the PDFs for arithmetic operations requires integral convolutions: e.g., for addition or multiplication of X, Y, having PDFs $p_X(x), p_Y(y)$, we have for $z \in \mathbb{R}$

$$p_{X+Y}(z) = \int_{-\infty}^{+\infty} p_X(x) p_Y(z-x) dx, \quad p_{XY}(z) = \int_{-\infty}^{+\infty} p_X(x) p_Y(\frac{z}{x}) \frac{1}{|x|} dx. \tag{7}$$

Consider addition $Z = X + Y$ and equation (3); being X, Y uniform on their supports $A = [a^-, a^+], B = [b^-, b^+]$, PDFs are $p_X(x) = \frac{1}{a^+ - a^-}, x \in A$, $p_Y(y) = \frac{1}{b^+ - b^-}, y \in B$ (and zero elsewhere). Integral (7) becomes $p_Z(z) = \frac{1}{(a^+ - a^-)(b^+ - b^-)} \int_{-\infty}^{+\infty} \mathbb{I}_A(x) \mathbb{I}_B(z-x) dx$ where $\mathbb{I}_A, \mathbb{I}_B$ are the set-indicator functions on A, B. By distinguishing the two cases $a^- + b^+ \leq a^+ + b^-$ or $a^- + b^+ > a^+ + b^-$, we have, respectively, that $c_Z - a_Z = b^+ - b^-$ or $c_Z - a_Z = a^+ - a^-$; it is then easy to verify that, denoting $A \oplus_M B = [a_Z, b_Z]$ and $A \oplus_{gH} B = [c_Z, d_Z]$, we have $a_Z \leq c_Z \leq d_Z \leq b_Z$ and, for $z \in [a_Z, b_Z]$ (otherwise $p_Z(z) = 0$),

$$p_Z(z) = \frac{1}{(a^+ - a^-)(b^+ - b^-)} \begin{cases} z - a_Z & \text{for } \ a_Z \leq z \leq c_Z \\ c_Z - a_Z & \text{for } c_Z \leq z \leq d_Z \\ b_Z - z & \text{for } d_Z < z \leq b_Z. \end{cases} \tag{8}$$

This completes the proof of (3). For the other operations, we proceed in analogous way, by distinguishing all possible positions of A with respect to B, each giving different expressions for the support and core of Z, and simple integrals for which the values can be obtained in terms of the extreme points of indicated interval supports and cores (for $Z = X - Y$ there are two different cases, nine for $Z = X \times Y$ and six for $Z = X/Y$). ■

3 Application to histogram-based probabilistic arithmetic

Arithmetic operations on random variables, in particular having different distributions, occur very frequently in scientific computations, especially in statistics and data analysis where the variables are known in terms of empirically based histograms and estimated density or cumulative distribution functions.

On the other hand, if we know the (finite) support $[a, b]$ and core $[c, d] \subseteq [a, b]$ of a PDF p_Z, we can always define a so-called trapezoidal probability distribution by

$$t(z; a, b, c, d) = \begin{cases} h \dfrac{z-a}{c-a} & \text{for } a \leq z \leq c \\ h & \text{for } c \leq z \leq d \\ h \dfrac{b-z}{b-d} & \text{for } d \leq z \leq b \end{cases} \tag{9}$$

expressed in terms of the four parameters $a, b, c, d \in \mathbb{R}$, having exactly the same support and core; the quantity $h = \dfrac{2}{b - a + d - c} > 0$ is the maximum value of $t(z)$ and is determined such that the integral of $t(z; a, b, c, d)$ on the support is equal to one.

Let $T(z; a, b, c, d)$ be the CDF of $t(z; a, b, c, d)$.

Our results can be summarized as follows: let X, Y be two uniform random variables on intervals A, B, respectively; then

– (Addition): the PDF of $Z = X + Y$ is the trapezoidal distribution (9) with $[a, b] = A \oplus_M B$ and $[c, d] = A \oplus_{gH} B$;
– (Subtraction): the PDF of $Z = X - Y$ is the trapezoidal distribution (9) with $[a, b] = A \ominus_M B$ and $[c, d] = A \ominus_{gH} B$;
– (Multiplication): the trapezoidal distribution (9) with $[a, b] = A \otimes_M B$ and $[c, d] = A \otimes_{gH} B$ is an approximation of the PDF of $Z = X \times Y$, preserving support and core.
– (Division): the trapezoidal distribution (9) with $[a, b] = A \oslash_M B$ and $[c, d] = A \oslash_{pgH} B$ (assuming $0 \notin B$) is an approximation of the PDF of $Z = X/Y$, preserving support and core.

Consider now two variables X, Y having (or considered on) compact supports $A = [a_X, b_X]$ and $B = [a_Y, b_Y]$, respectively; suppose that intervals A, B are partitioned into n_A, n_B sub-intervals, obtaining histograms $A_1, ..., A_{n_X}$ with associated probabilities $p_1, ..., p_{n_X}$ and $B_1, ..., B_{n_Y}$ with probabilities $q_1, ..., q_{n_Y}$. The probabilities p_i and q_j are assumed to be the marginals of joint probabilities $\pi_{i,j}$ corresponding to bi-variate bins $A_i \times B_j$ (Cartesian product). If X, Y are independent, then it is $\pi_{i,j} = p_i q_j$.

