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Glivenko sequent classes and constructive cut elimination in geometric logics

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Abstract

A constructivisation of the cut-elimination proof for sequent calculi for classical, intuitionistic and minimal infinitary logics with geometric rules—given in earlier work by the second author—is presented. This is achieved through a procedure where the non-constructive transfinite induction on the commutative sum of ordinals is replaced by two instances of Brouwer's Bar Induction. The proof of admissibility of the structural rules is made ordinal-free by introducing a new well-founded relation based on a notion of embeddability of derivations. Additionally, conservativity for classical over intuitionistic/minimal logic for the seven (finitary) Glivenko sequent classes is here shown to hold also for the corresponding infinitary classes.

Keywords: Geometric theories, Glivenko sequent classes, infinitary logic, conservativity, constructive cut elimination

MSC Classification: 03F05 , 03B20 , 18C10 , 18B15

1 Introduction

Notable parts of algebra and geometry can be formalised as *coherent theories* over first-order classical or intuitionistic logic. Their axioms are *coherent implications*, i.e., universal closures of implications $D_1 \supset D_2$, where both D_1 and D_2 are built up from atoms using conjunction, disjunction and existential quantification. Examples include all algebraic theories, such as group theory and the theory of rings, all essentially algebraic theories, such as category theory [9], the theory of fields, the theory of local rings, lattice theory [28], projective and affine geometry [22, 28], the theory of separably closed local rings (aka "strictly Henselian local rings") [12, 22, 31].

Although wide, the class of coherent theories leaves out certain axioms used in algebra—such as the axioms of torsion abelian groups or of Archimedean ordered fields, or in the theory of connected graphs, as well as in the modelling of epistemic social notions such as common knowledge. All the latter examples can however be axiomatised by means of *geometric axioms*: a generalisation of coherent axioms that admits infinitary disjunctions.

Coherent and geometric implications give a Glivenko sequent class [23], as shown by Barr's Theorem:

Theorem 1 (Barr's Theorem [3]) *If $\mathcal{T}$ is a coherent (geometric) theory and A is a coherent (geometric) sentence provable from $\mathcal{T}$ with (infinitary) classical logic, then A is provable from $\mathcal{T}$ with (infinitary) intuitionistic logic.*

Barr's Theorem[1] has its origin, through appropriate completeness results, in the theory of sheaf models, with the following formulation:

Theorem 2 ([15], Ch.9, Thm.2) *For every Grothendieck topos $\mathcal{E}$ there exists a complete Boolean algebra $\mathbf{B}$ and a surjective geometric morphism $Sh(\mathbf{B}) \longrightarrow \mathcal{E}$.*

Barr's theorem provides an important conservativity result for classical and intuitionistic geometric theories. Orevkov [23] has established some well-known conservativity results of classical logic over intuitionistic and minimal first-order logics with equality. These results generalise the finitary Barr's Theorem by considering further classes of sequents for which conservativity holds. In particular, [23] isolates seven classes of single-succedent sequents—the so-called

This paper is a revised and extended version of the conference paper [7]. The latter presented only the constructive cut elimination for classical and intuitionistic geometric logics based on Brouwer's Bar Induction. The main novelties of this paper are (i) that also minimal geometric logic is considered, (ii) that the notion of "proof-embeddability" is here introduced and transfinite inductions on ordinals are replaced by Noetherian induction with proof-embeddability, and (iii) that proofs of conservativity for the infinitary Glivenko classes are given.

[1] Barr's theorem is often alleged to achieve more in that it also allows to eliminate uses of the axiom of choice. That such formulations of Barr's theorem should be taken with caution is demonstrated in [25] where *internal* vs. *external* addition of the the axiom of choice is considered and it is shown that the latter preserves conservativity whereas the former does not.

Glivenko sequent classes—defined in terms of the absence of positive or negative occurrences of particular logical symbols (in a first-order language with equality) where classical derivability implies intuitionistic or even minimal derivability. The same article also shows that these classes are optimal:[2] any class of sequents for which classical derivability implies intuitionisitc derivability is contained in one of these seven classes. The interest of such conservativity results is twofold. First, since proofs in intuitionistic logic obtain a computational meaning via the Curry-Howard correspondence, such results identify some classical theories having a computational content. Second, since it may be easier to prove theorems in classical than in intuitionistic or minimal logic and since there are more well-developed automated theorem provers for classical than for sub-classical logics, such results simplify the search for theorems in intuitionistic (and minimal) theories.

Orevkov's results on Glivenko sequent classes have not received much attention despite their usefulness in analysing the computational content of classical theories. One of the main reasons for this is the complexity of Orevkov's [23] proofs. In recent years simpler proofs of conservativity results for some Glivenko sequent classes have been given [11, 17, 26]. An extremely simple and purely logical proof of the first-order Barr's Theorem for coherent theories has been given in [18] by means of **G3**-style sequent calculi: it is shown how to express coherent implications by means of rules that preserve the admissibility of the structural rules of inference. As a consequence, Barr's theorem is proved by simply noticing that a proof in **G3C**.**T**, i.e. the calculus for classical logic extended with rules expressing coherent implications—is also a proof in the intuitionistic multisuccedent calculus **G3I**.**T**. A purely logical proof of Barr's Theorem for infinitary geometric theories has been given [19].

This simple and purely logical proof of Barr's Theorem has been extended to geometric theories in [19]. This work considers the **G3**-style calculi for classical and intuitionistic infinitary logic $\mathbf{G3[CI]}_\omega$ (with finite sequents instead of countably infinite sequents) and their extension with rules expressing geometric implications $\mathbf{G3[CI]}_\omega.\mathbf{G}$. To illustrate, the geometric axiom of torsion abelian groups

$$\forall x. \bigvee_{n>0} nx = 0$$

is expressed by the infinitary rule:

$$\frac{\{nx = 0, \Gamma \Rightarrow \Delta \mid n > 0\}}{\Gamma \Rightarrow \Delta}$$

The main results in [19] are that in $\mathbf{G3[CI]}_\omega.\mathbf{G}$ all rules are height-preserving invertible, the structural rules of weakening and contraction are height-preserving admissible, and cut is admissible. Hence, Barr's Theorem for geometric theories is proved in [19] as it was done in [18] for coherent ones: a

[2] Barr's Theorem corresponds to Orevkov's first class.

proof in $\mathbf{G3C}_\omega.\mathbf{G}$ is also a proof in the intuitionistic multisuccedent calculus $\mathbf{G3I}_\omega.\mathbf{G}$.

The aforementioned proof of first-order Barr's Theorem has further been extended to cover all other first-order Glivenko sequent classes in [20]. In this paper we extend the purely logical proof of the infinitary Barr's Theorem given in [19] to cover all other infinitary Glivenko sequent classes: for each class we give a purely constructive proof of conservativity of classical infinitary logic and of a class of classical geometric theories over intuitionistic and minimal infinitary logics and geometric theories, respectively.

We observe that the cut-elimination procedure given in Sect. 4.1 of [19] is not constructive. This is an instance of a typical limitation of cut eliminations in infinitary logics [6, 14, 29] since these proofs use the "natural" (or "Hessenberg") commutative sum of ordinals $\alpha\#\beta$:

$$(\omega^{\alpha_m} + \cdots + \omega^{\alpha_0})\#(\omega^{\beta_n} + \cdots + \omega^{\beta_0}) = (\omega^{\gamma_{m+n+1}} + \cdots + \omega^{\gamma_0})$$

where $\gamma_{m+n+1}, \ldots, \gamma_0$ is a decreasing permutation of $\alpha_m, \ldots, \alpha_0, \beta_n, \ldots, \beta_0$; see [30, 10.1.2B]. The resort to the natural sum is inescapable for proofs using the cut-height—i.e., the sum of the derivation-height of the premisses of cut—as inductive parameter: it ensures that we can apply the inductive hypothesis when permuting the cut upwards in the derivation of one of the premisses. Nevertheless, it makes the proof non-constructive since

> [its] definition utilises the Cantor normal form of ordinals to base ω. This normal form is not available in **CZF** (or **IZF**) and thus a different approach is called for. [25, p. 369]

This makes the conservativity results about infinitary Glivenko classes less appealing from the perspective of constructivists: cut is necessary to prove completeness of geometric theories—since they are axiomatised via geometric rules—and a non-constructive proof of cut elimination implies that we are working in a classical meta-theory.

To overcome this drawback we constructivise[3] the proof of (height-preserving) admissibility of the structural rules for $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ by giving procedures that avoid completely the need for ordinal numbers: transfinite inductions on (sums of) ordinals are replaced by inductions on well-founded trees and by Brouwer's principle of Bar Induction—see Theorem 26.[4] In particular, we capture the fact that a derivation $\mathcal{D}_1$ is "smaller" than $\mathcal{D}_2$ if each branch of $\mathcal{D}_1$ is "smaller" than a branch of $\mathcal{D}_2$ by introducing a new well-founded relation based on the notion of *proof embeddability*. This allows us to compare derivations without explicitly giving them a height, and thus to replace the transfinite inductions on the height of derivations used in [7, 19] with well-founded inductions on this new relation. This will allow us to give

[3] By "constructive" here we mean not relying on classical logical principles such as excluded middle or linearity of ordinals but we do not mean acceptable in all schools of constructive mathematics.

[4] See [25, $\mathbf{S}_7$] for a different proof, based on constructive ordinals, of cut elimination in infinitary logic. The proof in [2] does not use ordinals, but it is inherently classical in that it uses a one-sided calculus based on De Morgan's dualities.

an ordinal-free proof of invertibility and of the admissibility of the structural rules of weakening and contraction.[5] Next, we build on these results to give a constructive and ordinal-free proof of cut-elimination for geometric logics. In order to do so, we replace the Dragalin-style proof adopted in [19] with a (modification of a) proof strategy introduced in [16] for fuzzy logics. This strategy replaces the induction on the natural sum of the heights of the derivations of the two premisses of Cut with two separate well-founded inductions with proof embeddability on the derivation of the right and left premiss, respectively. Finally, we use two instances of Brouwer's Bar Induction, the first to prove that an uppermost instance of Cut is admissible and the second to prove that all instances of Cut are admissible. Bar Induction is needed to avoid considering a Cut of maximal rank as in [16]—since this would require trichotomy of ordinals—and, hence, to obtain a constructive and ordinal-free proof of the admissibility of Cut in $\mathbf{G3[CIM]}_\omega.\mathbf{G}$.

This paper is organised as follows: Sections 2 and 3 introduce sequent calculi for infinitary logics and for geometric theories, respectively. Next, Section 4 introduces the notion of proof-embeddability and Section 5 proves that all rules of $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ are proof embeddable invertible and that the structural rules of weakening and contraction are proof embeddable admissible. Building on these results, Section 6 presents an ordinal-free and constructive proof of the admissibility of Cut. Finally, Section 7 proves conservativity results of classical logic (theories) over intuitionistic and minimal logics (theories) for the infinitary Glivenko sequent classes.

2 Syntax and sequent calculi for infinitary logics

Let $\mathcal{S}$ be a signature containing, for every $n \in \mathbb{N}$, a countable (i.e., finite, possibly empty, or countably infinite) set $REL_n^{\mathcal{S}}$ of n-ary predicate letters $P_1^n, P_2^n, \ldots$, and a countable set $FUN_n^{\mathcal{S}}$ of n-ary function letters $f_1^n, f_2^n, \ldots$. Let VAR be a denumerable set of variables $x_1, x_2, \ldots$. The language contains the following logical symbols: $=$, $\bot$, $\top$, $\wedge$, $\vee$, $\supset$, $\forall$, $\exists$, as well as countable conjunction $\bigwedge_{n>0}$ and countable disjunction $\bigvee_{n>0}$.

