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Parabolic De Giorgi classes with doubly nonlinear, nonstandard growth: local boundedness under exact integrability assumptions

Simone Ciani [a,*], Eurica Henriques [b], Mariia O. Savchenko [c], Igor I. Skrypnik [c]

[a] *Università di Bologna Alma Mater, Piazza Porta San Donato 5, Italy*
[b] *Centro de Matemática CMAT; Polo CMAT-UTAD Departamento de Matemática Universidade de Trás-os-Montes e Alto Douro, Vila Real, Portugal*
[c] *Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Gen. Batiouk Str. 19, 84116 Sloviansk, Ukraine*

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Abstract

We define a suitable class $\mathcal{PDG}$ of functions bearing unbalanced energy estimates, that are embodied by local weak subsolutions to doubly nonlinear, double-phase, Orlicz-type and fully anisotropic operators. Then we prove that members of $\mathcal{PDG}$ are locally bounded, under critical, sub-critical and limit growth conditions typical of singular and degenerate parabolic operators, with quantitative point-wise estimates that follow the lines of the pioneering work of Ladyzhenskaya, Solonnikov and Uraltseva [31]. These local bounds are new in the critical case and sub-critical cases, and have been obtained without any qualitative boundedness assumption. In particular, our proof of local boundedness in the critical case is valid disregarding of any additional integrability conditions and covers both the classical p-Laplacian and the porous medium equations.

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[*] Corresponding author.

E-mail addresses: simone.ciani3@unibo.it (S. Ciani), eurica@utad.pt (E. Henriques), shan_maria@ukr.net (M.O. Savchenko), ihor.skrypnik@gmail.com (I.I. Skrypnik).

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1. Introduction

In [14] Ennio De Giorgi proved that the Hölder continuity of weak solutions to

$$\sum_{i,j} D_i(a_{i,j}(x)D_j u)=0,$$

for $a_{i,j}$ measurable and bounded coefficients, is a property inherent to their energy, i.e. the satisfaction of Caccioppoli inequalities, rather than to the fact that u solves an equation. This concept has gained a parabolic flavor in the pioneering work [31], where the authors identify a suitable energy class $\mathcal{B}_2(\Omega_T)$, for $\Omega_T=\Omega\times(0,T]$, consisting of functions

$$u\in C_{loc}(0,T;L^2_{loc}(\Omega))\cap L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega)),\qquad \Omega\subset\mathbb{R}^N,$$

satisfying, for a fixed positive constant c_1, the energy inequalities

$$\begin{aligned}\mathcal{E}_2 = &\sup_{(\tau-\eta,\,\tau)}\int\limits_{K_\rho}(u-k)^2_\pm\zeta^2(x,t)\,dx+\int\limits_{\tau-\eta}^{\tau}\int\limits_{K_\rho}|\nabla[(u-k)_\pm\zeta]|^2\,dxdt\\ &\le c_1\int\limits_{K_\rho}(u-k)^2_\pm\,\zeta^2(x,\tau-\eta)\,dx+c_1\int\limits_{\tau-\eta}^{\tau}\int\limits_{K_\rho}(u-k)^2_\pm\Big[|\zeta_t|+|D\zeta|^2\Big]dxdt,\end{aligned}\tag{1.1}$$

for any $k\in\mathbb{R}$, any cylinder $K_\rho\times(\tau-\eta,\tau)\subset\Omega_T$ with $\rho,\eta>0$, and any suitable piecewise smooth cutoff function ζ vanishing on ∂K_ρ (see also [16,17,23,30] and references therein). In order to understand the extent of this definition, we observe that weak solutions to parabolic equations with measurable and bounded coefficients

$$u_t-\operatorname{div}(A(x,t)\nabla u)=0,$$

with $A(x,t)=(a_{i,j}(x,t))_{i,j=1}^N$ measurable and satisfying $\lambda|\xi|^2\le\xi^T A(x,t)\xi\le\Lambda|\xi|^2$, $0<\lambda\le\Lambda$, are members of $\mathcal{B}_2(\Omega_T)$. More in general the parabolic quasi-minima, as introduced by [39], which are functions

$$u\in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega)),$$

satisfying for some $Q\ge1$ the variational inequality

$$
\begin{aligned}
-\iint_Q u\,\phi_t\,dxdt + \iint_Q F(x,t,u,\nabla u)\,dxdt, \qquad &\text{being} \qquad Q=\operatorname{supp}(\phi)\,,\\
\le Q \iint_Q F(x,t,u-\phi,\nabla u-\nabla\phi)\,dxdt\,,
\end{aligned}
$$

with suitable assumptions on F as simply $\lambda|\xi|^2 \le F(x,t,u,\xi) \le \Lambda|\xi|^2$, can be shown to belong to $\mathcal{B}_2(\Omega_T)$. Hence, any property derived for the members of $\mathcal{B}_2(\Omega_T)$ passes to local weak solutions, minima, and parabolic quasi-minima.

Now, when the growth of the operator is unbalanced, as in the case of the parabolic p-Laplacian

$$u_t - \operatorname{div}(A(x,t)|\nabla u|^{p-2}\nabla u) = 0,$$

the energy estimates $\mathcal{E}_p$ suffer from an inhomogeneity on the parabolic and elliptic energy term:

$$
\begin{aligned}
\mathcal{E}_p = &\sup_{(\tau-\eta,\,\tau)} \int_{K_\rho} (u-k)_\pm^2\zeta^2(x,t)\,dx + \int_{\tau-\eta}^{\tau}\int_{K_\rho} |\nabla[(u-k)_\pm\zeta]|^p\,dxdt\\
&\le c_2\int_{K_\rho}(u-k)_\pm^2\,\zeta^2(x,\tau-\eta)\,dx + c_2\int_{\tau-\eta}^{\tau}\int_{K_\rho}(u-k)_\pm^2\Big[|\zeta_t|+|\nabla\zeta|^p\Big]dxdt, \qquad c_2>0,
\end{aligned}
\tag{1.2}
$$

and still nowadays a complete regularity theory for the class of functions satisfying these inequalities is not available, since more energy inequalities tied to the structure of the equation are needed (see [15], [16]) to control the oscillation, while a complete study of the local boundedness is still possible. The local boundedness of weak solutions, together with quantitative bounds on the essential suprema, is the starting point for the study of higher interior regularity such as $C^{0,\alpha}$, $C^{1,\alpha}$ interior regularity, Harnack-type inequalities and much more (see, for example [16,17,26,30,31]).

In this work we generalize this principle to operators that have a very wild unbalance in the energy, as for instance the doubly nonlinear parabolic equation with standard growth

$$u_t - div\Big(|\nabla u^m|^{p-2}\nabla u^m\Big) = 0, \quad m>0, \tag{1.3}$$

the doubly nonlinear parabolic equation with generalized Orlicz growth φ (see subsection 4.2)

$$u_t - div\Big(\varphi(x,t,u,|\nabla u^m|)\frac{\nabla u^m}{|\nabla u^m|}\Big) = 0, \tag{1.4}$$

and the doubly nonlinear anisotropic parabolic equation

$$u_t - \sum_{i=1}^{N} D_i\Big(u^{(m_i-m)(p_i-1)}|D_i u^m|^{p_i-2}D_i u^m\Big) = 0\,, \tag{1.5}$$

as well as their quasi-linear generalizations with bounded and measurable coefficients. In connection with the growth of these equations, we consider sets of numbers $\{p_i\}_{i=1}^N$, $\{q_i\}_{i=1}^N$, $\{m_i\}_{i=1}^N$, $\{n_i\}_{i=1}^N$, where the p_i's and q_i's are linked by

$$1 < p_i \leqslant q_i \quad \forall i = 1, 2, \dots, N,$$

and are meant to reflect the unbalance of the operator in its elliptic structure - such as double-phase (p, q) growth, fully anisotropic, Orlicz -, while m_i's and n_i's are meant to represent the unbalance offered from the point of view of the time derivative, when considering equation (1.3) as

$$(u^l)_t - \Delta_p u = 0\,, \qquad l = 1/m\,.$$

In this framework, the parameters q_i and n_i describe the generalized growth behavior of the operator, complementing the standard spatial and temporal structure captured by p_i and m_i, and may reflect, for example, double-phase or generalized Orlicz-type growth (see Subsection 4.2). We define the numbers

$$0 < m := \min(m_1, m_2, ..., m_N, n_1, n_2, ..., n_N), \qquad q := \max(q_1, ..., q_N), \tag{1.6}$$

the parabolic anisotropic energy spaces in a cylindrical domain $\Omega_T = \Omega \times (0, T]$ as

$$\begin{aligned}\mathcal{V}_{loc}(\Omega_T) := \Big\{ & u : \Omega_T \to \mathbb{R} \quad \text{measurable and such that for all } i = 1, \dots, N, \\ & u^{\frac{m_i(p_i-1)+m}{p_i}}, u^{\frac{n_i(q_i-1)+m}{p_i}}, D_i\big(u^{\frac{m_i(p_i-1)+m}{p_i}}\big) \in L^{p_i}(0, T; L^{p_i}_{loc}(\Omega))\Big\},\end{aligned} \tag{1.7}$$

and the geometric configurations

$$Q_{\vec{r},\eta}(y,\tau) := K_{\vec{r}}(y) \times (\tau - \eta, \tau), \quad \text{with} \quad K_{\vec{r}}(y) := \prod_{i=1}^N \Big\{ |x_i - y_i| < r_i, \Big\}. \tag{1.8}$$

With this notation, we can finally define the suitable parabolic De Giorgi class in the cylinder Ω_T, with $\Omega \subset \mathbb{R}^N$ open and bounded.

