Mathematical Logic

Model theory of monadic predicate logic with the infinity quantifier

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Abstract

This paper establishes model-theoretic properties of ME^∞, a variation of monadic first-order logic that features the generalised quantifier $\exists^\infty$ ('there are infinitely many'). We will also prove analogous versions of these results in the simpler setting of monadic first-order logic with and without equality (ME and M, respectively). For each logic $\mathrm{L} \in \{\mathrm{M}, \mathrm{ME}, \mathrm{ME}^\infty\}$ we will show the following. We provide syntactically defined fragments of L characterising four different semantic properties of L-sentences: (1) being monotone and (2) (Scott) continuous in a given set of monadic predicates; (3) having truth preserved under taking submodels or (4) being truth invariant under taking quotients. In each case, we produce an effectively defined map that translates an arbitrary sentence φ to a sentence φ^p belonging to the corresponding syntactic fragment, with the property that φ is equivalent to φ^p precisely when it has the associated semantic property. As a corollary of our developments, we obtain that the four semantic properties above are decidable for L-sentences.

Keywords Monadic first-order logic · Generalised quantifier · Infinity quantifier · Characterisation theorem · Preservation theorem · Continuity

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1 Introduction

Model theory investigates the relationship between formal languages and semantics. From this perspective, among the most important results are the so called *preservation*[1] or *characterisation theorems*, linking the syntactic shape of formulas to some semantic property. Typically, these results characterise a certain language as the fragment of another, richer language consisting of those formulas that satisfy the given model-theoretic property. In the case of classical first-order logic, notable examples are the Łoś–Tarski theorem, stating that a first-order formula is equivalent to a universal one if and only if the class of its models is closed under taking submodels, and Lyndon's theorem, stating that a first-order formula is equivalent to one for which each occurrence of a relation symbol R is positive if and only if it is monotone with respect to the interpretation of R (see e.g. [17]).

The aim of this paper is to show that similar results also hold for the predicate logic ME^∞ that allows only monadic predicate symbols and no function symbols, but that goes beyond standard first-order logic with equality in that it features the generalised quantifier 'there are infinitely many'.

Generalised quantifiers were introduced by Mostowski in [24], and in a more general sense by Lindström in [21], the main motivation being the observation that standard first-order quantifiers 'there are some' and 'for all' are not sufficient for expressing some basic mathematical concepts. Since then, they have attracted a lot of interest, insomuch that their study constitutes nowadays a well-established field of logic with important ramifications in disciplines such as linguistics and computer science.[2]

Despite the fact that the absence of polyadic predicates clearly restricts its expressive power, monadic first-order logic (with identity) displays nice properties, both from a computational and a model-theoretic point of view. Indeed, the satisfiability problem becomes decidable [4,22], and, in addition to an immediate application of Łoś–Tarski and Lyndon's theorems, one can also obtain a Lindström like characterisation result [26]. Moreover, adding the possibility of quantifying over predicates does not increase the expressiveness of the language [2], meaning that when restricted to monadic predicates, monadic second-order logic collapses into first-order logic.

Concerning monadic first-order logic extended with an infinity quantifier, Mostowski [24] already proved a decidability result, whereas from work of Väänänen [27] we know that its expressive power coincides with that of weak monadic second-order logic restricted to monadic predicates, that is monadic first-order logic extended with a second-order quantifier ranging over finite sets.[3]

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[1] Although it is quite commom to refer to these results as *preservation* theorems, in this paper we shall exclusively use the terminology *characterisation theorem*, reserving the term 'preservation' for the easier part of a characterisation result, which states that formulas in the given syntactic shape have the semantic property.

[2] For an overview see e.g. [6,28,33]. For an introduction to the model theory of generalised quantifiers, the interested reader can consult for instance [29, Chapter 10].

[3] Extensions of monadic first-order logic with other generalised quantifiers have also been studied (see e.g. [7,25]).

Characterisation results and proof outline

A characterisation result involves some fragment $\mathtt{L}_{\mathfrak{P}}$ of a given yardstick logic $\mathtt{L}$, related to a certain semantic property $\mathfrak{P}$. It is usually formulated as

$$\varphi \in \mathtt{L} \text{ has the property } \mathfrak{P} \text{ iff } \varphi \text{ is equivalent to some } \varphi' \in \mathtt{L}_{\mathfrak{P}}. \tag{1}$$

In this work, our main yardstick logic will be $\mathtt{ME}^{\infty}$. Table 1 summarises the semantic properties ($\mathfrak{P}$) we are going to consider, the corresponding expressively complete fragment ($\mathtt{L}_{\mathfrak{P}}$) and the actual characterisation theorem.

The proof of each characterisation theorem is composed of two parts. The first, simpler one concerns the claim that each sentence in the fragment satisfies the concerned property. It is usually proved by a straightforward induction on the structure of the formula. The other direction is the *expressive completeness statement*, stating that within the considered logic, the fragment is expressively complete for the property. Its verification generally requires more effort. In this paper, we will actually verify a stronger expressive completeness statement. Namely, for each semantic property $\mathfrak{P}$ and corresponding fragment $\mathtt{L}_{\mathfrak{P}}$ from Table 1, we are going to provide an effective translation operation $(\cdot)^{p} : \mathtt{ME}^{\infty} \to \mathtt{L}_{\mathfrak{P}}$ such that

$$\text{if } \varphi \in \mathtt{ME}^{\infty} \text{ has the property } \mathfrak{P} \text{ then } \varphi \text{ is equivalent to } \varphi^{p}. \tag{2}$$

The proof of each instance of (2) will follow a uniform pattern, analogous to the one employed in the aim of obtaining similar results in the context of the modal μ-calculus [11,15,19]. The crux of the adopted proof method is the following. Extending known results on monadic first-order logic and using an appropriate version of Ehrenfeucht–Fraïssé games, for each sentence φ in $\mathtt{ME}^{\infty}$ it is possible to compute a logically equivalent sentence in *basic normal norm*. Such normal forms will take the shape of a disjunction $\bigvee \nabla_{\mathtt{ME}^{\infty}}$, where each disjunct $\nabla_{\mathtt{ME}^{\infty}}$ characterises a class of models of φ satisfying the same set of $\mathtt{ME}^{\infty}$-sentences of equal quantifier rank as φ. Based on this, it will therefore be enough to define an effective translation $(\cdot)^{p}$ for sentences in

Table 1 A summary of our characterisation theorems

$\mathfrak{P}$	$\mathtt{L}_{\mathfrak{P}}$	Characterisation theorem
Monotonicity	Positive fragment	Theorem 3
(Definition 16)	$\mathtt{Pos}(\mathtt{ME}^{\infty})$	
Continuity	Continuous fragment	Theorem 4
(Definition 22)	$\mathtt{Con}(\mathtt{ME}^{\infty})$	
Preservation under submodels	Universal fragment	Theorem 5
(Definition 26)	$\mathtt{Univ}(\mathtt{ME}^{\infty})$	
Invariance under quotients	Monadic first-order logic	Theorem 6
(Definition 26)	$\mathtt{M}$	

Table 2 An overview of our expressive completeness and normal form results

	Normal forms	Language		
		M Fact 2	ME Theorem 1	ME^{∞} Theorem 2
Monotonicity	Completeness	Proposition 8	Proposition 9	Proposition 10
	Normal forms	Corollary 1	Corollary 2	Corollary 3
Continuity	Completeness	Proposition 14	Fact 11	Proposition 15
	Normal forms	Corollary 4	–	Corollary 5
Preservation under submodels	Completeness	Proposition 17		
	Normal forms	Corollary 6(1)	Corollary 6(2)	Corollary 6(3)
Invariance under quotients	Completeness	Proposition 18	Proposition 20	
	Normal forms	Fact 2	Corollary 7	

normal form, point-wise in each disjunct $\nabla_{\text{ME}^{\infty}}$, and then verify that it indeed satisfies (2).

