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On hyperbolic mixed problems with dynamic and Wentzell boundary conditions

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Abstract

We study mixed hyperbolic systems with dynamic and Wentzell boundary conditions. The boundary condition contains a tangential operator which is strongly elliptic on the boundary. We prove results of generation of strongly continuous groups and well-posedness.

Keywords: Hyperbolic problems, dynamic boundary conditions, Wentzell boundary conditions
2010 MSC: 35L53, 47D06.

1 Introduction

The aim of this paper is the study of a problem in the form

$$\begin{cases} D_t^2 u(t,x) = A(x,D_x)u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \Omega, \\ D_t^2 \gamma u(t,x') = \nabla_\tau \cdot (B(x')\nabla_\tau \gamma u)(t,x') + F(x',D_x)u(t,x') + h(t,x'), \\ (t,x') \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ D_t u(0,x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

eq3.1

Roughly speaking (precise assumptions will be given in the following), $A(x, D_x)$ is a strongly elliptic differential operator in divergence form in the bounded domain Ω with smooth boundary $\partial\Omega$; γf is the trace of f on $\partial\Omega$; ∇_τ is the tangential gradient in $\partial\Omega$, $\nabla_\tau \cdot$ is the divergence operator in $\partial\Omega$, $B(x')$ is a positive definite symmetric operator in the tangent space $T_{x'}(\partial\Omega)$, with $x' \in \partial\Omega$, $F(x', D_x)$ is a linear differential operator of order not exceeding one (not necessarily tangential) and coefficients defined in $\partial\Omega$.

(1.1) is strictly connected with the problem

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$$\begin{cases} D_t^2 u(t,x) = A(x,D_x)u(t,x) + f(t,x), \quad (t,x) \in (0,T)\times\Omega, \\ A(x',D_x)u(t,x') = \nabla_\tau \cdot (B(x')\nabla_\tau u)(t,x') + F(x',D_x)u(t,x') + h(t,x'), \quad (t,x') \in (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ D_t u(0,x) = u_1(x), \quad x\in\Omega. \end{cases} \tag{1.2}$$

eq0.2

formally obtained replacing in (1.1) $D_t^2 \gamma u(t,x')$ in the second equation with the trace of the second term in the first equation. In case (1.2) one usually speaks of Wentzell boundary conditions.

A physical interpretation of (1.2) is given in [10], Chapter 6.

In our knowledge, problems (1.1) and (1.2) have been always considered in the particular case that

$$F(x',D_x) = -\beta(x')\frac{\partial}{\partial\nu_A} - c(x'), \tag{1.3}$$

eq0.3

where we indicate with $\frac{\partial}{\partial\nu_A}$ the conormal derivative associated with $A(x,D_x)$. See, for example, [1], [9], [12], often connected with problems of control.

The most general results are contained in [2], where $F(x',D_x)$ is in the form (1.3) with $\beta(x')>0$ which is allowed (to some extent) to be unbounded and with infimum equal to 0. The authors do not even assume that the coefficients of $A(x,D_x)$ and $B(x')$ are continuous; they need to be just measurable and bounded. They work in the basic space $L^2(\overline{\Omega},d\mu) := L^2(\Omega)\times L^2(\partial\Omega, dS/\beta)$. with $F(x',D_x)$ as in (1.3). They show that a certain operator connected with (1.1) and (1.2) is self-adjoint and upper bounded. This allows to formulate theorems of well-posedness in a certain generalized sense. They consider also the case when D_t^2 is replaced by $D_t^2 + aD_t$ (this is the telegraph equation).

Roughly speaking, in this paper we want to show that, at least in case of "regular coefficients" for $A(x,D_x)$ and $B(x')$, (1.1) and (1.2) are well posed whenever the operator $F(x',D_x)$ has bounded and measurable coefficients in $\partial\Omega$.

This is the plan of this paper: Section 2 is dedicated to the proof of Theorem 2.1. We begin by considering a particular case, with $F(x',D_x) = -\frac{\partial}{\partial\nu_A} - \gamma$. In this situation the result is essentially known (see for this also [3]), but we have decided to give a complete proof in order to make the paper more or less self-contained. The general statement is obtained by combining an estimate of the conormal derivative of the solution to a hyperbolic Cauchy-Dirichlet system (see Theorem 2.10) with a perturbation theorem of Miyadera type (Theorem 2.12). The estimate is inspired by a nice result due to I. Lasiecka, J. L. Lions, R. Triggiani (see [8])

The final Section 3 contains developments and applications of Theorem 2.1 to a generalization of (1.1), and to (1.2).

To conclude this preliminary section, we describe some notations we are going to use.