Definition 1.1. A non-negative function u belongs to De Giorgi's class $\mathcal{PDG}^+(\Omega_T, C)$ if

$$u \in C_{loc}(0, T; L^{1+m}_{loc}(\Omega)) \cap \mathcal{V}_{loc}(\Omega_T),$$

and for every $k > 0$, every cylinder $Q_{8\vec{r},8\eta}(y,\tau) \subset \Omega_T$, and every piecewise smooth, cutoff function $\zeta(x,t)$ vanishing on $\partial K_{\vec{r}}(y)$ and such that $0 \leqslant \zeta(x,t) \leqslant 1$ the inequality

$$\mathcal{E}^m_p = \sup_{\tau-\eta\leqslant t\leqslant\tau} \int\limits_{K_{\vec{r}}(y)} g(u^m, k^m)\, \zeta^q \, dx + \frac{1}{C} \sum_{i=1}^N \iint\limits_{Q_{\vec{r},\eta}(y,\tau)} u^{(m_i-m)(p_i-1)} |D_i(u^m - k^m)_+|^{p_i} \zeta^q \, dx\, dt$$

$$\leqslant \int\limits_{K_{\vec{r}}(y)\times\{\tau-\eta\}} g(u^m,k^m)\,\zeta^q\,dx + C \iint\limits_{Q_{\vec{r},\eta}(y,\tau)} g(u^m,k^m)\,|\zeta_t|\,dxdt+$$

$$+C\sum_{i=1}^{N}\iint\limits_{Q_{\vec{r},\eta}(y,\tau)} u^{(m_i-m)(p_i-1)}(u^m-k^m)_+^{p_i}|D_i\zeta|^{p_i}\,dx\,dt+$$

$$+C\sum_{i=1}^{N}\iint\limits_{Q_{\vec{r},\eta}(y,\tau)} u^{(n_i-m)(q_i-1)}(u^m-k^m)_+^{q_i}|D_i\zeta|^{q_i}\,dx\,dt, \qquad (1.9)$$

is valid with a constant $C>0$ depending only on the data,[1] and

$$g(u^m,k^m):=\frac{1}{m}\int\limits_{k^m}^{u^m} z^{\frac{1}{m}-1}(z-k^m)_+\,dz.$$

Remark 1.1. In the case of the standard p-Laplacian equation $u_t-\Delta_p u=0$, i.e. in (1.3) with $m=1$, the function $g(u,k)$ above reduces to the classic parabolic term $g(u,k)=(u-k)_+^2/2$.

Remark 1.2. Since $u\in\mathcal{V}_{loc}(\Omega_T)$, then all the integral quantities in (1.9) are convergent, and the chain rule

$$\left|D_i\left([u^m]^{\left(\frac{m_i(p_i-1)+m}{p_i m}\right)}\right)\right|^{p_i}=\gamma_i|D_i[u^m]|^{p_i}\,u^{(m_i-m)(p_i-1)}, \qquad \gamma_i=\left(\frac{m_i(p_i-1)+m}{p_i m}\right),$$

will be tacitly assumed in the membership $u\in\mathcal{V}_{loc}(\Omega_T)$.

Novelty and Significance

Motivation of this paper relies on the relatively recent research in [34], where the regularity results for solutions of double phase elliptic equation with (p,q) growth were proved under additional assumptions that

$$q>\frac{Np}{N-p}, \quad p<N \quad \text{and} \quad u\in L^s_{loc}(\Omega) \quad \text{with} \quad s>\frac{N}{p}(q-p)>\frac{Np}{N-p}. \qquad (1.10)$$

The boundedness of solutions to elliptic equations with (p,q) growth in the case $q\leqslant\frac{Np}{N-p}$ (as well, to parabolic equations under condition $\frac{2N}{N+2}<p\leqslant q\leqslant p(1+\frac{2}{N})$) starting with the papers of Kolodii [28,29] is well known, see, for example [1,2,6,7,12,13,20,21,33,35,36]. At the same time, examples constructed by Marcellini [32] and Giaquinta [22] show that there exist unbounded solutions, provided that $q>\frac{Np}{N-p}$. In this case, additional assumptions are needed on the solution for its local boundedness, that are extra-integral conditions as (1.10).

---

[1] We say that a constant C depends only on the data if it depends only on $\{p_i,q_i,n_i,m_i,N\}$.

Similar questions arise for parabolic equations with nonstandard growth, but, unlike elliptic equations, in this case two special exponents show up: one of them (see case $\Lambda > 1$ in the next Section) is responsible for the behavior of subsolutions in the degenerate case, and the second (see case $\Lambda < 1$ in the next Section) in the singular case. Surprisingly, even in the case of standard growth, i.e. referring to $\mathcal{PDG}^+(\Omega_T)$ with $0 < m = m_i = n_i$, $p = p_i = q_i$, $i = 1, ..., N$ and $(m+1)p + N(m(p-1)-1) = 0$ (if $m = 1$ that is $p = \frac{2N}{N+2}$, or if $p = 2$ that is $m = \frac{N-2}{N+2}$, $N \geqslant 3$), local boundedness was known only under the additional condition that $u \in L^s_{loc}(\Omega_T)$ with sufficiently large s (see, for example [3–5,16,17], [24], [25]). As Theorem 1.1 shows, this condition can be removed.

About the known counter-examples (see for instance [5]), simple computations show that the known unbounded solutions in this borderline case do not belong to the corresponding Sobolev spaces defining the solutions, and so, the corresponding unbounded solutions fail to be weak solutions.

Here we also note that in the sub-critical case the quantitative point-wise estimate is usually proved under the additional condition that u is qualitatively locally bounded (see [3–5,16,17] for the standard case); while in [11] this assumption has been removed in a two-step proof. Theorem 1.2 shows that the qualitative boundedness condition can be discarded, for the whole class $\mathcal{PDG}^+(\Omega_T)$, by means of a direct method.

Main Results

In order to state our main results we need to define numbers $\{\lambda_i\}_{i=1}^N$, $\{\Lambda_i\}_{i=1}^N$ by

$$\lambda_i := m_i(p_i - 1) \leqslant \Lambda_i := n_i(q_i - 1) \qquad \forall i = 1, ..., N,$$

relative to the set of numbers $\{p_i\}_{i=1}^N$, $\{q_i\}_{i=1}^N$, $\{m_i\}_{i=1}^N$, $\{n_i\}_{i=1}^N$ of $\mathcal{PDG}^+(\Omega_T)$. Then we define the special indexes

$$\Lambda := \max(\Lambda_1, ..., \Lambda_N), \quad \frac{1}{p} := \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}, \quad \left|\frac{\lambda}{p}\right| := \frac{1}{N}\sum_{i=1}^N \frac{\lambda_i}{p_i}, \quad p < N.$$

When $\Lambda < 1$ we say that the equation has *singular* behavior, while when $\Lambda > 1$ we say that the equation has *degenerate* behavior. Clearly, we refer to the case $\Lambda = 1$ as the *limiting* case.

Theorem 1.1. *Let $u \in \mathcal{PDG}^+(\Omega_T)$ and let*

$$\mathcal{L} := \max(1, \Lambda) \leqslant p\left|\frac{\lambda}{p}\right| + (m+1)\frac{p}{N} =: \mathcal{M}, \tag{1.11}$$

then u is locally bounded. In particular, if

$$\mathcal{L} < p\left|\frac{\lambda}{p}\right| + (m+1)\frac{p}{N},$$

then for any cylinder $Q_{8\vec{\rho},8\theta}(x_0,t_0) \subset \Omega_T$ the following estimate holds true

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)} u \leqslant \gamma\left(\mathcal{H}(\theta,\vec{\rho})^{\frac{N+p}{p}} \iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^{m+\mathcal{L}}\, dxdt\right)^{\frac{p}{(\mathcal{M}-\mathcal{L})N}} + \mathcal{R}(\theta,\vec{\rho}), \tag{1.12}$$

for a constant $\gamma > 0$ that depends only on the data, while the functions $\mathcal{H}(\theta, \vec{\rho}), \mathcal{R}(\theta, \vec{\rho})$ are defined, respectively, in (3.4) *and* (3.5)*.*

Remark 1.3. In the critical case $\mathcal{L} = \mathcal{M}$, local boundedness is established without a quantitative point-wise bound, which can be obtained only under the additional assumption $u \in L^s_{\text{loc}}(\Omega_T)$, see Theorem 1.2 below.