As a corollary of the employed proof method, we obtain effective normal forms for sentences satisfying the considered property.

In addition to ME^{∞}, in this paper we also consider monadic first-order logic with and without equality, denoted by ME and M, respectively. Table 2 shows a summary of the expressive completeness and normal form results presented in this paper.

Since the satisfiability problem for ME^{∞} is decidable and the translation $(\cdot)^p$ is effectively computable, we obtain, as an immediate corollary of (2), that for each property $\mathfrak{P}$ listed in Table 1

$$\text{the problem whether a } \text{ME}^{\infty} - \text{sentence satisfies property} \mathfrak{P} \text{or not is decidable.} \quad (3)$$

We consider these decidability results as a byproduct of our characterisation results, and we do not explore, for instance, computational *complexity* questions. Addressing these would involve a study of the complexity of the procedure that brings a formula φ into normal form and then translates it into a formula $(\varphi)^p$ of the required shape. There are easier ways to prove the mentioned decidability results[4] and these may be useful as well to obtain complexity results.

Application of obtained results: the companion paper

Our original motivation to study characterisation results for these logics stems from our interest in so-called *parity automata*: these are finite-state systems that play a crucial role in obtaining decidability and expressiveness results in fixpoint logics and monadic second-order logics over trees and labelled transition systems (see e.g. [30]). Parity automata are specified by a finite set of states A, a distinguished, initial state

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[4] In particular, A. Rabinovitch suggested to us (in personal communication), that our decidability results can be obtained by formulating semantic properties of our monadic predicate logic formulas in certain propositional languages.

$a \in A$, a function Ω assigning to each states a priority (a natural number), and a transition map Δ. In various interesting cases, the co-domain of this transition map is given by a monadic logic in which the set of (monadic) predicates coincides with A. Hence, each monadic logic L induces its own class of automata Aut(L).

A landmark result in this area is Janin and Walukiewicz's theorem stating that the bisimulation-invariant fragment of monadic second-order logic coincides with the modal μ-calculus [19], and the proof of this result is an interesting mix of the theory of parity automata and the model theory of monadic predicate logic. First, normal forms results and characterisation theorems are used to verify that (on tree models) Aut(Pos(ME)) is the class of automata characterising the expressive power of monadic second-order logic [32], whereas Aut(Pos(M)) corresponds to the modal μ-calculus [18], where Pos(L) denote the positive fragment of the monadic logic L. Then, Janin and Walukiewicz' expressiveness theorem is a consequence of these automata characterisations and the fact that positive monadic first-order logic without equality provides the quotient-invariant fragment of positive monadic first-order logic with equality (see Theorem 7).

In our companion paper [10], among other things we provide a Janin–Walukiewicz type characterisation result for *weak* monadic second-order logic. Analogous to the case of full monadic second-order logic discussed previously, our proof crucially employs normal form results and characterisation theorems for ME$^\infty$, as listed in the Tables 1 and 2.

Other versions

Results in this paper first appeared in the first author's PhD thesis ([8, Chapter 5]); this journal version largely expands material first published as part of the conference papers [9,14]. In particular, the whole of Sect. 6 below contains new results.

2 Basics

In this section we provide the basic definitions of the monadic predicate logics that we study in this paper.

Throughout this paper we fix a finite set A of objects that we shall refer to as *(monadic) predicate symbols* or *names*. We shall also assume an infinite set iVar of *individual variables*.

Definition 1 Given a finite set A we define a *(monadic) model* to be a pair $\mathbb{D} = (D, V)$ consisting of a set D, which we call the *domain* of $\mathbb{D}$, and an interpretation or *valuation* $V : A \to \wp(D)$. The class of all models will be denoted by $\mathfrak{M}$.

Remark 1 Note that we make the somewhat unusual choice of allowing the domain of a monadic model to be *empty*. In view of the applications of our results to automata theory (see Sect. 1) this choice is very natural, even if it means that some of our proofs here become more laborious in requiring an extra check. Observe that there is exactly one monadic model based on the empty domain; we shall denote this model as $\mathbb{D}_\varnothing := (\varnothing, \varnothing)$.

Definition 2 Observe that a valuation $V : A \to \wp(D)$ can equivalently be presented via its associated *colouring* $V^\flat : D \to \wp(A)$ given by

$$V^\flat(d) := \{a \in A \mid d \in V(a)\}.$$

We will use these perspectives interchangeably, calling the set $V^\flat(d) \subseteq A$ the *colour* or *type* of d. In case $D = \varnothing$, $V^\flat$ is simply the empty map.

In this paper we study three languages of monadic predicate logic: the languages ME and M of monadic first-order logic with and without equality, respectively, and the extension ME^∞ of ME with the generalised quantifiers $\exists^\infty$ and $\forall^\infty$. Probably the most concise definition of the full language of monadic predicate logic would be given by the following grammar:

$$\varphi ::= a(x) \mid x \approx y \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists x.\varphi \mid \exists^\infty x.\varphi,$$

where $a \in A$ and x and y belong to the set iVar of individual variables. In this set-up we would need to introduce the quantifiers $\forall$ and $\forall^\infty$ as abbreviations of $\neg\exists\neg$ and $\neg\exists^\infty\neg$, respectively. However, for our purposes it will be more convenient to work with a variant of this language where all formulas are in negation normal form; that is, we only permit the occurrence of the negation symbol $\neg$ in front of an atomic formula. In addition, for technical reasons we will add $\bot$ and $\top$ as constants, and we will write $\neg(x \approx y)$ as $x \not\approx y$. Thus we arrive at the following definition of our syntax.

Definition 3 The set $\mathrm{ME}^\infty(A)$ of *monadic formulas* is given by the following grammar:

$$\begin{aligned} \varphi \ ::=\ &\top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \\ &\times \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi \end{aligned}$$

where $a \in A$ and $x, y \in$ iVar. The language $\mathrm{ME}(A)$ of *monadic first-order logic with equality* is defined as the fragment of $\mathrm{ME}^\infty(A)$ where occurrences of the generalised quantifiers $\exists^\infty$ and $\forall^\infty$ are not allowed:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \exists x.\varphi \mid \forall x.\varphi$$

Finally, the language $\mathrm{M}(A)$ of *monadic first-order logic* is the equality-free fragment of $\mathrm{ME}(A)$; that is, atomic formulas of the form $x \approx y$ and $x \not\approx y$ are not permitted either:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \exists x.\varphi \mid \forall x.\varphi$$

In all three languages we use the standard definition of free and bound variables, and we call a formula a *sentence* if it has no free variables. In the sequel we will often use the symbol L to denote either of the languages M, ME or ME^∞.