If Ω is a domain with smooth boundary and $x'\in\partial\Omega$, we shall indicate with $\nu(x')$ the unit normal vector to $\partial\Omega$ in x', pointing outside Ω, with $\frac{\partial}{\partial\nu}$ the corresponding normal derivative. $T_{x'}(\partial\Omega)$ will be the tangent space to $\partial\Omega$ in x' and $T(\partial\Omega)$ the tangent bundle. If A is the differential operator

$$\sum_{i=1}^n\sum_{j=1}^n D_{x_i}(a_{ij}(x)D_{x_j}\cdot) + \sum_{j=1}^n a_j(x)D_{x_j} + a_0(x),$$

and $x'\in\partial\Omega$, we set

$$D_{\nu_A}u(x') = \frac{\partial u}{\partial\nu_A}(x') = \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x')D_{x_j}u(x')\nu_i(x').$$

C will indicate a positive constant we are not interested to precise. In a sequence of formulas we shall write C_1, C_2, ... If the constants depend on T, we shall write $C(T)$, $C_1(T)$, ...

If X and Y are normed spaces, we shall indicate with $\mathcal{L}(X,Y)$ the space of linear bounded operators from X to Y. If $X=Y$, we shall write $\mathcal{L}(X)$. If V is a Hilbert space, we shall indicate with V^* the space of antilinear bounded functionals in V.

2 The main theorem

se2

As we said, in this section we shall study a simplified version of (1.1). We begin by stating our assumptions.

(A1) Ω *is an open bounded subset of* $\mathbb{R}^n$ *lying on one side of its boundary* $\partial\Omega$*, which is a submanifold of* $\mathbb{R}^n$ *of dimension* $n-1$ *and class* C^2*.*

(A2) $A_0(x, D_x) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{x_i}[a_{ij}(x) D_{x_j} \cdot]$,
with $a_{ij} \in C^1(\overline{\Omega})$ $(1 \leq i, j \leq n)$*, real valued,* $a_{ij} = a_{ji}$*,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2,$$

for any $x \in \overline{\Omega}$*,* $\xi \in \mathbb{R}^n$*, for some* ν *positive.*

(A3) $\forall x' \in \partial\Omega$ $B(x')$ *is is symmetric and positive definite element of* $\mathcal{L}(T_{x'}(\partial\Omega))$*.*

(A4) $B(x')$ *depends smoothly on* x'*, in the sense that, if* u *is a* C^1 *section with values in* $T(\partial\Omega)$*,* $B(\cdot)u(\cdot)$ *is a* C^1 *section.*

(A5) $F(x', D_x)u(x') = \sum_{j=1}^{n} f_j(x') D_{x_j} u(x') + f_0(x')u(x')$*, with* $f_j \in L^\infty(\partial\Omega)$ $(0 \leq j \leq n)$*.*

We set

$$H = L^2(\Omega) \times L^2(\partial\Omega), \tag{2.1}$$

Of course, H is a Hilbert space with the usual scalar product

$$((f_0, h_0), (f_1, h_1))_H = \int_\Omega f_0(x)\overline{f_1(x)}dx + \int_{\partial\Omega} h_0(x')\overline{h_1(x')}d\sigma,$$

where σ is the standard Riemannian measure in $\partial\Omega$. We set also

$$V = \{(\phi, \psi) \in H^1(\Omega) \times H^1(\partial\Omega) : \gamma\phi = \psi\}. \tag{2.2}$$

We equip V with the scalar product

$$((\phi_0, \psi_0), (\phi_1, \psi_1))_V$$

$$:= \int_\Omega \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) D_{x_j}\phi_0(x)\overline{D_{x_i}\phi_1(x)}dx + \int_{\partial\Omega} [B(x')\nabla_\tau\psi_0(x') \cdot \overline{\nabla_\tau\psi_1(x')} + \psi_0(x)\overline{\psi_1(x')}]d\sigma. \tag{2.3}$$

We introduce the following operator A_2 in $H \times H$:

$$\begin{cases} D(A_2) = W := \{(\phi, \psi) \in H^2(\Omega) \times H^2(\partial\Omega) : \psi = \gamma\phi\}, \\ \\ A_2(\phi, \psi) = (A_0(\cdot, D_x)\phi, \nabla_\tau \cdot (B(\cdot)\nabla_\tau\psi) + F(\cdot, D_x)\phi). \end{cases}$$

The main result if this section is the following

th0.1 **Theorem 2.1.** *Suppose that (A1)-(A5) are fulfilled. We introduce the following operator* M*:*

$$\begin{cases} D(M) = W \times V, \\ \\ M((\phi, \psi), (f, h)) = ((f, h), A_2(\phi, \psi)) \end{cases}$$

Then M *is the infinitesimal generator of a strongly continuous group in* $V \times H$*.*

We begin the proof of Theorem 2.1 by recalling the well known procedure of identifying the element (f, h) of H with the element $J(f, h)$ of V^* defined as

$$(J(f, h), (\phi, \psi)) = ((f, h), (\phi, \psi))_H = \int_\Omega f(x)\overline{\phi(x)}dx + \int_{\partial\Omega} h(x')\overline{\psi(x')}d\sigma, \quad (\phi, \psi) \in V.$$

