SUPPLEMENTARY MATERIALS: CONSTRUCTION AND EVALUATION OF PH CURVES IN EXPONENTIAL-POLYNOMIAL SPACES*

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SM1. Proof of Proposition 3.3.

Proof. (a) is a consequence of the fact that $\phi^{\omega}_{i,1}(t) \geq 0$ for all $t \in [0,1]$ and $\sum_{i=0}^{3} \phi^{\omega}_{i,1}(t) = 1$ for all $t \in [0,1]$;
(b) is due to the fact that $\phi^{\omega}_{i,1}(t) = \phi^{\omega}_{3-i,1}(1-t)$ for all $t \in [0,1]$, $i = 0, \dots, 3$;
(c) follows from the fact that

$$\begin{aligned}
\frac{d}{dt}\phi^{\omega}_{0,1}(t) &= -\frac{\varphi^{\omega}_{0,1}(t)}{\int_0^1 \varphi^{\omega}_{0,1}(x)\,dx}, \\
\frac{d}{dt}\phi^{\omega}_{i,1}(t) &= \frac{\varphi^{\omega}_{i-1,1}(t)}{\int_0^1 \varphi^{\omega}_{i-1,1}(x)\,dx} - \frac{\varphi^{\omega}_{i,1}(t)}{\int_0^1 \varphi^{\omega}_{i,1}(x)\,dx} \qquad i = 1,\, 2 \\
\frac{d}{dt}\phi^{\omega}_{3,1}(t) &= \frac{\varphi^{\omega}_{2,1}(t)}{\int_0^1 \varphi^{\omega}_{2,1}(x)\,dx};
\end{aligned}$$

(d) is a consequence of (c). ∎

SM2. Proof of Proposition 3.4.

Proof. By substituting (3.8) into (2.5) we obtain

$$\mathbf{r}'(t) = \mathbf{A}_0\mathbf{i}\mathbf{A}_0^* \left(\psi^{\omega}_{0,1}(t)\right)^2 + (\mathbf{A}_0\mathbf{i}\mathbf{A}_1^* + \mathbf{A}_1\mathbf{i}\mathbf{A}_0^*)\,\psi^{\omega}_{0,1}(t)\psi^{\omega}_{1,1}(t) + \mathbf{A}_1\mathbf{i}\mathbf{A}_1^* \left(\psi^{\omega}_{1,1}(t)\right)^2$$

and thus, in light of (3.3),

$$\mathbf{r}'(t) = \mathbf{A}_0\mathbf{i}\mathbf{A}_0^*\varphi^{\omega}_{0,1}(t) + (\mathbf{A}_0\mathbf{i}\mathbf{A}_1^* + \mathbf{A}_1\mathbf{i}\mathbf{A}_0^*)\frac{1}{2}c_1(\omega)\varphi^{\omega}_{1,1}(t) + \mathbf{A}_1\mathbf{i}\mathbf{A}_1^*\varphi^{\omega}_{2,1}(t). \tag{SM2.1}$$

Since $\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{r}'(x)dx$, by integrating the expression in (SM2.1) exploiting the formulae in (3.5), we obtain the Bézier-like form of $\mathbf{r}(t)$ with control points in (3.10). ∎

SM3. Proof of Proposition 3.6.

Proof. Since $\sigma(t) = \mathbf{A}(t)\mathbf{A}^*(t)$, in light of (3.8) we can write

$$\sigma(t) = |\mathbf{A}_0|^2 \left(\psi^{\omega}_{0,1}(t)\right)^2 + (\mathbf{A}_1\mathbf{A}_0^* + \mathbf{A}_0\mathbf{A}_1^*)\,\psi^{\omega}_{0,1}(t)\psi^{\omega}_{1,1}(t) + |\mathbf{A}_1|^2 \left(\psi^{\omega}_{1,1}(t)\right)^2.$$

Then, exploiting (3.3), the claimed result is obtained. ∎

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SM4. Proof of Proposition 3.7.

