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A MODEL OF SOLVABLE SECOND ORDER PDE WITH NON SMOOTH COEFFICIENTS

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Abstract. In this paper it is shown the L^2-local solvability of some second order partial differential operators (which are a variation of that introduced by Federico and Parmeggiani in [2]) with $C^{1,1}$ or $C^{0,1}$ coefficients and with multiple characteristics.

1. Introduction

Let us consider the second order partial differential operator on $\mathbb{R}^n$ of the form

$$P = \sum_{j=1}^{N} X_j^* g|g| X_j + iX_0 + a_0, \tag{1.1}$$

where $X_j = X_j(D)$, $1 \leq j \leq N$, are homogeneous first order differential operators (in other words they are vector fields) with real or complex constant coefficients (the two cases will be analyzed separately), $X_0 = X_0(x, D)$ is a homogeneous first order operator with *affine* real coefficients, g is an affine function, and a_0 is a continuous function on $\mathbb{R}^n$ with complex values.
The purpose here is to study the L^2-local solvability of P in a neighborhood of the zeros of the function g, where the principal symbol of the operator can possibly change sign. Why are we interested in the local solvability around these points? The motivation is related to the fact that a change of sign in the principal symbol can produce the non solvability of the associated operator (as in the Kannai example [4] and in [1]), therefore it is interesting to study what kind of operators with this property are still solvable. We will see that P is a model of solvable second order operator with $C^{1,1}$ coefficients if the vector fields X_j are tangent to $S = g^{-1}(0) \neq \emptyset$ for all index j, $1 \leq j \leq N$, or with $C^{0,1}$ coefficients if there is at least an index $k \neq 0$ such that X_k is transverse to S. The proof follows the approach used in [2] by Federico and Parmeggiani, in which a solvability result for degenerate second order operators with smooth coefficients analogous to P is proved.
The class considered here is an elaboration of that introduced in [2] and it differs from that class in the regularity assumption on the coefficients which

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are assumed to be less regular.
As we said before, we are interested in the L^2-local solvability, whose definition is given below (for more information about solvability see [3] and [5]).

Definition 1.1. *Given a partial differential operator P, defined on an open set $\Omega \subseteq \mathbb{R}^n$, and at least with L^∞ coefficients, we say that P is L^2-locally solvable in Ω if for any given $x_0 \in \Omega$ there is a compact set $K \subset \Omega$ with $x_0 \in U = \mathring{K}$ (where $\mathring{K}$ denotes the interior of K) such that for all $f \in L^2_{loc}(\Omega)$ there exists $u \in L^2(U)$ such that for every compact $K \subset U$*

$$(u, P^*\varphi) = (f, \varphi), \quad \forall \varphi \in C_0^\infty(K),$$

where $(\cdot, \cdot)$ is the L^2 inner product.

Throughout we shall refer to the previous definition when talking about L^2-local solvability and solution of the problem mentioned before.
Let us end this introduction by giving the plan of the paper.
In Section 2 we make the setting precise, introduce the main hypotheses, and give the proof of their invariance under affine changes of variables of the latter.
In Section 3 we prove a fundamental estimate, corresponding to the main estimate in [2], that will be the crucial step in the proof of the *solvability estimate* (3.2) below.
In Section 4 we prove a solvability result for the operator (1.1) in the real coefficients case. Here we shall use the estimate of Section 3 to derive the solvability estimate from which the result follows.
In Section 5 a first complex coefficients case is analyzed. Again, by using the estimate of Section 3, we obtain a solvability result.
In Section 6 we study a second kind of complex coefficients case, that is in some sense more "general" with respect to the case presented in Section 5.
Finally in Section 7 we look at another model operator that differs from the operator in (1.1) in that the function responsible for the extra degeneracy of the symbol *does not* change sign across its zero set but the coefficients are less regular. Here, unlike the other cases listed, the solvability result is not based on the fundamental estimate proved in Section 3, but it follows by using a Carleman estimate.

2. INVARIANCE WITH RESPECT TO AFFINE CHANGES OF VARIABLES

Let P be a linear second order partial differential operator as in the introduction, then the first order partial differential operators $X_j, 1 \leq j \leq N$, and X_0 in the expression of P are of the form

$$X_j(x, D) = X_j(D) = \langle \alpha_j, D \rangle, \quad X_0(x, D) = \langle \beta(x), D \rangle$$

where $D = (D_1, D_2, \ldots, D_n)$, $D_j = -i\partial_{x_j}$, $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbb{C}^n$, and $\beta(x) = (\beta_1(x), \ldots, \beta_n(x))$, where $\beta_j(x)$, $j = 1, \ldots, n$, are affine real functions of the form $\beta_j(x) = \sum_{k=1}^n \beta_{j,k}x_k + \beta_{j,0}$, and $\beta_{j,k}, \beta_{j,0} \in \mathbb{R}$ for all $j,k = 1,\dots,n$. Moreover g is an affine real function over $\mathbb{R}^n$, thus we have $g(x) = \sum_{j=1}^n g_j x_j + g_0$, with $g_j, g_0 \in \mathbb{R}$ for all $j = 1,\dots,n$, and g is such that $S = g^{-1}(0) \neq \emptyset$. Note also that the commutator $[X_j, X_0]$, for all $1 \leq j \leq N$, is a first order homogenous partial differential operator with complex *constant* coefficients. In addition we suppose:

(H1) $iX_0 g(x) > 0$ for all $x \in S := g^{-1}(0)$;
(H2) for all $1 \leq j \leq N$ there exists a constant $C > 0$ such that

$$|\{X_j, X_0\}(\xi)|^2 \leq C \sum_{j=1}^N |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

where the $\{X_j, X_0\}(\xi)$ and the $X_j(\xi)$ are the total (principal because of homegeneity) symbols of $[X_j, X_0]$ and X_j respectively, $\{\cdot,\cdot\}$ denoting the Poisson bracket.

First of all we show that the analysis of the local solvability of

$$P = \sum_{j=1}^N X_j^* g|g| X_j + iX_0 + a_0$$

can always be reduced, after a linear change of variables, to that of

$$\tilde{P} = \sum_{j=1}^N \tilde{X}_j^* y_1|y_1| \tilde{X}_j + i\tilde{X}_0 + \tilde{a}_0,$$

where $\tilde{P}$ (in the new variables) is of the same kind of P, and the new quantities still satisfy hypotheses (H1) and (H2).
After that we will focus our attention on the local solvability of P in a neighborhood of the points of $S = g^{-1}(0)$, where, by the previous argument, we can assume $g(x) = x_1$. In this way we deal with an operator which is simpler to study.