For each of the languages $\mathrm{L} \in \{\mathrm{M}, \mathrm{ME}, \mathrm{ME}^\infty\}$, we define the *positive fragment* L^+ of L as the language obtained by almost the same grammar as for L, but with the difference that we do not allow negative formulas of the form $\neg a(x)$.

To define the semantics of these languages we need to make a case distinction. For non-empty models we use the standard truth definition, which applies to arbitrary formulas since we can introduce the notion of an assignment, mapping individual variables to elements of the domain. In the case of the empty model, however, it is not possible to define assignments, so here we restrict the truth definition to sentences.

Definition 4 The meaning of sentences in the languages M, ME and ME$^\infty$ is given in the form of a truth relation $\models$. To define this truth relation on a model $\mathbb{D} = (D, V)$, we distinguish cases.

Case $D = \varnothing$: We define the *truth relation* $\models$ on the empty model $\mathbb{D}_\varnothing$ for all formulas that are Boolean combinations of sentences of the form $Qx.\varphi$, where $Q \in \{\exists, \exists^\infty, \forall, \forall^\infty\}$ is a quantifier. The definition is by induction on the complexity of such sentences; the "atomic" clauses, where the sentence is of the form $Qx.\varphi$, are as follows:

$$\mathbb{D}_\varnothing \not\models Qx.\varphi \text{ if } Q \in \{\exists, \exists^\infty\},$$
$$\mathbb{D}_\varnothing \models Qx.\varphi \text{ if } Q \in \{\forall, \forall^\infty\}.$$

The clauses for the Boolean connectives are standard.

Case $D \neq \varnothing$: In the case of a non-empty model $\mathbb{D}$, we extend the truth relation to arbitrary formulas in a standard way, involving assignments of individual variables to elements of the domain. That is, given a model $\mathbb{D} = (D, V)$, an assignment $g : \mathsf{iVar} \to D$ and a formula $\varphi \in \mathtt{ME}^\infty(A)$ we define the *truth* relation $\models$ by a straightforward induction on the complexity of φ. Below we explicitly provide the clauses of the quantifiers:

$$\mathbb{D}, g \models \exists x.\varphi \quad \text{iff} \quad \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for some } d \in D,$$
$$\mathbb{D}, g \models \forall x.\varphi \quad \text{iff} \quad \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all } d \in D,$$
$$\mathbb{D}, g \models \exists^\infty x.\varphi \text{ iff} \quad \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for infinitely many } d \in D,$$
$$\mathbb{D}, g \models \forall^\infty x.\varphi \text{ iff} \quad \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all but at most finitely many } d \in D.$$

The clauses for the atomic formulas and for the Boolean connectives are standard.

In what follows, when discussing the truth of φ on the empty model, we always implicitly assume that φ is a sentence.

As mentioned in the introduction, general quantifiers such as $\exists^\infty$ and $\forall^\infty$ were introduced by Mostowski [24], who proved the decidability for the language obtained by extending M with such quantifiers. The decidability of the full language ME$^\infty$ was then proved by Slomson in [25].[5] The case for M and ME goes back already to [4,22].

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[5] The argument in [25] is given in terms of the so called *Chang quantifier* Q_C given by $(D, V) \models Q_C x.\varphi$ iff the set $\{d \in D \mid (D, V) \models \varphi(d)\}$ of objects that satisfy φ has the same cardinality as D itself. The proof is easily seen to work also for $\exists^\infty$ and $\forall^\infty$, however. Both Mostowski's and Slomson's decidability results can be extended to the case of the empty domain.

Fact 1 *For each logic* $L \in \{M, ME, ME^\infty\}$, *the problem of whether a given* L-*sentence* φ *is satisfiable, is decidable.*

In the remainder of the section we fix some further definitions and notations, starting with some useful syntactic abbreviations.

Definition 5 Given a list $\overline{y} = y_1 \ldots y_n$ of individual variables, we use the formula

$$\text{diff}(\overline{y}) := \bigwedge_{1 \leq m < m' \leq n} (y_m \not\approx y_{m'})$$

to state that the elements $\overline{y}$ are all distinct. An A-*type* is a formula of the form

$$\tau_S(x) := \bigwedge_{a \in S} a(x) \wedge \bigwedge_{a \in A \setminus S} \neg a(x),$$

where $S \subseteq A$. Here and elsewhere we use the convention that $\bigwedge \varnothing = \top$ (and $\bigvee \varnothing = \bot$). The *positive* A-type $\tau_S^+(x)$ only bears positive information, and is defined as

$$\tau_S^+(x) := \bigwedge_{a \in S} a(x).$$

Given a monadic model $\mathbb{D} = (D, V)$ and a subset S of A, we define

$$|S|_\mathbb{D} := |\{d \in D \mid \mathbb{D} \models \tau_S(d)\}|$$

as the number of elements of $\mathbb{D}$ that realise the type τ_S.

We often blur the distinction between the formula $\tau_S(x)$ and the subset $S \subseteq A$, calling S an A-type as well. Note that we have $\mathbb{D} \models \tau_S(d)$ iff $V^\flat(d) = S$, so that we may refer to $V^\flat(d)$ as the *type of* $d \in D$ indeed.

Definition 6 The quantifier rank $\text{qr}(\varphi)$ of a formula $\varphi \in ME^\infty$ (hence also for M and ME) is defined as follows:

$$
\begin{array}{lll}
\text{qr}(\varphi) & := 0 & \text{if } \varphi \text{ is atomic,} \\
\text{qr}(\neg \psi) & := \text{qr}(\psi) & \\
\text{qr}(\psi_1 \heartsuit \psi_2) & := \max\{\text{qr}(\psi_1), \text{qr}(\psi_2)\} & \text{where } \heartsuit \in \{\wedge, \vee\} \\
\text{qr}(Qx.\psi) & := 1 + \text{qr}(\psi), & \text{where } Q \in \{\exists, \forall, \exists^\infty, \forall^\infty\}
\end{array}
$$

Given a monadic logic L we write $\mathbb{D} \equiv_k^L \mathbb{D}'$ to indicate that the models $\mathbb{D}$ and $\mathbb{D}'$ satisfy exactly the same sentences $\varphi \in L$ with $\text{qr}(\varphi) \leq k$. We write $\mathbb{D} \equiv^L \mathbb{D}'$ if $\mathbb{D} \equiv_k^L \mathbb{D}'$ for all k. When clear from context, we may omit explicit reference to L.

Definition 7 A *partial isomorphism* between two models (D, V) and (D', V') is a partial function $f : D \rightharpoonup D'$ which is injective and satisfies that $d \in V(a) \Leftrightarrow f(d) \in V'(a)$ for all $a \in A$ and $d \in \text{Dom}(f)$. Given two sequences $\overline{d} \in D^k$ and $\overline{d}' \in D'^k$ we use $f : \overline{d} \mapsto \overline{d}'$ to denote the partial function $f : D \rightharpoonup D'$ defined as $f(d_i) := d_i'$. We will take care to avoid cases where there exist d_i, d_j such that $d_i = d_j$ but $d_i' \neq d_j'$.

