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SIMPLICIAL VOLUME VIA NORMALISED CYCLES

CLARA LÖH AND MARCO MORASCHINI

ABSTRACT. We show that the Connes-Consani semi-norm on singular homology with real coefficients, defined via s -modules, coincides with the ordinary ℓ^1 -semi-norm on singular homology in all dimensions.

1. INTRODUCTION

Connes and Consani introduced a semi-norm on singular homology via s -modules and established that this semi-norm is equivalent to the ℓ^1 -semi-norm defined by Gromov [CC20]. Moreover, they proved that their semi-norm is equal to the ℓ^1 -semi-norm in the case of surfaces [CC20, Theorem 1.4], using a delicate construction specific to surfaces. In this note, we show that the two semi-norms agree in *all* dimensions, thereby confirming and extending a conjecture of Connes and Consani [CC20, p. 4]:

Theorem 1.1. *Let X be a topological space, let $n \in \mathbb{N}$, let $\alpha \in H_n(X; \mathbb{R})$, and let $\lambda \in \mathbb{R}_{>0}$. Then $\|\alpha\|_1 < \lambda$ if and only if α lies in the image of the canonical map $H_n(X; \|H\mathbb{R}\|_\lambda) \rightarrow H_n(X; \mathbb{R})$.*

In particular, the simplicial volume of closed manifolds can also be expressed in terms of homology of s -modules.

As explained by Connes and Consani, in order to show Theorem 1.1 it suffices to prove that the ℓ^1 -semi-norm on singular homology can be computed via *normalised* singular cycles (see Section 2.3):

Proposition 1.2. *Let X be a topological space and let $n \in \mathbb{N}$. Then, for all $\alpha \in H_n(X; \mathbb{R})$, we have*

$$\|\alpha\|_1 = \|\alpha\|_1^{\text{norm}}.$$

In Section 2, we recall basic definitions and notation. The proof of Proposition 1.2 is given in Section 3, based on a symmetrisation construction.

2. THE (NORMALISED) ℓ^1 -SEMI-NORM

2.1. The singular chain complex. Let $n \in \mathbb{N}$ and let Δ^n be the standard n -simplex. For $j \in \{0, \dots, n\}$, we denote by $\iota_j^n: \Delta^{n-1} \rightarrow \Delta^n$ the affine inclusion of the j -th facet of Δ^n .

Given a topological space X , we consider the singular simplicial set $S(X)$: For $n \in \mathbb{N}$, we have $S_n(X) := \text{map}(\Delta^n, X)$ and for $j \in \{0, \dots, n\}$, the face

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maps $\partial_j: S_n(X) \rightarrow S_{n-1}(X)$ are given by

$$\partial_j(\sigma) := \sigma \circ \iota_j^n$$

for all $\sigma \in \text{map}(\Delta^n, X)$. The singular chain complex $C_\bullet(X; \mathbb{R})$ with real coefficients is the free \mathbb{R} -chain complex associated with $S(X)$.

Furthermore, we have the *Moore normalisation* $NC_\bullet(X; \mathbb{R})$ of $C_\bullet(X; \mathbb{R})$, given by the submodules

$$NC_n(X; \mathbb{R}) := \bigcap_{j=0}^{n-1} \ker(\partial_j) \subseteq C_n(X; \mathbb{R})$$

and the boundary maps $d := \partial_n: NC_n(X; \mathbb{R}) \rightarrow NC_{n-1}(X; \mathbb{R})$.

Definition 2.1 (normalised chain). A singular chain $c \in C_n(X; \mathbb{R})$ is *normalised* if it lies in the submodule $NC_n(X; \mathbb{R})$.

2.2. The ℓ^1 -semi-norms. We briefly recall Gromov's ℓ^1 -semi-norm on singular homology [Gro82]: The ℓ^1 -norm $\|\cdot\|_1$ on $C_n(X; \mathbb{R})$ associated with the basis $S_n(X)$ induces a semi-norm on $H_n(X; \mathbb{R})$, the ℓ^1 -*semi-norm*, which we will also denote by $\|\cdot\|_1$.

Following Connes and Consani [CC20], one can also endow $H_n(X; \mathbb{R})$ with the semi-norm induced by the ℓ^1 -norm on the normalised complex $NC_\bullet(X; \mathbb{R})$: For $\alpha \in H_n(X; \mathbb{R})$ one sets

$$\|\alpha\|_1^{\text{norm}} := \inf\{\|c\|_1 \mid c \in C_n(X; \mathbb{R}) \text{ is a normalised cycle representing } \alpha\}.$$

Connes and Consani prove that the two semi-norms are equivalent [CC20, Lemma 3.4], namely, for every $\alpha \in H_n(X; \mathbb{R})$, we have

$$\|\alpha\|_1 \leq \|\alpha\|_1^{\text{norm}} \leq \max(1, 2^{n-1}) \cdot \|\alpha\|_1.$$

Proposition 1.2 states that they are in fact equal.

2.3. Deriving Theorem 1.1 from Proposition 1.2. Connes and Consani introduce a filtration of the s -module $H\mathbb{R}$ by a family $(\|H\mathbb{R}\|_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ of sub- s -modules and, for topological spaces X , associated singular homology objects $(H_n(X; \|H\mathbb{R}\|_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ [CC20]. Moreover, these come with canonical maps

$$\varrho_{n,\lambda}: H_n(X; \|H\mathbb{R}\|_\lambda) \rightarrow H_n(X; \mathbb{R})$$

to the singular homology of X [CC20, Section 3.4]. This filtration defines a semi-norm on $H_n(X; \mathbb{R})$ that is equivalent to $\|\cdot\|_1$ [CC20, Corollary 3.6]. More precisely, the image of $\varrho_{n,\lambda}$ coincides with the set of elements $\alpha \in H_n(X; \mathbb{R})$ with $\|\alpha\|_1^{\text{norm}} < \lambda$ [CC20, Theorem 3.5]. Therefore, Theorem 1.1 is a direct consequence of Proposition 1.2.