From Poincaré inequality, we deduce

$$|(J(f,h),(\phi,\psi))| \leq \|(f,h)\|_H \|(\phi,\psi)\|_H \leq C_0 \|(f,h)\|_H \|(\phi,\psi)\|_V,$$

for any (ϕ,ψ) in V. We deduce that $\|J(f,h)\|_{V^*} \leq C_0\|(f,h)\|_H$. So the identification of (f,h) with $J(f,h)$ carries to $\|(f,h)\|_{V^*} \leq C_0\|(f,h)\|_H$ and $H \hookrightarrow V^*$. We introduce the operator A_0, defined as follows:

$$\begin{cases} D(A_0) = \{(u,v) \in V : \exists (f,h) \in H : ((u,v),(\psi,\psi))_V = ((f,h),(\phi,\psi))_H \quad \forall (\phi,\psi) \in V\}, \\ A_0(u,v) = (f,h). \end{cases} \tag{2.4}$$ eq0.9

The following result is well known (for a proof, see [11], Chapter 2.2).

le2.2 **Lemma 2.2.** *If A_0 is the linear operator defined in (2.4), $D(A_0)$ is dense in H, A_0 is self-adjoint and positive and $D(A_0^{1/2}) = V$.*

Concerning $D(A_0)$, we have:

le0.2 **Lemma 2.3.** *Suppose that (A1)-(A4) hold. Then*

$$D(A_0) = W$$

and $\forall (u,v) \in W$

$$A_0(u,v) = (-A_0(\cdot,D_x)u, -\nabla_\tau \cdot (B(\cdot)\nabla_\tau v) + \frac{\partial u}{\partial \nu_A} + v).$$

Proof. We consider the operator $A_1 : W \to H$,

$$A_1(u,v) = (-A_0(\cdot,D_x)u, -\nabla_\tau \cdot (B(\cdot)\nabla_\tau v) + \frac{\partial u}{\partial \nu_A} + v).$$

It is immediately seen, employing Green's formula, that

$$A_1 \subseteq A_0. \tag{2.5}$$ eq0.10

On the other hand, as A_0 is self adjoint and positive, $-A_0$ is the infinitesimal generator of an analytic semigroup in H. By Theorem 4.1 in [6], the same happens for A_1. So $\rho(A_0) \cap \rho(A_1) \neq \emptyset$. This, together with (2.5), implies the conclusion.

□

Remark 2.4. By Lemma 2.3, in case $F(\cdot,D_x) = -\frac{\partial}{\partial \nu_A} - \gamma$, $A_2 = -A_0$.

Now we are able to employ the following result (see [7], Theorem 7.2):

th0.3 **Theorem 2.5.** *Let B be the infinitesimal generator of a strongly continuous group in the Banach space X. Assume that $0 \in \rho(B)$. Define*

$$\begin{cases} D(M_0) = D(B^2) \times D(B), \\ M_0(u,v) = (v, B^2 u). \end{cases}$$

Then M_0 is the infinitesimal generator of a strongly continuous group in the Banach space $D(B) \times X$.

co0.4 **Corollary 2.6.** *Suppose that (A1)-(A4) hold. We introduce the following operator M_0:*

$$\begin{cases} D(M_0) = W \times V, \\ M_0((\phi,\psi),(f,h)) = ((f,h), -A_0(\phi,\psi)). \end{cases} \tag{2.6}$$ eq0.11A

Then M_0 is the infinitesimal generator of a strongly continuous group in $V \times H$.

Proof. We set $B := iA_0^{1/2}$. Then B is skew-adjoint and $D(B) = V$. By Stone's theorem, B is the infinitesimal generator of a strongly continuous group of isometries in H. We have $B^2 = -A_0$. So the conclusion follows from Theorem 2.5. □

We shall indicate with $(e^{tM_0})_{t\in\mathbb{R}}$ the group generated by M_0 in $V \times H$.

Remark 2.7. If $(\phi, \psi) \in V$ and $(\alpha, \beta) \in H$, we shall often write $(\phi, \psi, \alpha, \beta)$ instead of $((\phi, \psi), (\alpha, \beta))$

eq2.8A **Remark 2.8.** If $(\phi, \psi, \alpha, \beta)$ belongs to $V \times H$ and its components are real valued, then the components of $e^{tM_0}(\phi, \psi, \alpha, \beta)$ are real valued. In case $(\phi, \psi, \alpha, \beta) \in W \times V$, This can be easily deduced from the uniqueness of the solution of the problem

$$\begin{cases} D_t^2 u(t,x) = A_0(x, D_x)u(t,x), \quad (t,x) \in \mathbb{R} \times \Omega, \\ D_t^2 \gamma u(t,x') = \nabla_\tau \cdot (B(x')\nabla_\tau \gamma u)(t,x') - \frac{\partial u}{\partial \nu_A}(t,x') - \gamma u(t,x'), \\ (t,x') \in \mathbb{R} \times \partial\Omega, \\ u(0,x) = \phi(x), \quad x \in \Omega, \\ D_t u(0,x) = \alpha(x), \quad x \in \Omega, \end{cases}$$

which follows from Corollary 2.6. The general case follows by continuity.