Finally, for future reference we briefly discuss the notion of *Boolean duals*. We first give a concrete definition of a dualisation operator on the set of monadic formulas.

Definition 8 The *(Boolean) dual* $\varphi^\delta \in \text{ME}^\infty(A)$ of $\varphi \in \text{ME}^\infty(A)$ is the formula given by:

$$(a(x))^\delta := a(x) \qquad\qquad (\neg a(x))^\delta := \neg a(x)$$
$$(\top)^\delta := \bot \qquad\qquad (\bot)^\delta := \top$$
$$(x \approx y)^\delta := x \not\approx y \qquad\qquad (x \not\approx y)^\delta := x \approx y$$
$$(\varphi \wedge \psi)^\delta := \varphi^\delta \vee \psi^\delta \qquad\qquad (\varphi \vee \psi)^\delta := \varphi^\delta \wedge \psi^\delta$$
$$(\exists x.\psi)^\delta := \forall x.\psi^\delta \qquad\qquad (\forall x.\psi)^\delta := \exists x.\psi^\delta$$
$$(\exists^\infty x.\psi)^\delta := \forall^\infty x.\psi^\delta \qquad\qquad (\forall^\infty x.\psi)^\delta := \exists^\infty x.\psi^\delta$$

Remark 2 Where $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^\infty\}$, observe that if $\varphi \in \text{L}(A)$ then $\varphi^\delta \in \text{L}(A)$. Moreover, the operator preserves positivity of the predicates, that is, if $\varphi \in \text{L}^+(A)$ then $\varphi^\delta \in \text{L}^+(A)$.

The following proposition states that the formulas φ and φ^δ are Boolean duals. We omit its proof, which is a routine check.

Proposition 1 *Let $\varphi \in \text{ME}^\infty(A)$ be a monadic formula. Then φ and φ^δ are indeed Boolean duals, in the sense that for every monadic model (D, V) we have that*

$$(D, V) \models \varphi \text{ iff } (D, V^c) \not\models \varphi^\delta,$$

where $V^c : A \to \wp(D)$ is the valuation given by $V^c(a) := D \setminus V(a)$.

3 Normal forms

In this section we provide, for each of the logics M, ME and ME$^\infty$, normal forms that will be pivotal for characterising the different fragments of these logics in later sections. Our approach will be game-theoretic, based on Ehrenfeucht–Fraïssé style model comparison games. These games were introduced by Ehrenfeucht [13] to study Fraïssé's analyis of first-order logic using so-called back-and-forth systems. Over the years, similar games have been introduced for various other logics, including extensions of first-order logic with generalised quantifiers [20]. As an important application of Ehrenfeucht–Fraïssé games one may use the notion of a winning strategy to obtain certain normal forms for formulas in the formalism under scrutiny. In the case of monadic first-order logic, one may extract relatively simple normal forms; this observation goes back to (at least) the work of Walukiewicz [32]. Our contribution here is that we use the method to obtain normal forms for the logic ME$^\infty$.

Convention Here and in the sequel it will often be convenient to blur the distinction between lists and sets. For instance, identifying the list $\overline{\mathbf{T}} = T_1 \ldots T_n$ with the set

$\{T_1, \ldots, T_n\}$, we may write statements like $S \in \overline{\mathbf{T}}$ or $\Pi \subseteq \overline{\mathbf{T}}$. Moreover, given a finite set $\Phi = \{\varphi_1, \ldots, \varphi_n\}$, we write $\varphi_1 \wedge \cdots \wedge \varphi_n$ as $\bigwedge \Phi$, and $\varphi_1 \vee \cdots \vee \varphi_n$ as $\bigvee \Phi$. If Φ is empty, we set as usual $\bigwedge \Phi = \top$ and $\bigvee \Phi = \bot$. Finally, notice that we write $\bigvee_{1 \le m < m' \le n}(y_m \approx y_{m'}) \vee \psi$ as $\mathrm{diff}(\overline{\mathbf{y}}) \rightarrow \psi$.

3.1 Normal form for M

We start by introducing a normal form for monadic first-order logic without equality.

Definition 9 Given sets of types $\Sigma, \Pi \subseteq \wp(A)$, we define the following formulas:

$$\nabla_{\mathrm{M}}(\Sigma, \Pi) := \bigwedge_{S \in \Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S(x)$$
$$\nabla_{\mathrm{M}}(\Sigma) \quad := \nabla_{\mathrm{M}}(\Sigma, \Sigma)$$

A sentence of $\mathrm{M}(A)$ is in *basic form* if it is a disjunction of formulas of the form $\nabla_{\mathrm{M}}(\Sigma)$.

Observe that $\nabla_{\mathrm{M}}(\Sigma, \Pi) \equiv \bot$ in case $\Sigma \nsubseteq \Pi$ and that $\nabla_{\mathrm{M}}(\Sigma, \Pi) = \nabla_{\mathrm{M}}(\Sigma) = \forall x.\bot$ if $\Sigma = \Pi = \varnothing$. The meaning of the formula $\nabla_{\mathrm{M}}(\Sigma)$ is that Σ is a complete description of the collection of types that are realised in a monadic model. The formula $\nabla_{\mathrm{M}}(\varnothing)$ distinghuishes the empty model from the non-empty ones.
Every M-formula is effectively equivalent to a formula in basic form.

Fact 2 *There is an effective procedure that transforms an arbitrary M-sentence φ into an equivalent formula φ^* in basic form.*

This observation is easy to prove using Ehrenfeucht–Fraïssé games (proof sketches can be found in [16, Lemma 16.23] and [31, Proposition 4.14]), and the decidability of the satisfiability problem for M (Fact 1). We omit a full proof because it is very similar to the following more complex cases.

3.2 Normal form for ME

Due to the additional expressive power provided by the (in-)equalities, the basic normal forms of ME take a more involved shape than those of M.

Definition 10 We say that a formula $\varphi \in \mathrm{ME}(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi)$ where each disjunct is of the form

$$\nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi) = \exists \overline{\mathbf{x}}.\left(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \rightarrow \bigvee_{S \in \Pi} \tau_S(z))\right)$$

with $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.

We prove that every sentence of monadic first-order logic with equality is equivalent to a formula in basic form. Although this result seems to be folklore, we provide a detailed proof because some of its ingredients will be used later, when we give a normal form for ME^∞. We start by defining the following relation between monadic models.

Definition 11 For every $k \in \mathbb{N}$ we define the relation $\sim_k^=$ on the class $\mathfrak{M}$ of monadic models by putting

$$\mathbb{D} \sim_k^= \mathbb{D}' \iff \forall S \subseteq A \left(|S|_{\mathbb{D}} = |S|_{\mathbb{D}'} < k \text{ or } |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} \geq k \right),$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models.

Intuitively, two models are related by $\sim_k^=$ when their type information coincides 'modulo k'. Later on we prove that this is the same as saying that they cannot be distinguished by a sentence of ME with quantifier rank at most k. As a special case, observe that any two monadic models are related by $\sim_0^=$.