Observe that hypothesis (H1) is explicitly stated as

$$iX_0 g(x) = \langle \beta(x), \bigtriangledown g \rangle = \textstyle\sum_{j=1}^n \beta_j(x) g_j > 0 \text{ on } S.$$

We may suppose that $\frac{\partial g}{\partial x_1} = g_1 \neq 0$. Under this assumption the function $\chi : (x_1,\dots,x_n) \mapsto (g(x), x_2,\dots,x_n)$ is an affine diffeomorphism of $\mathbb{R}^n$, and we can choose $(y_1,\dots,y_n) = \chi(x_1,\dots,x_n)$ as new coordinates.
Changing variables we have

$$\tilde{P} = \sum_{j=1}^N \tilde{X}_j^* y_1|y_1| \tilde{X}_j + i\tilde{X}_0 + \tilde{a}_0,$$

where

$$\begin{aligned}
\tilde{X}_j(D_y) &= \sum_{k=1}^{n} \alpha_{j,k} g_k D_{y_1} + \sum_{k=2}^{n} \alpha_{j,k} D_{y_k}, \\
\tilde{X}_0(y, D_y) &= \sum_{j=1}^{n} (\beta_j \circ \chi^{-1})(y) g_j D_{y_1} + \sum_{j=2}^{n} (\beta_j \circ \chi^{-1})(y) D_{y_j}, \\
\tilde{a}_0(y) &= (a_0 \circ \chi^{-1})(y), \\
\tilde{g}(y) &= (g \circ \chi^{-1})(y) = y_1.
\end{aligned}$$

It is important to note that $\tilde{X}_j$, $1 \leqslant j \leqslant N$, and $\tilde{X}_0$ are still first order homogeneous partial differential operators, and they still have, respectively, constant and affine coefficients.
Now we look at conditions (H1),(H2), and we see that if they are satisfied by X_j, $1 \leqslant j \leqslant N$, and X_0, then the same holds for $\tilde{X}_j$, $1 \leqslant j \leqslant N$, and $\tilde{X}_0$. In fact, since

$$\tilde{X}_j \tilde{g} = \tilde{X}_j y_1 = X_j g, \quad \tilde{X}_0 \tilde{g} = \tilde{X}_0 y_1 = X_0 g, \tag{2.2}$$

then our hypothesis (H1) is trivially invariant with respect to affine changes of variables. As for condition (H2), there is nothing to prove, since the principal symbol is an invariant of partial differential operators. Observe moreover that the first identity in (2.2) means that if X_j, $1 \le j \le N$, is tangent or transverse to S, then the same holds for $\tilde{X}_j$.
Summarizing, we have proved that, after performing an affine change of variables in P, what we get is an operator with a simpler expression and of the same type of P.

3. THE FUNDAMENTAL ESTIMATE

By the argument of Section 2, we can reduce our problem to the analysis of the local solvability of operators of the form

$$P = \sum_{j=1}^{N} X_j^* x_1 |x_1| X_j + i X_0 + a_0, \tag{3.1}$$

where X_j, $1 \leqslant j \leqslant N$, X_0 and a_0 are assumed to be as before.
To obtain a local L^2-solvability result for an operator P on $\mathbb{R}^n$ the main point is to obtain the following a priori estimate: there exist a compact set K and a positive constant C such that

$$\|P^* u\| \geqslant C \|u\|, \quad \forall u \in C_0^\infty(K), \tag{3.2}$$

where P^* is the formal adjoint of P, and $\|\cdot\|$ is the L^2-norm. If this inequality holds for P^* then, using standard arguments, we have for all $v \in L^2_{loc}(\mathbb{R}^n)$ the existence of $u \in L^2(\mathring{K})$ solving $Pu = v$ in $U = \mathring{K}$, where $\mathring{K}$ denotes the interior of K.
Consequently, our goal is to obtain the solvability estimates (3.2) for our operator P of the form (3.1) in a neighborhood of S. To this aim, we

need some further preliminary estimates. In particular we will derive in this section a fundamental estimate that will be useful both in the real and in the complex coefficients case.

Proposition 3.1. *Let $S = \{x \in \mathbb{R}^n;\ x_1 = 0\}$. Then for all $x_0 \in S$ there exist a compact set K_0 containing x_0 in its interior and three positive constants $C = C(K_0), c = c(K_0)$ and $\varepsilon_0 = \varepsilon_0(K_0)$, with $\varepsilon_0 \to 0$ as $K_0 \searrow \{x_0\}$, such that for all compact sets $K \subset K_0$*

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 + c(P_0 u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K), \tag{3.3}$$

where

$$P_0 = \sum_{j=1}^{N} (X_j^* |x_1| X_j - \varepsilon_0^2 [X_j, X_0]^* |x_1| [X_j, X_0]). \tag{3.4}$$

Proof. First of all we observe that $X_0^* = X_0 + d_{X_0}$, where $d_{X_0} = \sum_{k=1}^n D_k(\beta_k) \in i\mathbb{R}$, $D_k = -i\partial_{x_k}$, and $X_j^* = \langle \overline{\alpha}_j, D\rangle$, $1 \leqslant j \leqslant N$ (we are considering the general case in which $\alpha_j \in \mathbb{C}^n$).
Moreover, since $P^* = \sum_{j=1}^N X_j^* x_1|x_1| X_j - iX_0^* + \overline{a}_0$, for all compact $K \subset \mathbb{R}^n$ we have

$$\|P^* u\|^2 \geq \frac{1}{2}\|\sum_{j=1}^N (X_j^* x_1|x_1|X_j - iX_0^*)u\|^2 - \|a_0\|^2_{L^\infty(K)}\|u\|^2$$

for all $u \in C_0^\infty(K)$, where

$$\begin{aligned}
\frac{1}{2}\|\sum_{j=1}^N (X_j^* x_1|x_1|X_j - iX_0^*)u\|^2 &= \frac{1}{2}\Big(\|X_0^* u\|^2 + \|\sum_{j=1}^N X_j^* x_1|x_1|X_j u\|^2 \\
&- 2\sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u)\Big) \\
&\geq \frac{1}{2}\|X_0^* u\|^2 - \sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u).
\end{aligned}$$

Since

$$\sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u) = \sum_{j=1}^N \mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u),$$

we then estimate the imaginary part. Thus, for each index j, we have

$$\begin{aligned}
\mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u) &= \mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0 u) + \mathsf{Im}(X_j^* x_1|x_1|X_j u, d_{X_0} u) \\
&= \mathsf{Im}(x_1|x_1|X_j u, X_j X_0 u) + \mathsf{Im}(x_1|x_1|X_j u, (X_j d_{X_0})u) \\
&+ \mathsf{Im}(x_1|x_1|X_j u, d_{X_0} X_j u),
\end{aligned}$$

where

$$\begin{aligned}\mathsf{Im}(x_1|x_1|X_ju, X_jX_0u) &= \mathsf{Im}(x_1|x_1|X_ju, [X_j, X_0]u) + \mathsf{Im}(x_1|x_1|X_ju, X_0X_ju)\\ &= \mathsf{Im}(x_1|x_1|X_ju, [X_j, X_0]u) + \frac{1}{2i}\big[(x_1|x_1|X_ju, X_0X_ju)\\ &- (X_0X_ju, x_1|x_1|X_ju)\big]\\ &= \mathsf{Im}(x_1|x_1|X_ju, [X_j, X_0]u) + \frac{1}{2i}\big[(|x_1|(X_0x_1)X_ju, X_ju)\\ &+ (x_1(X_0|x_1|)X_ju, X_ju) + (x_1|x_1|X_0X_ju, X_ju)\\ &+ (d_{X_0}x_1|x_1|X_ju, X_ju) - (X_0X_ju, x_1|x_1|X_ju)\big]\\ &= \mathsf{Im}(x_1|x_1|X_ju, [X_j, X_0]u) - (|x_1|(iX_0x_1)X_ju, X_ju)\\ &+ \frac{1}{2}\mathsf{Im}(d_{X_0}x_1|x_1|X_ju, X_ju).\end{aligned}$$