Proof. Since

$$s(t) \;=\; \int_0^t \sigma(x)dx \;=\; \int_0^t \sum_{i=0}^{2} \sigma_i \varphi^{\omega}_{i,1}(x)dx \;=\; \sum_{i=0}^{2} \sigma_i \int_0^t \varphi^{\omega}_{i,1}(x)dx,$$

then, recalling formulae (3.5) we arrive at

$$s(t) \;=\; \sigma_2\, c_2(\omega)\phi^{\omega}_{3,1}(t) \;+\; \sigma_1 \frac{c_3(\omega)}{c_1(\omega)} \sum_{i=2}^{3} \phi^{\omega}_{i,1}(t) \;+\; \sigma_0\, c_2(\omega) \sum_{i=1}^{3} \phi^{\omega}_{i,1}(t)$$

and, by collecting the coefficients of each basis function $\phi^{\omega}_{i,1}(t)$, $i = 0, \ldots, 3$, we get the claimed result. ∎

SM5. Proof of Proposition 4.3.

Proof. (a) is a consequence of the fact that $\phi^{\omega}_{i,2}(t) \geq 0$ for all $t \in [0, 1]$ and $\sum_{i=0}^{5} \phi^{\omega}_{i,2}(t) = 1$ for all $t \in [0, 1]$;
(b) is due to the fact that $\phi^{\omega}_{i,2}(t) = \phi^{\omega}_{5-i,2}(1 - t)$ for all $t \in [0, 1]$, $i = 0, \ldots, 5$;
(c) follows from the fact that

$$\begin{aligned}
\frac{d}{dt}\phi^{\omega}_{0,2}(t) \;&=\; -\frac{\varphi^{\omega}_{0,2}(t)}{\int_0^1 \varphi^{\omega}_{0,2}(x)\, dx},\\
\frac{d}{dt}\phi^{\omega}_{i,2}(t) \;&=\; \frac{\varphi^{\omega}_{i-1,2}(t)}{\int_0^1 \varphi^{\omega}_{i-1,2}(x)\, dx} \;-\; \frac{\varphi^{\omega}_{i,2}(t)}{\int_0^1 \varphi^{\omega}_{i,2}(x)\, dx}, \qquad i = 1, \ldots, 4\\
\frac{d}{dt}\phi^{\omega}_{5,2}(t) \;&=\; \frac{\varphi^{\omega}_{4,2}(t)}{\int_0^1 \varphi^{\omega}_{4,2}(x)\, dx};
\end{aligned}$$

(d) is a consequence of (c). ∎

SM6. Proof of Proposition 4.4.

Proof. By substituting (4.6) into (2.5) we obtain

$$\begin{aligned}
\mathbf{r}'(t) = \sum_{j=0}^{2} \mathbf{A}_j \mathbf{i} \mathbf{A}_j^* \left(\psi^{\omega}_{j,2}(t)\right)^2 \;+\; \left(\mathbf{A}_0\mathbf{i}\mathbf{A}_1^* + \mathbf{A}_1\mathbf{i}\mathbf{A}_0^*\right) \psi^{\omega}_{0,2}(t)\psi^{\omega}_{1,2}(t)\\
+ \left(\mathbf{A}_1\mathbf{i}\mathbf{A}_2^* + \mathbf{A}_2\mathbf{i}\mathbf{A}_1^*\right) \psi^{\omega}_{1,2}(t)\psi^{\omega}_{2,2}(t) \;+\; \left(\mathbf{A}_0\mathbf{i}\mathbf{A}_2^* + \mathbf{A}_2\mathbf{i}\mathbf{A}_0^*\right) \psi^{\omega}_{0,2}(t)\psi^{\omega}_{2,2}(t)
\end{aligned}$$

and thus, in light of (4.1),

$$\begin{aligned}
\mathbf{r}'(t) = \mathbf{A}_0\mathbf{i}\mathbf{A}_0^*\, \varphi^{\omega}_{0,2}(t) \;+\; \left(\mathbf{A}_0\mathbf{i}\mathbf{A}_1^* + \mathbf{A}_1\mathbf{i}\mathbf{A}_0^*\right) \frac{1}{2}\varphi^{\omega}_{1,2}(t)\\
+\; \left(\mathbf{A}_1\mathbf{i}\mathbf{A}_1^*\, q_0(\omega) + \left(\mathbf{A}_0\mathbf{i}\mathbf{A}_2^* + \mathbf{A}_2\mathbf{i}\mathbf{A}_0^*\right)\frac{1}{2}\, q_1(\omega)\right) \varphi^{\omega}_{2,2}(t)\\
+\; \left(\mathbf{A}_1\mathbf{i}\mathbf{A}_2^* + \mathbf{A}_2\mathbf{i}\mathbf{A}_1^*\right) \frac{1}{2}\varphi^{\omega}_{3,2}(t) \;+\; \mathbf{A}_2\mathbf{i}\mathbf{A}_2^*\, \varphi^{\omega}_{4,2}(t).
\end{aligned} \tag{SM6.1}$$