For the moment, we record the following properties of these relations.

Proposition 2 *The following hold:*

1. *The relation $\sim_k^=$ is an equivalence relation of finite index.*
2. *Every $E \in \mathfrak{M}/\sim_k^=$ is characterised by a sentence $\varphi_E^= \in \text{ME}(A)$ with $\text{qr}(\varphi_E^=) = k$.*

Proof We only prove the second statement, and first we consider the case where $k = 0$. The equivalence relation $\sim_0^=$ has the class $\mathfrak{M}$ of all monadic models as its unique equivalence class, so here we may define $\varphi_{\mathfrak{M}}^= := \top$.

From now on we assume that $k > 0$. Take some equivalence class $E \in \mathfrak{M}/\sim_k^=$, and some representative $\mathbb{D} \in E$. Let $S_1, \ldots, S_n \subseteq A$ be the types such that $|S_i|_{\mathbb{D}} = l_i < k$ and let $S_1', \ldots, S_m' \subseteq A$ be those satisfying $|S_i'|_{\mathbb{D}} \geq k$. Note that the union of all the S_i and S_i' yields all the possible A-types, and that if a type S_j is not realised at all, we take $l_j = 0$. Now define

$$\varphi_E^= := \bigwedge_{i \leq n} \left(\exists x_1, \ldots, x_{l_i}.\text{diff}(x_1, \ldots, x_{l_i}) \wedge \bigwedge_{j \leq l_i} \tau_{S_i}(x_j) \right.$$

$$\wedge \, \forall z.\text{diff}(x_1, \ldots, x_{l_i}, z) \rightarrow \neg\tau_{S_i}(z) \bigg)$$

$$\wedge \bigwedge_{i \leq m} \left(\exists x_1, \ldots, x_k.\text{diff}(x_1, \ldots, x_k) \wedge \bigwedge_{j \leq k} \tau_{S_i'}(x_j) \right),$$

where we understand that any conjunct of the form $\exists x_1, \ldots, x_l.\psi$ with $l = 0$ is simply omitted (or, to the same effect, defined as $\top$). It is easy to see that $\text{qr}(\varphi_E^=) = k$ and that $\mathbb{D}' \models \varphi_E^=$ iff $\mathbb{D}' \in E$. Intuitively, $\varphi_E^=$ gives a specification of E "type by type"; in particular observe that $\varphi_{\mathbb{D}_\emptyset}^= \equiv \forall x.\bot$. $\qquad\square$

Next we recall a (standard) notion of Ehrenfeucht–Fraïssé game for ME which will be used to establish the connection between $\sim_k^=$ and $\equiv_k^{\text{ME}}$.

Definition 12 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\text{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. If $\mathbb{D}_i$ is one of the models we use $\mathbb{D}_{-i}$ to denote the other model. A position in this game is a pair of sequences $\overline{s_0} \in D_0^n$ and $\overline{s_1} \in D_1^n$ with $n \leq k$. The game consists of k rounds. To describe a single round of the game, assume that n rounds have passed (where $0 \leq n < k$); round $n + 1$ then consists of the following steps:

1. $\forall$ chooses an element d_i in one of the $\mathbb{D}_i$;
2. $\exists$ responds with an element d_{-i} in the model $\mathbb{D}_{-i}$.

In this way, the sequences $\overline{\mathbf{s_i}} \in D_i^n$ of elements chosen up to round n are extended to $\overline{\mathbf{s_i}}' := \overline{\mathbf{s_i}} \cdot d_i \in D_i^{n+1}$. Player $\exists$ survives the round iff she does not get stuck and the function $f_{n+1} : \overline{\mathbf{s_0}}' \mapsto \overline{\mathbf{s_1}}'$ is a partial isomorphism of monadic models. Finally, player $\exists$ wins the match iff she survives all k rounds.

Given $n \leq k$ and $\overline{\mathbf{s_i}} \in D_i^n$ such that $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism, we write $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ to denote the (initialised) game where n moves have been played and $k - n$ moves are left to be played.

Proposition 3 *The following are equivalent:*

(1) $\mathbb{D}_0 \equiv_k^{\mathrm{ME}} \mathbb{D}_1$,
(2) $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$,
(3) $\exists$ *has a winning strategy in* $EF_k^=(\mathbb{D}_0, \mathbb{D}_1)$.

Proof The implication from (1) to (2) is direct by Proposition 2. For the implication from (2) to (3) we give a winning strategy for $\exists$ in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ by showing the following claim.

Claim 1 Let $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$ and $\overline{\mathbf{s_i}} \in D_i^n$ be such that $n < k$ and $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism; then $\exists$ can survive one more round in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$.

Proof of Claim 1 Let $\forall$ pick $d_i \in D_i$ such that the type of d_i is $T \subseteq A$. If d_i had already been played then $\exists$ picks the same element as before and $f_{n+1} = f_n$. If d_i is new and $|T|_{\mathbb{D}_i} \geq k$ then, as at most $n < k$ elements have been played, there is always some new $d_{-i} \in D_{-i}$ that $\exists$ can choose to match d_i. If $|T|_{\mathbb{D}_i} = m < k$ then we know that $|T|_{\mathbb{D}_{-i}} = m$. Therefore, as d_i is new and f_n is injective, there must be a $d_{-i} \in D_{-i}$ that $\exists$ can choose. $\square$

The implication from (3) to (1) is a standard result [12, Corollary 2.2.9] which we prove anyway because we will need to extend it later. We prove the following loaded statement.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \ldots, z_n) \in \mathrm{ME}(A)$ be such that $\mathrm{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in the game $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim 2 If φ is atomic the claim holds because of $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ being a partial isomorphism. The Boolean cases are straightforward. Let $\varphi(z_1, \ldots, z_n) = \exists x.\psi(z_1, \ldots, z_n, x)$ and suppose $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, there exists $d_0 \in D_0$ such that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$. By hypothesis we know that $\exists$ has a winning strategy for $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ picks $d_0 \in D_0$ she can respond with some $d_1 \in D_1$ and have a winning strategy for $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}} \cdot d_0, \overline{\mathbf{s_1}} \cdot d_1)$. By induction hypothesis, because $\mathrm{qr}(\psi) \leq k - (n + 1)$, we have that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$ iff $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ and hence $\mathbb{D}_1 \models \exists x.\psi(\overline{\mathbf{s_1}}, x)$. The opposite direction is proved by a symmetric argument. $\square$

We finish the proof of the proposition by combining these two claims. $\square$

Theorem 1 *There is an effective procedure that transforms an arbitrary ME-sentence φ into an equivalent formula φ^* in basic form.*

Proof Let $\operatorname{qr}(\psi) = k$ and let $[\![\psi]\!]$ be the class of models satisfying ψ. As $\mathfrak{M}/{\equiv_k^{\mathrm{ME}}}$ is the same as $\mathfrak{M}/{\sim_k^=}$ by Proposition 3, it is easy to see that ψ is equivalent to $\bigvee\{\varphi_E^= \mid E \in [\![\psi]\!]/{\sim_k^=}\}$. Now it only remains to see that each $\varphi_E^=$ is equivalent to the sentence $\nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi)$ for some $\overline{\mathbf{T}}, \Pi \subseteq \wp(A)$ with $\Pi \subseteq \overline{\mathbf{T}}$.