Now, in order to formulate the respective of Theorem 1.1 in the sub-critical case $\mathcal{L} \geq \mathcal{M}$, we need to define the non-degeneracy number

$$\varkappa_s := p(s+1-\mathcal{L}) + N(p\Big|\frac{\lambda}{p}\Big| - \mathcal{L}) = \begin{cases} ps + N(p|\frac{\lambda}{p}| - 1), & \text{if} \quad \Lambda < 1, \\ p(s+1-\Lambda) + N(p|\frac{\lambda}{p}| - \Lambda), & \text{if} \quad \Lambda > 1. \end{cases} \tag{1.13}$$

Theorem 1.2. *Let $u \in \mathcal{PDG}^+(\Omega_T)$, let $\mathcal{L} \geq \mathcal{M}$ and let $s > m+1$ be any number such that*

$$\varkappa_s > 0, \quad i.e. \quad s > \mathcal{L} - 1 + \frac{N}{p}(\mathcal{L} - p\Big|\frac{\lambda}{p}\Big|). \tag{1.14}$$

If we assume additionally that $u \in L^s_{loc}(\Omega_T)$, then u is locally bounded, and moreover, the following inequality holds

$$\sup_{Q_{\frac{\vec{\rho}}{2}, \frac{\theta}{2}}(x_0,t_0)} u \leqslant \gamma \Bigg(\mathcal{H}(\theta, \vec{\rho})^{\frac{N+p}{p}} \iint_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^s \, dxdt \Bigg)^{\frac{p}{\varkappa_s}} + \mathcal{R}(\theta, \vec{\rho}), \tag{1.15}$$

for a constant $\gamma > 0$ depending only on the data.

Remark 1.4. The condition $\varkappa_s > 0$ allows us to control the behavior of u both in the singular case ($\Lambda < 1$) and in the degenerate case ($\Lambda > 1$). Our approach allows us to prove the $L^\infty_{loc}(\Omega_T)$ bound in both of these cases simultaneously.

By simple computations on the exponents one sees that for the p-Laplacian equations (1.2) the condition $\mathcal{L} < \mathcal{M}$ corresponds to $p > 2N/(N+2)$, so that, coherently with the study of [16], we refer to Theorem 1.1 as the *super-critical* case, since $\mathcal{L} < \mathcal{M}$, and Theorem 1.2 as the *critical/sub-critical case*, since $\mathcal{L} \geq \mathcal{M}$.

Structure of the Proof

The proofs of Theorems 1.1-1.2 are based on the method of De Giorgi [14], for an exhaustive overview on the subject, see, for example [16,17,23,30,31]. Although the general technique for establishing boundedness is similar to the standard p-growth, we had to overcome the difficulties offered by the presence of (m_i, p_i)-(n_i, q_i) growth coming from the unbalanced energy inequality (1.9) defining $\mathcal{PDG}^+(\Omega_T)$. In the limiting case $\mathcal{L} = \mathcal{M}$ we also explore the ideas from [2,12,20,21]. Here below we explain, in the simplified case of the p-Laplacian equations, the strategy of the proof for the critical case.

-Critical case: The Underlying Strategy -

The usual L^∞-bounds machine in the variational case consists in chaining an embedding

$$\|v\|_{L^{p^*}(K)} \leq \gamma \ \|\nabla v|_{L^p(K)}, \quad \text{for} \quad v \in W_o^{1,p}(K),$$

locally with the energy estimates - constructing so a sequence of numbers y_j, which are weighted integral averages of the solution in some shrinking compact sets, and which must satisfy the assumptions of the technical iterative Lemma 2.2.

Suppose for the sake of clarity that we are in the p-Laplacian case (i.e. (1.3), $m = 1$), which is, function $u \geq 0$ satisfies the energy inequality (1.2) that we rewrite on ideally designed shrinking sets (as in Section 3) $Q_j = B_j \times I_j$ and increasing levels k_j as

$$\sup_{t\in I_j} \int_{B_j} (u-k_{j+1})_+^2 \zeta_j \, dx + \iint_{Q_j} |\nabla (u-k_{j+1})_+|^p \zeta_j \, dx\, dt \leqslant \frac{\gamma 2^{j\gamma}}{\theta} \iint_{Q_j} (u-k_j)_+^2 \, dx\, dt, \tag{1.16}$$

having chosen $k \geq (\theta/\rho^p)^{\frac{1}{2-p}}$ and being in this case $p = \frac{2N}{N+2}$ the critical case. Now, setting

$$y_j := \iint_{Q_j} (u-k_j)_+^2 \, dxdt,$$

and using Lemma 2.3 and (1.16), we obtain

$$y_{j+1} \leqslant \gamma 2^{j\gamma} \, \theta^{-\frac{N+p}{N}} \, y_j^{\frac{N+p}{N}}. \tag{1.17}$$

Lemma 2.2 can be applied to inequality (3.14) yielding $\lim_{j\to\infty} y_j = 0$ (and therefore the local boundedness of u), provided that

$$y_0 = \iint_{Q_{\rho,\theta}} (u - \frac{k}{2})_+^2 \, dxdt \leqslant \frac{1}{\gamma} \, \theta^{-\frac{N+p}{p}}. \tag{1.18}$$

Since

$$\lim_{k\to\infty} \iint_{Q_{\bar{\rho},\theta}} (u - \frac{k}{2})_+^2 \, dxdt \leqslant \lim_{k\to\infty} \iint_{Q_{\rho,\theta}\cap\{u\geqslant k/2\}} u^2 \, dxdt = 0,$$

we can choose k large enough to satisfy (1.18). Observe that in this case the value of k is only qualitative. When more information, such as $u \in L^s_{loc}(\Omega_T)$ is available, then this machinery can be refined to expliciting k and having a quantitative upper bound, see Subsection 3.2.2 for further details.

Comparison with known literature

Theorem 1.1 generalizes Ok's result [34] for elliptic equations with (p, q) growth and, in addition, refines known results in the so-called sub-critical case for doubly nonlinear parabolic equations with standard (m, p)-growth, [5]. In particular, if

$$(m+1)p + N(m(p-1)-1) = 0, \qquad m > 0,$$

we obtain the local boundedness of non-negative sub-solutions without the assumption that $u \in L^s_{loc}(\Omega_T)$ with some sufficiently large $s > 1$. Our results also cover the new cases of parabolic equations with nonstandard $(m, p) - (n, q)$ and anisotropic (m_i, p_i) growth, generalizing the parabolic studies of [8], [9], [18], [19], and literature therein. See subsection 4.3 for more details. The case $p \geq N$ will be the aim of further investigations.

Structure of the paper: In Section 2 we collect some preliminary and technical properties, Section 3 is devoted to the proofs Theorems 1.1-1.2. Finally, Section 4 contains several examples illustrating that solutions to equations (1.3)-(1.4)-(1.5) belong to $\mathcal{PDG}^+(\Omega_T)$, and therefore Theorems 1.1, 1.2 apply to them.

2. Auxiliary material

In this brief section we collect our main tools of the trade. First, we start with a Lemma that allows to estimate more precisely function g in (1.9) of Definition 1.1. For $u, k, m \in \mathbb{R}_+$ set

$$g_\pm(u,k) := \pm \frac{1}{m} \int_k^u z^{\frac{1}{m}-1}(z-k)_\pm \, dz.$$

Then, truncations $g_\pm(u,k)$ are bounded above and below.

Lemma 2.1 *(e.g. [5], Lemma 3.2). There exists constant $\gamma > 0$ depending only on m such that*

$$\frac{1}{\gamma}\,(k+u)^{\frac{1}{m}-1}(u-k)^2_\pm \leqslant g_\pm(u,k) \leqslant \gamma\,(k+u)^{\frac{1}{m}-1}(u-k)^2_\pm.$$

The next Lemma is called in the literature The Geometric Convergence Lemma, it is a classic tool for *a priori* estimates for which we refer to [16], Chapter 2.

Lemma 2.2. *Let $\{y_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers satisfying the recursive inequalities*

$$y_{j+1} \leqslant A\,B^j\,y_j^{1+\delta}, \quad j = 0, 1, ...,$$

where $A,\ B > 1$ and $\delta > 0$ are given numbers. If

$$y_0 \leqslant A^{-\frac{1}{\delta}}\,B^{-\frac{1}{\delta^2}},$$

then $\lim_{j\to\infty} y_j = 0$.

Finally, we will need the following anisotropic embedding at its full strength.

Lemma 2.3 *(e.g. [37] or [19] Prop. 2.3.). Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $u \in W^{1,1}_0(\Omega)$ and*

$$\sum_{i=1}^N \int_\Omega |D_i u^{\alpha_i}|^{p_i} dx < \infty, \quad \alpha_i > 0, \quad p < N.$$

Then there exists c, depending on $N, p_1, ..., p_N, \alpha_1, ..., \alpha_N$ *such that*

$$\Big(\int_{\Omega} |u|^{\bar{p}} dx\Big)^{\frac{N-p}{N}} \leqslant c \prod_{i=1}^{N} \Big(\int_{\Omega} |D_i u^{\alpha_i}|^{p_i} dx\Big)^{\frac{p}{Np_i}}, \quad \bar{p} = \frac{N\alpha p}{N-p}, \quad \alpha = \frac{1}{N}\sum_{i=1}^{N} \alpha_i .$$

2.1. Notation

We refer to the parameters $C, N, m_1, ..., m_N, n_1, ..., n_N, p_1, ..., p_N, q_1, ..., q_N$ as our structural data, and we write γ if it can be quantitatively determined a priori only in terms of the above quantities. The generic constant γ may change from line to line.