The sets of *terms* and *formulas* of the language $\mathscr{L}_\omega^{\mathcal{S}}$ are generated, respectively, by:

$$t ::= x \mid f^n t_1, \ldots t_n$$

$$A ::= P^n t_1, \ldots, t_n \mid t_1 = t_2 \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \supset A \mid \forall x A \mid \exists x A \mid \bigwedge_{n>0} A_n \mid \bigvee_{n>0} A_n$$

where $f^n \in FUN_n^{\mathcal{S}}$, $P_n \in REL_n^{\mathcal{S}}$, and $x, x_1, \ldots, x_n \in VAR$.
We use the following metavariables:

- x, y, z for variables and $\vec{x}, \vec{y}, \vec{z}$ for lists thereof;

[5] Even if all proofs in [7] make no use of non-constructive assumptions about ordinals, we prefer to avoid completely the assumption of total ordering.

- a, b, c for 0-ary functions (aka individual constants);
- t, s, r for terms;
- P, Q, R for atomic formulas;
- A, B, C for formulas.

We use $A(\vec{x})$ to say that the variables having free occurrences in A are included in $\vec{x}$. We follow the standard conventions for parentheses. The formulas $\neg A$ and $A \supset\subset B$ are defined as expected. When considering (infinitary) classical logic we can shrink the set of primitive logical symbols by means of the well-known De Morgan's dualities (including $\bigvee_{n>0} A_n \supset\subset \neg \bigwedge_{n>0} \neg A$), however also in the classical case we consider a language where all operators (excluding $\neg$ and $\supset\subset$) are taken as primitive. This is not just useful but even necessary since our purpose is to extract the constructive content of classical proofs and many of the interdefinabilities do not hold in intuitionistic logic.

The notions of *free* and *bound occurrences* of a variable in a formula are the usual ones. We posit that no formula may have infinitely many free variables. A *sentence* is a formula without free occurrences of variables. Given a formula A, we use $A[t/x]$ to denote the formula obtained by replacing each free occurrence of x in A with an occurrence of t, provided that t is free for x in A—i.e., no new occurrence of t is bound by a quantifier.

Sequents $\Gamma \Rightarrow \Delta$ have a finite multiset of formulas on each side. The inference rules for $\bigvee_{n>0}$ are thus:

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee{}_k \quad .$$

Observe that L$\bigvee$ has countably many premisses, one for each $n > 0$ and that there are infinitely many R$\bigvee$ rules, the subscript of which will be usually omitted. The rules for $\bigwedge_{n>0}$ are dual to the above ones.

Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length. The *leaves* of the trees are those where the two sides have an atomic formula (or $\bot$ for minimal logic) in common, and also instances of rules L$\bot$, R$\top$. To make this precise, we give a formal definition of the notion of *derivation* $\mathcal{D}$ and of its *end-sequent*.

Definition 3 (Derivations and their end-sequent)

1. Any sequent $\Gamma \Rightarrow \Delta$, where some atomic formula occurs in both Γ and Δ, is a derivation with end-sequent $\Gamma \Rightarrow \Delta$.

   In minimal logic, any sequent $\bot, \Gamma \Rightarrow \Delta, \bot$, is a derivation with end-sequent $\bot, \Gamma \Rightarrow \Delta, \bot$.
2. Let $\beta \leq \omega$. If each $\mathcal{D}_n$, for $0 < n < \beta$, is a derivation with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

$$\frac{\dots \quad \Gamma_n \Rightarrow \Delta_n \quad \dots}{\Gamma \Rightarrow \Delta}\ \mathrm{R}$$

is an instance of a rule with β premisses, then

$$\dfrac{\dots \quad \mathcal{D}_n \left\{ \begin{array}{c} \vdots \\ \Gamma_n \Rightarrow \Delta_n \end{array} \right. \quad \dots}{\Gamma \Rightarrow \Delta} \, R$$

is a derivation with end-sequent $\Gamma \Rightarrow \Delta$.[6]

If **X** is a calculus, we use $\mathbf{X} \vdash \Gamma \Rightarrow \Delta$ to say that $\Gamma \Rightarrow \Delta$ is derivable in the calculus **X**. Derivations and formulas can be associated with ordinals, but we don't need this association here and actually depart from the ordinal approach for the reasons explained above. For the definition of ordinal height of a derivation and ordinal depth of a formula in infinitary logic we refer the reader to [19].

Definition 4 (Sequent calculi for infinitary logics with equality)

1. $\mathbf{G3C}_\omega$ is defined by the rules in Table 1;
2. $\mathbf{G3I}_\omega$ is defined as $\mathbf{G3C}_\omega$ with the exception of rules L$\supset$, R$\supset$, R$\forall$, and R$\bigwedge$ that are defined as in Table 3;
3. $\mathbf{G3M}_\omega$ is defined as $\mathbf{G3I}_\omega$ with the exception of rules L$\bot$ that is replaced by intial sequents of the shape $\bot, \Gamma \Rightarrow \Delta, \bot$.

By $\mathbf{G3}[\mathbf{CIM}]_\omega$ we denote any one of the three calculi above. Observe that a multi-succedent intuitionistic calculus as the one we use is closer to a classical calculus than the usual calculus with the restriction that the succedent of sequents should consist of at most one formula (used, for example in [25]). As in the finitary case such a multi-succedent choice is particularly useful for proving Glivenko-style results [20].

As usual, we consider only derivations of *pure sequents*, i.e., sequents where no variable has both free and bound occurrences. We say that $\Gamma \Rightarrow \Delta$ is $\mathbf{G3[CIM]}_\omega$-*derivable*—and write $\mathbf{G3}[\mathbf{CIM}]_\omega \vdash \Gamma \Rightarrow \Delta$—if there is a $\mathbf{G3[CIM]}_\omega$-derivation of $\Gamma \Rightarrow \Delta$ or of an alphabetic variant of $\Gamma \Rightarrow \Delta$. A rule is said to be *admissible* in $\mathbf{G3[CIM]}_\omega$ if, whenever its premisses are $\mathbf{G3[CIM]}_\omega$-derivable, also its conclusion is $\mathbf{G3[CIM]}_\omega$-derivable. A rule is said to be *invertible* in $\mathbf{G3[CIM]}_\omega$ if, whenever its conclusion is $\mathbf{G3[CIM]}_\omega$-derivable, also its premisses are $\mathbf{G3[CIM]}_\omega$-derivable. In each rule depicted in Tables 1, 2, and 3 the multisets Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and the formulas occurring only in the premiss(es) are called *active*.

[6] Derivations can thus be represented as (infinite) trees, where the nodes are the sequents in the derivation, and a nodes that corresponds to a premiss of a rule is an immediate successor of the node that corresponds to the conclusion of such rule. Therefore, a node that corresponds to the conclusion of a rule with β premisses has β immediate successors.

Table 1 Rules of the calculus $\mathbf{G3C}_\omega$, $(*)$ z fresh in rules $L\exists$ and $R\forall$

Initial sequents: $P, \Gamma \Rightarrow \Delta, P$	$\dfrac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$	$\dfrac{}{\Gamma \Rightarrow \Delta, \top}\ R\top$
$\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$	$\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$
$\dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge$	$\dfrac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$	$\dfrac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$
$\dfrac{\Gamma \Rightarrow \Delta, A(t/x), \exists x A}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$	$\dfrac{\Gamma \Rightarrow \Delta, A(z/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall\ (*)$	
$\dfrac{A(z/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists\ (*)$	$\dfrac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall$	
$\dfrac{\{\Gamma \Rightarrow \Delta, A_i \mid i > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge A_n}\ R\bigwedge$	$\dfrac{\Gamma \Rightarrow \Delta, \bigvee A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee A_n}\ R\bigvee_k$	
$\dfrac{A_k, \bigwedge A_n, \Gamma \Rightarrow \Delta}{\bigwedge A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge_k$	$\dfrac{\{A_i, \Gamma \Rightarrow \Delta \mid i > 0\}}{\bigvee A_n, \Gamma \Rightarrow \Delta}\ L\bigvee$	

Table 2 Rules for equality in $\mathbf{G3[CIM]}_\omega$

$\dfrac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref$	$\dfrac{P(t/x), s = t, P(s/x), \Gamma \Rightarrow \Delta}{s = t, P(s/x), \Gamma \Rightarrow \Delta}\ Repl$
$\dfrac{t = f(\dots, f(\dots, t, \dots), \dots), t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}{t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}\ Repl^c$	

Table 3 Non-classical rules for $\mathbf{G3[IM]}_\omega$, $(\star)$ z fresh in $R\forall$

$\dfrac{A \supset B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$	$\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$
$\dfrac{\Gamma \Rightarrow A(z/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall\ (*)$	$\dfrac{\{\Gamma \Rightarrow A_i \mid i > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge A_n}\ R\bigwedge$

Table 4 Geometric rule L_G expressing the geometric sentence (G)

$$\frac{\dots \quad Q_{n_1}(\vec{x},\vec{y}_n),\dots,Q_{n_m}(\vec{x},\vec{y}_n),P_1(\vec{x}),\dots,P_k(\vec{x}),\Gamma\Rightarrow\Delta \quad \dots}{P_1(\vec{x}),\dots,P_k(\vec{x}),\Gamma\Rightarrow\Delta}\ L_G$$

3 From geometric implications to geometric rules

By a *geometric implication* we mean the universal closure of an implicative formula whose antecedent and consequent are positive formulas (i.e., formulas constructed from atomic formulas and $\bot,\top$ using only $\wedge$, $\vee$, $\exists$, and $\bigvee_{n>0}$). More precisely

Definition 5 (Geometric implication)

- A formula is *Horn* iff it is built from atoms and $\top$ using only $\wedge$;
- A formula is *geometric* iff it is built from atoms and $\top$, $\bot$ using only $\wedge$, $\vee$, $\exists$, and $\bigvee_{n>0}$;
- A sentence is a *geometric implication* iff it is of the form $\forall\vec{x}(A\supset B)$ where A and B are geometric formulas.

By a *coherent implication* we mean a geometric implication without occurrences of $\bigvee_{n>0}$.

As is well known, for geometric implications we have a normal form theorem.

Theorem 6 (Geometric normal form (GNF)) *Any geometric implication is equivalent to a possibly infinite conjunction of sentences of the form*

$$\forall\vec{x}(A\supset B)$$

where A is Horn and B is a possibly infinite disjunction of existentially quantified Horn formulas.

This normal form theorem is important because, as shown in [18] for coherent implications and in [19] for geometric ones, we can extract from a sentence G in GNF a *geometric rule* L_G (where the name L_G indicates that it is a *left rule*) that can be added to a sequent calculus without altering its structural properties. To be more precise, let us consider the following sentence G in GNF:

$$\forall\vec{x}(P_1(\vec{x})\wedge\cdots\wedge P_k(\vec{x})\supset\bigvee_{n>0}\exists\vec{y}(Q_{n_1}(\vec{x},\vec{y})\wedge\cdots\wedge Q_{n_m}(\vec{x},\vec{y}))) \qquad (G)$$

Such a sentence G determines the (finitary or infinitary) *geometric rule* given in Table 4 with one premiss for each of the countably many disjuncts in

$\bigvee_{n>0}(Q_{n_1}(\vec{x},\vec{y}) \wedge \dots \wedge Q_{n_m}(\vec{x},\vec{y}))$. The variables in $\vec{y}_n$ are chosen to be *fresh*, i.e. they are not in the conclusion—and without loss of generality they are all distinct. The list $\vec{y}_n$ of variables may vary as n varies, and maybe no finite list suffices for all the countably many cases. The variables $\vec{x}$ (finite in number) may be instantiated with arbitrary terms. Henceforth we shall normally omit mention of the variables.