re2.8 **Remark 2.9.** Suppose that the assumptions (A1)-(A4) are satisfied. Let $u_0 \in H^2(\Omega))$, $\gamma u_0 \in H^2(\partial\Omega)$, $u_1 \in H^1(\Omega)$, $\gamma u_1 \in H^1(\partial\Omega)$, so that $(u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V$. Let

$$(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1). \tag{2.7}$$ eq0.14

Then $\phi \in \cap_{j=0}^2 C^{2-j}(\mathbb{R}; H^j(\Omega))$, $\psi \in \cap_{j=0}^2 C^{2-j}(\mathbb{R}; H^j(\partial\Omega))$, $\psi = \gamma\phi$, $\alpha = D_t\phi$, $\beta = D_t\psi = \gamma\alpha$. Moreover,

$$D_t^2 \phi(t,x) = A_0(x, D_x)\phi(t,x), \quad (t,x) \in \mathbb{R} \times \Omega.$$

So ϕ is also the solution of the mixed Cauchy-Dirichlet problem

$$\begin{cases} D_t^2 \phi(t,x) = A_0(x, D_x)u(t,x), \quad (t,x) \in (a,b) \times \Omega, \\ \phi(t,x') = \psi(t,x'), \quad (t,x') \in (a,b) \times \partial\Omega, \\ \phi(0,x) = u_0(x), \quad x \in \Omega, \\ D_t\phi(0,x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{2.8}$$ eq2.8

Now we want to replace $-\frac{\partial \cdot}{\partial \nu_A} - \gamma$ with an essentially arbitrary differential operator of order not exceeding one. To this aim, the key fact is the following result, following the idea of [8], Theorem 2.1. For a slightly different situation, see also [5].

th2.10 **Theorem 2.10.** *Suppose that the conditions (A1)-(A4) are fulfilled. Let $a, b \in \mathbb{R}$, with $a < b$. Then there exists $C(a,b)$ positive such that, $\forall (u_0, \gamma u_0, u_1, \gamma u_1)$ belonging to $W \times V$,*

$$\|\frac{\partial \phi}{\partial \nu}\|_{L^2((a,b)\times\partial\Omega)} \leq C\|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V\times H},$$

with $\phi(t)$ first component of $e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)$ (see Remark 2.9).

Proof. We continue to employ the notation (2.7). Concerning the proof, it suffices to consider the case that the components of $(u_0, \gamma u_0, u_1, \gamma u_1)$ are real valued, so that, by Remark 2.8, ϕ is real valued.

We set

$$N := \|u_0\|^2_{H^1(\Omega)} + \|\gamma u_0\|^2_{H^1(\partial\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|\gamma u_1\|^2_{L^2(\partial\Omega)}.$$

By simplicity of notation, we shall write, given a certain expression E, that $E = O(N)$ if there exists C positive, possibly dependending on (a, b), but not $(u_0, \gamma u_0, u_1, \gamma u_1)$, such that

$$|E| \le C\|(u_0, \gamma u_0, u_1, \gamma u_1)\|^2_{V\times H}.$$

For example,

$$\max_{t\in[a,b]} \{\phi(t)\|_{H^1(\Omega)} + \|\psi(t)\|_{H^1(\partial\Omega)} + \|\alpha(t)\|_{L^2(\Omega)} + \|\beta(t)\|_{L^2(\partial\Omega)}\} = O(N).$$

We introduce a function $h \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that, for each $j \in \{1, \dots, n\}$, if $x' \in \partial\Omega$,

$$h_j(x') = \sum_{i=1}^n a_{i,j}(x')\nu_i(x').$$

If $b \in H^1(\Omega)$, we have, by Green's formula,

$$\int_\Omega h(x)\cdot\nabla_x b(x)dx = \int_{\partial\Omega} A(x')b(x')d\sigma(x') - \int_\Omega div_x(h)(x)b(x)dx, \tag{2.9}$$ eq3.15

with

$$A(x') = \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x')\nu_i(x')\nu_j(x'), \quad x' \in \partial\Omega.$$

We have also that in $(a, b)\times\partial\Omega$, for $j = 1, \dots, n$, employing the notation (2.7),

$$D_{x_j}\phi(t, x') = \nu_j(x')\frac{\partial\phi}{\partial\nu}(t, x') + T_j\psi(t, x'), \tag{2.10}$$ eq3.16A

with T_j differential operator of order one in $\partial\Omega$, with coefficients in $C^1(\partial\Omega)$. Then, by Remark 2.9,

$$\begin{aligned} &\int_{(a,b)\times\Omega} D_t^2\phi(t,x)h(x)\cdot\nabla_x\phi(t,x)dtdx \\ &= \int_{(a,b)\times\Omega} A_0(x, D_x)\phi(t,x)h(x)\cdot\nabla_x\phi(t,x)dtdx. \end{aligned} \tag{2.11}$$ eq3.16