3. Local boundedness: proof of Theorems 1.1, 1.2

The first step is a common energy estimate. We set up the iterative geometry: let us fix $(x_0, t_0) \in \Omega_T$ such that $Q_{8\vec{\rho},8\theta}(x_0, t_0) \subset \Omega_T$. For $k > 0$ to be determined, we define the increasing sequence of levels

$$k_j^m = k^m - \frac{k^m}{2^{j+1}}, \qquad \bar{k}_j^m := \frac{1}{2}(k_j^m + k_{j+1}^m), \qquad [k_j']^m := \frac{1}{2}(\bar{k}_j^m + k_{j+1}^m),$$

and the decreasing geometric sequences

$$\vec{r}_j := \frac{\vec{\rho}}{2}\Big(1 + \frac{1}{2^{j+1}}\Big), \qquad \eta_j := \frac{\theta}{2}\Big(1 + \frac{1}{2^{j+1}}\Big),$$
$$K_j := K_{\vec{r}_j}(x_0), \qquad Q_j := K_j \times I_j, \qquad I_j := (t_0 - \eta_j, t_0)\,.$$

Now, let $\zeta_j^{(1)}(x) \in C_0^1(K_j)$ be a cut-off function between K_j and K_{j+1}, i.e.

$$\zeta_j^{(1)}(x) = 1 \quad \text{in} \quad K_{j+1}, \quad 0 \leqslant \zeta_j^{(1)}(x) \leqslant 1,$$

and therefore obliged to satisfy the decays

$$|D_i \zeta_j^{(1)}(x)| \leqslant \gamma \frac{2^j}{\rho_i}, \qquad \forall i = 1, ..., N\,.$$

Moreover, in order to localize with respect to time, let us consider $\zeta_j^{(2)}(t) \in C^1(\mathbb{R})$ such that

$$\zeta_j^{(2)}(t) = 0 \quad \text{for} \quad t \leqslant t_0 - \eta_j, \quad \zeta_j^{(2)}(t) = 1 \quad \text{for} \quad t \geqslant t_0 - \eta_{j+1},$$
$$1 \leqslant \zeta_j^{(2)}(t) \leqslant 1, \quad |\zeta_{j,t}^{(2)}(t)| \leqslant \gamma \frac{2^j}{\theta},$$

and finally set $\zeta_j := \zeta_j^{(1)} \zeta_j^{(2)}$. In order to control the parabolic energy terms, we observe that on the set $\{u \geqslant k_{j+1}\}$ one has

$$1 \leqslant \frac{u^m + [k_j']^m}{u^m - [k_j']^m} \leqslant \frac{2u^m}{u^m - [k_j']^m} \leqslant \frac{2k_{j+1}^m}{k_{j+1}^m - [k_j']^m} \leqslant \gamma 2^{j\gamma},$$

and hence Lemma 2.1 yields

$$g(u^m, [k_j']^m) \geqslant \frac{1}{\gamma 2^{j\gamma}} (u^m - k_{j+1}^m)^{1+\frac{1}{m}}, \quad \text{if} \quad u \geqslant k_{j+1}. \tag{3.1}$$

Moreover, on the set $\{u \geqslant k_j'\}$

$$1 \leqslant \frac{u^m + [k_j']^m}{u^m - \bar{k}_j^m} \leqslant \frac{2u^m}{u^m - \bar{k}_j^m} \leqslant \frac{2[k_j']^m}{[k_j']^m - \bar{k}_j^m} \leqslant \gamma 2^{j\gamma},$$

and hence, using the fact that $(u^m - [k_j']^m)_+ \leqslant (u^m - \bar{k}_j^m)_+$, from Lemma 2.1 we get

$$g(u^m, [k_j']^m) \leqslant \gamma 2^{j\gamma} (u^m - \bar{k}_j^m)_+^{1+\frac{1}{m}}. \tag{3.2}$$

Therefore, setting

$$\alpha_i := \frac{\lambda_i + m}{m\, p_i}, \quad i = 1, ..., N,$$

by (3.1), (3.2), inequality (1.9) can be rewritten as

$$\begin{aligned}
&\sup_{t \in I_j} \int\limits_{K_j} (u^m - k_{j+1}^m)_+^{1+\frac{1}{m}} \zeta_j^q \, dx + \sum_{i=1}^N \iint\limits_{Q_j} |D_i (u^m - k_{j+1}^m)_+^{\alpha_i}|^{p_i} \zeta_j^q \, dx\, dt \\
&\qquad \leqslant \frac{\gamma 2^{j\gamma}}{\theta} \iint\limits_{Q_j} (u^m - \bar{k}_j^m)_+^{1+\frac{1}{m}} \, dx dt + \\
&\qquad + \sum_{i=1}^N \frac{\gamma 2^{j\gamma}}{\rho_i^{p_i}} \iint\limits_{Q_j} u^{(m_i - m)(p_i - 1)} (u^m - \bar{k}_j^m)_+^{p_i} \, dx\, dt + \\
&\qquad + \sum_{i=1}^N \frac{\gamma 2^{j\gamma}}{\rho_i^{q_i}} \iint\limits_{Q_j} u^{(n_i - m)(q_i - 1)} (u^m - \bar{k}_j^m)_+^{q_i} \, dx\, dt \\
&\qquad \leqslant \gamma 2^{j\gamma} \left[\frac{k^{1-\mathcal{L}}}{\theta} + \sum_{i=1}^N \frac{k^{\lambda_i - \mathcal{L}}}{\rho_i^{p_i}} + \sum_{i=1}^N \frac{k^{\Lambda_i - \mathcal{L}}}{\rho_i^{q_i}} \right] \iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}} \, dx\, dt.
\end{aligned} \tag{3.3}$$

By the simple arithmetic fact that $\sum_{i=1}^S a_i \geqslant \left(\sum_{i=1}^S \frac{1}{a_i} \right)^{-1}$ for any $S \geqslant 1$ and $a_i > 0$, the terms in brackets in (3.3) are estimated by stipulating to take

$$k \geqslant \mathcal{R}(\theta, \vec{\rho}) := \begin{cases} \sum_{i=1}^{N} \big(\frac{\theta}{\rho_i^{p_i}}\big)^{\frac{1}{1-\lambda_i}} + \sum_{i=1}^{N} \big(\frac{\theta}{\rho_i^{q_i}}\big)^{\frac{1}{1-\Lambda_i}}, \quad \text{if} \quad \Lambda < 1 = \mathcal{L}, \\ \sum_{i:\lambda_i<1} \big(\frac{\theta}{\rho_i^{p_i}}\big)^{\frac{1}{1-\lambda_i}} + \sum_{i:\Lambda_i<1} \big(\frac{\theta}{\rho_i^{q_i}}\big)^{\frac{1}{1-\Lambda_i}}, \quad \text{if} \quad \Lambda = 1 = \mathcal{L}, \\ \sum_{i:\lambda_i=\Lambda} \big(\frac{\rho_i^{p_i}}{\theta}\big)^{\frac{1}{\Lambda-1}} + \sum_{i:\Lambda_i=\Lambda} \big(\frac{\rho_i^{q_i}}{\theta}\big)^{\frac{1}{\Lambda-1}} + \sum_{i:\lambda_i<\Lambda} \Big(\frac{1}{\rho_i^{p_i}} \sum_{l:\lambda_l=\Lambda} \rho_l^{p_l}\Big)^{\frac{1}{\Lambda-\lambda_i}} + \\ + \sum_{i:\Lambda_i<\Lambda} \Big(\frac{1}{\rho_i^{q_i}} \sum_{l:\Lambda_l=\Lambda} \rho_l^{q_l}\Big)^{\frac{1}{\Lambda-\Lambda_i}}, \quad \text{if} \quad 1 < \Lambda = \mathcal{L}. \end{cases} \tag{3.4}$$

Therefore, setting

$$\mathcal{H}(\theta, \vec{\rho}) := \begin{cases} \frac{1}{\theta}, \quad \text{if} \quad \Lambda < 1, \\ \frac{1}{\theta} + \sum_{i:\lambda_i=1} \frac{1}{\rho_i^{p_i}} + \sum_{i:\Lambda_i=1} \frac{1}{\rho_i^{q_i}}, \quad \text{if} \quad \Lambda = 1, \\ \sum_{i:\lambda_i=\Lambda} \frac{1}{\rho_i^{p_i}} + \sum_{i:\Lambda_i=\Lambda} \frac{1}{\rho_i^{q_i}}, \quad \text{if} \quad \Lambda > 1, \end{cases} \tag{3.5}$$

the energy inequality (3.3) gets simplified into

$$\sup_{t \in I_j} \int\limits_{B_j} (u^m - k_{j+1}^m)_+^{1+\frac{1}{m}} \zeta_j^q \, dx + \sum_{i=1}^{N} \iint\limits_{Q_j} |D_i (u^m - k_{j+1}^m)_+^{\alpha_i}|^{p_i} \zeta_j^q \, dx\, dt$$
$$\leqslant \gamma 2^{j\gamma} \, \mathcal{H}(\theta, \vec{\rho}) \iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}} \, dx\, dt. \tag{3.6}$$

Here the proof of the two Theorems splits.