Since $\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{r}'(x)dx$, by integrating the expression in (SM6.1) exploiting the formulae in (4.3), we obtain the Bézier-like form of $\mathbf{r}(t)$ with control points in (4.8). ∎

SM7. Proof of Proposition 4.6.

Proof. Since $\sigma(t) = \mathbf{A}(t)\mathbf{A}^*(t)$, in light of (4.6) we can write

$$\sigma(t) = \sum_{j=0}^{2} |\mathbf{A}_j|^2 \left(\psi^{\omega}_{j,2}(t)\right)^2 \; + \; (\mathbf{A}_1\mathbf{A}_0^* + \mathbf{A}_0\mathbf{A}_1^*)\, \psi^{\omega}_{0,2}(t)\psi^{\omega}_{1,2}(t)$$
$$+ (\mathbf{A}_1\mathbf{A}_2^* + \mathbf{A}_2\mathbf{A}_1^*)\, \psi^{\omega}_{1,2}(t)\psi^{\omega}_{2,2}(t) \; + \; (\mathbf{A}_2\mathbf{A}_0^* + \mathbf{A}_0\mathbf{A}_2^*)\, \psi^{\omega}_{0,2}(t)\psi^{\omega}_{2,2}(t).$$

Then, exploiting (4.1), we obtain the claimed result. ∎

SM8. Proof of Proposition 4.7.

Proof. Since

$$s(t) \; = \; \int_0^t \sigma(x)dx \; = \; \int_0^t \sum_{i=0}^{4} \sigma_i \varphi^{\omega}_{i,2}(x)dx \; = \; \sum_{i=0}^{4} \sigma_i \int_0^t \varphi^{\omega}_{i,2}(x)dx,$$

recalling formulae (4.3), we then arrive at

$$s(t) = \sigma_0\, q_2(\omega) \sum_{i=1}^{5} \phi^{\omega}_{i,2}(t) \; + \; \sigma_1\, q_3(\omega) \sum_{i=2}^{5} \phi^{\omega}_{i,2}(t) \; + \; \sigma_2\, \frac{q_4(\omega)}{q_1(\omega)} \sum_{i=3}^{5} \phi^{\omega}_{i,2}(t)$$
$$+ \sigma_3\, q_3(\omega) \sum_{i=4}^{5} \phi^{\omega}_{i,2}(t) \; + \; \sigma_4\, q_2(\omega)\, \phi^{\omega}_{5,2}(t).$$

By collecting the coefficients of each basis function $\phi^{\omega}_{i,2}(t)$, $i = 0, \ldots, 5$, we get the claimed result. ∎

SM9. 5^{th} order Taylor expansions at $\omega = 0$ of $\{\phi^{\omega}_{j,m}(t)\}_{j=0}^{2m+1}$, $m \in \{1, 2\}$.

For $\{\phi^{\omega}_{j,1}(t)\}_{j=0}^{3}$,

$$T^{\omega}_{0,1}(t) = \; T^{\omega}_{3,1}(1-t), \qquad T^{\omega}_{1,1}(t) \; = \; T^{\omega}_{2,1}(1-t),$$

$$T^{\omega}_{2,1}(t) = \; 3t^2\,(1-t) \; \frac{(30t^4 - 40t^3 + 23t^2 - 12t - 3)\omega^4 \; + \; 420(3t^2 - 2t + 1)\omega^2 \; + \; 25200}{25200},$$

$$T^{\omega}_{3,1}(t) = \; t^3 \; \frac{(10t^4 - 21t^2 + 11)\omega^4 \; + \; 420(t^2 - 1)\omega^2 \; + \; 8400}{8400},$$

and, for $\{\phi^{\omega}_{j,2}(t)\}_{j=0}^{5}$,

$$T^{\omega}_{0,2}(t) \; = \; T^{\omega}_{5,2}(1-t), \qquad T^{\omega}_{1,2}(t) \; = \; T^{\omega}_{4,2}(1-t), \qquad T^{\omega}_{2,2}(t) \; = \; T^{\omega}_{3,2}(1-t),$$