We need also a further condition:

Definition 7 (Closure condition) Given a calculus with geometric rules, if it has a rule with an instance with repetition of some principal formula such as:

$$\frac{\dots \qquad Q_1,\dots,Q_n,P_1,\dots,P_{k-2},P,P,\Gamma\Rightarrow\Delta \qquad \dots}{P_1,\dots,P_{k-2},P,P,\Gamma\Rightarrow\Delta}\ \mathrm{L}^c_G$$

then also the *contracted instance*

$$\frac{\dots \qquad Q_1,\dots,Q_m,P_1,\dots,P_{k-2},P,\Gamma\Rightarrow\Delta \qquad \dots}{P_1,\dots,P_{k-2},P,\Gamma\Rightarrow\Delta}\ \mathrm{L}^c_G$$

has to be included in the calculus.

As for the finitary case [18], also in the infinitary case the condition is unproblematic, since each atomic formula contains only a finite number of variables and therefore so are the instances; it follows that, for each geometric rule, the number of rules that have to be added is finite. Moreover, in many cases contracted instances need not be added since they are already admissible in the calculus. To illustrate, we consider the coherent rule Repl for equality given in Table 1:

$$\frac{P[t/x],s=t,P[s/x],\Gamma\Rightarrow\Delta}{s=t,P[s/x],\Gamma\Rightarrow\Delta}\ \mathrm{Repl}$$

This rules generates contracted instances when its two principal formulas are two copies of the same equality atom $s=t$ and some function of arity greater than zero occur in s,t. For example, the following valid sequent:

$$x=f(x)\Rightarrow x=f(f(x)) \tag{1}$$

would not be 1-derivable in $\mathbf{G3[cim]}_\omega$ if it weren't for the presence of rule *Repl*^{c}, see Figures 1 and 2. Nevertheless (1) is a contracted instance of the following 1-derivable sequent: $x=f(x),x=f(x)\Rightarrow x=f(f(x))$.[7]

When both s ant t are variables or individual constants, instead, we don't need contracted instances (this is why they were not considered in [21]): s and t must be the same term and hence we can obtain the conclusion of the contracted instance by applying an instance of rule *Ref*.

[7]This example is due to Parlamento and Previale [24].

$$\frac{x = f(f(x)), x = f(x) \Rightarrow x = f(f(x))}{x = f(x) \Rightarrow x = f(f(x))}\; Repl^c$$

Fig. 1 Minimal $Repl^c$-derivation of (1).

$$\frac{\dfrac{x = f(f(x)), f(x) = f(f(x)), x = f(x) \Rightarrow x = f(f(x))}{f(x) = f(f(x)), x = f(x) \Rightarrow x = f(f(x))}\; Trans}{x = f(x) \Rightarrow x = f(f(x))}\; Sub$$

Fig. 2 $Repl^c$-free-derivation of (1) (Rules *Sub* and *Trans* are derivable in **G3**[**CIM**]$_\omega$ [20]).

Theorem 8 ([19]) *If we add to the calculus* **G3**[**CIM**]$_\omega$ *a finite or infinite family of geometric rules* L_G*, then we can prove all of the geometric sentences* G *from which they were determined.*

In the following, we shall denote with **G3**[**CIM**]$_\omega$.**G** any extension of **G3[CIM]**$_\omega$ with a finite or infinite family of geometric rules L_G (together with all needed contracted instances thereof).

Before proceeding with the structural properties, we give some examples of geometric axioms and their corresponding rules.

Example 1 (Geometric axioms and rules)

1. The axiom of **torsion Abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule
$$\frac{\ldots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\; \mathrm{R}_{Tor}$$
2. The axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule
$$\frac{\ldots \quad x < n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\; \mathrm{R}_{Arc}$$
3. The axiom of **connected graphs**,
$$\forall xy.x = y \vee \bigvee_{n\geq 1} \exists z_0 \ldots \exists z_n (x = z_0 \wedge y = z_n \wedge z_0 R z_1 \wedge \cdots \wedge z_{n-1} R z_n)$$
becomes the rule
$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \ldots \quad x = z_0, y = z_n, z_0 R z_1, \ldots, z_{n-1} R z_n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\; \mathrm{R}_{Conn}$$

4 Embeddable derivation

The proofs given in [7, 19] make use of transfinite inductions on the height of derivations, which are quite powerful tools. We claim, however, that they are in a certain sense too powerful: they are often non-constructive and, as it will be shown, can be avoided.

Usually, in order to compare two derivations, one assigns ordinal numbers, called *heights*, to them, then compares these parameters. As heights are inductively defined by means of the branches of the derivation, this becomes a comparison between branches. Our main observation is that, in order to compare two derivations, what we actually need is just the fact that $\mathcal{D}$ is "smaller" than $\mathcal{D}'$ if each branch of $\mathcal{D}$ is "smaller" than a branch of $\mathcal{D}'$, without explicitly "measuring" them.

We make this precise by inductively defining simultaneously the relations $\prec$ and $\preccurlyeq$ between derivations. We read $\mathcal{D} \preccurlyeq \mathcal{D}'$ as "$\mathcal{D}$ is (proof) embeddable in $\mathcal{D}'$" and $\mathcal{D} \prec \mathcal{D}'$ as "$\mathcal{D}$ is strictly embeddable in $\mathcal{D}'$".

In what follows, we say that a derivation $\mathcal{D}$ is

- *trivial* if it is an initial or empty sequent;
- *composite*, or *nontrivial*, if has the following form:

$$\mathcal{D}\left\{\frac{\{\mathcal{D}_i\}}{\Gamma \Rightarrow \Delta}\right.$$

It is decidable whether $\mathcal{D}$ is trivial or composite, and the two properties are mutually exclusive.

Definition 9 (proof-embeddability)

(i) If $\mathcal{D}$ and $\mathcal{D}'$ are trivial, then $\mathcal{D} \preccurlyeq \mathcal{D}'$;
(ii) If

$$\mathcal{D}\left\{\frac{\{\mathcal{D}_i\}}{\Gamma \Rightarrow \Delta}\right. \qquad \text{and} \qquad \mathcal{D}'\left\{\frac{\{\mathcal{D}'_j\}}{\Gamma' \Rightarrow \Delta'}\right.$$

and for each $\mathcal{D}_i$ there is $\mathcal{D}'_j$ such that $\mathcal{D}_i \preccurlyeq \mathcal{D}'_j$, then $\mathcal{D} \preccurlyeq \mathcal{D}'$;[8]
(iii) If $\mathcal{D} \preccurlyeq \mathcal{D}'$ and

$$\mathcal{D}''\left\{\frac{\dots \quad \mathcal{D}' \quad \dots}{\Gamma \Rightarrow \Delta}\right.$$

then $\mathcal{D} \prec \mathcal{D}''$;
(iv) If $\mathcal{D} \prec \mathcal{D}'$ then $\mathcal{D} \preccurlyeq \mathcal{D}'$.

This is a compact but unusual way to do parallel inductive definitions. An equivalent, more standard way to do this is to first define $\preccurlyeq$ by taking clauses (i)–(iii), where in the latter $\prec$ is replaced by $\preccurlyeq$, and then to define $\prec$ by taking clause (iii) alone. In this way, clause (iv) becomes automatic.

Remark 1

[8] One may be mislead here by assuming that the correspondence between branches implies that the two derivations have the same structure. However, this is not the case as the correspondence is not required to be injective nor surjective.

1. By definition, $\mathcal{D} \prec \mathcal{D}'$ implies $\mathcal{D} \preccurlyeq \mathcal{D}'$.
2. Note that, in general, $\mathcal{D} \prec \mathcal{D}'$ is not the same as the conjunction of $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D} \neq \mathcal{D}'$; and similarly $\mathcal{D} \preccurlyeq \mathcal{D}'$ is not the same as the disjunction of $\mathcal{D} \prec \mathcal{D}$ and $\mathcal{D} = \mathcal{D}'$. However, it can be shown that $\mathcal{D} \prec \mathcal{D}'$ if and only if $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D}' \not\preccurlyeq \mathcal{D}$.

Lemma 10 *Let $\mathcal{D}$ be a trivial derivation.*

1. $\mathcal{D}' \preccurlyeq \mathcal{D}$ if and only if $\mathcal{D}'$ is trivial, and there is no $\mathcal{D}''$ such that $\mathcal{D}'' \prec \mathcal{D}$.
2. $\mathcal{D} \preccurlyeq \mathcal{D}'$ for every $\mathcal{D}'$, and $\mathcal{D} \prec \mathcal{D}''$ for every nontrivial $\mathcal{D}''$.

Proof Straightforward. □

Lemma 11 *The relation $\preccurlyeq$ is a (non-strict) preorder, i.e. it is reflexive and transitive.*

Proof Reflexivity: Take a derivation $\mathcal{D}$. We prove that

$$\mathcal{D} \preccurlyeq \mathcal{D} \tag{2}$$

by structural induction on $\mathcal{D}$. If $\mathcal{D}$ is trivial, then $\mathcal{D} \preccurlyeq \mathcal{D}$ by clause (i) of the definition. If

$$\mathcal{D}\left\{\frac{\{\mathcal{D}_i\}}{\Gamma \Rightarrow \Delta}\right.$$

with each $\mathcal{D}_i$ satisfying (2), then $\mathcal{D} \preccurlyeq \mathcal{D}$ by clause (ii) of the definition.

Transitivity: Take a derivation $\mathcal{D}$. We prove that

$$\forall \mathcal{D}' \forall \mathcal{D}''.(\mathcal{D} \preccurlyeq \mathcal{D}' \,\&\, \mathcal{D}' \preccurlyeq \mathcal{D}'') \Rightarrow \mathcal{D} \preccurlyeq \mathcal{D}'' \tag{3}$$

by structural induction on $\mathcal{D}$. If $\mathcal{D}$ is trivial, see Lemma 10. Suppose that

$$\mathcal{D}\left\{\frac{\{\mathcal{D}_i\}}{\Gamma \Rightarrow \Delta}\right.$$

with each $\mathcal{D}_i$ satisfying (3). Consider $\mathcal{D}', \mathcal{D}''$ such that $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D}' \preccurlyeq \mathcal{D}''$. By Lemma 10, since $\mathcal{D}$ is composite, then so must be $\mathcal{D}'$, and similarly since $\mathcal{D}'$ is composite, then so must be $\mathcal{D}''$:

$$\mathcal{D}'\left\{\frac{\{\mathcal{D}'_j\}}{\Gamma' \Rightarrow \Delta'}\right. \qquad \text{and} \qquad \mathcal{D}''\left\{\frac{\{\mathcal{D}''_k\}}{\Gamma'' \Rightarrow \Delta''}\right.$$

For every $\mathcal{D}_{i^*}$, we show that there is a (finite) chain

$$\mathcal{D}_{i^*} \preccurlyeq \ldots \preccurlyeq \mathcal{D}''_{k^*}$$

for some $\mathcal{D}''_{k^*}$. We do a proof by cases, depending on whether $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D}' \preccurlyeq \mathcal{D}''$ are witnessed by clause (ii) or (iii):