3.1. *Proof of Theorem 1.1*

3.1.1. *Case $\mathcal{L} < \mathcal{M}$*

Choose

$$\begin{cases} \alpha := \frac{1}{N} \sum_{i=1}^{N} \alpha_i = \frac{1}{p}\big(1 + \frac{p}{m}\big|\frac{\lambda}{p}\big|\big), \\ p_* := \alpha p + (1 + \frac{1}{m})\frac{p}{N} > 1 + \frac{\mathcal{L}}{m}. \end{cases}$$

Let us set

$$\alpha^- := \min_{1 \leqslant i \leqslant N} \alpha_i, \quad p^- := \min_{1 \leqslant i \leqslant N} p_i, \quad \gamma_0 := q\left(1 + \frac{1}{p^- \alpha^-}\right).$$

By Hölder's inequality and Lemma 2.3, from (3.6) we obtain

$$
\begin{aligned}
&\iint\limits_{Q_{j+1}} (u^m - k_{j+1}^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt \\
&\quad\leqslant \Big(\iint\limits_{Q_j} \big[(u^m - k_{j+1}^m)_+ \zeta_j^{\gamma_0}\big]^{p_*}\, dxdt \Big)^{\frac{1+\frac{\mathcal{L}}{m}}{p_*}} |Q_j \cap \{u \geqslant k_{j+1}\}|^{1-\frac{1+\frac{\mathcal{L}}{m}}{p_*}} \\
&\quad\leqslant \gamma \Big(\sup_{t\in I_j} \int\limits_{B_j} (u^m - k_j^m)_+^{1+\frac{1}{m}} \zeta_j^q\, dx \Big)^{\frac{p(1+\frac{\mathcal{L}}{m})}{Np_*}} \times \\
&\qquad \times \Big(\int\limits_{I_j} \Big[\int\limits_{B_j} \big[(u^m - k_{j+1}^m)_+ \zeta_j^{\frac{q}{p^- \alpha^-}}\big]^{\frac{N\alpha p}{N-p}}\, dx \Big]^{\frac{N-p}{N}} dt \Big)^{\frac{1+\frac{\mathcal{L}}{m}}{p_*}} |Q_j \cap \{u \geqslant k_{j+1}\}|^{1-\frac{1+\frac{\mathcal{L}}{m}}{p_*}} \\
&\quad\leqslant \gamma \Big(\sup_{t\in I_j} \int\limits_{B_j} (u^m - k_j^m)_+^{1+\frac{1}{m}} \zeta_j^q\, dx \Big)^{\frac{p(1+\frac{\mathcal{L}}{m})}{Np_*}} \times \\
&\qquad \times \Big(\sum_{i=1}^N \iint\limits_{Q_j} \big| D_i \big[(u^m - k_{j+1}^m)_+^{\alpha_i} \zeta_j^{\frac{q}{p^-}} \big] \big|^{p_i}\, dxdt \Big)^{\frac{1+\frac{\mathcal{L}}{m}}{p_*}} |Q_j \cap \{u \geqslant k_{j+1}\}|^{1-\frac{1+\frac{\mathcal{L}}{m}}{p_*}},
\end{aligned}
\tag{3.7}
$$

and by the common energy estimates (3.6) we obtain

$$
\begin{aligned}
&\iint\limits_{Q_{j+1}} (u^m - k_{j+1}^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt \\
&\qquad \leqslant \gamma 2^{j\gamma} k^{-(m+\mathcal{L})(1-\frac{1+\frac{\mathcal{L}}{m}}{p_*})} \Big(\iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt \Big)^{1-\frac{1+\frac{\mathcal{L}}{m}}{p_*}} \times \\
&\qquad\qquad \times \Big[\mathcal{H}(\theta, \vec{\rho}) \iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt \Big]^{\frac{(1+\frac{\mathcal{L}}{m})(N+p)}{p_* N}}.
\end{aligned}
\tag{3.8}
$$

Therefore, if we define

$$
y_j := \iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt,
$$

then from (3.7) we obtain

$$
y_{j+1} \leqslant \gamma 2^{j\gamma} k^{-(m+\mathcal{L})(1-\frac{1+\frac{\mathcal{L}}{m}}{p_*})} \big[\mathcal{H}(\theta, \vec{\rho})\big]^{\frac{(1+\frac{\mathcal{L}}{m})(N+p)}{p_* N}} y_j^{1+\frac{(1+\frac{\mathcal{L}}{m})p}{p_* N}}.
\tag{3.9}
$$

By Lemma 2.2 $\lim_{j\to\infty} y_j = 0$, provided that k is chosen to satisfy

$$k^{\frac{(m+1)p+N(p|\frac{\lambda}{p}|-\mathcal{L})}{p}} \geqslant \gamma \big[\mathcal{H}(\theta,\vec{\rho})\big]^{\frac{N+p}{p}} y_0. \tag{3.10}$$

From this, taking into account (3.4) and (3.5), we arrive at the required estimate (1.12).

Remark 3.1 *(Explicit formulas).* In the singular case $\Lambda < 1 = \mathcal{L}$, formula (1.12) takes the explicit shape

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)} u \leqslant \gamma \Big(\frac{1}{\theta^{\frac{N+p}{p}}} \iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^{m+1}\, dxdt \Big)^{\frac{p}{(m+1)p+N(p|\frac{\lambda}{p}|-1)}} +$$

$$+ \gamma \Big[\sum_{i=1}^{N} \Big(\frac{\theta}{\rho_i^{p_i}}\Big)^{\frac{1}{1-\lambda_i}} + \sum_{i=1}^{N} \Big(\frac{\theta}{\rho^{q_i}}\Big)^{\frac{1}{1-\Lambda_i}} \Big], \tag{3.11}$$

while in the limiting case $\Lambda = 1 = \mathcal{L}$ the expression (1.12) is

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)} u \leqslant \gamma \Big(\Big[\frac{1}{\theta} + \sum_{i:\lambda_i=1} \frac{1}{\rho_i^{p_i}} + \sum_{i:\Lambda_i=1} \frac{1}{\rho_i^{q_i}} \Big]^{\frac{N+p}{p}} \iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^{m+1}\, dxdt \Big)^{\frac{p}{(m+1)p+N(p|\frac{\lambda}{p}|-1)}} +$$

$$+ \gamma \Big[\sum_{i:\lambda_i<1} \Big(\frac{\theta}{\rho_i^{p_i}}\Big)^{\frac{1}{1-\lambda_i}} + \sum_{i:\Lambda_i<1} \Big(\frac{\theta}{\rho_i^{q_i}}\Big)^{\frac{1}{1-\Lambda_i}} \Big], \tag{3.12}$$

and finally in the degenerate case $1 < \Lambda = \mathcal{L}$ we have

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)} u \leqslant \gamma \Big(\Big[\sum_{i:\lambda_i=\Lambda} \frac{1}{\rho_i^{p_i}} + \sum_{i:\Lambda_i=\Lambda} \frac{1}{\rho_i^{q_i}} \Big]^{\frac{N+p}{p}} \iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^{m+\Lambda}\, dxdt \Big)^{\frac{p}{(m+1)p+N(p|\frac{\lambda}{p}|-\Lambda)}} +$$

$$+ \gamma \Big[\sum_{i:\lambda_i=\Lambda} \Big(\frac{\rho_i^{p_i}}{\theta}\Big)^{\frac{1}{\Lambda-1}} + \sum_{i:\Lambda_i=\Lambda} \Big(\frac{\rho_i^{q_i}}{\theta}\Big)^{\frac{1}{\Lambda-1}} + \sum_{i:\lambda_i<\Lambda} \Big(\frac{1}{\rho_i^{p_i}} \sum_{l:\lambda_l=\Lambda} \rho_l^{p_l}\Big)^{\frac{1}{\Lambda-\lambda_i}} +$$

$$+ \sum_{i:\Lambda_i<\Lambda} \Big(\frac{1}{\rho_i^{q_i}} \sum_{l:\Lambda_l=\Lambda} \rho_l^{q_l}\Big)^{\frac{1}{\Lambda-\Lambda_i}} \Big]. \tag{3.13}$$

3.1.2. Case $\mathcal{L} = \mathcal{M}$

Let us set

$$y_j := \iint\limits_{Q_j} (u^m - k_j^m)_+^{p_*}\, dxdt,$$

$$p_* = \alpha p + (1 + \frac{1}{m})\frac{p}{N} = 1 + \frac{\mathcal{L}}{m}, \quad \alpha p = 1 + \frac{p}{m}\Big|\frac{\lambda}{p}\Big|.$$

Using Lemma 2.3 and (3.6), similarly to (3.9) we obtain

$$y_{j+1} \leqslant \gamma 2^{j\gamma} \left[\mathcal{H}(\theta, \vec{\rho})\right]^{\frac{N+p}{N}} y_j^{\frac{N+p}{N}}. \tag{3.14}$$