Now,

$$\begin{aligned} &\int_{(a,b)\times\Omega} D_t^2\phi(t,x)h(x)\cdot\nabla_x\phi(t,x)dtdx \\ &= \int_\Omega \alpha(b,x)h(x)\cdot\nabla_x\phi(b,x)dx - \int_\Omega \alpha(a,x)h(x)\cdot\nabla_x\phi(a,x)dx \\ &- \int_{(a,b)\times\Omega} \alpha(t,x)h(x)\cdot\nabla_x\alpha(t,x)dtdx. \end{aligned}$$

We have

$$\begin{aligned} &|\int_\Omega \alpha(b,x)h(x)\cdot\nabla_x\phi(b,x)dx - \int_\Omega \alpha(a,x)h(x)\cdot\nabla_x\phi(a,x)dx| \\ &\le C(a,b)(\|\alpha(b,\cdot)\|_{L^2(\Omega)}\|\phi(b,\cdot)\|_{H^1(\Omega)} + \|\alpha(a,\cdot)\|_{L^2(\Omega)}\|\phi\|_{C([a,b];H^1(\Omega))}) \\ &= O(N). \end{aligned}$$

Moreover, by (2.9),

$$\begin{aligned} &|\int_{(a,b)\times\Omega} \alpha(t,x)h(x)\cdot\nabla_x\alpha(t,x)dtdx| \\ &= \frac{1}{2}|\int_{(a,b)\times\Omega} h(x)\cdot\nabla_x\alpha^2(t,x)dtdx| \\ &= \frac{1}{2}|\int_a^b(\int_{\partial\Omega} A(x')\beta(t,x')^2d\sigma)dt - \int_a^b(\int_\Omega div_x h(x)\alpha(t,x)^2dx)dt| \\ &\le C((\|\beta\|^2_{C([a,b];L^2(\partial\Omega))} + \|\alpha\|^2_{C([a,b];L^2(\Omega))}) \\ &= O(N). \end{aligned}$$

So, by (2.11),

$$\int_{(a,b)\times\Omega} A_0(x,D_x)\phi(t,x)h(x)\cdot\nabla_x\phi(t,x)dtdx = O(N).$$

We have

$$\begin{aligned}
&\textstyle\int_{(a,b)\times\Omega} A_0(x,D_x)\phi(t,x)h(x)\cdot\nabla_x\phi(t,x)dtdx\\
&= \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_i}(a_{ij}(x)D_{x_j}\phi(t,x)h_k(x)D_{x_k}\phi(t,x))dtdx\\
&- \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ij}(x)D_{x_j}\phi(t,x)D_{x_i}(h_k(x)D_{x_k}\phi(t,x))dtdx\\
&= \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_i}(a_{ij}(x)D_{x_j}\phi(t,x)h_k(x)D_{x_k}\phi(t,x))dtdx\\
&- \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ij}(x)D_{x_j}\phi(t,x)h_k(x)D_{x_ix_k}\phi(t,x)h_k(x)dtdx\\
&- \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ij}(x)D_{x_j}\phi(t,x)D_{x_k}\phi(t,x)D_{x_i}h_k(x)dtdx\\
&:= I_1 - I_2 + O(N).
\end{aligned}$$

Moreover,

$$\begin{aligned}
-I_2 &= -\textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_k}(a_{ij}(x)D_{x_j}\phi(t,x)D_{x_i}\phi(t,x)h_k(x))dtdx\\
&+ \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_k}(a_{ij}(x)D_{x_j}\phi(t,x)h_k(x))D_{x_i}\phi(t,x)dtdx\\
&= -\textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_k}(a_{ij}(x)D_{x_j}\phi(t,x)D_{x_i}\phi(t,x)h_k(x))dtdx\\
&+ \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ij}(x)D_{x_kx_j}\phi(t,x)h_k(x)D_{x_i}\phi(t,x)dtdx\\
&+ \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_j}\phi(t,x)D_{x_i}\phi(t,x)D_{x_k}(a_{ij}(x)h_k(x))dtdx\\
&:= -I_3 + I_2 + O(N).
\end{aligned}$$

So

$$I_2 = \frac{I_3}{2} + O(N).$$

We deduce that

$$I_1 - \frac{I_3}{2} = O(N). \tag{2.12}$$

eq3.18

We have

$$\begin{aligned}
I_1 &= \textstyle\int_{(a,b)\times\Omega}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n D_{x_i}(a_{ij}(x)D_{x_j}\phi(t,x)h_k(x)D_{x_k}\phi(t,x))dtdx\\
&= \textstyle\int_{(a,b)\times\partial\Omega}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x')D_{x_j}\phi(t,x')\nu_i(x')\sum_{k=1}^n h_k(x')D_{x_k}\phi(t,x')dtd\sigma\\
&= \textstyle\int_{(a,b)\times\partial\Omega}(D_{\nu_A}\phi(t,x'))^2dtd\sigma.
\end{aligned}$$