- Suppose that both $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D}' \preccurlyeq \mathcal{D}''$ are witnessed by clause (ii). In particular, there is $\mathcal{D}'_{j^*}$ such that $\mathcal{D}_{i^*} \preccurlyeq \mathcal{D}'_{j^*}$, for which in turn there is $\mathcal{D}''_{k^*}$ such that

$$\mathcal{D}_{i^*} \preccurlyeq \mathcal{D}'_{j^*} \preccurlyeq \mathcal{D}''_{k^*}.$$

- Suppose that $\mathcal{D} \preccurlyeq \mathcal{D}'$ is witnessed by clause (iii) and that $\mathcal{D}' \preccurlyeq \mathcal{D}''$ is witnessed by clause (ii). This means that there is $\mathcal{D}_{j^*}$ such that $\mathcal{D} \preccurlyeq \mathcal{D}'_{j^*}$ and for each $\mathcal{D}'_j$ there is $\mathcal{D}''_k$ such that $\mathcal{D}'_j \preccurlyeq \mathcal{D}''_k$. In particular, there is $\mathcal{D}''_{k^*}$ such that
$$\mathcal{D}_{i^*} \preccurlyeq \mathcal{D} \preccurlyeq \mathcal{D}'_{j^*} \preccurlyeq \mathcal{D}''_{k^*},$$
where $\mathcal{D}_{i^*} \preccurlyeq \mathcal{D}$ because of reflexivity and clause (iii).
- Suppose that $\mathcal{D}' \preccurlyeq \mathcal{D}''$ is witnessed by clause (iii). This means that there is $\mathcal{D}''_{k^*}$ such that $\mathcal{D}' \preccurlyeq \mathcal{D}''_{k^*}$. It follows that
$$\mathcal{D}_{i^*} \preccurlyeq \mathcal{D} \preccurlyeq \mathcal{D}' \preccurlyeq \mathcal{D}''_{k^*},$$
where $\mathcal{D}_{i^*} \preccurlyeq \mathcal{D}$ because of reflexivity and clause (iii).

We apply (3) to the chain, possibly multiple times, and get $\mathcal{D}_{i^*} \preccurlyeq \mathcal{D}''_{k^*}$. We can now apply clause (ii) to conclude that $\mathcal{D} \preccurlyeq \mathcal{D}''$. □

Lemma 12

1. *If $\mathcal{D} \preccurlyeq \mathcal{D}'$ and $\mathcal{D}' \prec \mathcal{D}''$, then $\mathcal{D} \prec \mathcal{D}''$.*
2. *If $\mathcal{D} \prec \mathcal{D}'$ and $\mathcal{D}' \prec \mathcal{D}''$, then $\mathcal{D} \prec \mathcal{D}''$.*
3. *If $\mathcal{D} \prec \mathcal{D}'$ and $\mathcal{D}' \preccurlyeq \mathcal{D}''$, then $\mathcal{D} \prec \mathcal{D}''$.*

Proof

1. By definition of $\prec$, we have $\mathcal{D}^*$ such that $\mathcal{D}' \preccurlyeq \mathcal{D}^*$ and
$$\mathcal{D}'' \left\{ \frac{\ldots \quad \mathcal{D}^* \quad \ldots}{\Gamma \Rightarrow \Delta} \right.$$
By transitivity of $\preccurlyeq$ (Lemma 11), we have that $\mathcal{D} \preccurlyeq \mathcal{D}^*$. We conclude that $\mathcal{D} \prec \mathcal{D}''$ by clause (iii) of the definition.
2. If $\mathcal{D} \prec \mathcal{D}'$, then in particular $\mathcal{D} \preccurlyeq \mathcal{D}'$, so the claim follows from (i).
3. We do a proof by cases, depending on whether $\mathcal{D}' \preccurlyeq \mathcal{D}''$ is witnessed by clause (i), (ii) or (iv), whereas clause (iii) does not apply:
   - If it is witnessed by clause (i)—i.e., $\mathcal{D}'$ and $\mathcal{D}''$ are trivial—then there is no such $\mathcal{D}$, and the claim is vacuously satisfied.
   - Suppose that it is witnessed by clause (ii), i.e.
$$\mathcal{D}' \left\{ \frac{\{\mathcal{D}'_j\}}{\Gamma' \Rightarrow \Delta'} \right. \qquad \text{and} \qquad \mathcal{D}'' \left\{ \frac{\{\mathcal{D}''_k\}}{\Gamma'' \Rightarrow \Delta''} \right.$$
and for each $\mathcal{D}'_j$ there is $\mathcal{D}''_k$ such that $\mathcal{D}'_j \preccurlyeq \mathcal{D}''_k$. By definition of $\mathcal{D} \prec \mathcal{D}'$, we have $\mathcal{D} \preccurlyeq \mathcal{D}'_{j^*}$ for some $\mathcal{D}'_{j^*}$, hence by transitivity $\mathcal{D} \preccurlyeq \mathcal{D}''_{k^*}$ for the corresponding $\mathcal{D}''_{k^*}$. It follows that $\mathcal{D} \prec \mathcal{D}''$.

- If it is witnessed by clause (iv)—i.e., $\mathcal{D}' \prec \mathcal{D}''$—then $\mathcal{D} \prec \mathcal{D}''$ follows from (ii). □

We say that a property E of derivations is *progressive*, if

$$\text{for every } \mathcal{D}, \quad \forall \mathcal{D}' \prec \mathcal{D}(E\mathcal{D}') \text{ implies } E\mathcal{D}.$$

Theorem 13 *The relation of strict proof embeddability* $\prec$ *satisfies Noetherian induction, i.e. it satisfies* $\forall \mathcal{D}(E\mathcal{D})$ *for every progressive property* E.

Proof Consider a progressive property E. It is enough to show that

$$\forall \mathcal{D}' \prec \mathcal{D}(E\mathcal{D}') \tag{4}$$

for every derivation $\mathcal{D}$. We proceed by structural induction on $\mathcal{D}$. If $\mathcal{D}$ is trivial, then it has no predecessors (Lemma 10) and the claim holds. Suppose that

$$\mathcal{D}\left\{ \frac{\{\mathcal{D}_i\}}{\Gamma \Rightarrow \Delta} \right.$$

with each $\mathcal{D}_i$ satisfying (4). Consider $\mathcal{D}' \prec \mathcal{D}$. By definition, $\mathcal{D}' \preccurlyeq \mathcal{D}_{i^*}$ for some $\mathcal{D}_{i^*}$. We claim that $E\mathcal{D}''$ for each $\mathcal{D}'' \prec \mathcal{D}'$. In fact, given any such $\mathcal{D}''$, by Lemma 12 we have that $\mathcal{D}'' \prec \mathcal{D}_{i^*}$. The claim follows by the fact that $\mathcal{D}_{i^*}$ satisfies (4). Since E is progressive, we get $E\mathcal{D}'$. □

Corollary 14 *The relation* $\prec$ *is a strict partial order, i.e. it is irreflexive and transitive.*

Proof Transitivity is Lemma 12(ii), while irreflexivity follows from Noetherian induction (see e.g. [8, Lemma 4.1]). □

Given a calculus $\mathbf{G}$, by $\mathbf{G} \vdash^{\mathcal{D}} \Gamma \Rightarrow \Delta$ we mean that there is a derivation $\mathcal{D}$ with end-sequent $\Gamma \Rightarrow \Delta$ in calculus $\mathbf{G}$.

We say that a rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}$$

is *proof embeddable* admissible (for short pe-admissible) if for each derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ there is a derivation $\mathcal{D}'$ of $\Gamma' \Rightarrow \Delta'$ such that $\mathcal{D}' \preccurlyeq \mathcal{D}$.

The notion of pe-admissibility is used in place of hp-admissibility for the calculi $\mathbf{G3}[\mathbf{CIM}]_\omega.\mathbf{G}$, and is studied in the following sections.

5 Structural rules

We present here the results concerning the admissibility of the structural rules, cut excluded, in the calculi $\mathbf{G3}[\mathbf{CIM}]_\omega.\mathbf{G}$. All these results have been proved in Sect. 4 of [19] by simple transfinite induction on ordinals, either on the depth of a formula or on the height of a derivation, here replaced by proof-embeddability both in the statement of the results and in their proofs.

Lemma 15 (α-conversion) *If* $\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_1} \Gamma \Rightarrow \Delta$ *then*

$$\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_2} \Gamma' \Rightarrow \Delta'$$

with $\mathcal{D}_2 \preccurlyeq \mathcal{D}_1$, *for* $\Gamma' \Rightarrow \Delta'$ *a bound alphabetic variant of* $\Gamma \Rightarrow \Delta$.

Proof Similar to the proof of hp-α-conversion in [26]. □

Lemma 16 (Substitution) *If* $\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_1} \Gamma \Rightarrow \Delta$ *then*

$$\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_2} \Gamma[t/x] \Rightarrow \Delta[t/x]$$

(for t *free for* x *in* Γ, Δ*) with* $\mathcal{D}_2 \preccurlyeq \mathcal{D}_1$.

Proof Similar to the proof of hp-substitution in [26]. □

Theorem 17 (Weakening) *The left and right rules of weakening:*

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW$$

are pe-admissible in $\mathbf{G3[CIM]}_\omega.\mathbf{G}$.

Proof Similar to the proof of hp-weakening in [26]. □

Lemma 18 (Invertibility)

1. *Each rule of* $\mathbf{G3C}_\omega.\mathbf{G}$ *is pe-invertible.*
2. *Each rule of* $\mathbf{G3[IM]}_\omega.\mathbf{G}$ *except* $R\supset$, $R\forall$, *and* $R\bigwedge$ *is pe-invertible.*

Proof The proof for rules L$\forall$, R$\exists$, L$\bigwedge$ and R$\bigvee$ follows from Theorem 17. For the other rules we proceed by Noetherian induction with proof-embeddability.

We consider the case of L$\bigvee$, i.e. a sequent $\bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta$. If it is an initial sequent, then each $A_n, \Gamma \Rightarrow \Delta$ is also an initial sequent. If it is an instance of L$\bot$, then there's nothing to prove. Let us consider the last (proper) rule and distinguish the case in which $\bigvee_{n>0} A_n$ is a side formula and the case in which it is the principal formula. In the former case the last rule can have one, two or denumerably many premisses. The derivation $\mathcal{D}$ has the form

$$\mathcal{D}_m \left\{ \begin{array}{c} \vdots \\ \{\bigvee_{n>0} A_n, \Gamma_m \Rightarrow \Delta_m \mid m \in I\} \end{array} \right. \over \bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta \quad rule$$

where I is either $\{1\}, \{1,2\}$ or $\mathbb{N}$. Clearly $\mathcal{D}_m \prec \mathcal{D}$ for each m. By inductive hypothesis, we have derivations $\mathcal{D}_{mn} \preccurlyeq \mathcal{D}_m$ of $A_n, \Gamma_m \Rightarrow \Delta_m$. Then we get derivations

$$\dfrac{\mathcal{D}_{mn}\left\{\begin{array}{c}\vdots\\ \{A_n, \Gamma_m \Rightarrow \Delta_m \mid m \in I\}\end{array}\right.}{A_n, \Gamma \Rightarrow \Delta}\ rule$$

which are embeddable in $\mathcal{D}$. If instead $\bigvee_{n>0} A_n$ is principal, the derivation $\mathcal{D}$ has the form

$$\dfrac{\mathcal{D}_{n}\left\{\begin{array}{c}\vdots\\ \{A_n, \Gamma \Rightarrow \Delta \mid n > 0\}\end{array}\right.}{\bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigvee$$

and we just need to observe that $\mathcal{D}_n \preccurlyeq \mathcal{D}$.

The proof for other rules is similar. □

Theorem 19 (Contraction) *The left and right rules of contraction:*

$$\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

are pe-admissible in $\mathbf{G3[CIM]}_\omega.\mathbf{G}$.