3. PROOF OF PROPOSITION 1.2

3.1. Symmetrisation of chains. We recall the *symmetrisation map* on singular chains, which is given by averaging singular simplices over all vertex-permutations of the standard simplex: In the following, let X be a topological space and $n \in \mathbb{N}$. Let Σ_{n+1} denote the symmetric group on $\{0, \dots, n\}$ and $\text{sgn}: \Sigma_{n+1} \rightarrow \{\pm 1\}$ the sign function. For a map $\pi: \{0, \dots, k\} \rightarrow \{0, \dots, n\}$, we write $\Delta(\pi) := [\pi(0), \dots, \pi(k)]: \Delta^k \rightarrow \Delta^n$ for the affine map that extends the map π on the vertices.

Definition 3.1 (symmetrisation map). The *symmetrisation map*

$$\text{symm}_n: C_n(X; \mathbb{R}) \rightarrow C_n(X; \mathbb{R})$$

is the \mathbb{R} -linear map defined on each singular n -simplex σ as

$$\text{symm}_n(\sigma) := \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi) \cdot \sigma \circ \Delta(\pi).$$

Lemma 3.2 ([FM11, Lemma 2.6]). *The symmetrisation map symm_\bullet is a chain map $C_\bullet(X; \mathbb{R}) \rightarrow C_\bullet(X; \mathbb{R})$ that is chain homotopic to the identity. Moreover, for all $c \in C_n(X; \mathbb{R})$, we have*

$$\|\text{symm}_n(c)\|_1 \leq \|c\|_1.$$

For us, the key observation is that symmetrisation enforces normalisation on cycles:

Lemma 3.3 (normalisation via symmetrisation).

(1) *For all $j \in \{0, \dots, n\}$, we have*

$$\partial_j \circ \text{symm}_n = (-1)^j \cdot \partial_0 \circ \text{symm}_n.$$

(2) *In particular: If $c \in C_n(X; \mathbb{R})$ is a cycle, then $\partial_j(\text{symm}_n(c)) = 0$ for all $j \in \{0, \dots, n\}$.*

Proof. *Ad 1.* Using the cyclic permutation $\tau_j := (j \ j-1 \ \dots \ 1 \ 0) \in \Sigma_{n+1}$, we can re-write $\partial_j \circ \text{symm}_n$ as follows: Each permutation $\pi \in \Sigma_{n+1}$ satisfies

$$\begin{aligned} \Delta(\pi) \circ \iota_j^n &= [\pi(0), \dots, \pi(j-1), \pi(j+1), \dots, \pi(n)] \\ &= [\pi \circ \tau_j(1), \dots, \pi \circ \tau_j(j), \pi \circ \tau_j(j+1), \dots, \pi \circ \tau_j(n)] \\ &= \Delta(\pi \circ \tau_j) \circ \iota_0^n. \end{aligned}$$

Therefore, for all singular n -simplices σ on X we have

$$\begin{aligned} \partial_j \circ \text{symm}_n(\sigma) &= \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \text{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi \circ \tau_j) \circ \iota_0^n \\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\eta \in \Sigma_{n+1}} \text{sgn}(\eta) \cdot \sigma \circ \Delta(\eta) \circ \iota_0^n \\ &= (-1)^j \cdot \partial_0 \circ \text{symm}_n(\sigma). \end{aligned}$$

Ad 2. As symm_\bullet is a chain map (Lemma 3.2), if $c \in C_n(X; \mathbb{R})$ is a cycle, then $\text{symm}_n(c)$ is a cycle, and in combination with the first part we see that

$$\begin{aligned} 0 &= \partial(\text{symm}_n(c)) = \sum_{j=0}^n (-1)^j \cdot \partial_j(\text{symm}_n(c)) = \sum_{j=0}^n (-1)^{2j} \cdot \partial_0(\text{symm}_n(c)) \\ &= (n+1) \cdot \partial_0(\text{symm}_n(c)). \end{aligned}$$

Therefore, $\partial_0(\text{symm}_n(c)) = 0$. Applying the first part once more shows that $\partial_j(\text{symm}_n(c)) = 0$ for all $j \in \{0, \dots, n\}$. \square

3.2. Proof of Proposition 1.2. We already know that $\|\alpha\|_1 \leq \|\alpha\|_1^{\text{norm}}$ for every $\alpha \in H_n(X; \mathbb{R})$. Let us prove the opposite inequality. Let $c \in C_n(X; \mathbb{R})$ be a cycle representing $\alpha \in H_n(X; \mathbb{R})$. Then, we can consider $\text{symm}_n(c) \in C_n(X; \mathbb{R})$. By Lemma 3.2 we know that $\text{symm}_n(c)$ is homologous to c and satisfies $\|\text{symm}_n(c)\|_1 \leq \|c\|_1$. Moreover, Lemma 3.3 implies that $\text{symm}_n(c)$ is normalised. This shows that

$$\|\alpha\|_1^{\text{norm}} \leq \|\text{symm}_n(c)\|_1 \leq \|c\|_1.$$

Taking the infimum over all cycles representing α completes the proof.

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