Putting the last expression inside the term $\mathsf{Im}(X_j^*x_1|x_1|X_ju, X_0^*u)$ gives

$$-\mathsf{Im}(X_j^*x_1|x_1|X_ju, X_0^*u) = -\mathsf{Im}(x_1|x_1|X_ju, [X_j, X_0]u) + (|x_1|(iX_0x_1)X_ju, X_ju)$$

$$+\frac{1}{2}\mathsf{Im}(d_{X_0}x_1|x_1|X_ju, X_ju) - \mathsf{Im}(x_1|x_1|X_ju, (X_jd_{X_0})u).$$

By hypothesis (H1), for each $x_0 \in S$ we can find a compact set K_1 such that $x_0 \in \mathring{K}_1$ and $iX_0g(x) > c_0$ in K_1, with $c_0 > 0$ and $g(x) = x_1$. We then work in a fixed compact set K_1 containing the point $x_0 \in S$ in its interior, and we get

$$-\mathsf{Im}(X_j^*x_1|x_1|X_ju, X_0^*u) \geq c_0(|x_1|^{1/2}X_ju, |x_1|^{1/2}X_ju)$$

$$-\mathsf{Im}(x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}[X_j, X_0]u) + \frac{1}{2}\mathsf{Im}(d_{X_0}x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}X_ju)$$

$$-\mathsf{Im}(x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}(X_jd_{X_0})u),$$

for all $u \in C_0^\infty(K_1)$. Therefore, for each $K \subset K_1$ with $x_0 \in \mathring{K}$, we have

$$-\mathsf{Im}(X_j^*x_1|x_1|X_ju, X_0^*u) \geq c_0\||x_1|^{1/2}X_ju\|^2$$

$$-2\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}X_ju\|\||x_1|^{1/2}[X_j, X_0]u\|$$

$$-\frac{1}{2}\|x_1\|_{L^\infty(K)}\|d_{X_0}\|_{L^\infty(K_1)}\||x_1|^{1/2}X_ju\|^2$$

$$-2\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\|u\|\||x_1|^{1/2}X_ju\|$$

$$\geq \||x_1|^{1/2}X_ju\|^2\Big(c_0 - \|x_1\|_{L^\infty(K)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K_1)} + \||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\big)\Big)$$

$$-\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}[X_j, X_0]u\|^2 - \|x_1\|_{L^\infty(K)}\||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\|u\|^2.$$

Since $\|x_1\|_{L^\infty(K)} \longrightarrow 0$ when $K \searrow \{x_0\}$, $x_0 \in S$, we can find a compact set $K_0 \subset K_1$ containing x_0 in its interior such that, for all $K \subset K_0$ with $x_0 \in \overset{\circ}{K}$, we have

$$\begin{aligned} c_0 - \|x_1\|_{L^\infty(K)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K)} + \||x_1|^{1/2}(X_j d_{X_0})\|_{L^\infty(K)}\big) &\geqslant \\ c_0 - \|x_1\|_{L^\infty(K_0)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K_0)} + \||x_1|^{1/2}(X_j d_{X_0})\|_{L^\infty(K_0)}\big) &\geqslant \frac{c_0}{2} \end{aligned}$$

since $\|x_1\|_{L^\infty(K)} \leq \|x_1\|_{L^\infty(K_0)}$ for every compact set $K \subset K_0$. Then, calling $c = c(K_0) := c_0/2$, we may choose

$$\varepsilon_0 = \varepsilon_0(K_0) := (\|x_1\|_{L^\infty(K_0)}/c)^{1/2}, \quad \text{with} \quad \varepsilon_0 \to 0 \quad \text{as} \quad K_0 \searrow \{x_0\},$$

which depends only on K_0. By the previous arguments we get that, for all $K \subset K_0$, and all $u \in C_0^\infty(K)$

$$\|P^*u\|^2 \geq \frac{1}{2}\|X_0^*u\|^2 + c(P_0u, u) - c\varepsilon_0^2\||x_1|^{1/2}(X_j d_{X_0})\|^2_{L^\infty(K_0)}\|u\|^2 - \|a_0\|^2_{L^\infty(K_0)}\|u\|^2.$$

Finally, since $\|X_0^*u\|^2 \geq (1/2)\|X_0u\|^2 - \|d_{X_0}\|^2_{L^\infty(K_0)}\|u\|^2$ for all $u \in C_0^\infty(K)$, for all $K \subset K_0$ (containing x_0 in its interior), we obtain inequality (3.3). □

Looking at (3.3), it is obvious that we need to estimate (P_0u, u) to have the solvability estimate (3.2). For this reason we have to distinguish between the real and the complex coefficients case, since we need different hypotheses on the vector fields X_j's in order to obtain the appropriate estimate for the term (P_0u, u).

4. LOCAL SOLVABILITY RESULT IN THE REAL COEFFICIENTS CASE

Let us start with the real case, that is, we assume that $X_j = X_j(D)$, for each $j \neq 0$, is a vector field with real constant coefficients, and $X_0 = X_0(x, D)$ is a vector field with real affine coefficients. The plan is to use (3.3) to derive (3.2) by estimating P_0 from below in L^2 and then by using a Poincaré inequality for X_0. Before proving an estimate for P_0, we give the following consequence of hypothesis (H2).

Lemma 4.1. *If condition (H2) holds, then, for each index $j \in \{1, \cdots, N\}$, we have*

$$[X_j, X_0] = \sum_{k=1}^{N} c_k X_k, \quad c_k \in \mathbb{R}. \tag{4.1}$$

Proof. Recall that X_j and $[X_j, X_0]$, $1 \leq j \leq N$, have real constant coefficients, and that $\alpha_j \in \mathbb{R}^n$ is the vector associated to X_j, $1 \leq j \leq N$. Now we consider two cases. The first one is when there exist n linear independent elements $\alpha_{j_1}, \cdots, \alpha_{j_n}$ of $\mathbb{R}^n$ (associated to $X_{j_1}, \cdots, X_{j_n}$), with $j_1, \cdots, j_n \in$

$\{1,\cdots,N\}$. This means essentially that $\mathbb{R}^n=\mathrm{Span}\{\alpha_{j_1},\cdots,\alpha_{j_n}\}$, and thus, for each index j, we have

$$[X_j,X_0]=\sum_{k=1}^{n}c_kX_{j_k}=\sum_{k=1}^{N}c_kX_k,\quad c_k\in\mathbb{R},\, c_k=0,\ \forall k\notin\{j_1,\cdots,j_n\}.$$