Proof By simultaneous Noetherian induction with $\prec$ on the left and right contraction rule. Consider the left rule. If it is an initial sequent, then the conclusion is also an initial sequent and is embeddable. If the contraction formula A is not principal in the last rule, we have the derivation $\mathcal{D}$

$$\dfrac{\mathcal{D}_{m}\left\{\begin{array}{c}\vdots\\ \{A, A, \Gamma_m \Rightarrow \Delta_m \mid m \in I\}\end{array}\right.}{A, A, \Gamma \Rightarrow \Delta}\ rule$$

where I is either $\{1\}, \{1,2\}$ or $\mathbb{N}$. Clearly $\mathcal{D}_m \prec \mathcal{D}$ for each m. By induction hypothesis we have derivations $\mathcal{D}'_m \preccurlyeq \mathcal{D}_m$ of $A, \Gamma_m \Rightarrow \Delta_m$. Then the derivation

$$\dfrac{\mathcal{D}'_{m}\left\{\begin{array}{c}\vdots\\ \{A, \Gamma_m \Rightarrow \Delta_m \mid m \in I\}\end{array}\right.}{A, \Gamma \Rightarrow \Delta}\ rule$$

is as wanted.

We're left with the case in which the contraction formula is principal in the last rule. Consider the case of $\bigvee_{n>0} A_n$ in $\mathrm{L}\bigvee$. We have the derivation $\mathcal{D}$

$$\dfrac{\mathcal{D}_{n}\left\{\begin{array}{c}\vdots\\ \{\bigvee_{n>0} A_n, A_n, \Gamma \Rightarrow \Delta \mid n > 0\}\end{array}\right.}{\bigvee_{n>0} A_n, \bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigvee$$

where clearly $\mathcal{D}_n \prec \mathcal{D}$. By pe-invertibility of L$\bigvee$ we obtain derivations $\mathcal{D}'_n \preccurlyeq \mathcal{D}_n$ of $A_n, A_n, \Gamma \Rightarrow \Delta$, and thus $\mathcal{D}'_n \prec \mathcal{D}$ by clause (ii) of proof-embeddability, cf. Def. 9. By induction hypothesis, we now get derivations $\mathcal{D}''_n \preccurlyeq \mathcal{D}'_n$ of $A_n, \Gamma \Rightarrow \Delta$. By transitivity, $\mathcal{D}''_n \preccurlyeq \mathcal{D}_n$. In conclusion, we get the derivation $\mathcal{D}'$

$$\mathcal{D}''_n \left\{ \begin{array}{c} \vdots \\ \{A_n, \Gamma \Rightarrow \Delta \mid n > 0\} \end{array} \right. \over \bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta \quad \text{L}\bigvee$$

which is embeddable in $\mathcal{D}$. The proof for other invertible rules is similar.

Consider the case of $A \supset B$ principal in intuitionistic R$\supset$. We have the derivation $\mathcal{D}$

$$\mathcal{D}^- \left\{ \begin{array}{c} \vdots \\ A, \Gamma \Rightarrow B \end{array} \right. \over \Gamma \Rightarrow \Delta, A \supset B, A \supset B \quad \text{R}\supset$$

where clearly $\mathcal{D}^- \prec \mathcal{D}$. We easily get the derivation $\mathcal{D}'$

$$\mathcal{D}^- \left\{ \begin{array}{c} \vdots \\ A, \Gamma \Rightarrow B \end{array} \right. \over \Gamma \Rightarrow \Delta, A \supset B \quad \text{R}\supset$$

which is embeddable in $\mathcal{D}$. Again, the proof for other non-invertible rules is similar. □

6 Constructive cut-elimination

We are now ready to prove that the following context-sharing rule of cut

$$\frac{\Gamma \Rightarrow \Delta, C \quad C, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{Cut}$$

is eliminable in the calculus $\mathbf{G3}[\mathbf{CIM}]_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ obtained by extending $\mathbf{G3}[\mathbf{CIM}]_\omega.\mathbf{G}$ with Cut. In order to give a proof of cut elimination that uses only constructively admissible proof-theoretical tools we must avoid the "natural" (or Hessenberg) commutative sum of ordinals: we cannot use the cut-height as inductive parameter as done in Gentzen- and Dragalin-style proofs. In order to avoid it, we make use of a proof strategy introduced in [16] for fuzzy logics that has been extensively used in the context of hypersequent calculi; see [4, 10, 13]. This proof strategy can be seen as a simplified and local version of the proof given by H.B. Curry in [5]. The proof is based on two main lemmas (Lemmas 23 and 24 below) that are proved by induction (with proof-embeddability) on the derivation of the right and of the left premiss of cut, respectively. Moreover, (almost) all non-principal instances of cut are taken care by separate lemmas (Lemmas 21 and 22) which shows that Cut can be permuted upwards with respect to rule instances not having the cut formula among their principal formulas.

Observe that, differently from [4, 13, 16], we will not consider an arbitrary instance of Cut of maximal rank (i.e., such that its cut formula has maximal depth among the cut formulas occurring in the derivation), but we will always consider an uppermost instance of Cut, i.e. a cut the premisses of which are cut-free derivations. Otherwise, in Lemmas 23 and 24 as well as in Theorem 26, we would have to assume that ordinals are linearly/totally ordered; but in a constructive setting this assumption implies the law of excluded middle [1]. In Theorem 26 we will proceed, instead, by using two instances of Brouwer's principle of Bar Induction: the first will be used to show that an uppermost instance of Cut is eliminable and the second to show that all instances of Cut are eliminable. Note that although it is a constructively admissible principle, Bar Induction increases the proof-theoretic strength of **CFZ**, cf. [25].

Definition 20 (Cut-substitutive rule) A sequent rule *Rule* is *cut-substitutive* if each instance of cut with cut formula not principal in the last rule instance *Rule* of one of the premisses of cut can be permuted upwards w.r.t. *Rule* as in the following example:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta, B, C}{\Gamma \Rightarrow \Delta, A \supset B, C}\,\mathrm{R}\supset \qquad C, \Gamma \Rightarrow \Delta, A \supset B}{\Gamma \Rightarrow \Delta, A \supset B}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta, B, C \qquad \dfrac{C, \Gamma \Rightarrow \Delta, A \supset B}{A, C, \Gamma \Rightarrow \Delta, B}\,\text{pe-inv}}{A, \Gamma \Rightarrow \Delta, B}\,\mathrm{Cut}}{\Gamma \Rightarrow \Delta, A \supset B}\,\mathrm{R}\supset$$

Lemma 21 *Each rule of* $\mathbf{G3C}_\omega.\mathbf{G}$ *is cut-substitutive.*

Proof By inspecting the rules in Tables 1 it is immediate to realise that each of them is cut-substitutive because they are all pe-invertible (using Lemma 16 for rules L∃, R∀, and for geometric rules with a variable condition). □

Lemma 22 *Each rule of* $\mathbf{G3[IM]}_\omega.\mathbf{G}$ *except R*$\supset$*, R*$\forall$ *and R*$\bigwedge$ *is cut-substitutive.*

Proof Same as for $\mathbf{G3C}_\omega.\mathbf{G}$. □

Lemma 23 (Right reduction) *If we are in* $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ *and all of the following hold:*

1. $\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_1} \Gamma \Rightarrow \Delta, A$
2. $\mathbf{G3[CIM]}_\omega.\mathbf{G} \vdash^{\mathcal{D}_2} A, \Gamma \Rightarrow \Delta$
3. *A is principal in the last rule instance applied in* $\mathcal{D}_1$

4. A is not of shape $\exists x B$ or $\bigvee_{n>0} B_n$.

Then there is a $\mathbf{G3[CIM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$*-derivation* $\mathcal{D}$ *concluding* $\Gamma \Rightarrow \Delta$ *containing only cuts on proper subformulas of* A*.*

Proof By Noetherian induction with proof-embeddability in the derivation $\mathcal{D}_2$ of $A, \Gamma \Rightarrow \Delta$.

If $\mathcal{D}_2$ is a one node tree, since A cannot be principal in the initial sequent, then the conclusion of Cut is also initial.

Else, we have two cases depending on whether A is principal in the last rule instance applied in $\mathcal{D}_2$ or not.

In the latter case, if we are in $\mathbf{G3C}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$, the lemma holds thanks to Lemma 21. If we are in $\mathbf{G3[IM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ and the last step of $\mathcal{D}_2$ is not by one of R$\supset$, $R\forall$, and R$\bigwedge$ then it holds by Lemma 22. In the remaining three cases, we have two cases according to whether $\mathcal{D}_1$ ends with a step by an invertible rule or not. In the latter case, $\mathcal{D}_1$ ends with one of R$\supset$, $R\forall$, and R$\bigwedge$. We permute the cut upwards in the right premiss. To illustrate, we consider the case of R$\bigwedge$. We transform

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow B[y/x]\end{array}\right.}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, \forall x B}\,\mathrm{R}\forall \qquad \dfrac{\mathcal{D}_{2i}\left\{\begin{array}{c}\vdots\\ \{\forall x B, \Gamma \Rightarrow A_i \mid i>0\}\end{array}\right.}{\forall x B, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\mathrm{R}\bigwedge}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\mathrm{Cut}$$

into

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow B[y/x]\end{array}\right.}{\Gamma \Rightarrow \forall x B}\,\mathrm{R}\forall}{\{\Gamma \Rightarrow \forall x B, A_i \mid i>0\}}\,\mathrm{RW} \qquad \mathcal{D}_{2i}\left\{\begin{array}{c}\vdots\\ \{\forall x B, \Gamma \Rightarrow A_i \mid i>0\}\end{array}\right.}{\{\Gamma \Rightarrow A_i \mid i>0\}}\,\mathrm{I.H.}_i,\ i>0}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\mathrm{R}\bigwedge$$

If, instead, $\mathcal{D}_1$ ends by an invertible rule, then we apply invertibility, thus transforming the derivation into one having only cuts on proper subformulas of A. For example, if $\mathcal{D}_1$ ends with a step by R$\wedge$, we transform

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, B\end{array}\right. \qquad \mathcal{D}_{12}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, C\end{array}\right.}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, B \wedge C}\,\mathrm{R}\wedge \qquad \mathcal{D}_2\left\{\begin{array}{c}\vdots\\ B \wedge C, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n\end{array}\right.}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\mathrm{Cut}$$

into

$$
\dfrac{\mathcal{D}_{12}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, C\end{array}\right.
\qquad
\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, B\end{array}\right.}{C, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n, B}\,\text{LW}
\quad
\dfrac{\mathcal{D}_{2}\left\{\begin{array}{c}\vdots\\ B \wedge C, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n\end{array}\right.}{B, C, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\text{pe-inv}}{C, \Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\text{Cut}}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} A_n}\,\text{Cut}
$$

Next, we consider the case with A principal in the last rule instance applied in $\mathcal{D}_2$. We have cases according to the shape of A.

If $A \equiv P$ for some atomic formula P, then the last rule instance in $\mathcal{D}_2$ is by a geometric rule (rules for equality included) L_G concluding $P_1, \ldots, P, \ldots, P_k, \Gamma'' \Rightarrow \Delta', P$ and $\mathcal{D}_1$ is the one node tree $P, \Gamma' \Rightarrow \Delta', P$. The conclusion of cut is the initial sequent $P, \Gamma' \Rightarrow \Delta', P$ which is cut-free derivable.

The cases with $A \equiv \bot$, $A \equiv \top$ or $A \equiv B \circ C$, for ($\circ \in \{\wedge \vee, \supset\}$), are left to the reader.