By (2.10), we have

$$D_{\nu_A}\phi(t,x') = A(x')\frac{\partial\phi}{\partial\nu}(t,x') + \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x')\nu_i(x')T_j\psi(t,x')$$

so that

$$\begin{aligned}
I_1 &= \textstyle\int_{(a,b)\times\partial\Omega} A(x')^2\frac{\partial\phi}{\partial\nu}(t,x')^2dtd\sigma\\
&+ \textstyle\int_{(a,b)\times\partial\Omega}\frac{\partial\phi}{\partial\nu}(t,x')S_1\psi(t,x')dtd\sigma + O(N),
\end{aligned}$$

with S_1 differential operator of order one in $\partial\Omega$, while

$$I_3 = \int_{(a,b)\times\partial\Omega} A(x') \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') D_{x_j}\phi(t,x') D_{x_i}\phi(t,x') dt d\sigma$$

$$= \int_{(a,b)\times\partial\Omega} A(x') \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') (\nu_j(x') \tfrac{\partial\phi}{\partial\nu}(t,x') + T_j\psi(t,x'))$$

$$\times (\nu_i(x') \tfrac{\partial\phi}{\partial\nu}(t,x') + T_i\psi(t,x')) dt d\sigma$$

$$= \int_{(a,b)\times\partial\Omega} A(x')^2 \tfrac{\partial\phi}{\partial\nu}(t,x')^2 dt d\sigma$$

$$+ \int_{(a,b)\times\partial\Omega} \tfrac{\partial\phi}{\partial\nu}(t,x') S_2\psi(t,x') dt d\sigma + O(N),$$

with S_2 differential operator of order one in $\partial\Omega$. From (2.12) we deduce

$$\frac{1}{2}\int_{(a,b)\times\partial\Omega} A(x')^2 \frac{\partial\phi}{\partial\nu}(t,x')^2 dt d\sigma + \int_{(a,b)\times\partial\Omega} \frac{\partial\phi}{\partial\nu}(t,x')(S_1 - \frac{1}{2}S_2)\psi(t,x') dt d\sigma = O(N),$$

and, as $A(x')^2$ is lower bounded by a positive constant, for some C_0 positive independent of (u_0, ξ, v_0, η_0), s,

$$\int_{(a,b)\times\partial\Omega} \frac{\partial u}{\partial\nu}(t,x')^2 dt d\sigma \leq C_0[N + N^{1/2}(\int_{(s,T)\times\partial\Omega} \frac{\partial u}{\partial\nu}(t,x')^2 dt d\sigma)^{1/2}],$$

implying

$$\int_{(a,b)\times\partial\Omega} \frac{\partial u}{\partial\nu}(t,x')^2 dt d\sigma \leq \frac{(C_0 + \sqrt{C_0^2 + 4C_0})^2}{4} N.$$

□

co0.9 **Corollary 2.11.** *Suppose (A1)-(A4) hold. Let* $T \in \mathbb{R}^+$, $u_0 \in H^2(\Omega)$ *with* $\gamma u_0 \in H^2(\partial\Omega)$, $u_1 \in H^1(\Omega)$ *with* $\gamma u_1 \in H^1(\partial\Omega)$, *so that* $(u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V$. *Let*

$$(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1) \tag{2.13}$$ eq2.9

Moreover, if $x' \in \partial\Omega$, *let* $G(x', D_x)u(x') = \sum_{j=1}^n g_j(x') D_{x_j} u(x') + g_0(x')u(x')$, *with* $g_j \in L^\infty(\partial\Omega)$. *Then there exists* $C(T)$ *positive, independent of* u_0 *and* u_1, *such that*

$$\|G(\cdot, D_x)\phi\|_{L^2((-T,T)\times\partial\Omega)} \leq C(T)\|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V\times H}.$$

Proof. Let

$$g(x') = (g_1(x'), \dots, g_n(x')).$$

If we set

$$k(x') := g(x') \cdot \nu(x'),$$

then $t(x') := g(x') - k(x')\nu(x')$ is tangential to $\partial\Omega$ in x'. So

$$G(x', D_x)\phi(t,x') = k(x') \frac{\partial\phi}{\partial\nu}(t,x') + t(x') \cdot \nabla_\tau \psi(t,x') + g_0(x')\psi(t,x').$$

and

$$\|G(\cdot, D_x)\phi(t,\cdot)\|_{L^2(\partial\Omega)} \leq C_0(\|\frac{\partial\phi}{\partial\nu}(t,\cdot)\|_{L^2(\partial\Omega)} + \|\psi(t,\cdot)\|_{H^1(\partial\Omega)}) \tag{2.14}$$ eq0.16A

So the conclusion follows from Theorem 2.10. □

Now we recall the following perturbation result of Miyadera type (see [4], Corollary 3.16):

th0.10 **Theorem 2.12.** *Let* A *be the infinitesimal generator of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *and let* $B \in \mathcal{L}(D(A), X)$ *satisfy, for some* $t_0 > 0$, $q \in [0,1)$,

$$\int_0^{t_0} \|BT(t)x\| dt \leq q\|x\|, \quad \forall x \in D(A).$$

Then $A + B$, *with domain* $D(A)$, *is the infinitesimal generator of a strongly continuous semigroup in* X.

co2.13 **Corollary 2.13.** *Let A be the infinitesimal generator of a strongly continuous group $(T(t))_{t\in\mathbb{R}}$ on a Banach space X and let $B \in \mathcal{L}(D(A), X)$ satisfy, for some $t_0 > 0$, $q \in [0, 1)$,*

$$\int_{-t_0}^{t_0} \|BT(t)x\| dt \leq q\|x\|, \quad \forall x \in D(A).$$

Then $A + B$, with domain $D(A)$, is the infinitesimal generator of a strongly continuous group in X.