The second case is that in which there are $m<n$ linear independent elements $\alpha_{j_1},\cdots,\alpha_{j_m}$ of $\mathbb{R}^n$ (associated to $X_{j_1},\cdots,X_{j_m}$), with $j_1,\cdots,j_m\in\{1,\cdots,N\}$.
Since $X_j(x,D)=X_j(D)=\langle\alpha_j,D\rangle$ and $[X_j,X_0](x,D)=[X_j,X_0](D)=\langle\gamma_j,D\rangle$, we shall denote by V_k and W_k the sets

$$V_k=\{\xi\in\mathbb{R}^n;\ X_k(\xi)=0\}=\mathrm{Span}_{\mathbb{R}}\{\alpha_k\}^{\perp},$$
$$W_k=\{\xi\in\mathbb{R}^n;\ \{X_k,X_0\}(\xi)=0\}=\mathrm{Span}_{\mathbb{R}}\{\gamma_k\}^{\perp},$$

and also by Σ_{X_k}, $\Sigma_{[X_k,X_0]}$ the characteristic sets of X_k, $[X_k,X_0]$ respectively, so that

$$\Sigma_{X_k}=V_k\setminus\{0\},\quad \Sigma_{[X_k,X_0]}=W_k\setminus\{0\}.$$

In this situation condition (H2) states that, for all $1\le j\le N$, there exists a constant $C>0$ such that

$$|\langle\gamma_j,\xi\rangle|^2\le C\sum_{k=1}^{m}|\langle\alpha_{j_k},\xi\rangle|^2,\quad \forall\xi\in\mathbb{R}^n\setminus\{0\},$$

which implies

$$\bigcap_{k=1}^{m}V_{j_k}\subseteq W_j,\quad 1\le j\le N.$$

The latter inclusion shows that, passing to the orthogonal complements, we have

$$\Big[\bigcap_{k=1}^{m}V_{j_k}\Big]^{\perp}\supseteq W_j^{\perp}. \tag{4.2}$$

Now, applying in (4.2) the well-known relations

$$\Big(\bigcap_{i=1}^{m}V_i\Big)^{\perp} = V_1^{\perp}+\cdots+V_m^{\perp} \tag{4.3}$$

(4.4)

we have

$$V_{j_1}^{\perp}+\cdots+V_{j_m}^{\perp}\supseteq W_j^{\perp},\quad \forall j=1,\cdots,N,$$

which is equivalent to

$$\mathrm{Span}_{\mathbb{R}}\{\alpha_{j_1},\cdots,\alpha_{j_m}\}\supseteq\mathrm{Span}_{\mathbb{R}}\{\gamma_j\},\quad \forall j=1,\cdots,N.$$

Finally, by the latter inclusion, we have

$$[X_j,X_0]=\langle\gamma,D\rangle=\langle\sum_{k=1}^{m}c_k\alpha_{j_k},D\rangle=\sum_{k=1}^{m}c_kX_{j_k}=\sum_{k=1}^{N}c_kX_k,$$

where $c_k \in \mathbb{R}$, and $c_k = 0,\ \forall k \notin \{j_1, \cdots, j_m\}$. □

Next we prove the following lemma.

Lemma 4.2. *Consider $x_0 \in S$ and K_0 as in Proposition 3.1. Then, suitably shrinking K_0 to a compact set containing x_0 in its interior, and that we still denote by K_0, we have that for all $K \subset K_0$, with $x_0 \in \overset{\circ}{K}$, we have*

$$(P_0 u, u) \geqslant 0, \quad \forall u \in C_0^\infty(K).$$

Proof. Recall that

$$(P_0 u, u) = \sum_{j=1}^{N} \| |x_1|^{1/2} X_j u \|^2 - \varepsilon_0^2 \sum_{j=1}^{N} \| |x_1|^{1/2} [X_j, X_0] u \|^2,$$

where $K \subseteq K_0$ and $u \in C_0^\infty(K)$.
Observe now, in view of Lemma 4.1, that

$$\| |x_1|^{1/2} [X_j, X_0] u \|^2 = \int |x_1| |[X_j, X_0] u|^2 dx = \int |x_1| |\sum_{k=1}^{N} c_k X_k u|^2 dx \leq$$

$$\leq N \sum_{k=1}^{N} \int |x_1| |c_k X_k u|^2 dx \leq N (\max_k c_k^2) \sum_{k=1}^{N} \int |x_1| |X_k u|^2 dx$$

$$= C(j) \sum_{k=1}^{N} \| |x_1|^{1/2} X_k u \|^2,$$

where $C(j) = N \max_k c_k^2 > 0$.
The latter inequality yields

$$\begin{aligned}
\sum_{j=1}^{N} \| |x_1|^{1/2} [X_j, X_0] u \|^2 &\leq \sum_{j=1}^{N} C(j) \sum_{k=1}^{N} \| |x_1|^{1/2} X_k u \|^2 \\
&\leq N (\max_j C(j)) \sum_{k=1}^{N} \| |x_1|^{1/2} X_k u \|^2 \\
&= C \sum_{k=1}^{N} \| |x_1|^{1/2} X_k u \|^2,
\end{aligned}$$

where $C = N \max_j C(j) > 0$.
Therefore

$$\begin{aligned}
(P_0 u, u) &= \sum_{j=1}^{N} \| |x_1|^{1/2} X_j u \|^2 - \varepsilon_0^2 \sum_{j=1}^{N} \| |x_1|^{1/2} [X_j, X_0] u \|^2 \\
&\geqslant (1 - \varepsilon_0^2 C) \sum_{j=1}^{N} \| |x_1|^{1/2} X_j u \|^2, \qquad \forall K \subset K_0,\ \forall u \in C_0^\infty(K).
\end{aligned}$$

and since $\varepsilon_0^2 = \varepsilon_0(K_0)^2$, we can shrink K_0 to a compact set K_0' containing x_0 in its interior in such a way that $C\varepsilon_0(K_0')^2 \leqslant 1/2$. Finally, denoting K_0' by K_0 again, the result follows. □

Remark 4.3. *Lemma 4.2 still holds for all compact $K \subset K_0$ not necessarily containing x_0 in its interior.*

Remark 4.4. *Summarizing, by Proposition 3.1 and Lemma 4.2, for every compact $K \subset K_0$ (containing x_0 in its interior) we have that there exists a positive constant $C = C(K_0)$ such that*

$$\|P^*u\|^2 \geqslant \frac{1}{4}\|X_0u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K). \tag{4.5}$$

To conclude the solvability estimate (3.2) we need the following Poincaré inequality for $X_0 \neq 0$ near S.