If $A \equiv \forall x B$ we transform (if we are in $\mathbf{G3[IM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$, Δ is not in the premiss of R$\forall$)

$$
\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, B[y/x]\end{array}\right.}{\Gamma \Rightarrow \Delta, \forall x B}\,\text{R}\forall
\quad
\dfrac{\mathcal{D}_{21}\left\{\begin{array}{c}\vdots\\ B[t/x], \forall x B, \Gamma \Rightarrow \Delta\end{array}\right.}{\forall x B, \Gamma \Rightarrow \Delta}\,\text{L}\forall}{\Gamma \Rightarrow \Delta}\,\text{Cut}
$$

into the following derivation having only cuts on proper subformulas of A (if we are in $\mathbf{G3[IM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ then Δ is introduced in $\mathcal{D}_{11}$ by pe-weakenings):

$$
\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, B[y/x]\end{array}\right.}{\Gamma \Rightarrow \Delta, B[t/x]}\,\text{Subs}
\quad
\dfrac{\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, B[y/x]\end{array}\right.}{\Gamma \Rightarrow \Delta, \forall x B}\,\text{R}\forall}{B[t/x], \Gamma \Rightarrow \Delta, \forall x B}\,\text{LW}
\quad
\mathcal{D}_{21}\left\{\begin{array}{c}\vdots\\ B[t/x], \forall x B, \Gamma \Rightarrow \Delta\end{array}\right.}{B[t/x], \Gamma \Rightarrow \Delta}\,\text{I.H.}}{\Gamma \Rightarrow \Delta}\,\text{Cut}
$$

If $A \equiv \bigwedge_{n>0} B_n$ we transform (Δ not in the premisses of R$\bigwedge$ if we are in $\mathbf{G3[IM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$)

$$
\dfrac{\dfrac{\mathcal{D}_{1i}\left\{\begin{array}{c}\vdots\\ \{\Gamma \Rightarrow \Delta, B_i \mid i > 0\}\end{array}\right.}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} B_n}\,\text{R}\bigwedge
\quad
\dfrac{\mathcal{D}_{21}\left\{\begin{array}{c}\vdots\\ B_k, \bigwedge_{n>0} B_n, \Gamma \Rightarrow \Delta\end{array}\right.}{\bigwedge_{n>0} B_n, \Gamma \Rightarrow \Delta}\,\text{L}\bigwedge}{\Gamma \Rightarrow \Delta}\,\text{Cut}
$$

into the following derivation having only cuts on proper subformulas of A (if we are in $\mathbf{G3[IM]}_\omega.\mathbf{G}+\{\mathbf{Cut}\}$ then Δ is introduced in $\mathcal{D}_{1k}$ by pe-weakenings):

$$\dfrac{\mathcal{D}_{1k}\left\{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,B_k\end{array}\right. \qquad \dfrac{\dfrac{\dfrac{\mathcal{D}_{1i}\left\{\begin{array}{c}\vdots\\ \{\Gamma\Rightarrow\Delta,B_i \mid i>0\}\end{array}\right.}{\Gamma\Rightarrow\Delta,\bigwedge_{n>0}B_n}\,\text{R}\bigwedge}{B_k,\Gamma\Rightarrow\Delta,\bigwedge_{n>0}B_n}\,\text{LW} \qquad \mathcal{D}_{21}\left\{\begin{array}{c}\vdots\\ B_k,\bigwedge_{n>0}B_n,\Gamma\Rightarrow\Delta\end{array}\right.}{B_k,\Gamma\Rightarrow\Delta}\,\text{I.H.}}{\Gamma\Rightarrow\Delta}\,\text{Cut}$$

□

Lemma 24 (Left reduction) *If we are in* $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ *and all of the following hold:*

1. $\mathbf{G3[CIM]}_\omega.\mathbf{G}\vdash^{\mathcal{D}_1}\Gamma\Rightarrow\Delta,A$
2. $\mathbf{G3[CIM]}_\omega.\mathbf{G}\vdash^{\mathcal{D}_2}A,\Gamma\Rightarrow\Delta$

Then there is a $\mathbf{G3[CIM]}_\omega.\mathbf{G}$*-derivation* $\mathcal{D}$ *concluding* $\Gamma\Rightarrow\Delta$ *containing only cuts on proper subformulas of* A*.*

Proof By Noetherian induction with proof-embeddability in the derivation $\mathcal{D}_1$ of $\Gamma\Rightarrow\Delta,A$.

If $\mathcal{D}_1$ is a one node tree, the lemma obviously holds.

Else, we have two cases depending on whether A is principal in the last rule instance applied in $\mathcal{D}_1$ or not.

In the latter case, the lemma holds thanks to Lemma 21 or 22 (if the last step of $\mathcal{D}_1$ is by an intuitionistic non-invertible rule we proceed as in the analogous case of Lemma 23). In the former case we have cases according to the shape of A.

If A is an atomic formula, or $\bot$, or $\top$ or $B\circ C$ ($\circ\in\{\wedge,\vee\supset\}$), or $\forall xB$, or $\bigwedge B_n$, the lemma holds thanks to Lemma 23.

If $A\equiv\exists xB$ we transform:

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\exists xB,B[t/x]\end{array}\right.}{\Gamma\Rightarrow\Delta,\exists xB}\,\text{R}\exists \qquad \mathcal{D}_{2}\left\{\begin{array}{c}\vdots\\ \exists xB,\Gamma\Rightarrow\Delta\end{array}\right.}{\Gamma\Rightarrow\Delta}\,\text{Cut}$$

into the following derivation having only cuts on proper subformulas of A:

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\exists xB,B[t/x]\end{array}\right. \qquad \dfrac{\mathcal{D}_{2}\left\{\begin{array}{c}\vdots\\ \exists xB,\Gamma\Rightarrow\Delta\end{array}\right.}{\exists xB,\Gamma\Rightarrow\Delta,B[t/x]}\,\text{RW}}{\Gamma\Rightarrow\Delta,B[t/x]}\,\text{I.H.} \qquad \dfrac{\mathcal{D}_{2}\left\{\begin{array}{c}\vdots\\ \exists xB,\Gamma\Rightarrow\Delta\end{array}\right.}{B[t/x],\Gamma\Rightarrow\Delta}\,\text{pe-inv}}{\Gamma\Rightarrow\Delta}\,\text{Cut}$$

If $A \equiv \bigvee B_n$ we transform:

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \bigvee_{n>0} B_n, B_k\end{array}\right.}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} B_n}\,\text{R}\bigvee \qquad \mathcal{D}_2\left\{\begin{array}{c}\vdots\\ \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta\end{array}\right.}{\Gamma \Rightarrow \Delta}\,\text{Cut}$$

into the following derivation:

$$\dfrac{\dfrac{\mathcal{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \bigvee_{n>0} B_n, B_k\end{array}\right. \qquad \dfrac{\mathcal{D}_2\left\{\begin{array}{c}\vdots\\ \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta\end{array}\right.}{\bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta, B_k}\,\text{RW}}{\Gamma \Rightarrow \Delta, B_k}\,\text{I.H.} \qquad \dfrac{\mathcal{D}_2\left\{\begin{array}{c}\vdots\\ \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta\end{array}\right.}{B_k, \Gamma \Rightarrow \Delta}\,\text{pe-inv}}{\Gamma \Rightarrow \Delta}\,\text{Cut}$$

□

In order to prove Cut elimination in a constructive way we use Bar Induction as done in [27, p. 18] for ω-arithmetic. This strategy avoids the assumption of total ordering of ordinal numbers. Before proving the theorem we introduce Brouwer's principle of (decidable) Bar Induction.

Definition 25 (Bar Induction) Let B and I be unary predicates (the so-called "base predicate" and "inductive predicate", respectively) of finite lists of natural numbers (to be denoted by $u, v, \ldots$). If:

1. B is decidable;
2. Every infinite sequence of natural numbers has a finite initial segment satisfying B;
3. $B(u)$ implies $I(u)$ for every finite list u;
4. If $I(u * n)$ holds for all $n \in \mathbb{N}$ then $I(u)$ holds;

Then I holds for the empty list of natural numbers.

Theorem 26 (Cut elimination) *Cut is admissible in* $\mathbf{G3[CIM]}_\omega\mathbf{.G}$.

Proof Throughout this proof, we use finite lists of natural numbers to index (partial) branches of trees, i.e. directed paths from the root to a node, possibly a leaf. Consider a tree such that each node has at most countable immediate successors, i.e. that are either indexed by ω or else by some natural number k, and such that each branch has finite length, then:

- The empty list $\{\}$ indexes the root of the tree.

- Suppose that u indexes a partial branch $\mathcal{R}$ of the tree and that the last node a has immediate successor nodes indexed by $k < \omega$, and let a natural number n be given. Let $m = n \mod k$: that is, m is the remainder of n after division by k. Then $u * n$ indexes $\mathcal{R}$ extended with the m^{th} immediate successor node of a. For example, in the case of a 2-premiss rule, odd numbers index the left premiss, even numbers the right premiss.
- Suppose that u indexes a partial branch $\mathcal{R}$ of the tree and that the last node a has immediate successor nodes indexed by ω, then $u * n$ indexes $\mathcal{R}$ extended with the n^{th} immediate successor node of a.

Notice that the above gives a partial surjective map, with decidable domain, from sequences of natural numbers to branches in the given tree. Moreover, this ensures that every infinite sequence has an initial segment that indexes a branch of the tree.[9]

Let $\mathcal{D}$ be a derivation in the calculus $\mathbf{G3[CIM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$. The proof consists of two parts, each building on an appropriate instance of Bar Induction.

- **Part 1.** We use Bar Induction to show that an uppermost instance of Cut with cut-formula C occurring in $\mathcal{D}$ is admissible. We use the method defined above to index the branches of the formation tree of the formula C—where C is the root of the tree and atomic formulas or $\top$ or $\bot$ are its leaves. Let $B(u)$ hold if u indexes a branch whose last element is an atom or $\bot$ or $\top$; let $I(u)$ hold if u indexes a partial branch whose last element is a formula D such that an uppermost cut on D or on some proper subformula theoreof in $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ is eliminable.
  The following hold:

  1. $B(u)$ is decidable by simply comparing the list with the formation tree;
  2. By definition of the indexing, the n^{th} element of the sequence identifies the n^{th} node in a branch of the formation tree of a formula. After a finite number of steps from the root we find an atom or $\bot$ or $\top$ since all branches of the tree are finite and this identifies an initial segment of the infinite sequence that satisfies B.
  3. $B(u)$ implies $I(u)$ since cuts on atomic formulas, $\top$, or $\bot$ are eliminable;
  4. $I(u * n)$ for all n implies $I(u)$: by Lemma 24 an uppermost cut on some formula E can be reduced to cuts on proper subformulas of E.

  By Bar Induction we conclude that the uppermost cut with cut-formula C is eliminable from $\mathbf{G3[CIM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$.
- **Part 2.** We show that all cuts can be eliminated from $\mathcal{D}$. We consider a derivation $\mathcal{D}$ in $\mathbf{G3[CIM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ and, as above, we use lists of natural numbers to index branches of $\mathcal{D}$. Let $B(u)$ hold if u indexes a branch ending in a leaf of $\mathcal{D}$; let $I(u)$ hold if u indexes a partial branch whose last element has a cut-free derivation (i.e., it is $\mathbf{G3[CIM]}_\omega.\mathbf{G}$-derivable). All conditions of Bar Induction are satisfied by this choice of B and I:

  1. $B(u)$ is decidable;

[9]Since the number of nodes of the tree is at most countable, one may also define an encoding such that the correspondence is unique. This however would require more effort and we would lose the property that every infinite sequence has an initial segment that indexes a branch of the tree.