$$T^{\omega}_{3,2}(t) \; = \; 10t^3(1-t)^2 \; \frac{(245t^4 - 392t^3 + 253t^2 - 82t + 3)\omega^4 \; + \; 420(10t^2 - 8t + 3)\omega^2 \; + \; 35280}{35280},$$

$$T^{\omega}_{4,2}(t) \; = \; 5t^4\,(1-t) \; \frac{(245t^4 - 196t^3 - 96t^2 + 44t - 1)\omega^4 \; + \; 840(5t^2 - 2t - 1)\omega^2 \; + \; 35280}{35280},$$

$$T^{\omega}_{5,2}(t) \; = \; t^5 \; \frac{(49t^4 - 100t^2 + 51)\omega^4 \; + \; 840(t^2 - 1)\omega^2 \; + \; 7056}{7056}.$$

SM10. Stable expressions of $\{\phi^{\omega}_{j,m}(t)\}_{j=0}^{2m+1}$, $m \in \{1, 2\}$, for large ω.

For $m = 1$,

$$\phi^{\omega}_{0,1}(t) \; = \; \phi^{\omega}_{3,1}(1-t), \qquad \phi^{\omega}_{1,1}(t) \; = \; \phi^{\omega}_{2,1}(1-t),$$

$$\phi^\omega_{2,1}(t) \;=\; \frac{N\phi^\omega_{2,1}(t)}{D\phi^\omega_{2,1}}, \qquad \phi^\omega_{3,1}(t) \;=\; \frac{e^{-2\omega t} \;+\; 2\omega t e^{-\omega t} \;-\; 1}{e^{-2\omega} \;+\; 2\omega e^{-\omega} \;-\; 1}\; e^{\omega(t-1)},$$

where

$$\begin{aligned} N\phi^\omega_{2,1}(t) = {} & \left(\frac{1}{\omega}+t\right)e^{-3\omega} \;-\; \frac{1}{\omega}e^{\omega(t-3)} \;-\; \left(\frac{1}{\omega}+1\right)e^{-\omega(t+2)} \;-\; \left(\frac{1}{\omega}+3t-2\right)e^{-2\omega} \\ & +\; \left(\frac{2}{\omega}-1\right)e^{\omega(t-2)} \;+\; \left(\frac{2}{\omega}+1\right)e^{-\omega(t+1)} \\ & -\; \left(\frac{1}{\omega}-3t+2\right)e^{-\omega} \;-\; \left(\frac{1}{\omega}-1\right)e^{\omega(t-1)} \;-\; \frac{1}{\omega}e^{-t\omega} \;+\; \frac{1}{\omega}-t, \end{aligned}$$

$$D\phi^\omega_{2,1} \;=\; \left(\frac{2}{\omega}+1\right)e^{-3\omega} \;-\; \left(\frac{2}{\omega}-5-2\omega\right)e^{-2\omega} \;-\; \left(\frac{2}{\omega}+5-2\omega\right)e^{-\omega} \;+\; \frac{2}{\omega}-1.$$

For $m = 2$,

$$\phi^\omega_{0,2}(t) \;=\; \phi^\omega_{5,2}(1-t), \qquad \phi^\omega_{1,2}(t) \;=\; \phi^\omega_{4,2}(1-t), \qquad \phi^\omega_{2,2}(t) \;=\; \phi^\omega_{3,2}(1-t),$$

$$\phi^\omega_{3,2}(t) \;=\; \frac{N\phi^\omega_{3,2}(t)}{D\phi^\omega_{3,2}}, \qquad \phi^\omega_{4,2}(t) \;=\; \frac{N\phi^\omega_{4,2}(t)}{D\phi^\omega_{4,2}}\; e^{\omega(t-1)},$$

$$\phi^\omega_{5,2}(t) \;=\; \frac{e^{-4\omega t} - 8e^{-3\omega t} - 12\omega t e^{-2\omega t} + 8e^{-\omega t} - 1}{e^{-4\omega} - 8e^{-3\omega} - 12\omega e^{-2\omega} + 8e^{-\omega} - 1}\; e^{2\omega(t-1)},$$