The crucial observation is that we will use $\overline{\mathbf{T}}$ and Π to give a specification of the types "element by element". Take some representative $\mathbb{D}$ of the equivalence class E. Let $S_1, \ldots, S_n \subseteq A$ be the types such that $|S_i|_{\mathbb{D}} = l_i < k$ and $S'_1, \ldots, S'_m \subseteq A$ those satisfying $|S'_j|_{\mathbb{D}} \geq k$. The size of the sequence $\overline{\mathbf{T}}$ is defined to be $(\sum_{i=1}^n l_i) + k \times m$ where $\overline{\mathbf{T}}$ contains exactly l_i occurrences of type S_i and at least k occurrences of each S'_j. On the other hand we set $\Pi := \{S'_1, \ldots, S'_m\}$. It is straightforward to check that $\Pi \subseteq \overline{\mathbf{T}}$ and $\varphi_E^=$ is equivalent to $\nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi)$. (Observe however, that the quantifier rank of the latter is only bounded by $k \times 2^{|A|} + 1$.) In particular $\varphi_{\mathbb{D}_\varnothing}^= \equiv \nabla_{\mathrm{ME}}(\varnothing, \varnothing) = \forall x. \bot$.

The effectiveness of the procedure follows from the fact that, given the previous bound on the size of a normal form, it is possible to non-deterministically guess the number of disjuncts, types and associated parameters for each conjunct and repeatedly check whether the formulas φ and $\bigvee \nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi)$ are equivalent, this latter problem being decidable by Fact 1. □

3.3 Normal form for ME$^\infty$

The logic ME$^\infty$ extends ME with the capacity to tear apart finite and infinite sets of elements. This is reflected in the normal form for ME$^\infty$ by adding extra information to the normal form of ME.

Definition 13 We say that a formula $\varphi \in \mathrm{ME}^\infty(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where each disjunct is of the form

$$\nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi) \wedge \nabla_\infty(\Sigma)$$

where $\nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi)$ is as in Definition 10, and

$$\nabla_\infty(\Sigma) := \bigwedge_{S \in \Sigma} \exists^\infty y. \tau_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y).$$

Here $\overline{\mathbf{T}} \in \wp(A)^k$ for some k, and $\Pi, \Sigma \subseteq \wp(A)$ are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$.

Observe that basic formulas of ME are *not* basic formulas of ME$^\infty$.

Intuitively, the formula $\nabla_\infty(\Sigma)$ says that (1) for every type $S \in \Sigma$, there are infinitely many elements satisfying S and (2) only finitely many elements do not satisfy any type in Σ. As a special case, the formula $\nabla_\infty(\varnothing)$ expresses that the model is finite. A short argument reveals that, intuitively, every disjunct of the form $\nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ expresses that any monadic model satisfying it admits a partition of its domain in three parts:

(i)   distinct elements $t_1, \ldots, t_n$ with respective types $T_1, \ldots, T_n$,
(ii) finitely many elements whose types belong to Π, and
(iii) for each $S \in \Sigma$, infinitely many elements with type S.

Note that this partition is not necessarily unique, unless we modify item (ii) so that it mentions finitely many elements whose type belongs to $\Pi \setminus \Sigma$.

  In the same way as before, we define an equivalence relation $\sim_k^\infty$ on monadic models which refines $\sim_k^=$ by adding information about the (in-)finiteness of the types.

Definition 14 For every $k \in \mathbb{N}$ we define the relation $\sim_k^\infty$ on the class $\mathfrak{M}$ of monadic models by putting

$$\mathbb{D} \sim_k^\infty \mathbb{D}' \iff \forall S \subseteq A \left(|S|_\mathbb{D} = |S|_{\mathbb{D}'} < k \text{ or } k \leq |S|_\mathbb{D}, |S|_{\mathbb{D}'} < \omega \text{ or } |S|_\mathbb{D}, |S|_{\mathbb{D}'} \geq \omega \right),$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models.

  As before, with this definition we find that $\mathbb{D} \sim_0^\infty \mathbb{D}'$ holds always.

Proposition 4 *The following hold, for every $k \in \mathbb{N}$:*

1. *The relation $\sim_k^\infty$ is an equivalence relation of finite index.*
2. *The relation $\sim_k^\infty$ is a refinement of $\sim_k^=$.*
3. *Every $E \in \mathfrak{M}/\sim_k^\infty$ is characterised by a sentence $\varphi_E^\infty \in \mathrm{ME}^\infty(A)$ with $\mathrm{qr}(\varphi) = k$.*

Proof We only prove the last item, for $k > 0$. Let $E \in \mathfrak{M}/\sim_k^\infty$ and let $\mathbb{D} \in E$ be a representative of the class. Let $E' \in \mathfrak{M}/\sim_k^=$ be the equivalence class of $\mathbb{D}$ with respect to $\sim_k^=$. Let $S_1, \ldots, S_n \subseteq A$ be all the types such that $|S_i|_\mathbb{D} \geq \omega$, and define

$$\varphi_E^\infty := \varphi_{E'}^= \wedge \nabla_\infty(\{S_1, \ldots, S_n\}).$$

It is not difficult to see that $\mathrm{qr}(\varphi_E^\infty) = k$ and that $\mathbb{D}' \models \varphi_E^\infty$ iff $\mathbb{D}' \in E$. In particular $\varphi_{\mathbb{D}_\varnothing}^\infty \equiv \nabla_{\mathrm{ME}^\infty}(\varnothing, \varnothing, \varnothing) = \forall x.\bot \wedge \forall^\infty y.\bot$. $\qquad\square$

  Now we give a version of the Ehrenfeucht–Fraïssé game for ME^∞. This game, which extends $\mathrm{EF}_k^=$ with moves for $\exists^\infty$, is the adaptation of the Ehrenfeucht–Fraïssé game for monotone generalised quantifiers found in [20] to the case of full monadic first-order logic.

Definition 15 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. A position in this game is a pair of sequences $\overline{\mathbf{s_0}} \in D_0^n$ and $\overline{\mathbf{s_1}} \in D_1^n$ with $n \leq k$. The game consists of k rounds. To describe a single round of the game, assume that n rounds have passed (where $0 \leq n < k$); round $n+1$ then consists of the following steps.

  First $\forall$ chooses to perform one of the following types of moves:

(a) second-order move:

    1. $\forall$ chooses an infinite set $X_i \subseteq D_i$;
    2. $\exists$ responds with an infinite set $X_{-i} \subseteq D_{-i}$;

Fig. 1 Elements of type S have coloured background
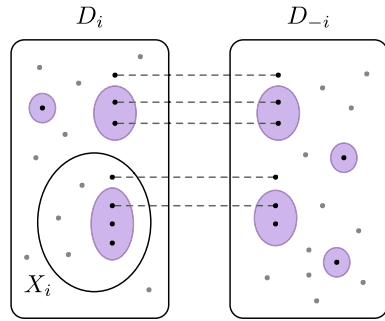
3. $\forall$ chooses an element $d_{-i} \in X_{-i}$;
4. $\exists$ responds with an element $d_i \in X_i$.