By Lemma 2.2, inequality (3.14) yields $\lim_{j\to\infty} y_j = 0$, provided that

$$y_0 = \iint\limits_{Q_{\vec{\rho},\theta}} (u^m - \frac{k^m}{2})_+^{p_*}\, dxdt \leqslant \frac{1}{\gamma} \left[\mathcal{H}(\theta, \vec{\rho})\right]^{-\frac{N+p}{p}}. \tag{3.15}$$

Since

$$\iint\limits_{Q_{\vec{\rho},\theta}} (u^m - \frac{k^m}{2})_+^{p_*}\, dxdt \leqslant \iint\limits_{Q_{\vec{\rho},\theta}\cap\{u \geqslant k 2^{-\frac{1}{m}}\}} u^{mp_*}\, dxdt$$

and using Chebyshev's inequality together with the absolute continuity of the integral with respect to the domain

$$\lim_{k\to\infty} |[u \geq k2^{-1/m}]| \leq \lim_{k\to\infty} \frac{1}{k^{mp^*}2^{-p^*}} \iint\limits_{Q_{\vec{\rho},\theta}} u^{mp^*}\, dxdt = 0 \quad \Rightarrow$$

$$\Rightarrow \lim_{k\to\infty} \iint\limits_{Q_{\vec{\rho},\theta}\cap\{u \geqslant k2^{-\frac{1}{m}}\}} u^{mp_*}\, dxdt = 0,$$

we choose k large enough that (3.15) holds, and hence

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)} u \leqslant k,$$

completing the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2

3.2.1. Case $\mathcal{L} > \mathcal{M}$

Due to the fact that $\mathcal{L} > p\left|\frac{\lambda}{p}\right| + (m+1)\frac{p}{N}$ we cannot follow the proof of Theorem 1.1. Let us estimate the integral on the right-hand side of (3.6), for this set

$$\epsilon = \frac{s - m - \mathcal{L}}{s - mp_*}, \quad p_* = \alpha p + (1 + \frac{1}{m})\frac{p}{N}, \quad \alpha p = 1 + \frac{p}{m}\left|\frac{\lambda}{p}\right|.$$

Note that by assumptions $\mathcal{L} > \mathcal{M}$ and (1.14) one can estimate

$$s > \mathcal{L} - 1 + \frac{N}{p}(\mathcal{L} - p\left|\frac{\lambda}{p}\right|) > \mathcal{L} + m > p\left|\frac{\lambda}{p}\right| + m + (m+1)\frac{p}{N} = m\alpha p + (m+1)\frac{p}{N} = mp_*,$$

so that $\epsilon \in (0, 1)$ is well defined. Fix number d such that

$$d := \Big(\iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^s dxdt \Big)^{\frac{1}{s}},$$

then by the Hölder inequality, and using the fact that $\dfrac{m+\mathcal{L}-m\epsilon p_*}{1-\epsilon} = s$, we obtain

$$\begin{aligned}\iint\limits_{Q_j} (u^m - k_j^m)_+^{1+\frac{\mathcal{L}}{m}}\, dxdt &\leqslant \gamma \iint\limits_{Q_j} (u^m - k_j^m)_+^{\epsilon p_*}\, u^{m+\mathcal{L}-m\epsilon p_*}\, dxdt \\ &\leqslant \Big(\iint\limits_{Q_j} (u^m - k_j^m)_+^{p_*}\, dxdt \Big)^{\epsilon} \Big(\iint\limits_{Q_j} u^s\, dxdt \Big)^{1-\epsilon} \\ &\leqslant \gamma 2^{j\gamma} d^{s(1-\epsilon)} \Big(\iint\limits_{Q_j} (u^m - k_j^m)_+^{p_*}\, dxdt \Big)^{\epsilon}. \end{aligned} \tag{3.16}$$

Inequalities (3.6), (3.16) yield

$$\begin{aligned}&\sup_{t\in I_j} \int\limits_{B_j} (u^m - k_{j+1}^m)_+^{1+\frac{1}{m}} \zeta_j^q\, dx + \sum_{i=1}^N \iint\limits_{Q_j} |D_i (u^m - k_{j+1}^m)_+^{\alpha_i}|^{p_i} \zeta_j^q\, dx\, dt \\ &\qquad \leqslant \gamma 2^{j\gamma}\, d^{s(1-\epsilon)}\, \mathcal{H}(\theta,\vec{\rho}) \Big(\iint\limits_{Q_j} (u^m - k_j^m)_+^{p_*}\, dxdt \Big)^{\epsilon}. \end{aligned} \tag{3.17}$$

Similarly to (3.14), by Lemma 2.3, using the Hölder inequality and (3.17), we have

$$y_{j+1} := \iint\limits_{Q_{j+1}} (u^m - k_{j+1}^m)_+^{p_*}\, dx\, dt \leqslant \gamma 2^{j\gamma} \Big[d^{s(1-\epsilon)}\, \mathcal{H}(\theta,\vec{\rho}) \Big]^{\frac{N+p}{N}} y_j^{\epsilon \frac{N+p}{N}}. \tag{3.18}$$

Calculations give

$$\epsilon(N+p) - N = \frac{p(s-\mathcal{L}+1) + N(p\big|\frac{\lambda}{p}\big| - \mathcal{L})}{s - mp_*} = \frac{\varkappa_s}{s-mp_*} > 0, \tag{3.19}$$

and hence $\epsilon\frac{N+p}{N} > 1$, so, from (3.18), by Lemma 2.2 we obtain that $\lim_{j\to\infty} y_j = 0$, provided that

$$y_0 \leqslant \frac{1}{\gamma} \Big[d^{s(1-\epsilon)} \mathcal{H}(\theta,\vec{\rho}) \Big]^{-\frac{N+p}{\epsilon(N+p)-N}}. \tag{3.20}$$

By the Hölder inequality

$$y_0=\iint\limits_{Q_{\vec{\rho},\theta}}(u^m-\frac{k^m}{2})_+^{p_*}\,dxdt\leqslant\iint\limits_{Q_{\vec{\rho},\theta}\cap\{u\geqslant k2^{-\frac{1}{m}}\}}u^{mp_*}\,dxdt\leqslant$$

$$\leqslant d^{mp_*}\big|Q_{\vec{\rho},\theta}(x_0,t_0)\cap\{u\geqslant k2^{-\frac{1}{m}}\}\big|^{1-\frac{mp_*}{s}}.$$

Evidently we have

$$\big|Q_{\vec{\rho},\theta}(x_0,t_0)\cap\{u\geqslant k2^{-\frac{1}{m}}\}\big|\leqslant\Big(\frac{2^{\frac{1}{m}}d}{k}\Big)^s,$$

and hence

$$y_0\leqslant\frac{2^{\frac{s}{m}(1-\frac{mp_*}{s})}d^s}{k^{s-mp_*}}. \tag{3.21}$$

Therefore, (3.21) implies (3.20), provided that we choose k large enough such that

$$k^{s-mp_*}=\gamma\,d^{s+\frac{s(1-\epsilon)(N+p)}{\epsilon(N+p)-N}}\,\big[\mathcal{H}(\theta,\vec{\rho})\big]^{\frac{N+p}{\epsilon(N+p)-N}}=\gamma d^{\frac{sp}{\epsilon(N+p)-N}}\big[\mathcal{H}(\theta,\vec{\rho})\big]^{\frac{N+p}{\epsilon(N+p)-N}}. \tag{3.22}$$

Taking into account (3.19) and the definition of d, (3.22) can be rewritten as

$$k=\gamma d^{\frac{sp}{\varkappa s}}\,\big[\mathcal{H}(\theta,\vec{\rho})\big]^{\frac{N+p}{\varkappa s}}=\gamma\big[\mathcal{H}(\theta,\vec{\rho})\big]^{\frac{N+p}{\varkappa s}}\Big(\iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)}u^s\,dxdt\Big)^{\frac{p}{\varkappa s}},$$

and hence

$$\sup_{Q_{\frac{\vec{\rho}}{2},\frac{\theta}{2}}(x_0,t_0)}u\leqslant k,$$

which, taking into account (3.4), (3.5), proves Theorem 1.2 in the case $\mathcal{L}>p\big|\frac{\lambda}{p}\big|+(m+1)\frac{p}{N}$.

3.2.2. *Case $\mathcal{L}=\mathcal{M}$*

Set

$$y_j:=\iint\limits_{Q_j}(u^m-k_j^m)_+^{p_*}\,dxdt,\quad p_*=\alpha p+(1+\frac{1}{m})\frac{p}{N}=1+\frac{\mathcal{L}}{m},\quad \alpha p=1+\frac{p}{m}\Big|\frac{\lambda}{p}\Big|,$$

by (3.14), (3.15) $\lim_{j\to\infty}y_j=0$, provided that

$$y_0\leqslant\frac{1}{\gamma}\big[\mathcal{H}(\theta,\vec{\rho})\big]^{-\frac{N+p}{p}}, \tag{3.23}$$

with $\mathcal{H}(\theta,\vec{\rho})$ defined in (3.5). By (3.21)

$$y_0 \leqslant \frac{2^{\frac{s}{m}(1-\frac{mp_*}{s})} d^s}{k^{s-mp_*}}, \quad d = \Big(\iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^s\, dxdt \Big)^{\frac{1}{s}}.$$

Therefore, inequality (3.23) holds, provided that k is chosen to satisfy

$$k^{s-mp_*} = \gamma \big[\mathcal{H}(\theta, \vec{\rho})\big]^{\frac{N+p}{p}} \iint\limits_{Q_{\vec{\rho},\theta}(x_0,t_0)} u^s\, dxdt.$$

By our choice of $\mathcal{L}$, calculations give $s - m\, p_* = \frac{\varkappa_s}{p}$. Taking into account (3.4), we arrive at the required (3.14), (3.15), this completes the proof of Theorem 1.2.