where

$$\begin{aligned} N\phi^\omega_{3,2}(t) = {} & \left(\frac{3}{\omega}+2t\right)e^{-5\omega} - \frac{4}{\omega}e^{\omega(t-5)} + \frac{1}{\omega}e^{\omega(2t-5)} - 4\left(\frac{2}{\omega}+1\right)e^{-\omega(t+4)} \\ & + \left(\frac{9}{\omega}-2(5t-6)\right)e^{-4\omega} - 4\left(\frac{1}{\omega}+3\right)e^{\omega(t-4)} + \left(\frac{3}{\omega}+4\right)e^{2\omega(t-2)} \\ & + \left(\frac{5}{\omega}+2\right)e^{-\omega(2t+3)} + 4\left(\frac{1}{\omega}-1\right)e^{-\omega(t+3)} - 4\left(\frac{3}{\omega}-5t\right)e^{-3\omega} \\ & + 4\left(\frac{3}{\omega}+1\right)e^{\omega(t-3)} - \left(\frac{9}{\omega}+2\right)e^{\omega(2t-3)} - \left(\frac{9}{\omega}-2\right)e^{-2\omega(t+1)} + 4\left(\frac{3}{\omega}-1\right)e^{-\omega(t+2)} \\ & - 4\left(\frac{3}{\omega}+5t\right)e^{-2\omega} + 4\left(\frac{1}{\omega}+1\right)e^{\omega(t-2)} + \left(\frac{5}{\omega}-2\right)e^{2\omega(t-1)} \\ & + \left(\frac{3}{\omega}-4\right)e^{-\omega(2t+1)} - 4\left(\frac{1}{\omega}-3\right)e^{-\omega(t+1)} + \left(\frac{9}{\omega}+2(5t-6)\right)e^{-\omega} \\ & - 4\left(\frac{2}{\omega}-1\right)e^{\omega(t-1)} + \frac{1}{\omega}e^{-2\omega t} - \frac{4}{\omega}e^{-\omega t} + \frac{3}{\omega} - 2t, \end{aligned}$$

$$\begin{aligned} D\phi^\omega_{3,2} = 2\Bigg[& \left(\frac{3}{\omega}+1\right)e^{-5\omega} + \left(\frac{27}{\omega}+31+6\omega\right)e^{-4\omega} - 2\left(\frac{15}{\omega}-23-15\omega\right)e^{-3\omega} \\ & -2\left(\frac{15}{\omega}+23-15\omega\right)e^{-2\omega} + \left(\frac{27}{\omega}-31+6\omega\right)e^{-\omega} + \frac{3}{\omega}-1\Bigg], \end{aligned}$$

$$
\begin{aligned}
N\phi^{\omega}_{4,2}(t) = {} & 2e^{-\omega(2t+5)} + 3(1+2\omega t)e^{-\omega(t+5)} - 6e^{-5\omega} + e^{\omega(t-5)} - 2e^{-\omega(3t+4)} \\
& - 2e^{-2\omega(t+2)} - 3(9+10\omega t)e^{-\omega(t+4)} + 38e^{-4\omega} - 7e^{\omega(t-4)} \\
& - (1+6\omega)e^{-3\omega(t+1)} + 24(1+\omega)e^{-\omega(2t+3)} + 12(2+\omega(5t-3))e^{-\omega(t+3)} \\
& - 8(7-3\omega)e^{-3\omega} + 3(3-2\omega)e^{\omega(t-3)} + 3(3+2\omega)e^{-\omega(3t+2)} - 8(7+3\omega)e^{-2\omega(t+1)} \\
& + 12(2-\omega(5t-3))e^{-\omega(t+2)} + 24(1-\omega)e^{-2\omega} - (1-6\omega)e^{\omega(t-2)} \\
& - 7e^{-\omega(3t+1)} + 38e^{-\omega(2t+1)} - 3(9-10\omega t)e^{-\omega(t+1)} - 2e^{-\omega} \\
& - 2e^{\omega(t-1)} + e^{-3\omega t} - 6e^{-2\omega t} + 3(1-2\omega t)e^{-\omega t} + 2,
\end{aligned}
$$

$$
\begin{aligned}
D\phi^{\omega}_{4,2} = {} & e^{-7\omega} + (1+6\omega)e^{-6\omega} - 27(3+2\omega)e^{-5\omega} + (79-156\omega-72\omega^2)e^{-4\omega} \\
& + (79+156\omega-72\omega^2)e^{-3\omega} - 27(3-2\omega)e^{-2\omega} + (1-6\omega)e^{-\omega} + 1.
\end{aligned}
$$