Proof. As $((T(t))_{t\in\mathbb{R}}$ is a group, if we set $T_-(t) := T(-t)$, with $t \geq 0$, $(T_-(t))_{t\geq 0}$ is a strongly continuous semigroup with infinitesimal generator $-A$. By Theorem 2.12, $-A - B$, with domain $D(A)$, is the infinitesimal generator of a strongly continuous semiproup. As both $\pm(A+B)$ are infinitesimal generators of strongly continuous semigroups, $A + B$ is the infinitesimal generator of a strongly continuous group. □

Now we are able to prove Theorem 2.1.

Proof of Theorem 2.1 We set $X = V \times H$, $A = M_0$ and we introduce the following operator B:

$$\begin{cases} B : W \times V \to V \times H, \\ B(u_0, \gamma u_0, u_1, \gamma u_1) = (0, 0, 0, F(\cdot, D_x)u_0 + \frac{\partial u_0}{\partial \nu_A} + \gamma u_0). \end{cases}$$

Setting $G(\cdot, D_x) = F(\cdot, D_x) + \frac{\partial}{\partial \nu_A} + \gamma$, we have, taking $t_0 \in (0, 1]$, with the position (2.13),

$$\int_{-t_0}^{t_0} \|Be^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V\times H} dt$$

$$\leq (2t_0)^{1/2} (\int_{-t_0}^{t_0} \|G(\cdot, D_x)\phi(t)\|^2_{L^2(\partial\Omega)} dt)^{1/2} \leq C(1)(2t_0)^{1/2}\|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V\times H}.$$

So the assumptions of Corollary 2.13 are satisfied and the conclusion follows from the fact that $M = M_0 + B$.

3 Developments of Theorem 2.1

se3

We shall employ the following well known fact, concerning strongly continuous semigroups:

pr3.1A **Proposition 3.1.** *Let A be the infinitesimal generator of a strongly continuos semigroup $(T(t))_{t\geq 0}$ in the Banach space X. Let $x \in D(A)$ and $f \in W^{1,1}(0, T; X) + C([0, T]; X) \cap L^1(0, T; D(A))$. Then the Cauchy problem*

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t \in [0, T], \\ u(0) = x \end{cases}$$

has a unique solution u in $C^1([0, T]; X) \cap C([0, T]; D(A))$ given by the variation of parameter formula

$$u(t) = T(t)x + \int_0^t T(t - s)f(s)ds.$$

We consider the following problem:

$$\begin{cases} D_t^2 u(t, x) + a(x)D_t u(t, x) = A(x, D_x)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \Omega, \\ D_t^2 \gamma u(t, x') + b(x')D_t \gamma u(t, x') = \nabla_\tau \cdot (B(x')\nabla_\tau \gamma u)(t, x') + F(x', D_x)u(t, x') + h(t, x'), \\ (t, x') \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ D_t u(0, x) = u_1(x), \quad x \in \Omega. \end{cases} \quad (3.1)$$

eq3.1A

We introduce the following assumptions:

(B1) (A1), (A3), (A4), (A5) hold;

$$A(x, D_x) = A_0(x, D_x) + \sum_{j=1}^{n} a_j(x) D_{x_j} + a_0(x),$$

with A_0 as in (A2), $a_j \in L^\infty(\Omega)$ $(0 \le j \le n)$;

(B2) $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$.

Then we have:

pr3.1 **Proposition 3.2.** *Suppose that (B1)-(B2) hold. We introduce the following operator M_1:*

$$\begin{cases} M_1 : V \times H \to V \times H, \\ M_1(v_0, v_1, w_0, w_1) = (0, 0, \sum_{j=1}^{n} a_j(\cdot) D_{x_j} v_0 + a_0(\cdot) v_0 - a(\cdot) w_0, -b(\cdot) w_1). \end{cases}$$

Then

(I) $M + M_1$, with domain $W \times V$, is the infinitesimal generator of a strongly continuous group in $V \times H$;

(II) consider the problem (3.1), with $T \in \mathbb{R}^+$. Suppose, moreover, that:

(a) $u_0 \in H^2(\Omega)$, $\gamma u_0 \in H^2(\partial\Omega)$, $u_1 \in H^1(\Omega)$, $\gamma u_1 \in H^1(\partial\Omega)$;

(b) $f(t, x) = f_1(t, x) + f_2(t, x)$, with $f_1 \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega))$, $\gamma f_1 \in C([0, T]; L^2(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega))$, $f_2 \in W^{1,1}(0, T; L^2(\Omega))$;

(c) $h(t, x') = \gamma f_1(t, x') + h_1(t, x')$, with $h_1 \in W^{1,1}(0, T; L^2(\partial\Omega))$.