2. Given any infinite sequence of numbers, we have $B(u)$ for every finite initial segment u that represents a full branch $\mathcal{R}$ of the tree, i.e., a root-to-leaf path; and by construction of the representation there are such u.
3. $B(u)$ implies $I(u)$ since the leaves of $\mathcal{D}$ trivially have a cut-free derivation;
4. $I(u * n)$ for all n implies $I(u)$: having shown in part 1 that uppermost instances of Cut are eliminable, if all the premisses of a rule instance in $\mathcal{D}$ have a cut-free derivation, then also its conclusion has a cut-free derivation.

By Bar Induction we conclude that the conclusion of $\mathcal{D}$ has a cut-free derivation. □

Corollary 27 *The rule of context-free cut:*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma} \; Cut_{cf}$$

is admissible in $\mathbf{G3[CI]}_\omega.\mathbf{G}$.

Proof This is an immediate consequence of Theorem 26 since rules Cut and Cut_{cf} are equivalent when weakening and contraction are admissible. □

7 Orevkov's theorems on infinitary Glivenko classes

We follow Orevkov's notation and denote by $\circ^+$ positive and by $\circ^-$ negative occurrences of the connective or quantifier $\circ$ in a sequent.

Theorem 28 (Glivenko Class 1) *If neither* $\supset^+$*, nor* $\forall^+$*, nor* $\bigwedge^+$ *occurs in* $\Gamma \Rightarrow \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{G} \vdash^{\mathcal{D}} \Gamma \Rightarrow \Delta$*, then* $\mathbf{G3I}_\omega.\mathbf{G} \vdash^{\mathcal{D}'} \Gamma \Rightarrow \Delta$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$*.*
If, moreover, $\bot^-$ *does not occur in* $\Gamma \Rightarrow \Delta$*, then* $\mathbf{G3M}_\omega.\mathbf{G} \vdash^{\mathcal{D}'} \Gamma \Rightarrow \Delta$*.*

Proof Any derivation in $\mathbf{G3C}_\omega.\mathbf{G}$ uses only rules that follow the (infinitary) geometric rule scheme and logical rules. Observe that geometric implications contain no $\supset$, nor $\forall$, nor $\bigwedge$ in the scope of $\vee$ nor of $\bigvee$, which means that no instance of the rules that violates the intuitionistic restrictions is used, so the derivation directly gives (through the addition, where needed, of the missing implications in steps of $L\supset$) a derivation in $\mathbf{G3I}_\omega.\mathbf{G}$ of the same conclusion. Moreover, if $\bot^-$ does not occur in $\Gamma \Rightarrow \Delta$ the derivation is a $\mathbf{G3M}_\omega.\mathbf{G}$-derivation. □

This is actually Barr's theorem.

Orevkov's theorem for most other Glivenko classes works only if we restrict ourselves to geometric rules with at most one premiss, i.e., rules expressing geometric implications without disjunction in the succedent. Hence we introduce the following piece of notation.

Definition 29 L_{GS} stands for a one premiss geometric rule and $\mathbf{G3[CIM]}_\omega.\mathbf{S}$ stands for any extension of $\mathbf{G3[CIM]}_\omega$ with a finite or infinite family of such rules L_{GS}.

Lemma 30 *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow \Delta$ *and neither* $\supset^+$, *nor* $\vee^-$, *nor* $\bigvee^-$ *occurs in* $\Gamma \Rightarrow \Delta$, *then*

- *if* Δ *is inhabited, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *for some* $A \in \Delta$*;*
- *if* Δ *is empty, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow \Delta$*;*

with $\mathcal{D}' \preccurlyeq \mathcal{D}$.

The same holds with respect to $\mathbf{G3M}_\omega.\mathbf{S}$ *if we assume additionally that no instance of* $\bot^-$ *occurs in* $\Gamma \Rightarrow \Delta$.

Proof By induction with proof-embeddability.

If $\Gamma \Rightarrow \Delta$ is an initial sequent with principal formula some atomic formula P, the lemma holds by taking $A \equiv P$. If $\mathcal{D}$ ends with an instance of $\mathrm{L}\bot$, we have two cases: if $\Delta \neq \emptyset$ we take $A \equiv D$ for some $D \in \Delta$; else we have that $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow \Delta$.

If the last step of $\mathcal{D}$ is an instance of $\mathrm{L}\wedge$, then $\mathcal{D}$ has the form:

$$\dfrac{\mathcal{D}_1 \left\{ \begin{array}{c} \vdots \\ B, C, \Gamma' \Rightarrow \Delta \end{array} \right.}{B \wedge C, \Gamma' \Rightarrow \Delta} \ \mathrm{L}\wedge$$

We apply the inductive hypothesis to $\mathcal{D}_1$ and we obtain a derivation $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$ in $\mathbf{G3I}_\omega.\mathbf{S}$ of either $B, C, \Gamma' \Rightarrow A$ for some $A \in \Delta$ or $B, C, \Gamma' \Rightarrow \Delta$, depending on whether Δ is inhabited or not. In both cases, we get a derivation $\mathcal{D}' \preccurlyeq \mathcal{D}$ via an application of $\mathrm{L}\wedge$:

$$\dfrac{\mathcal{D}_1' \left\{ \begin{array}{c} \vdots \\ B, C, \Gamma' \Rightarrow A \end{array} \right.}{B \wedge C, \Gamma' \Rightarrow A} \ \mathrm{L}\wedge \qquad \dfrac{\mathcal{D}_1' \left\{ \begin{array}{c} \vdots \\ B, C, \Gamma' \Rightarrow \Delta \end{array} \right.}{B \wedge C, \Gamma' \Rightarrow \Delta} \ \mathrm{L}\wedge$$

If the last step of $\mathcal{D}$ is an instance of $\mathrm{R}\wedge$, then $\mathcal{D}$ has the form:

$$\dfrac{\mathcal{D}_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta', B \end{array} \right. \quad \mathcal{D}_2 \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta', C \end{array} \right.}{\Gamma \Rightarrow \Delta', B \wedge C} \ \mathrm{R}\wedge$$

By applying the inductive hypothesis to $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1'} \Gamma \Rightarrow B'$ with $B' \in \Delta', B$ and $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_2'} \Gamma \Rightarrow C'$ with $C' \in \Delta', C$, such that $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}_2' \preccurlyeq \mathcal{D}_2$. If $B' \equiv B$ and $C' \equiv C$ we get the following derivation $\mathcal{D}' \preccurlyeq \mathcal{D}$ in $\mathbf{G3I}_\omega.\mathbf{S}$:

$$\dfrac{\mathcal{D}_1' \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta', B \end{array} \right. \quad \mathcal{D}_2' \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta', C \end{array} \right.}{\Gamma \Rightarrow \Delta', B \wedge C} \ \mathrm{R}\wedge$$

Else we set $A \equiv B'$ or, if $B \equiv B'$, $A \equiv C'$, and we are done.

If $\mathcal{D}$ ends with an instance of $\mathrm{R}\vee$ with premiss $\Gamma \Rightarrow \Delta', B, C$ and conclusion $\Gamma \Rightarrow \Delta', B \vee C$, then we have $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1} \Gamma \Rightarrow \Delta', B, C$ with $\mathcal{D}_1 \prec \mathcal{D}$. By induction hypothesis, we get $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1'} \Gamma \Rightarrow D$ with $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$ and D is either A, B, or in Δ. If $D \in \Delta$, then we're already done. If D is A or B we conclude by applying pe-weakening and R$\vee$.

If $\mathcal{D}$ ends with the following instance of L$\supset$:

$$\dfrac{\mathcal{D}_1\left\{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow \Delta, B\end{array}\right. \qquad \mathcal{D}_2\left\{\begin{array}{c}\vdots\\ C, \Gamma' \Rightarrow \Delta\end{array}\right.}{B \supset C, \Gamma' \Rightarrow \Delta}\,\text{L}\supset$$

and Δ is inhabited, by inductive hypothesis we have $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1'} \Gamma' \Rightarrow B'$ with $B' \in \Delta, B$ and $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_2'} C, \Gamma' \Rightarrow D$ with $D \in \Delta$, such that $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}_2' \preccurlyeq \mathcal{D}_2$. When $B' \equiv B$ we use the left and right right rules of weakening to obtain a derivation $\mathcal{D}_1'' \preccurlyeq \mathcal{D}_1'$ of $B \supset C, \Gamma' \Rightarrow D, B$ and we obtain $\mathcal{D}' \preccurlyeq \mathcal{D}$ as follows:

$$\dfrac{\mathcal{D}_1''\left\{\begin{array}{c}\vdots\\ B \supset C, \Gamma' \Rightarrow D, B\end{array}\right. \qquad \mathcal{D}_2'\left\{\begin{array}{c}\vdots\\ C, \Gamma' \Rightarrow D\end{array}\right.}{B \supset C, \Gamma' \Rightarrow D}\,\text{L}\supset$$

When, instead, $B' \equiv E$ for some $E \not\equiv B$, we conclude $C \supset B, \Gamma \Rightarrow E$ by applying an instance of left weakening to the first derivation.

Next we consider the case with $\Delta = \emptyset$. By induction we have derivations $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1'} \Gamma' \Rightarrow B$ and $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{D_2'} C, \Gamma' \Rightarrow \Delta$, such that $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}_2' \preccurlyeq \mathcal{D}_2$. By weakening to obtain a derivation $\mathcal{D}_1'' \preccurlyeq \mathcal{D}_1'$ of $B \supset C, \Gamma' \Rightarrow \Delta, B$ and we obtain $\mathcal{D}' \preccurlyeq \mathcal{D}$ as follows:

$$\dfrac{\mathcal{D}_1''\left\{\begin{array}{c}\vdots\\ B \supset C, \Gamma' \Rightarrow \Delta, B\end{array}\right. \qquad \mathcal{D}_2'\left\{\begin{array}{c}\vdots\\ C, \Gamma' \Rightarrow \Delta\end{array}\right.}{B \supset C, \Gamma' \Rightarrow \Delta}\,\text{L}\supset$$

The cases with $\mathcal{D}$ ending by a rule for the quantifiers are straightforward and can thus be omitted.

If $\mathcal{D}$ ends with an instance of L$\bigwedge$, we have simply to apply the inductive hypothesis to its premiss and an instance of L$\bigwedge$ to get a derivation (in $\mathbf{G3I}_\omega.\mathbf{S}$) $\mathcal{D}' \preccurlyeq \mathcal{D}$ of either $\Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow A$ for $A \in \mathcal{D}$ (depending on whether Δ is empty or not).

