ABOUT THE EXISTENCE OF AN ALT-CAFFARELLI-FRIEDMAN MONOTONICITY FORMULA IN THE HEISENBERG GROUP

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Abstract: We introduce some necessary conditions about the existence of an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group.

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1 INTRODUCTION

In free boundary problems, Alt-Caffarelli-Friedman monotonicity formula introduced in [1] plays a fundamental role in studying regularity properties of the solutions of two-phase free boundary problems.

In $\mathbb{R}^3$ the result states that there exists $r_0 > 0$ such that for every non-negative $u_1, u_2 \in C(B_1(0)) \cap H^1(B_1(0))$, if $\Delta u_i \geq 0$, $i = 1, 2$, $u_1(0) = u_2(0) = 0$ and $u_1 u_2 = 0$ in $B_1(0)$, where $B_1(0)$ is the Euclidean ball centered at 0 of radius 1 in $\mathbb{R}^3$, then

$$\Phi(r) := r^{-4} \int_{B_r(0)} \frac{|\nabla u_1(x)|^2}{|x|} dx \int_{B_r(0)} \frac{|\nabla u_2(x)|^2}{|x|} dx \tag{1}$$

is well defined, bounded and monotone increasing in $[0, r_0)$.

In [1] the authors applied monotonicity formula for proving the Lipschitz continuity of critical points to a functional like the following one

$$\mathcal{E}(v) := \int_\Omega \left(|\nabla v|^2 + \chi_{\{v>0\}}\right) dx \tag{2}$$

defined on a set $K \subset H^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a given bounded open set, K is determined by some given conditions on v on $\partial\Omega$ and $\chi_{\{v>0\}}$ denotes the characteristic function of the set $\{v > 0\}$.

The critical points of the previous functional $\mathcal{E}$ satisfy the following two-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta u = 0 & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{3}$$

The interested reader may find in [6] further details about this result, as well as in [12].

Few important topics about this kind of monotonicity formula can be found in [4], [5], [20] and [19].

In this note, we introduce a necessary condition about the existence of a monotonicity formula in the Heisenberg group, where the intrinsic laplacian is given by a degenerate elliptic operator with variable coefficients, see [12] and [13]. We did not obtain a monotonicity formula yet, but we proved that if such result holds, it has to fit the hypotheses stated in the following Theorem 1, see Section 3. In fact, in the long run, possibly knowing a monotonicity form in this framework, it will be possible to face the study of free boundary problems in this setting following the strategy introduced in [3], [7] and then continued in [8], [9]. In order to correctly state our result, in Section 2 we introduce the main tools useful in the Heisenberg group for working on this subject and in Section 3 we state our main result.

2 THE HEISENBERG FRAMEWORK

We denote by $\mathbb{H}^1$ the set $\mathbb{R}^3$ endowed with the following non-commutative inner law, in such a way that for every $P \equiv (x_1, y_1, t_1) \in \mathbb{R}^3$, $M \equiv (x_2, y_2, t_2) \in \mathbb{R}^3$:

$$P \circ M := (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2y_1 - x_1y_2)).$$

Let $X = (1, 0, 2y)$ and $Y = (0, 1, -2x)$.

We use the same symbols to denote the vector fields associated with the previous vectors, so that:

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t.$$

The commutator between the vector fields is

$$[X, Y] = -4\partial_t.$$

The intrinsic gradient of a smooth function u in a point P is

$$\nabla_{\mathbb{H}^1} u(P) = Xu(P)X(P) + Yu(P)Y(P).$$

There exists a unique metric on $H\mathbb{H}^1_P = \text{span}\{X(P), Y(P)\}$ which makes orthonormal the set of vectors $\{X, Y\}$. Thus, for every $P \in \mathbb{H}^1$ and for every $U, V \in H\mathbb{H}^1_P$, $U = \alpha_1 X(P) + \beta_1 Y(P)$, $V = \alpha_2 X(P) + \beta_2 Y(P)$, we have

$$\langle U, V\rangle = \alpha_1\alpha_2 + \beta_1\beta_2.$$

In particular, we have a norm associated with the metric on the space $H\mathbb{H}^1_P$ which is, for every $U \in H\mathbb{H}^1_P$:

$$\mid U \mid = \sqrt{\alpha_1^2 + \beta_1^2}.$$

Hence, the norm of the intrinsic gradient of a smooth function u in P is

$$\mid \nabla_{\mathbb{H}^1} u(P) \mid = \sqrt{(Xu(P))^2 + (Yu(P))^2}.$$

Moreover, if $\nabla_{\mathbb{H}^1} u(P) \neq 0$, then

$$\left| \frac{\nabla_{\mathbb{H}^1} u(P)}{\mid \nabla_{\mathbb{H}^1} u(P) \mid} \right| = 1.$$