4. Examples

In this Section we show that local weak sub-solutions to many evolutionary nonlinear equations belong to our parabolic class $\mathcal{PDG}^+(\Omega_T)$. In this context, two new points should be highlighted for the equations presented in the next subsection: doubly nonlinear equations (1.3) cover both the parabolic p-Laplacian and the porous medium equations, for which local boundedness is established in the critical case.

4.1. *Example (1.3) - doubly nonlinear equations*

Local weak solutions to doubly nonlinear parabolic equations with standard growth as (1.3) satisfy inequality (1.9) in the cylinder $Q_{r,\eta}(y,\tau) = B_r(y) \times (\tau - \eta, \tau)$ with

$$m = m_1 = \ldots = m_N = n_1 = \ldots = n_N, \quad p = p_1 = \ldots = p_N = q_1 = \ldots = q_N,$$

see, for example [5]. Conditions (1.11) are translated into

$$\mathcal{L} = \max\{1, m(p-1)\} \leqslant m(p-1) + (m+1)\frac{p}{N} = \mathcal{M}.$$

We specify Theorems 1.1-1.2 under the dichotomy *degenerate/singular*.

Let $(x_0, t_0) \in \Omega_T$ and $\rho > 0$ be such that the sub-cylinder $Q_{8\rho,8\theta}(x_0,t_0) \subset \Omega_T$ is properly contained in the domain.

In the degenerate and limiting case $\Lambda = m(p-1) \geq 1$ the condition $\mathcal{L} < \mathcal{M}$ is always satisfied, hence the quantitative bound is always at hands.

Corollary 4.1 *(Degenerate and Limiting case $\Lambda \geq 1$). Let $u \geq 0$ be a local weak subsolution to (1.3) with $\Lambda \geq 1$ and $p < N$. Then there exists a constant $\gamma > 1$ depending only on the data such that*

$$\sup_{Q_{\rho,\theta}(x_0,t_0)} u \leqslant \gamma \Big(\Big[\frac{1}{\theta} + \frac{1}{\rho^p}\Big] \iint\limits_{Q_{2\rho,2\theta}(x_0,t_0)} u^{mp}\, dxdt \Big)^{\frac{1}{m+1}} + \gamma \Big(\frac{\theta}{\rho^p}\Big)^{\frac{1}{1-m(p-1)}}.$$

On the other hand, in the singular case $\Lambda < 1$ then $\mathcal{L} = 1$ and we have to distinguish between super and sub-critical: here, the non-degeneracy number is $\varkappa_s = ps + N(m(p-1) - 1)$.

Corollary 4.2 *(Singular case $\Lambda < 1$). Let $u \geq 0$ be a local weak subsolution to (1.3) with $\Lambda < 1$ and $p < N$. There exists a constant $\gamma > 1$ depending only on the data such that:*

- *If p, m are in the super-critical range*

  $$p > \frac{(m+1)N}{m(N+1)+1}\,,$$

  then the following quantitative upper bound is valid

  $$\sup_{Q_{\rho,\theta}(x_0,t_0)} u \leqslant \gamma\left(\theta^{-\frac{N+p}{N}} \iint_{Q_{2\rho,2\theta}(x_0,t_0)} u^{m+1}\, dxdt\right)^{\frac{p}{N[m(p-1)-1]+(m+1)p}} + \gamma\left(\frac{\theta}{\rho^p}\right)^{\frac{1}{1-m(p-1)}}.$$

- *In the critical case $p = (m+1)N/[m(N+1)+1]$ the local weak sub-solution u is qualitatively locally bounded. Such a qualitative property can be made quantitative by assuming extra local integrability, as in the next point.*
- *Otherwise, in the critical and sub-critical range*

  $$p \leq \frac{(m+1)N}{m(N+1)+1}\,,$$

  any extra integrability condition $u \in L^s_{loc}(\Omega_T)$ for any number s such that

  $$\varkappa_s = ps + N(m(p-1)-1) > 0, \qquad s > m+1\,,$$

  results in the quantitative upper bound

  $$\sup_{Q_{\rho,\theta}(x_0,t_0)} u \leqslant \gamma\left(\theta^{-\frac{N+p}{p}} \iint_{Q_{2\rho,2\theta}(x_0,t_0)} u^{s}\, dxdt\right)^{\frac{p}{\varkappa_s}} + \gamma\left(\frac{\theta}{\rho^p}\right)^{\frac{1}{1-m(p-1)}}.$$

4.2. Example (1.4) - generalized Orlicz growth

Doubly nonlinear parabolic equations with generalized Orlicz growth as (1.4), i.e.

$$u_t - div\Big(\varphi(x,t,u,|\nabla u^{m^-}|)\frac{\nabla u^{m^-}}{|\nabla u^{m^-}|}\Big) = 0, \quad (x,t) \in \Omega_T,$$

where $\varphi(x,t,u,v)$ is increasing in v for all $(x,t) \in \Omega_T$ and $u > 0$ and satisfies the conditions

$$K_1 u^{(m-m^-)(p-1)} v^p \leqslant \Phi(x,t,u,v) := \varphi(x,t,u,v)\, v \leqslant K_2 u^{(n-m^-)(q-1)} v^q, \tag{4.1}$$

where $m^- := \min(m,n) > 0$, $p \leqslant q$, $m(p-1) \leqslant n(q-1)$.

Remark 4.1. The case of double-phase equations is covered by this one; the only difference being that the constants K_1, K_2 above and therefore the constant C of the definition of $\mathcal{PDG}^+(\Omega_T)$ (see (1.9)) depend on the local maximum and on the minimum value of the phase (see [10], Lemma 3.1).

Definition 4.1. A function

$$\begin{cases} u \in C_{loc}(0,T; L^{1+m^-}_{loc}(\Omega_T)), \quad u^{m^-} \in L^q(0,T; W^{1,q}_{loc}(\Omega)), \\ u^{\frac{m(p-1)+m^-}{p_1}} \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega)), \quad u^{\frac{n(q-1)+m^-}{q}} \in L^q_{loc}(0,T; L^q_{loc}(\Omega)) \end{cases}$$

is a local, weak subsolution to (1.4) if for every compact set $K \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$ the inequality

$$\int_K u\,\zeta\,dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big\{ -u\,\zeta_t + \varphi(x,t,u,|\nabla u^{m^-}|)\frac{\nabla u^{m^-}}{|\nabla u^{m^-}|}\nabla\zeta \Big\}\,dxdt \leqslant 0, \tag{4.2}$$

is valid for all test functions $0 \le \zeta \in W^{1,1+m^-}_{loc}(0,T; L^{1+m^-}(K)) \cap L^q_{loc}(0,T; W^{1,q}_0(K))$.

We show that local weak sub-solutions to (1.4) belong to $\mathcal{PDG}^+(\Omega_T)$.

Proof. First we observe that the following inequality holds true:

$$\varphi(x,t,u,a)\,b \leqslant \epsilon\varphi(x,t,u,a)\,a + \varphi(x,t,u,\frac{b}{\epsilon})\,b, \quad (x,t)\in\Omega_T, \quad \epsilon, u, a, b > 0. \tag{4.3}$$

Indeed, if $b \leqslant \epsilon a$, then $\varphi(x,t,u,a)\,b \leqslant \epsilon\varphi(x,t,u,a)\,a$, and if $b \geqslant \epsilon a$, by the fact that $\varphi(\cdot,\cdot,\cdot,a)$ is increasing, we obtain $\varphi(x,t,u,a)\,b \leqslant \varphi(x,t,u,\frac{b}{\epsilon})\,b$, from which the claim follows.