Lemma 4.5. *We may shrink K_0 around x_0 (in such a way that X_0 is non-degenerate in K_0) so that there exists $C_2 = C_2(K_0) > 0$ such that for all compact $K \subset K_0$*

$$\|u\| \leq C_2 \operatorname{diam}(K)\|X_0u\|, \quad \forall u \in C_0^\infty(K). \tag{4.6}$$

In view of (4.6) and (4.5), for all $K \subset K_0$ (K_0 suitably shrunk so that Lemma 4.5 holds)

$$\|P^*u\|^2 \geqslant \big(\frac{1}{4} - CC_2^2\operatorname{diam}(K)^2\big)\|X_0u\|^2, \quad \forall u \in C_0^\infty(K).$$

We finally choose a compact set $\tilde{K} \subset K_0$ (which is a shrinking of K_0 containing x_0 in its interior) such that

$$\operatorname{diam}(\tilde{K}) \leqslant \big(\frac{1}{8CC_2^2}\big)^{1/2},$$

and we obtain the solvability estimate

$$\|P^*u\|^2 \geq \frac{1}{8}\|X_0u\|^2 \geq \frac{1}{C_2^2\operatorname{diam}(\tilde{K})^2}\|u\|^2, \quad \forall u \in C_0^\infty(\tilde{K}). \tag{4.7}$$

We have essentially proved the following result.

Theorem 4.6. *Let P be of the form* (1.1) *such that all the vector fields X_j's have real coefficients and hypotheses (H1),(H2) are satisfied, and let S be the zero set of g. Then for all $x_0 \in S$ there exists a compact set $\tilde{K} \subset \mathbb{R}^n$ with $U = \mathring{\tilde{K}}$ and $x_0 \in U$, such that for all $v \in L^2_{loc}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U.*

Proof. After reducing P of the form (1.1) to the form (3.1) (see Section 2) the proof follows directly by the solvability estimate (4.7) using classical arguments. □

Remark 4.7. *Theorem 4.6 means that, for all* $v \in L^2_{loc}(\mathbb{R}^n)$*, there exists a solution* $u \in L^2(U)$ *of the equation* $Pu = v$ *in* U *in the sense of Definition 1.1, that is, for all compact* $K \subset U$*,*

$$(u, P^*\varphi) = (v, \varphi), \quad \forall \varphi \in C_0^\infty(K).$$

5. THE COMPLEX COEFFICIENTS CASE ($N = 1$)

Let us now consider a first kind of complex coefficients case, that is

$$P = X^* g|g| X + iX_0 + c \tag{5.1}$$

where g is an affine real function, c is a continuous function with complex values, X is a vector field with constant complex coefficients of the form

$$X = \langle \alpha, D \rangle, \quad \alpha \in \mathbb{C}^n,$$

and X_0 is a real vector field with *real* affine coefficients of the form

$$X_0 = \langle \beta(x), D \rangle, \quad \beta(x) \in \mathbb{R}^n,$$

in which

$$\beta_j(x) = \beta_{j,0} + \sum_{i=1}^n \beta_{j,i} x_i, \quad \beta_{j,i} \in \mathbb{R} \quad \forall i, j = 1, ...n.$$

We again reduce the study of this operator to that of the form (3.1) with $N = 1$, thus we consider the latter.
We assume now hypotheses (H1) to (H3):

(H1) $iX_0 x_1 > 0$;
(H2) there exists a constant $C > 0$ such that

$$|\{X, X_0\}(\xi)|^2 \leqslant C|X(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n;$$

(H3) the vector field $X = \langle \alpha, D \rangle$, $\alpha \in \mathbb{C}^n$, is such that

$$\dim\Big(\mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\alpha), \mathsf{Im}(\alpha)\}\Big) = 1.$$

Recall that hypotheses (H1),(H2) are invariant with respect to affine changes of variables (see Section 2), and the same holds for (H3).
We will show that, under hypotheses (H1) to (H3), the operator P is L^2-locally solvable in the sense od Definition 1.1.
To this aim we prove two preliminary lemmas.

Lemma 5.1. *Let* $X = \langle \alpha, D \rangle$ *and* $X_0 = \langle \beta(x), D \rangle$ *two vector fields on* $\mathbb{R}^n$*,* $n \geqslant 2$*, as before, i.e respectively with constant complex coefficients and with affine real coefficients, and suppose that either* $n > 2$ *and hypothesis (H2) is satisfied or* $n = 2$ *and (H2)-(H3) are satisfied. Then the commutator* $[X, X_0]$ *is of the form*

$$[X, X_0] = \langle \gamma, D \rangle, \quad \gamma \in \mathbb{C}^n,$$

and one has the following inclusion

$$\mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\gamma), \mathsf{Im}(\gamma)\} \subseteq \mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\alpha), \mathsf{Im}(\alpha)\}.$$

Proof. Since X has constant coefficients and X_0 has affine coefficients, it is trivial to see that their commutator is a vector field with constant complex coefficients, thus we pass to proving the inclusion between the subspaces. Recall that hypothesis (H2) is a condition on the principal symbols $X(\xi) = \langle \alpha, \xi \rangle$ and $\{X, X_0\}(\xi) = \langle \gamma, \xi \rangle$ of X and $[X, X_0]$ respectively.
We rephrase (H2) as

$$|\langle \gamma, \xi \rangle|^2 \leqslant C |\langle \alpha, \xi \rangle|^2, \quad \forall \xi \in \mathbb{R}^n,$$

which implies that, if $\langle \alpha, \xi \rangle = 0$ for some $\xi \in \mathbb{R}^n$, then also $\langle \gamma, \xi \rangle = 0$. More precisely, because of (H2), we have that

$$\begin{cases} \langle \mathsf{Re}(\alpha), \xi \rangle = 0 \\ \langle \mathsf{Im}(\alpha), \xi \rangle = 0 \end{cases} \Longrightarrow \begin{cases} \langle \mathsf{Re}(\gamma), \xi \rangle = 0 \\ \langle \mathsf{Im}(\gamma), \xi \rangle = 0 \end{cases},$$

which means exactly

$$\mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\gamma), \mathsf{Im}(\gamma)\}^{\perp} \supseteq \mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\alpha), \mathsf{Im}(\alpha)\}^{\perp} \neq \{0\},$$

where $\mathrm{Span}_{\mathbb{R}}\{\mathsf{Re}(\alpha), \mathsf{Im}(\alpha)\}^{\perp} \neq \{0\}$ because of (H3) when $n = 2$.
Therefore, passing to the orthogonal complements we have the statement. □

Remark 5.2. *Note that if $\Sigma_X = \emptyset$ (recall that Σ_X is the characteristic set of X) then Lemma 5.1 cannot hold except for the one dimensional case.*
In fact if $n = 1$ we have trivially the result since in this case $\mathbb{C} \ni \gamma = \beta'(x)\alpha$, and thus $[X, X_0](D) = \beta'(x)X(D)$, and this is true in general only by requiring that X and X_0 are respectively with constant complex coefficients and with affine real coefficients.
When $n = 2$ then $\Sigma_X \neq \emptyset$ if and only if condition (H3) holds, while when $n > 2$ Σ_X is always non-empty without asking for (H3).