SM11. Stable expressions of $\{\tau^{\omega}_{j,m}(t)\}_{j=0}^{2m}$, $m \in \{1,2\}$, for large ω.
For $m = 1$,

$$
\tau^{\omega}_{0,1}(t) \;=\; \frac{e^{\omega\,(t-2)} - 2\;(t-1)\;\omega\,e^{-\omega} - e^{-t\,\omega}}{(t-1)^2\;(e^{-2\,\omega} + 2\,\omega\,e^{-\omega} - 1)}, \qquad \tau^{\omega}_{1,1}(t) \;=\; \frac{N\tau^{\omega}_{1,1}(t)}{D\tau^{\omega}_{1,1}(t)},
$$

$$
\tau^{\omega}_{2,1}(t) \;=\; \frac{t^2\,e^{-2\,\omega} - e^{-\omega\,(t+1)} + 2\,t\;(t-1)\;\omega\,e^{-\omega} + e^{\omega\,(t-1)} - t^2}{t^2\;(e^{-2\,\omega} + 2\,\omega\,e^{-\omega} - 1)},
$$

where

$$
\begin{aligned}
N\tau^{\omega}_{1,1}(t) \;=\; & \left(\left(\frac{2}{\omega}+1\right) t^2 - \left(\frac{4}{\omega}+1\right) t + \frac{1}{\omega} \right) \mathrm{e}^{-2\,\omega} \;-\; \frac{1}{\omega}\,\mathrm{e}^{\omega\,(t-1)} \\
& -\; \frac{1}{\omega}\,\mathrm{e}^{\omega\,(t-2)} \;+\; 2t(t-1)\,\mathrm{e}^{-\omega} \;+\; \frac{1}{\omega}\,\mathrm{e}^{-t\,\omega} \\
& +\; \frac{1}{\omega}\,\mathrm{e}^{-\omega\,(t+1)} \;-\; \left(\frac{2}{\omega}-1\right) t^2 + \left(\frac{4}{\omega}-1\right) t - \frac{1}{\omega},
\end{aligned}
$$

$$
D\tau^{\omega}_{1,1}(t) \;=\; 2\,t\;(t-1)\;\left(e^{-\omega}+1\right)\;\left(\left(\frac{2}{\omega}+1\right) e^{-\omega} - \frac{2}{\omega} + 1 \right).
$$

For $m = 2$, $\tau^{\omega}_{j,2}(t) = N\tau^{\omega}_{j,2}(t)/D\tau^{\omega}_{j,2}(t)$, $j = 0, \ldots, 4$, where

$$
N\tau^{\omega}_{0,2}(t) \;=\; -\,e^{2\,\omega\,(t-2)} \;+\; 8\,e^{\omega\,(t-3)} \;-\; 12\;(t-1)\;\omega\,e^{-2\,\omega} \;-\; 8\,e^{-\omega\,(t+1)} \;+\; e^{-2\,t\,\omega},
$$

$$
D\tau^{\omega}_{0,2}(t) \;=\; (t-1)^4 \left(\;-\,e^{-4\,\omega} \;+\; 8\,e^{-3\,\omega} \;+\; 12\,\omega\,e^{-2\,\omega} \;-\; 8\,e^{-\omega} \;+\; 1 \;\right),
$$

$$
\begin{aligned}
N\tau^{\omega}_{1,2}(t) \;=\; & (t-1)^4\,e^{-3\,\omega} \;-\; 2\,e^{\omega\,(t-3)} \;+\; e^{\omega\,(2\,t-3)} \\
& +\; 3\left(\,3t^4 - 12\,t^3 + 18\,t^2 - 12\,t + 2 + 2t\omega\left(t^3 - 4\,t^2 + 6\,t - 3\right)\,\right)\,e^{-2\,\omega} \\
& -\; 6\,e^{\omega\,(t-2)} \;+\; 6\,e^{-\omega\,(t+1)} \\
& -\; 3\left(\,3\,t^4 - 12\,t^3 + 18\,t^2 - 12\,t + 2 - 2t\omega\left(t^3 - 4\,t^2 + 6\,t - 3\right)\,\right)\,e^{-\omega} \\
& -\; e^{-2\,t\,\omega} \;+\; 2\,e^{-t\,\omega} \;-\; (t-1)^4,
\end{aligned}
$$