On the contrary, if $\nabla_{\mathbb{H}^1} u(P) = 0$ we say that the point P is characteristic for the smooth surface $\{u = u(P)\}$. Hence, for every point $M \in \{u = u(P)\}$, which is not characteristic, that is $\nabla_{\mathbb{H}^1} u(P) \neq 0$, it is well defined the intrinsic normal to the surface $\{u = u(P)\}$:

$$\nu(M) = \frac{\nabla_{\mathbb{H}^1} u(M)}{\mid \nabla_{\mathbb{H}^1} u(M) \mid}.$$

In the Heisenberg group $\mathbb{H}^1$ the following gauge norm is defined:

$$|(x, y, t)|_{\mathbb{H}^1} := \sqrt[4]{(x^2 + y^2)^2 + t^2}.$$

The function $d_K : \mathbb{H}^1 \times \mathbb{H}^1 \to [0, \infty[$, such that for every $P, T \in \mathbb{H}^1$

$$d_K(P, T) = |P^{-1} \circ T|_{\mathbb{H}^1},$$

is a distance on the Heisenberg group $\mathbb{H}^1$. It is well known as the Koranyi distance. This distance is left invariant, that is for every $P, T, R \in \mathbb{H}^1$

$$d_K(R \circ P, R \circ T) = d_K(P, T).$$

In particular, it results that, see [11], $\Gamma(P,R) = c\left|P^{-1}\circ R\right|_{\mathbb{H}^1}^{-2}$ is the fundamental solution of the sub-laplacian $\Delta_{\mathbb{H}^1}$. We refer to [2] for a comprehensive discussion of the subject as well.

Concerning the natural Sobolev spaces to consider in the Heisenberg group $\mathbb{H}^1$, we refer to the literature, see for instance [18]. We simply recall that

$$\mathcal{L}^{1,2}(\Omega) := \{f \in L^2(\Omega) : Xf,\ Yf \in L^2(\Omega)\}$$

is a Hilbert space with respect to the norm

$$|f|_{\mathcal{L}^{1,2}(\Omega)} = \left(\int_\Omega ((Xf)^2 + (Yf)^2 + |f|^2)dx\right)^{\frac{1}{2}}.$$

Moreover

$$H^1_{\mathbb{H}^1}(\Omega) = \overline{C^\infty(\Omega)\cap\mathcal{L}^{1,2}(\Omega)}^{|\cdot|_{\mathcal{L}^{1,2}(\Omega)}}.$$

See [18], [15], [17], [16] for a detailed presentation of this topic.

3 MAIN RESULTS

Let

$$\mathcal{E}_{\mathbb{H}^1}(v) := \int_\Omega \left(|\nabla_{\mathbb{H}^1} v|^2 + \chi_{\{v>0\}}\right) dx,$$

$\Omega \subset \mathbb{H}^1$, be the functional associated with the intrinsic functional governing a free boundary problem in an appropriate subset of $K \subset H^1_{\mathbb{H}^1}(\Omega)$ determined by fixing some conditions on the boundary of Ω. Then, see [10] and [14], the variation domain solutions satisfy the following problem:

$$\begin{cases} \Delta_{\mathbb{H}^1} u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega :\ u(x) > 0\}, \\ \Delta_{\mathbb{H}^1} u = 0 & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega :\ u(x) \le 0\}), \\ |\nabla_{\mathbb{H}^1} u^+|^2 - |\nabla_{\mathbb{H}^1} u^-|^2 = 1 & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{4}$$

As a consequence, it is natural to consider, as a candidate for an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group, the following function:

$$J_{\beta,\mathbb{H}^1}(r) := r^{-\beta}\int_{B_r^{\mathbb{H}^1}(0)} \frac{|\ \nabla_{\mathbb{H}^1} u^+\ |^2}{|\zeta|^2_{\mathbb{H}^1}} d\zeta \int_{B_r^{\mathbb{H}^1}(0)} \frac{|\ \nabla_{\mathbb{H}^1} u^-\ |^2}{|\zeta|^2_{\mathbb{H}^1}} d\zeta, \tag{5}$$

where $\beta > 0$ is a suitable fixed exponent and $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$, being $0 \in \mathcal{F}(u)$.

In particular, we obtained the following result, see [12].

Theorem 1 *If there exists a positive number β for which $J_{\beta,\mathbb{H}^1}$ is monotone increasing for every $u_1, u_2 \in H^1_{\mathbb{H}^1}(B_1^{\mathbb{H}^1}(0))$, such that $\Delta_{\mathbb{H}^1} u_i \ge 0$, $u_i(0) = 0$, $i = 1, 2$ and $u_1u_2 = 0$, then $\beta \le 4$.*

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