Lemma 5.3. *Consider X, X_0 vector fields as before, and suppose that hypotheses (H2)-(H3) are satisfied. Then there exists a constant $z_0 \in \mathbb{C}$ such that*

$$[X, X_0](D) = z_0 X(D).$$

Proof. Note that, by Lemma 5.1, we have

$$\begin{array}{l} \mathsf{Re}(\gamma) = c_1 \mathsf{Re}(\alpha) + c_2 \mathsf{Im}(\alpha) \\ \mathsf{Im}(\gamma) = c_1' \mathsf{Re}(\alpha) + c_2' \mathsf{Im}(\alpha) \end{array}, \quad c_1, c_2, c_1', c_2' \in \mathbb{R}.$$

Moreover (H3) states that $\mathsf{Re}(\alpha)$ and $\mathsf{Im}(\alpha)$ are two *linearly dependent* vectors in $\mathbb{R}^n$, thus $\mathrm{Im}(\alpha) = c\mathrm{Re}(\alpha)$, for some $c \in \mathbb{R}$, and

$$\begin{array}{l} \mathsf{Re}(\gamma) = (c_1 + c c_2) \mathsf{Re}(\alpha) \\ \mathsf{Im}(\gamma) = (c_1' + c c_2') \mathsf{Re}(\alpha). \end{array}$$

Since $\alpha = (1 + ic)\mathrm{Re}(\alpha)$ then, we get $\gamma = z_0 \alpha$, where $z_0 \in \mathbb{C}$ is explicitly given by $z_0 = \Big((c_1 + cc_2) + i(c_1' + cc_2')\Big)/(1 + ic)$, and therefore $[X, X_0](D) = z_0 \langle \alpha, D \rangle = z_0 X(D)$. □

We can now show that for all $x_0 \in S$ there exist a positive constant C and a compact set K containing x_0 in its interior such that $\|P^*u\| \geqslant C\|u\|$ for all $u \in C_0^\infty(K)$.

Theorem 5.4. *Let P be of the form* (5.1) *such that hypotheses (H1) to (H3) are satisfied, and let S be the zero set of g. Then for all $x_0 \in S$ there exists a compact set $\tilde{K} \subset \mathbb{R}^n$ with $U = \mathring{\tilde{K}}$ and $x_0 \in U$, such that for all $v \in L^2_{loc}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U in the sense of Definition 1.1.*

Proof. First of all, we may assume $g(x) = x_1$ in (5.1). Note also that Proposition 3.1 still holds for an operator P of the form (5.1) (with $g(x) = x_1$), in which we have a unique vector field X with complex constant coefficients. Thus we have that, for all $x_0 \in S$ there exist a compact set K_0, containing x_0 in its interior, and three positive constants $C(K_0), c(K_0), \varepsilon_0(K_0)$ such that, for every compact set $K \subset K_0$

$$\|P^*u\|^2 \geqslant \frac{1}{4}\|X_0u\|^2 + c(P_0u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K),$$

where now P_0 is given by

$$P_0 = X^*|x_1|X - \varepsilon_0^2[X, X_0]^*|x_1|[X, X_0].$$

To prove the theorem it is sufficient to prove the solvability estimate (3.2). Thus, by showing that $(P_0u, u) \geqslant 0$ and by using the Poincaré inequality for X_0, the estimate (3.2) will follow. So we start by looking at

$$\begin{aligned}(P_0u, u) &= \||x_1|^{1/2}Xu\|^2 - \varepsilon_0^2\||x_1|^{1/2}[X, X_0]u\|^2 \\ &\underset{\text{Lemma 5.3}}{=} (1 - \varepsilon_0^2|z_0|^2)\||x_1|^{1/2}Xu\|^2\end{aligned}$$

for all $u \in C_0^\infty(K)$. Since $\varepsilon_0 = \varepsilon_0(K_0)$, and in particular ε_0 shrinks when K_0 is shrunk (see Proposition 3.1), we can then suitably shrink K_0 to a compact set that we still denote by K_0 and which contains x_0 in its interior, so that $\varepsilon_0 \leqslant 1/(2|z_0|^2)$.
Choosing K_0 in this way we have, for all compact $K \subset K_0$, that $(P_0u, u) \geqslant 0$ for all $u \in C_0^\infty(K)$, and moreover

$$\|P^*u\|^2 \geqslant \frac{1}{4}\|X_0u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K).$$

Then one ends the proof using Lemma 4.5 as before. □

6. A MORE GENERAL COMPLEX COEFFICIENTS CASE

Even in the complex coefficients case it is possible to prove a solvability result when P is given in the general form

$$P = \sum_{j=1}^{N} X_j^* g|g| X_j + iX_0 + a_0, \tag{6.1}$$

when $X_j = X_j(D)$ are vector fields with complex constants coefficients, and $X_0 = X_0(x, D)$ is a vector field as before, that is with real affine coefficients, but this time with $X_j g = 0$, $1 \leq j \leq N$. Once more we may reduce matters to the case $g(x) = x_1$.
We assume now the following hypotheses, which we state for $g(x) = x_1$, since they are invariant:

(H1) $iX_0 x_1 > 0$;
(H2) for all $1 \leq j \leq N$ there exists a constant $C > 0$ such that

$$|\{X_j, X_0\}(\xi)|^2 \leq C \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

(H3) $X_j g = X_j x_1 = 0, \quad \forall j = 1, ..., N,$

where (H3) means that each vector field X_j, with $j \neq 0$, is tangent to $S = \{x \in \mathbb{R}^n;\, x_1 = 0\} = g^{-1}(0)$, while (H1) states that X_0 is transverse to S.
Our goal now is to prove the analogue of Theorem 4.6 in this case.
The solvability result still follows by an a priori estimate: for all $x_0 \in S$ there exist a compact set K which contains x_0 in its interior and a positive constant C such that

$$\|P^* u\| \geqslant C\|u\|, \quad \forall u \in C_0^\infty(K).$$

First of all note that the main estimate (3.3) still holds for P even if P has complex coefficients in the second order part, thus we have that for all $x_0 \in S$ there exist a compact set K_0 containing x_0 in its interior and three positive constants $C = C(K_0), c = c(K_0)$ and $\varepsilon_0 = \varepsilon_0(K_0)$, with $\varepsilon_0 \to 0$ as $K_0 \searrow \{x_0\}$, such that for all compact $K \subset K_0$

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 + c(P_0 u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K),$$

where

$$P_0 = \sum_{j=1}^{N} (X_j^* |x_1| X_j - \varepsilon_0^2 [X_j, X_0]^* |x_1| [X_j, X_0]).$$

Since we need to control the term $(P_0 u, u)$ from below we will use hypotheses (H2),(H3) to obtain some useful results to conclude the desired estimate.

Corollary 6.1. *Consider $x_0 \in S$ and K_0 ($x_0 \in \mathring{K}_0$) as in Proposition 3.1. We then can shrink K_0 to a compact set that we keep denoting by K_0, with $x_0 \in \mathring{K}_0$, so that*

$$\varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(\xi)|^2 \leq \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \pi^{-1}(K_0), \tag{6.2}$$

where $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection.