Let $C_{i,j} = A_i \circ B_j = \{z \mid z = x \circ y, x \in A_i, y \in B_j\}$ so that the probabilities $Prob\,(C_{i,j})$ are well defined. Denote $C_{i,j} = \left[\underline{c}_{i,j}, \overline{c}_{i,j}\right]$.

Finally, for $Z = X \circ Y$, the support $S = [\underline{s}, \overline{s}]$ of Z is a subset of $\bigcup_{i=1}^{n_x} \bigcup_{j=1}^{n_y} C_{i,j}$ (and generally coincides with it); considering a partition of S into sub-intervals $S_k, k = 1, ..., n_Z$ with $\bigcup_{k=1}^{n_Z} S_k = S$ it also holds that each S_k intersects a subset of the family $\{C_{i,j}\}$ in a way that $S_k \cup C_{i,j} \neq \varnothing$ only if it exists $(x, y) \in C_{i,j}$ such that $x \circ y \in S_k$.

Let $F_Z\,(z) = Prob\,(Z \leq z)$ be the CDF of Z on its support, i.e., for $z \in S$; we know that for $z \in C_{i,j} = \left[\underline{c}_{i,j}, \overline{c}_{i,j}\right]$ each CDF $F_{i,j}(z)$ can be approximated through the (linear) trapezoidal distribution $T_{i,j}\,(z; a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j})$ and $F_Z(z)$ can be approximated by the mixture

$$T_Z(z) = \sum_{i=1}^{n_X} \sum_{j=1}^{n_Y} \pi_{i,j} T_{i,j}(z). \tag{10}$$

With the given $\pi_{i,j}$ we can approximate the CDF F_Z (or equivalently the PDF p_Z) without the need to separate the independent and dependent cases. Remark that in both cases the PDF of resulting t_Z is piece-wise linear and the CDF T_Z is the mixture (weighted average) of 'quadratic' $T_{i,j}$. With respect to the original Moore's approximation, we gain exact calculation for addition (and difference) and higher precision for multiplication.

Example (from [4]). Variables X, Y have supports $A = [1, 4]$ and $B = [2, 5]$, decomposed into $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3$, $A_1 = [1, 2]$, $A_2 = [2, 4]$, $B_1 = [2, 3]$, $B_2 = [3, 4]$, $B_3 = [4, 5]$, and (marginal) histograms (1) for X with bins A_1, A_2 and $Prob(X \in A_1) = v_1 = 1/2$, $Prob(X \in A_2) = v_2 = 1/2$; (2) for Y with bins B_1, B_2, B_3 and $Prob(Y \in B_1) = w_1 = 1/4$, $Prob(Y \in B_2) = w_2 = 1/2$, $Prob(Y \in$

$B_3) = w_3 = 1/4$. With this information, X, Y are mixtures $X = v_1X_1 + v_2X_2$, $Y = w_1Y_1 + w_2Y_2 + w_3Y_3$ with uniform X_1, X_2 on A_1, A_2 and Y_1, Y_2, Y_3 on B_1, B_2, B_3.

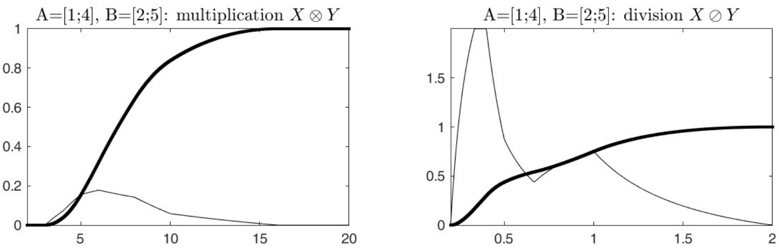

Fig. 1. PDF and CDF of product $X \times Y$ (left) and division X/Y (right), obtained by the described histogram-based trapezoidal approximations of the mixture components.

Suppose, for example, a positive dependence between X and Y, with joint distribution identified by the (2,3)-matrix

$$\mathbb{P} = \begin{bmatrix} P_{X,Y}(A_1, B_1) & P_{X,Y}(A_1, B_2) & P_{X,Y}(A_1, B_3) \\ P_{X,Y}(A_2, B_1) & P_{X,Y}(A_2, B_2) & P_{X,Y}(A_2, B_3) \end{bmatrix} = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 \end{bmatrix}$$

and bi-variate uniformity on each $A_i \times B_j$, $i = 1, 2, j = 1, 2, 3$.

The computations for addition and difference give exact results: the resulting PDFs are trapezoidal with linear shapes, precisely identified by support and core. For multiplication and division, we apply a refinement of the approximations by supplementary bins decomposition of the histogram; e.g., interval A_2 into 8 sub-bins and others into 4. The probability value $P(2, 1) = P_{X,Y}(A_2, B_1)$, is substituted by an 8×4 sub-matrix with elements $\frac{1/4}{32} = 1/128$; similarly for all other elements of $\mathbb{P}$. The resulting distributions are given in Figure 1; precision is high at low computational cost.

By interpreting Minkowski operations as exact support computations and generalized Hukuhara operations as exact core, we provide interval arithmetic with a new probabilistic semantics. This perspective not only clarifies the role of gH-based arithmetic but also offers a robust methodological foundation for uncertainty quantification, numerical probability, and data-driven probabilistic computation.

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