(b) first-order move:

1. $\forall$ chooses an element $d_i \in D_i$;
2. $\exists$ responds with an element $d_{-i} \in D_{-i}$.

The sequences $\overline{s_i} \in D_i^n$ of elements chosen up to round n are then extended to $\overline{s_i}' := \overline{s_i} \cdot d_i \in D_i^{n+1}$. $\exists$ survives the round iff she does not get stuck and the function $f_{n+1} : \overline{s_0}' \mapsto \overline{s_1}'$ is a partial isomorphism of monadic models.

Note that the only items that are recorded in a play of this game are the *objects* picked by the players; the subsets that are picked in a round starting with a second-order move by $\forall$, are forgotten as soon as the players have selected inhabitants of these sets (Fig. 1).

Proposition 5 *The following are equivalent:*

(1) $\mathbb{D}_0 \equiv_k^{\mathrm{ME}^\infty} \mathbb{D}_1$,
(2) $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$,
(3) $\exists$ *has a winning strategy in* $EF_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$.

Proof The implication from (1) to (2) is direct by Proposition 4. For the implication from (2)) to (3) we show the following.

Claim 1 Let $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ and $\overline{s_i} \in D_i^n$ be such that $n < k$ and $f_n : \overline{s_0} \mapsto \overline{s_1}$ is a partial isomorphism. Then $\exists$ can survive one more round in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1) @ (\overline{s_0}, \overline{s_1})$.

Proof of Claim 1 We focus on the second-order moves because the first-order moves are the same as in the corresponding Claim of Proposition 3. Let $\forall$ choose an infinite set $X_i \subseteq D_i$, we would like $\exists$ to choose an infinite set $X_{-i} \subseteq D_{-i}$ such that the following conditions hold:

(a) The map f_n is a well-defined partial isomorphism between the restricted monadic models $\mathbb{D}_0 {\restriction} X_0$ and $\mathbb{D}_1 {\restriction} X_1$,
(b) For every type S there is an element $d \in X_i$ of type S which is *not* connected by f_n iff there is such an element in X_{-i}.

First we prove that such a set X_{-i} exists. To satisfy item (a) $\exists$ just needs to add to X_{-i} the elements connected to X_i by f_n; this is not a problem.

For item (b) we proceed as follows: for every type S such that there is an element $d \in X_i$ of type S, we add a new element $d' \in D_{-i}$ of type S to X_{-i}. To see that this is always possible, observe first that $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ implies $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$. Using the properties of this relation, we divide in two cases:

- If $|S|_{D_i} \geq k$ we know that $|S|_{D_{-i}} \geq k$ as well. From the elements of D_{-i} of type S, at most $n < k$ are used by f_n. Hence, there is at least one $d' \in D_{-i}$ of type S to choose from.
- If $|S|_{D_i} < k$ we know that $|S|_{D_i} = |S|_{D_{-i}}$. From the elements of D_i of type S, at most $|S|_{D_i} - 1$ are used by f_n. (The reason for the '-1' is that we are assuming that we have just chosen a $d \in X_i$ which is not in f_n.) Using that $|S|_{D_i} = |S|_{D_{-i}}$ and that f_n is a partial isomorphism we can again conclude that there is at least one $d' \in D_{-i}$ of type S to choose from.

Finally, we need to show that $\exists$ can choose X_{-i} to be infinite. To see this, observe that X_i is infinite, while there are only finitely many types. Hence there must be some S such that $|S|_{X_i} \geq \omega$. It is then safe to add infinitely many elements for S in X_{-i} while considering point (b). Moreover, the existence of infinitely many elements satisfying S in D_{-i} is guaranteed by $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$.

Having shown that $\exists$ can choose a set X_{-i} satisfying the above conditions, it is now clear that using point (b) $\exists$ can survive the "first-order part" of the second-order move we were considering. This finishes the proof of the claim. $\square$

Returning to the proof of Proposition 5, for the implication from (3) to (1) we prove the following.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \ldots, z_n) \in \mathtt{ME}^\infty(A)$ be such that $\mathtt{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1) @ (\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim 2 All the cases involving operators of $\mathtt{ME}$ are the same as in Proposition 3. We prove the inductive case for the generalised quantifier. Let $\varphi(z_1, \ldots, z_n)$ be of the form $\exists^\infty x . \psi(z_1, \ldots, z_n, x)$ and let $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, the set $X_0 := \{d_0 \in D_0 \mid \mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)\}$ is infinite.

By assumption $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1) @ (\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ plays a second-order move by picking $X_0 \subseteq D_0$ she can respond with some infinite set $X_1 \subseteq D_1$. We claim that $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ for every $d_1 \in X_1$. First observe that if this holds then the set $X_1' := \{d_1 \in D_1 \mid \mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)\}$ must be infinite, and hence $\mathbb{D}_1 \models \exists^\infty x . \psi(\overline{\mathbf{s_1}}, x)$.

Assume, for a contradiction, that $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ for some $d_1' \in X_1$. Let $\forall$ play this d_1' as the second part of his move. Then, as $\exists$ has a winning strategy, she will respond with some $d_0' \in X_0$ for which she has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1) @ (\overline{\mathbf{s_0}} \cdot d_0', \overline{\mathbf{s_1}} \cdot d_1')$. But then by our induction hypothesis, which applies since $\mathtt{qr}(\psi) \leq k - (n+1)$, we may infer from $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ that $\mathbb{D}_0 \not\models \psi(\overline{\mathbf{s_0}}, d_0')$. This clearly contradicts the fact that $d_0' \in X_0$. $\square$

Combining the claims finishes the proof of the proposition. $\square$

Theorem 2 *There is an effective procedure that transforms an arbitrary* ME^∞*-sentence* φ *into an equivalent formula* φ^* *in basic form.*

Proof This can be proved using the same argument as in Theorem 1 but based on Proposition 5. Hence we only focus on showing that $\varphi_E^\infty \equiv \nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for some $\overline{\mathbf{T}}, \Pi, \Sigma \subseteq \wp(A)$ such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$, where φ_E^∞ is the sentence characterising $E \in \mathfrak{M}/\sim_k^\infty$ from Proposition 4(2). Recall that

$$\varphi_E^\infty = \varphi_{E'}^{\overline{=}} \wedge \nabla_\infty(\Sigma)$$

where Σ is the collection of types that are realised by infinitely many elements. Using Theorem 1 on $\varphi_{E'}^{\overline{=}}$ we know that this is equivalent to

$$\varphi_E^\infty = \nabla_{\mathrm{ME}}(\overline{\mathbf{T}}, \Pi) \wedge \nabla_\infty(\Sigma)$$

where $\Pi \subseteq \overline{\mathbf{T}}$. Observe that we may assume that $\Sigma \subseteq \Pi$, otherwise the formula would be inconsistent. We can then conclude that $\varphi_E^\infty \equiv \nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. □

4 Monotonicity

In this section we provide our first characterisation result, which concerns the notion of monotonicity.

Definition 16 Let V and V' be two A-valuations on the same domain D, and $B \subseteq A$. Then we say that V' is a *B-extension of* V, notation: $V \leq_B V'$, if $V(b) \subseteq V'(b)$ for every $b \in B$, and $V(a) = V'(a)$ for every $a \in A \setminus B$.