Proof. By condition (H2) we have

$$\sum_{j=1}^{N}|X_j(\xi)|^2-\varepsilon_0^2\sum_{j=1}^{N}|\{X_j,X_0\}(\xi)|^2\geq(1-CN\varepsilon_0^2)\sum_{j=1}^{N}|X_j(\xi)|^2,\quad\forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

By Proposition 3.1 we can shrink K_0 to a compact set, that we keep denoting by K_0, with $x_0\in\mathring{K}_0$, so that $CN\varepsilon_0(K_0)^2\leqslant 1/2$ and (6.2) holds. □

We shall work throughout in the compact set K_0 of Corollary 6.1, and we shall consequentely fix $\varepsilon_0=\varepsilon_0(K_0)$.

Remark 6.2. *Recall that by (H3) we have $X_jg(x)=0$ for each index $1\leqslant j\leqslant N$, where $g(x)=x_1$. Therefore, if we write $\xi=(\xi_1,\xi')$, $\xi'\in\mathbb{R}^{n-1}\setminus\{0\}$, we have $X_j(\xi)=X_j(\xi_1,0)+X_j(0,\xi')=X_j(0,\xi')$. Moreover, since condition (H2) holds, we even have $\{X_j,X_0\}(\xi)=\{X_j,X_0\}(0,\xi')$. Then, by Corollary 6.1,*

$$\varepsilon_0^2\sum_{j=1}^{N}|\{X_j,X_0\}(0,\xi')|^2\leq\sum_{j=1}^{N}|X_j(0,\xi')|^2,\quad\forall\xi'\in\pi_{\xi'}(\pi^{-1}(K_0')),$$

where $\pi_{\xi'}$ is the projection on the component ξ'.

Now we prove the following lemma.

Lemma 6.3. *Consider $x_0\in S$ and K_0 as in Corollary 6.1. Then for all $K\subset K_0$ with $x_0\in\mathring{K}$ we have*

$$(P_0u,u)\geqslant 0,\quad\forall u\in C_0^\infty(K).$$

Proof. Observe that

$$(P_0u,u)\geqslant 0,\quad\forall u\in C_0^\infty(K)$$

is equivalent to

$$\sum_{j=1}^{N}\||x_1|^{1/2}X_ju\|^2-\varepsilon_0^2\sum_{j=1}^{N}\||x_1|^{1/2}[X_0,X_j]u\|^2\geq 0,\quad\forall u\in C_0^\infty(K),\tag{6.3}$$

therefore we prove the latter.
We write $x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1}$. Since for all $K\subset K_0$ we have

$$\begin{aligned}\sum_{j=1}^{N}\||x_1|^{1/2}X_ju\|^2 &= \sum_{j=1}^{N}\int|x_1|\big|X_ju(x_1,x')\big|^2dx\\ &= \sum_{j=1}^{N}\int|x_1|\|X_ju(x_1,\cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})}dx_1,\quad u\in C_0^\infty(K),\end{aligned}$$

to have (6.3) it suffices to prove the pointwise estimate

$$\sum_{j=1}^{N}\|X_ju(x_1,\cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})}\geq\varepsilon_0^2\sum_{j=1}^{N}\|[X_j,X_0]u(x_1,\cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})},\tag{6.4}$$

where x_1 is thought of as a parameter.
Denoting by $\widehat{f}(x_1, \xi')$ the Fourier transform in the x' variable of a function $f(x) = f(x_1, x')$ then, by the Plancherel theorem and (6.2), we get

$$\begin{aligned}
\sum_{j=1}^{N} \|X_j u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})} &= \frac{1}{(2\pi)^{n-1}} \sum_{j=1}^{N} \|\widehat{X_j u}(x_1, \xi')\|^2_{L^2(\mathbb{R}^{n-1}_{\xi'})} \\
&= \frac{1}{(2\pi)^{n-1}} \sum_{j=1}^{N} \int |X_j(0, \xi')|^2 |\widehat{u}(x_1, \xi')|^2 d\xi' \\
&\underset{(6.2)}{\geq} \frac{\varepsilon_0^2}{(2\pi)^{n-1}} \sum_{j=1}^{N} \int |\{X_j, X_0\}(0, \xi')|^2 |\widehat{u}(x_1, \xi')|^2 d\xi' \\
&= \varepsilon_0^2 \sum_{j=1}^{N} \|[X_j, X_0] u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})},
\end{aligned}$$

which is exactly (6.4), whence (6.3) holds. □

Remark 6.4. *Summarizing, since Proposition 3.1 holds in K_0, $x_0 \in \overset{\circ}{K}_0$, $x_0 \in S$, and since we have shrunk K_0 in such a way that Lemma 6.3 holds, then for all $K \subset K_0$ (containing x_0 in its interior) we have that there exists a positive constant $C = C(K_0)$ such that*

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K). \tag{6.5}$$

Now, exactly as in the real case, by applying the Poincaré inequality (4.6) on X_0 we get the solvability estimate.
We have therefore proved the following theorem.

Theorem 6.5. *Let P be of the form* (6.1) *such that hypotheses (H1) to (H3) are satisfied, and let S be the zero set of g. Then for all $x_0 \in S$ there exist a compact set $\tilde{K} \subset \mathbb{R}^n$ with $U = \overset{\circ}{\tilde{K}}$ and $x_0 \in U$, such that for all $v \in L^2_{loc}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U in the sense of Definition 1.1.*

7. A FURTHER MODEL

Inspired by [2], in this final section we will study the solvability of a model operator similar to the previous one, that is

$$P = \sum_{j=1}^{N} X_j^* |f| X_j + i X_0 + a_0, \tag{7.1}$$

where $X_j = X_j(x, D)$, $0 \leq j \leq N$, are homogeneous first order differential operators with smooth coefficients (in other words they are smooth vector fields) defined on an open set $\Omega \subset \mathbb{R}^n$ and with a real principal symbol, $f : \Omega \longrightarrow \mathbb{R}^n$ is a C^1 function with $f^{-1}(0) \neq \emptyset$, and a_0 is a continuous possibly complex valued function.

This model is more "general" in the sense that the vector fields X_j's, $1 \leq j \leq N$, are not necessarily with *constant* coefficients but they are given in general with variable coefficients, and X_0 is not required to be with affine real coefficients but with smooth variable coefficients. Moreover note that, in this case, the coefficients of our operator P could have $C^{0,1}$ or L^∞ regularity depending on the tangency or transversality, respectively, to the zero set of f of the vector fields X_j's, $1 \leq j \leq N$, which is less demanding as far as the regularity of the coefficients in the preceding examples is concerned.
Our purpose is still to prove an L^2 local solvability result in a neighborhood of the zero set of the function f, that we keep denoting by S and which is non-empty by hypothesis.
The method used here is that of Carleman estimates.
We assume now only the following assumption

(H1) $X_0 f \neq 0$ for all $x \in S := f^{-1}(0) \neq \emptyset$.