Now, let us test (4.2) by $(u^{m^-} - k^{m^-})_+\zeta^q$, where $\zeta(x,t)$ is a piecewise smooth cutoff function, vanishing on $\partial B_r(y)\times(\tau-\eta,\tau)$, $0 \leqslant \zeta(x,t) \leqslant 1$ and integrate over $B_r(y)\times(\tau-\eta, t_1)$ with $t_1 \in (\tau-\eta,\tau)$. The use of such a test function is justified, modulus a mollified process firstly presented by Kinnunen and Lindqvist [27] for the parabolic p-Laplacian and considered also in the study of doubly nonlinear equations by several other authors (as in [5], [11] and [38] just to name a few), we obtain

$$\begin{aligned} &\int_{B_r(y)\times\{t_1\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \int_{\tau-\eta}^{t_1}\int_{B_r(y)} \Phi(x,t,u,|\nabla(u^{m^-}-k^{m^-})_+|)\,\zeta^q\,dxdt \\ &\quad\leqslant \int_{B_r(y)\times\{\tau-\eta\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \int_{\tau-\eta}^{t_1}\int_{B_r(y)} g_+(u^{m^-}, k^{m^-})\,|\zeta_t|\,dxdt + \\ &\qquad + q\int_{\tau-\eta}^{t_1}\int_{B_r(y)} \varphi(x,t,u,|\nabla(u^{m^-}-k^{m^-})_+|)(u^{m^-}-k^{m^-})_+|\nabla\zeta|\,\zeta^{q-1}dxdt. \end{aligned}$$

We estimate the last term of this inequality using (4.3) with

$$a = |\nabla(u^{m^-} - k^{m^-})_+|, \qquad b = q(u^{m^-} - k^{m^-})_+ |\nabla\zeta|\,\zeta^{-1}, \quad \text{and} \quad \epsilon = \frac{1}{2},$$

to get

$$\begin{aligned}
&\int\limits_{B_r(y)\times\{t_1\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \frac{1}{2}\int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} \Phi(x,t,u,|\nabla(u^{m^-} - k^{m^-})_+|)\,\zeta^q\,dxdt \\
&\quad\leqslant \int\limits_{B_r(y)\times\{\tau-\eta\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} g_+(u^{m^-}, k^{m^-})\,|\zeta_t|\,dxdt+ \\
&\quad\quad + q\int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} \varphi(x,t,u,2q(u^{m^-} - k^{m^-})_+||\nabla\zeta|\zeta^{-1})(u^{m^-} - k^{m^-})_+|\nabla\zeta|\,\zeta^{q-1}dxdt.
\end{aligned}$$

From this by (4.1)

$$\begin{aligned}
&\int\limits_{B_r(y)\times\{t_1\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \frac{K_1}{2}\int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} u^{(m-m^-)(p-1)}|\nabla(u^{m^-} - k^{m^-})_+|^p\,\zeta^q\,dxdt \\
&\quad\leqslant \int\limits_{B_r(y)\times\{\tau-\eta\}} g_+(u^{m^-}, k^{m^-})\,\zeta^q\,dx + \int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} g_+(u^{m^-}, k^{m^-})\,|\zeta_t|\,dxdt+ \\
&\quad\quad + \gamma(q,K_2)\int\limits_{\tau-\eta}^{t_1}\int\limits_{B_r(y)} u^{(n-m^-)(q-1)}(u^{m^-} - k^{m^-})_+^q\,|\nabla\zeta|^q\,dxdt,
\end{aligned} \tag{4.4}$$

that is inequality (1.9) with

$$m_1 = \ldots = m_N = m, \quad n_1 = \ldots = n_N = n, \quad p_1 = \ldots = p_N = p, \quad q_1 = \ldots = q_N = q\,.$$

Conditions (1.11) and $\mathcal{L} > \mathcal{M}$ are translated into

$$\max(1, n(q-1)) \leqslant m(p-1) + (m^- + 1)\frac{p}{N}, \quad \text{or} \quad \max(1, n(q-1)) > m(p-1) + (m^- + 1)\frac{p}{N}.$$

The numbers $\varkappa_s$ defined in (1.13) can be rewritten as

$$\varkappa_s = \begin{cases} ps + N(m(p-1) - 1), & \text{if} \quad n(q-1) < 1, \\ p(s + 1 - n(q-1)) + N(m(p-1) - n(q-1)), & \text{if} \quad n(q-1) > 1. \quad \square \end{cases}$$

4.3. *Example* (1.5) *- doubly nonlinear anisotropic equations*

Here we cover the case of doubly nonlinear anisotropic parabolic equations such as (1.5) in Ω_T, with $p_i > 1$ for all $i = 1 \dots, N$ and $m = \min(m_1, ..., m_N) > 0$.

Definition 4.2. A function

$$\begin{cases} u \in C_{loc}(0, T; L^{1+m}_{loc}(\Omega_T)), \quad u^m \in L^{\vec{p}}(0, T; W^{1,\vec{p}}_{loc}(\Omega)), \\ u^{\frac{m_i(p_i-1)+m}{p_i}}, D_i\big(u^{\frac{m_i(p_i-1)+m}{p_i}}\big) \in L^{p_i}(0, T; L^{p_i}_{loc}(\Omega)), \quad i = 1, ..., N, \end{cases}$$

is a local, weak subsolution to (1.5) if for every compact set $K \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$ the inequality

$$\int_K u\,\zeta\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big\{ - u\,\zeta_t + \sum_{i=1}^N (u^{(m_i-m)(p_i-1)} |D_i u^m|^{p_i-2} D_i u^m\, D_i\zeta \Big\}\, dxdt \leqslant 0, \tag{4.5}$$

is valid for all non-negative testing functions $\zeta \in W^{1,1+m}_{loc}(0, T; L^{1+m}(K)) \cap L^{\vec{p}}_{loc}(0, T; W^{1,\vec{p}}_0(K))$.

We show that local weak sub-solutions to (1.5) belong to $\mathcal{PDG}^+(\Omega_T)$.

Proof. Test (4.5) by $(u^m - k^m)_+\,\zeta^{p^+}$, where $p^+ := \max(p_1, ..., p_N)$ and $\zeta(x, t)$ is a piecewise smooth, cutoff function, vanishing on $\partial K_{\vec{r}}(y) \times (\tau - \eta, \tau)$, $0 \leqslant \zeta(x, t) \leqslant 1$ and integrate over $K_{\vec{r}}(y) \times (\tau - \eta, t_1)$ with $t_1 \in (\tau - \eta, \tau)$. The use of such a test function is justified, modulus a standard averaging process (see [27] and [11] for the details), we obtain

$$\int\limits_{K_{\vec{r}}(y)\times\{t_1\}} g(u^m, k^m)\,\zeta^{p^+}\,dx + \sum_{i=1}^N \int_{\tau-\eta}^{t_1}\int\limits_{K_{\vec{r}}(y)} u^{(m_i-m)(p_i-1)} |D_i(u^m - k^m)_+|^{p_i}\,\zeta^{p^+}\,dxdt$$

$$\leqslant \int\limits_{K_{\vec{r}}(y)\times\{\tau-\eta\}} g(u^m, k^m)\,\zeta^{p^+}\,dx + \int_{\tau-\eta}^{t_1}\int\limits_{K_{\vec{r}}(y)} g(u^m, k^m)\,|\zeta_t|\,dxdt +$$

$$+ \sum_{i=1}^N p_i \int_{\tau-\eta}^{t_1}\int\limits_{K_{\vec{r}}(y)} u^{(m_i-m)(p_i-1)} |D_i(u^m - k^m)_+|^{p_i-1}\,(u^m - k^m)_+\,|D_i\zeta|\,\zeta^{p^+-1} dxdt,$$

from which by the Young inequality we arrive at

$$\int\limits_{K_{\vec{r}}(y)\times\{t_1\}} g(u^m, k^m)\,\zeta^{p^+}\,dx + \frac{1}{2}\sum_{i=1}^N \int_{\tau-\eta}^{t_1}\int\limits_{K_{\vec{r}}(y)} u^{(m_i-m)(p_i-1)} |D_i(u^m - k^m)_+|^{p_i}\,\zeta^{p^+}\,dxdt$$

$$\leqslant \int\limits_{K_{\vec r}(y)\times\{\tau-\eta\}} g(u^m,k^m)\,\zeta^{p^+}\,dx + \int\limits_{\tau-\eta}^{t_1}\int\limits_{K_{\vec r}(y)} g(u^m,k^m)\,|\zeta_t|\,dxdt +$$

$$+\gamma(p_1,\dots,p_N)\sum_{i=1}^{N}\int\limits_{\tau-\eta}^{t_1}\int\limits_{K_{\vec r}(y)} u^{(m_i-m)(p_i-1)}\,(u^m-k^m)_+^{p_i}\,|D_i\zeta|^{p_i}\,dxdt, \tag{4.6}$$

that is inequality (1.9) with $m_i = n_i$, $p_i = q_i$, $i = 1, \dots, N$. Conditions (1.11), $\mathcal{L} > \mathcal{M}$ are translated into

$$\max(1,\Lambda) \leqslant p\Big|\frac{\lambda}{p}\Big| + (m+1)\frac{p}{N}, \quad \text{or} \quad \max(1,\Lambda) > p\Big|\frac{\lambda}{p}\Big| + (m+1)\frac{p}{N},$$

$\Lambda := \max(m_1(p_1-1), \dots, m_N(p_N-1))$. The numbers $\varkappa_s$ defined in (1.13) can be rewritten as

$$\varkappa_s = \begin{cases} ps + N(p|\frac{\lambda}{p}| - 1), & \text{if} \quad \Lambda < 1, \\ p(s+1-\Lambda) + N(p|\frac{\lambda}{p}| - \Lambda), & \text{if} \quad \Lambda > 1. \quad \square \end{cases}$$

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No data was used for the research described in the article.

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