$$D\tau^{\omega}_{1,2}(t) \;=\; 4t(t-1)^3 \left(\, e^{-3\omega} \;+\; 3\,(2\omega+3)\,e^{-2\omega} \;+\; 3\,(2\omega-3)\,e^{-\omega} \;-\; 1 \,\right),$$

$$\begin{aligned} N\tau^{\omega}_{2,2}(t) \;=\; -\, &\left(\frac{3(6\,t^4-16\,t^3+12\,t^2-1)}{\omega} \;+\; 2t(3\,t^3-8\,t^2+6\,t-1) \right) e^{-2\,\omega} \\ &+\; \frac{4}{\omega}e^{\omega\,(t-2)} \;-\; \frac{1}{\omega}e^{2\,\omega\,(t-1)} \;-\; \frac{4}{\omega}e^{-\omega\,(t+1)} \\ &+\; 8\,t\,\left(3\,t^3-8\,t^2+6\,t-1\right)\,e^{-\omega} \\ &+\; \frac{4}{\omega}e^{\omega\,(t-1)} \;+\; \frac{1}{\omega}e^{-2\,t\,\omega} \;-\; \frac{4}{\omega}e^{-t\,\omega} \\ &-\; \frac{3(6\,t^4-16\,t^3+12\,t^2-1)}{\omega} \;+\; 2t(3\,t^3-8\,t^2+6\,t-1), \end{aligned}$$

$$D\tau^{\omega}_{2,2}(t) \;=\; 12\,t^2\,(t-1)^2 \left(\left(\frac{3}{\omega}+1\right) e^{-2\,\omega} \;+\; 4\,e^{-\omega} \;-\; \frac{3}{\omega} \;+\; 1 \right),$$

$$\begin{aligned} N\tau^{\omega}_{3,2}(t) \;=\; & t^3\;(3\,t-4)\;e^{-3\,\omega} \;+\; 2\,e^{-\omega\,(t+2)} \\ &+\; 3\left(2\,t\,\omega-8\,t^3\,\omega+6\,t^4\,\omega-12\,t^3+9\,t^4+1\right)e^{-2\,\omega} \\ &-\; 6\,e^{\omega\,(t-2)} \;+\; e^{2\,\omega\,(t-1)} \;-\; e^{-\omega\,(2\,t+1)} \;+\; 6\,e^{-\omega\,(t+1)} \\ &+\; 3\left(2\,t\,\omega-8\,t^3\,\omega+6\,t^4\,\omega+12\,t^3-9\,t^4-1\right)e^{-\omega} \\ &\qquad -\; 2\,e^{\omega\,(t-1)} \;-\; t^3\;(3\,t-4)\,, \end{aligned}$$

$$D\tau^{\omega}_{3,2}(t) \;=\; 4\,t^3\;(t-1)\;\left(\, e^{-3\,\omega} \;+\; 3\,(2\,\omega+3)\,e^{-2\,\omega} \;+\; 3\,(2\,\omega-3)\,e^{-\omega} \;-\; 1\,\right),$$

$$\begin{aligned} N\tau^{\omega}_{4,2}(t) \;=\; & -\,t^4\,e^{-4\,\omega} \;+\; 8\,t^4\,e^{-3\,\omega} \;+\; e^{-2\,\omega\,(t+1)} \;-\; 8\,e^{-\omega\,(t+2)} \\ &+\; 12\,t\,\left(t^3-1\right)\,\omega\,e^{-2\,\omega} \;+\; 8\,e^{\omega\,(t-2)} \;-\; e^{2\,\omega\,(t-1)} \;-\; 8\,t^4\,e^{-\omega} \;+\; t^4, \end{aligned}$$

$$D\tau^{\omega}_{4,2}(t) \;=\; t^4\left(\,-e^{-4\,\omega} \;+\; 8\,e^{-3\,\omega} \;+\; 12\,\omega\,e^{-2\,\omega} \;-\; 8\,e^{-\omega} \;+\; 1\,\right).$$