Theorem 7.1. *Let the operator P in* (7.1) *satisfy hypothesis (H1). Then for all $x_0 \in S$ there exists a compact set $\tilde{K} \subset \Omega$ with $U = \mathring{\tilde{K}}$ and $x_0 \in U$, such that for all $v \in L^2_{loc}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U in the sense of Definition 1.1.*

Proof. We take $e^{2\lambda f}$, where f is the function appearing in P and λ is a real number that we will choose later. Observe that, for all $u \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
2\mathrm{Re}(P^* u, e^{2\lambda f} u) &= \sum_{j=1}^N \mathrm{Re}(X_j^* |f| X_j u, e^{2\lambda f} u) \\
&\quad -\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u) \\
= 2\lambda \sum_{j=1}^N \mathrm{Re}(|f| X_j u, (X_j f) e^{2\lambda f} u) &+ \sum_{j=1}^N \mathrm{Re}(|f| X_j u, e^{2\lambda f} X_j u) \\
&\quad -\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u) \\
= \sum_{j=1}^N \| |f|^{1/2} e^{\lambda f} X_j u\|^2 &+ 2\lambda \sum_{j=1}^N \mathrm{Re}(e^{\lambda f} |f|^{1/2} X_j u, e^{\lambda f} |f|^{1/2} (X_j f) u) \\
&\quad -\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u).
\end{aligned}$$

Furthermore, since $X_0^* = X_0 + d_{X_0}$, where $d_{X_0} = -i\,\mathrm{div}(X_0)$,

$$\begin{aligned}
\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) &= \frac{1}{2}\big[(iX_0^* u, e^{2\lambda f} u) + (e^{2\lambda f} u, iX_0^* u)\big] \\
&= \frac{1}{2}\big[2\lambda(u, (-iX_0 f) e^{2\lambda f} u) + (e^{2\lambda f} u, -iX_0 u) \\
&+ (e^{2\lambda f} u, iX_0 u) + (e^{2\lambda f} u, i d_{X_0} u)\big] \\
&= \lambda\big((iX_0 f) e^{\lambda f} u, e^{\lambda f} u\big) + \frac{1}{2}(e^{\lambda f} u, i e^{\lambda f} d_{X_0} u),
\end{aligned}$$

and therefore

$$2\mathrm{Re}(P^*u, e^{2\lambda f}u) \geq \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2$$
$$-2|\lambda| \sum_{j=1}^{N} \Big| \mathrm{Re}(e^{\lambda f}|f|^{1/2} X_j u, |f|^{1/2} e^{\lambda f}(X_j f)u) \Big| - \lambda\big((iX_0 f)e^{\lambda f}u, e^{\lambda f}u\big)$$
$$-\frac{1}{2}(e^{\lambda f}u, ie^{\lambda f} d_{X_0} u) - \|a_0\|\|e^{\lambda f}u\|^2, \quad \forall u \in C_0^\infty(\Omega).$$

Recall that (H1) states that $iX_0 f \neq 0$ on S, which yields that for all $x_0 \in S$ there exists a compact $K_0 \subset \Omega$ containing x_0 in its interior such that $iX_0 f \neq 0$ on K_0, and in particular it has a constant positive or negative sign in K_0. Hence, if $iX_0 f > 0$ in K_0, then we choose λ negative so that $-\lambda(iX_0 f) = |\lambda||iX_0 f| > c_0$ in K_0 for some positive constant c_0, otherwise if $iX_0 f < 0$ in K_0 we choose λ positive so that $-\lambda(iX_0 f) = |\lambda||iX_0 f| > c_0$ in K_0. Thus, by choosing λ having the appropriate sign, we have

$$2\mathrm{Re}(P^*u, e^{2\lambda f}u) \geq \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2 - \delta|\lambda| \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2$$
$$-\frac{|\lambda|}{\delta} \||f|^{1/2}\|^2_{L^\infty(K_0)} \sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(K_0)} \|e^{\lambda f}u\|^2 + c_0|\lambda|\|e^{\lambda f}u\|^2$$
$$-\|d_{X_0}\|_{L^\infty(K_0)}\|e^{\lambda f}u\|^2 - \|a_0\|_{L^\infty(K_0)}\|e^{\lambda f}u\|^2$$
$$= (1 - \delta|\lambda|) \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2 + |\lambda|\Big(c_0$$
$$-\frac{1}{\delta}\||f|^{1/2}\|^2_{L^\infty(K_0)} \sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(K_0)} - \frac{\|d_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{|\lambda|}\Big)\|e^{\lambda f}u\|^2,$$

for all $u \in C_0^\infty(K_0)$. Now we fix $\lambda := \lambda_0$ (with the sign previously chosen) such that $|\lambda_0|$ is so big that

$$c_0 - \frac{\|d_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{|\lambda|} \geq \frac{c_0}{2}.$$

In addition we choose $\delta := 1/(2|\lambda_0|)$ so that

$$\mathrm{Re}(P^* u, e^{2\lambda_0 f} u) \geq \frac{1}{2}\sum_{j=1}^{N} \||f|^{1/2} e^{\lambda_0 f} X_j u\|^2 + |\lambda_0|\Big(\frac{c_0}{2}$$

$$-2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(K_0)} \sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(K_0)}\Big)\|e^{\lambda_0 f} u\|^2$$

$$\geq |\lambda_0|\Big(\frac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(K_0)} \sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(K_0)}\Big)\|e^{\lambda_0 f} u\|^2,$$

for all $u \in C_0^\infty(K_0)$, and in particular for all $u \in C_0^\infty(K)$ for every compact $K \subset K_0$.

Since $x_0 \in K_0$ and $f(x_0) = 0$, we can find a compact set $\tilde{K} \subset K_0$ sufficiently small and containig x_0 in its interior such that

$$\begin{aligned} &\tfrac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(\tilde{K})} \textstyle\sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(\tilde{K})} \\ &\geq \tfrac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(\tilde{K})} \textstyle\sum_{j=1}^{N} \|X_j f\|^2_{L^\infty(K_0)} \geq \tfrac{c_0}{4}, \end{aligned}$$

whence

$$|\mathrm{Re}(P^* u, e^{2\lambda_0 f} u)| \geq \mathrm{Re}(P^* u, e^{2\lambda_0 f} u) \geq |\lambda_0| \frac{c_0}{4} \|e^{\lambda_0 f} u\|^2, \quad \forall u \in C_0^\infty(\tilde{K}),$$

and finally

$$e^{2\lambda_0 \|f\|_{L^\infty(\tilde{K})}} \|P^* u\| \|u\| \geq |\lambda_0| \frac{c_0}{4} e^{-2\lambda_0 \|f\|_{L^\infty(\tilde{K})}} \|u\|^2, \quad \forall u \in C_0^\infty(\tilde{K}).$$

In conclusion, we have shown that for all $x_0 \in S$ there exist a compact set $\tilde{K}$ containing x_0 in its interior and a positive constant (depending on the compact $\tilde{K}$) $C = |\lambda_0| \frac{c_0}{4} e^{-4\lambda_0 \|f\|_{L^\infty(\tilde{K})}}$ such that

$$\|P^* u\| \geq C\|u\|, \quad \forall u \in C_0^\infty(\tilde{K}).$$

The latter inequality is exactly the solvability estimate (3.2) that we were searching for, thus the proof follows directly by once more using standard functional analysis arguments. □

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