Then (3.1) has a unique solution u belonging to $\cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\Omega))$, with γu belonging to $\cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\partial\Omega))$.

Proof. (I) follows from Theorem 2.1 and the fact that M_1 belongs to $\mathcal{L}(V \times H)$.

(II) We set $\phi := u$, $\psi := \gamma u$, $\alpha := D_t u$, $\beta := \gamma\alpha = D_t\psi$. Then (3.1) can be written in the equivalent form

$$\begin{cases} (\phi'(t), \psi'(t), \alpha'(t), \beta'(t)) = (M + M_1)(\phi(t), \psi(t), \alpha(t), \beta(t)) + (0, 0, f(t), h(t)), \quad t \in [0, T], \\ (\phi(0), \psi(0), \alpha(0), \beta(0)) = (u_0, \gamma u_0, u_1, \gamma u_1). \end{cases} \tag{3.2}$$ eq3.2

Then $(u_0, \gamma u_0, u_1, \gamma u_1)$ belongs to $W \times V$, while

$$(0, 0, f(t), h(t)) = (0, 0, f_1(t), \gamma f_1(t)) + (0, 0, f_2(t), h_1(t)),$$

with the first summand in

$$C([0, T]; V \times H) \cap L^1(0, T; W \times V) = C([0, T]; V \times H) \cap L^1(0, T; D(M + M_1)),$$

the second summand in $W^{1,1}(0, T; V \times H)$. By Proposition 3.1, (3.2) has a unique solution in $C^1([0, T]; V \times H) \cap C([0, T]; W \times V))$. □

We conclude with an application to (1.2).

Proposition 3.3. *Consider the problem (1.2), with the assumption (B1) and $T \in \mathbb{R}^+$. Suppose, moreover, that:*

(a) $u_0 \in H^2(\Omega)$, $\gamma u_0 \in H^2(\partial\Omega)$, $u_1 \in H^1(\Omega)$, $\gamma u_1 \in H^1(\partial\Omega)$;

(b) $f \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega))$, $\gamma f \in C([0, T]; L^2(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega))$;

(c) $h \in W^{1,1}(0, T; L^2(\partial\Omega))$.

Then (1.2) has a unique solution u belonging to $\cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\Omega))$, with γu belonging to $\cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\partial\Omega))$. Here $A(x', D_x)u(t, x')$ is intended as $D_t^2 \gamma u - \gamma f$.

Proof. The problem is equivalent to (3.1) with $a \equiv 0$, $b \equiv 0$ and h replaced by $\gamma f + h$. So the conclusion follows from Proposition 3.2. □

References

CaKhMe1 [1] M. M. Cavalcanti, A. Khemmoudj, M. Medjden, "Uniform stabilisation of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions", J. Math. Anal. Appl. **328** (2007), 900-930.

ClGoGo1 [2] R. Clendenen, G. R. Goldstein, J. A. Goldstein, "Degenerate flux for dynamic boundary conditions in parabolic and hyperbolic equations", Discr. Cont. Dynam. Syst. Ser. S 9 (2016), 651-660.

CoFaGoGoRo1 [3] G. M. Coclite, A. Favini, G. R. Goldstein, J. A. Goldstein, S. Romanelli, "Continuous dependence in hyperbolic problems with Wentzell boundary conditions", Commun. Pure Applied Anal. **13** (2004), 419-433.

EnNa1 [4] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer (2000).

GiGu1 [5] C. Giorgi, D. Guidetti, Reconstruction of kernel depending also on a space variable, Math. Methods Appl. Sci. **41** (2018), 4560-4588.

GoGoGuRo1 [6] G. R. Goldstein, J. A. Goldstein, D. Guidetti, S. Romanelli, "Maximal regularity, analytic semigroups, and dynamic and general Wentzell boundary conditions with a diffusion term on the boundary", Annali di Matematica Pura ed Applicata, https://doi.org/10.1007/s10231-019-00868-3.

Go1 [7] J. A. Goldstein, *Semigroups of Linear Operators & Applications*, Dover Publications, Inc. (Second Edition) (2017).

LaLiTr1 [8] I. Lasiecka, J. L. Lions, R. Triggiani, "Non homogeneous boundary value problems for second order hyperbolic operators", J Math Pures et Appl. **65** (1986), 149 -192.

NiLa1 [9] S. Nicaise, K. Laoubi, "Polynomial stabilization of the wave equation with Ventcel's boundary conditions", Math. Nachr. **283** (2010), 1428-1438.

Ru1 [10] G. Ruiz Goldstein, "Derivation and physical interpretation of general boundary conditions", Adv. Diff. Eq. **11** (2006), 457-480.

Ta1 [11] H. Tanabe, *Equations of evolution*, Pitman (1975).

Vi1 [12] E. Vitillaro, "On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources", Journ. Diff. Eq. **265** (2018), 4873-4941.