If $\mathcal{D}$ ends with the following instance of R$\bigwedge$:

$$\mathcal{D}_i\left\{\begin{array}{c}\vdots\\ \dfrac{\{\Gamma \Rightarrow \Delta', B_i \mid i > 0\}}{\Gamma \Rightarrow \Delta', \bigwedge_{n>0} B_n}\,\text{R}\bigwedge\end{array}\right.$$

we apply the inductive hypothesis to obtain, for each $i > 0$, $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_i'} \Gamma \Rightarrow C_i$ with $C_i \in \Delta', B_i$, such that $\mathcal{D}_i' \preccurlyeq \mathcal{D}_i$. If for some $j > 0$ we have $C_j \in \Delta'$, then by taking $A \equiv C_j$ we observe that $\mathcal{D}_i'$ is as wanted. Else $\mathcal{D}_i'$ is an intuitionistic derivation of $\Gamma \Rightarrow B_i$ for all $i > 0$ and we conclude by an intuitionistic instance of R$\bigwedge$:

$$\mathcal{D}_i'\left\{\begin{array}{c}\vdots\\ \dfrac{\{\Gamma \Rightarrow B_i \mid i > 0\}}{\Gamma \Rightarrow \bigwedge_{n>0} B_n}\,\text{R}\bigwedge\end{array}\right.$$

If $\mathcal{D}$ ends with the following instance of R$\bigvee$:

$$\dfrac{\mathcal{D}_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta', \bigvee_{n>0} B_n, B_k \end{array} \right.}{\Gamma \Rightarrow \Delta', \bigvee_{n>0} B_n} \text{R}\bigvee$$

by inductive hypothesis $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}_1'} \Gamma \Rightarrow D$ with $D \in \Delta, \bigvee B_n, B_k$ and $\mathcal{D}_1' \preccurlyeq \mathcal{D}_1$. If $D \in \Delta, \bigvee B_n$, we conclude by taking $A \equiv D$. Else $\mathcal{D}_1'$ is a derivation of $\Gamma \Rightarrow B_k$ and, by right weakening we get a derivation $\mathcal{D}_1'' \preccurlyeq \mathcal{D}_1'$ of $\Gamma \Rightarrow B_k, \bigvee B_n$. Finally, we get a derivation $\mathcal{D}' \preccurlyeq \mathcal{D}$ in $\mathbf{G3I}_\omega.\mathbf{S}$:

$$\dfrac{\mathcal{D}_1'' \left\{ \begin{array}{c} \vdots \\ \Gamma \Rightarrow \bigvee_{n>0} B_n, B_k \end{array} \right.}{\Gamma \Rightarrow \bigvee_{n>0} B_n} \text{R}\bigvee$$

If $\mathcal{D}$ ends with a one-premiss geometric rule (rules for equality included) L_{GS}, then we have simply to apply the inductive hypothesis to the premiss and then an instance of L_{GS} to obtain the desired conclusion. Observe that we had to exclude geometric rules with more than one premiss since the inductive hypothesis would have given us sequents with a possibly different succedent (for the same reason we had to exclude $\vee^-$ and $\bigvee^-$).

The proof for $\mathbf{G3M}_\omega.\mathbf{S}$ is as for $\mathbf{G3I}_\omega.\mathbf{S}$ (assuming no instance of $\bot^-$ occurs in $\Gamma \Rightarrow \Delta$) and can thus be omitted. □

Theorem 31 (Glivenko Class 2) *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$ *and neither* $\supset^+$, *nor* $\vee^-$, *nor* $\bigvee^-$ *occurs in* $\Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \Rightarrow \Delta$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$.

Proof An immediate corollary of Lemma 30. □

We list here two other corollaries of Lemma 30, the latter being an infinitary version of the result proved in [26].

Corollary 32 *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow \Delta$ *and neither* $\supset^+$, *nor* $\vee^-$, *nor* $\bigvee^-$ *occurs in* $\Gamma \Rightarrow \Delta$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow \Delta$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \Rightarrow \Delta$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow \Delta$.

Corollary 33 *Assume that no instance of* $\supset$ *occurs in* A *and that no instance of* $\bot^+$, $\vee^+$, $\bigvee^+$, *and* $\supset^-$ *occurs in* Γ. *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$, *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.

Lemma 34 *If neither* $\supset^+$, *nor* $\forall^-$ *occurs in* $\Gamma \Rightarrow \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow \Delta$, *then there is a classical derivation of* $\Gamma \Rightarrow \Delta$ *such that all instances of rules in* $G_1 = \{L_{G^S}, R\wedge, R\vee, R\forall, R\exists, R\bigwedge, R\bigvee\}$ *precede all instances of rules in* $G_2 = \{L\wedge, L\vee, L\supset, L\exists, L\bigwedge, L\bigvee\}$.

Proof First notice that in general it is possible to permute rules in $G2$ below rules in $G1$ since, having excluded instances of rule R⊃, the principal formula of rules in $G2$ cannot be active in rules in $G1$. In particular, instances of geometric rules have atomic formulas as active and, having excluded instances of R⊃, all active formulas of logical rules in $G1$ occur in the antecedents while principal formulas of rules in $G2$ occur in the succedents. Moreover instances of rule L∃ can be permuted down with respect to instances of R∀ and of geometric rules with a variable condition since their *eigenvariables* are necessarily distinct. □

Theorem 35 (Glivenko class 3)

1. *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow A$ *and neither* $\supset^+$, *nor* $\forall^-$ *occurs in* $\Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow A$.
   If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \Rightarrow A$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow A$.
2. *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow \Delta$ *and neither* $\supset^+$, *nor* $\forall^-$ *occurs in* $\Gamma \Rightarrow \Delta$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow \Delta$.
   If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \Rightarrow \Delta$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow \Delta$.

Proof We begin with item 1. By Lemma 34 we can transform the classical derivation of $\Gamma \Rightarrow \Delta$ into a classical derivation where all instances of rules in group $G1$ precede instances of rules in $G2$. Then the lemma holds for the upper $G1$-component of this derivation by Theorem 31 and it holds for the lower $G2$-component since all rules instances applied therein are instances of rules identical in classical, intuitionistic and minimal logics.

Item 2 can be proved analogously using Corollary 32 instead of Theorem 31. □

Theorem 36 (Glivenko class 4) *If* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$ *and neither* $\supset^-$, *nor* $\vee^+$, *nor* $\exists^+$ *nor* $\bigvee^+$ *occurs in* $\Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \Rightarrow A$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$.

Proof Rules L⊃, $\vee^+$, $\exists^+$, and $\bigvee^+$ are the only rules of $\mathbf{G3C}_\omega.\mathbf{G}$ having instances with a single-succedent conclusion and a multi-succedent premiss. This implies that all sequents in the classical derivation $\mathcal{D}$ of $\Gamma \Rightarrow A$ are single-succedent ones, hence all rule instances occurring in $\mathcal{D}$ satisfy the intuitionistic (and minimal) restriction. □

Next we move to Glivenko classes 5, 6, and 7 which, roughly, are versions of classes 1,2, and 3 where the restriction on occurrences of $\supset^+$ is relaxed by allowing occurrences of $\supset^+$ having $\bot$ as succedent.

Lemma 37 *If* $\Gamma \Rightarrow \Delta$ *does not contain* $\bot^-$, $\vee^+$, $\supset^+$, *or* $\bigvee^+$ *and* Δ *is either empty or* $\bot^+$ *occurs in each one of its formulas, then* $\mathbf{G3C}_\omega.\mathbf{G} \nvdash \Gamma \Rightarrow \Delta$.

Proof Since $\bot^-$ cannot occur in $\Gamma \Rightarrow \Delta$ and since all formulas in Δ must contain an occurrence of $\bot^+$, $\Gamma \Rightarrow \Delta$ cannot be the conclusion of an instance of L$\bot$ nor an initial sequent (atomic formulas do not contain occurrences of $\bot^+$). Having excluded the applicability of rules R$\vee$, R$\supset$ and $\bigvee^+$, we know that at least one branch of a proof-search tree for $\Gamma \Rightarrow \Delta$ is such that $\bot^+$ occurs in each formula occurring in its succedents, hence that branch cannot reach an inital sequent. The lemma follows by the invertibility of the rules of $\mathbf{G3C}_\omega.\mathbf{G}$. □

Corollary 38 *If* $\Gamma \Rightarrow A$ *does not contain* $\neg^-$, $\vee^+$, $\supset$, *or* $\bigvee^+$, Γ *does not contain* $\bot$, *and* $\Gamma \Rightarrow A$ *contains an occurrence of* $\neg^+$, *then* $\mathbf{G3C}_\omega.\mathbf{G} \nvdash \Gamma \Rightarrow A$.

Proof $\Gamma \Rightarrow A$ satisfies the conditions of Lemma 37 since, by the restiction on implications, $\bot^+$ occurs in A. □

Corollary 39 *If* $\Gamma \Rightarrow A$ *does not contain* $\neg^-$, $\vee^+$, $\supset$, *or* $\bigvee^+$, *but it contains an occurrence of* $\neg^+$, *if* $\mathbf{G3C}_\omega.\mathbf{G} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{G} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$. *If, moreover, no instance of* $\bot^-$ *occurs in* $\Gamma \Rightarrow A$, *then* $\mathbf{G3M}_\omega.\mathbf{S}^{\mathcal{D}'} \Gamma \Rightarrow A$.

Proof If $\mathcal{D}$ is an initial sequent, then there's nothing to prove. Suppose it's not. If $\bot$ occurs in Γ, then the corollary holds because $\Gamma \Rightarrow A$ is a conclusion of L$\bot$, thus we obtain a one-step derivation $\mathcal{D}'$ which is embeddable in any nontrivial derivation. Else it follows from Corollary 38. If no instance of $\bot^-$ occurs in $\Gamma \Rightarrow A$, then we're always in the latter case. □

Theorem 40 (Glivenko class 5) *If neither* $\supset^-$, *nor* $\vee^+$, *nor* $\forall^+$ *nor* $\bigvee^+$ *occurs in* $\Gamma \Rightarrow A$, *and* $\supset^+$ *occurs in* $\Gamma \Rightarrow A$ *only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.

Proof If $\bot^+$ does not occur in $\Gamma \Rightarrow A$, then the sequent is in Glivenko class 1 (see Theorem 28), else the theorem follows by Corollary 39. □

Theorem 41 (Glivenko class 6) *If neither* $\supset^-$, *nor* $\vee$, *nor* $\bigvee$ *occurs in* $\Gamma \Rightarrow A$, *and* $\supset^+$ *occurs in* $\Gamma \Rightarrow A$ *only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathcal{D}} \Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathcal{D}'} \Gamma \Rightarrow A$ *with* $\mathcal{D}' \preccurlyeq \mathcal{D}$.

Proof If $\bot^+$ does not occur in $\Gamma \Rightarrow A$, then the sequent is in Glivenko class 2 (see Theorem 31), else the theorem follows by Corollary 39. □

Theorem 42 (Glivenko class 7) *If neither* $\supset^-$, *nor* $\vee^+$, *nor* $\forall^-$ *nor* $\bigvee^+$ *occurs in* $\Gamma \Rightarrow A$, *and* $\supset^+$ *occurs in* $\Gamma \Rightarrow A$ *only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow A$, *then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \Rightarrow A$.

Proof If $\bot^+$ does not occur in $\Gamma \Rightarrow A$, then the sequent is in Glivenko class 3 (see Theorem 35), else the theorem follows by Corollary 39. □

8 Conclusion

We have proved that classical derivability entails intuitionistic or even minimal derivability for seven infinitary Glivenko sequent classes. This result naturally extends the results presented in [20] for the finitary Glivenko sequent classes and in [19] for Barr's theorem for infinitary geometric theories. Moreover, we have also shown how to constructivise the cut-elimination procedure for geometric logics given in [19]: by introducing the notion of proof embeddability and by making use of Brouwer's principle of Bar Induction we have given an ordinal-free proof of cut elimination that works within **IZF** (but not within **CZF**). The present proof strategy should allow to constructivise the cut-elimination procedure for other infinitary calculi such as those in [6, 14, 29].

One question that remains open is whether the seven infinitary Glivenko sequent classes considered here are optimal for conservativity.[10] We leave this question for future research.

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[10] Orevkov [23] proved optimality for the finitary case by classifying the other possible classes of sequents and exhibiting for each of them a sequent that is classically but not intuitionistically derivable.

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