Given a monadic logic $\mathtt{L}$ and a formula $\varphi \in \mathtt{L}(A)$ we say that φ is *monotone in* $B \subseteq A$ if

$$(D, V), g \models \varphi \text{ and } V \leq_B V' \text{ imply } (D, V'), g \models \varphi, \tag{4}$$

for every pair of monadic models (D, V) and (D, V') and every assignment $g : \mathsf{iVar} \to D$.

Remark 3 It is easy to prove that a formula is monotone in $B \subseteq A$ if and only if it is monotone in every $b \in B$.

The semantic property of monotonicity can usually be linked to the syntactic notion of positivity. Indeed, for many logics, a formula φ is monotone in $a \in A$ iff φ is equivalent to a formula where all occurrences of a have a positive polarity, that is, they are situated in the scope of an even number of negations.

Definition 17 For $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$ we define the fragment of A-formulas that are *positive* in all predicates in B, in short: the *B-positive formulas* by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x.\varphi \mid \forall x.\varphi,$$

where $b \in B$ and $\psi \in \text{L}(A \setminus B)$ (that is, there are no occurrences of any $b \in B$ in ψ). Similarly, the B-positive fragment of ME^∞ is given by

$$\varphi ::= \psi \mid b(x) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi,$$

where $b \in B$ and $\psi \in \text{ME}^\infty(A \setminus B)$.

In all three cases, we let $\text{Pos}_B(\text{L}(A))$ denote the set of B-positive sentences.

Note that the difference between the fragments $\text{Pos}_B(\text{M}(A))$ and $\text{Pos}_B(\text{ME}(A))$ lies in the fact that in the latter case, the 'B-free' formulas ψ may contain the equality symbol, both positively ($\approx$) and negatively ($\not\approx$). Clearly $\text{Pos}_A(\text{L}(A)) = \text{L}^+$.

Remark 4 Perhaps a more natural presentation of the fragment $\text{Pos}_B(\text{L}(A))$ would be via the following grammar (in the case of M, the other cases would be similar):

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid b(x) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x.\varphi \mid \forall x.\varphi, \tag{5}$$

where $a \in A \setminus B$ and $b \in B$. It is not difficult to see that the above grammar produces the same formulas as the one in Definition 17. The latter presentation, however, is more convenient in the context of our companion paper [10], and in line with the definition of the fragments $\text{Con}_B(\text{ME}^\infty(A))$ studied in the next section.

Theorem 3 *Let φ be a sentence of the monadic logic $\text{L}(A)$, where $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^\infty\}$. Then φ is monotone in a set $B \subseteq A$ if and only if there is an equivalent formula $\varphi^\oslash \in \text{Pos}_B(\text{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \text{L}(A)$ has this property or not.*

The 'easy' direction of the first claim of the theorem is taken care of by the following proposition.

Proposition 6 *Every formula $\varphi \in \text{Pos}_B(\text{L}(A))$ is monotone in B, where L is one of the logics $\{\text{M}, \text{ME}, \text{ME}^\infty\}$.*

Proof The case for $D = \varnothing$ being immediate, we assume $D \neq \varnothing$. The proof is a routine argument by induction on the complexity of φ. That is, we show by induction, that any formula φ in the B-positive fragment (which may not be a sentence) satisfies (4), for every monadic model (D, V), valuation $V' \geq_B V$ and assignment $g : \text{iVar} \to D$. We focus on the generalised quantifiers. Let $(D, V), g \models \varphi$ and $V \leq_B V'$.

- Case $\varphi = \exists^\infty x.\varphi'(x)$. By definition there exists an infinite set $I \subseteq D$ such that for all $d \in I$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in I$. Therefore $(D, V'), g \models \exists^\infty x.\varphi'(x)$.
- Case $\varphi = \forall^\infty x.\varphi'(x)$. Hence there exists $C \subseteq D$ such that for all $d \in C$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$ and $D \setminus C$ is *finite*. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in C$. Therefore $(D, V'), g \models \forall^\infty x.\varphi'(x)$.

This finishes the proof.   □

The 'hard' direction of the first claim of Theorem 3 states that the fragment $\textsc{Pos}_B(\textsc{M})$ is complete for monotonicity in B. In order to prove this, we need to show that every sentence which is monotone in B is equivalent to some formula in $\textsc{Pos}_B(\textsc{M})$. We actually are going to prove a stronger result.

Proposition 7 *Let $\textsc{L}$ be one of the logics $\{\textsc{M}, \textsc{ME}, \textsc{ME}^\infty\}$. There exists an effective translation $(-)^\oslash : \textsc{L}(A) \to \textsc{Pos}_B(\textsc{L}(A))$ such that a sentence $\varphi \in \textsc{L}(A)$ is monotone in $B \subseteq A$ only if $\varphi \equiv \varphi^\oslash$.*

We prove the three manifestations of Proposition 7 separately, in three respective subsections.

Proof of Theorem 3 The first claim of the Theorem is an immediate consequence of Proposition 7. By effectiveness of the translation and Fact 1, it is decidable whether a sentence $\varphi \in \textsc{L}(A)$ is monotone in $B \subseteq A$ or not. □

The following definition will be used throughout in the remainder of the section.

Definition 18 Given $S \subseteq A$ and $B \subseteq A$ we use the following notation

$$\tau_S^B(x) := \bigwedge_{b \in S} b(x) \wedge \bigwedge_{b \in A \backslash (S \cup B)} \neg b(x),$$

for what we call the *B-positive A-type* τ_S^B.

Intuitively, τ_S^B works almost like the *A*-type τ_S, the difference being that τ_S^B discards the negative information for the names in B. If $B = \{a\}$ we write τ_S^a instead of $\tau_S^{\{a\}}$. Observe that with this notation, τ_S^+ is equivalent to τ_S^A.

4.1 Monotone fragment of M

In this subsection we prove the M-variant of Proposition 7. That is, we give a translation that constructively maps arbitrary sentences into $\textsc{Pos}_B(\textsc{M})$ and moreover preserves truth iff the given sentence is monotone in B. To formulate the translation we need to introduce some new notation.

Definition 19 Let $B \subseteq A$ be a finite set of names and $\Sigma \subseteq \wp(A)$ be types. The *B*-positive variant of $\nabla_\textsc{M}(\Sigma)$ is given as follows:

$$\nabla_\textsc{M}^B(\Sigma) := \bigwedge_{S \in \Sigma} \exists x. \tau_S^B(x) \wedge \forall x. \bigvee_{S \in \Sigma} \tau_S^B(x).$$

We also introduce the following generalised forms of the above notation, with types $\Pi \subseteq \wp(A)$:

$$\nabla_\textsc{M}^B(\Sigma, \Pi) := \bigwedge_{S \in \Sigma} \exists x. \tau_S^B(x) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S^B(x).$$

The *positive* variants of the above notations are defined as $\nabla_{\mathtt{M}}^{+}(\Sigma):=\nabla_{\mathtt{M}}^{A}(\Sigma)$ and $\nabla_{\mathtt{M}}^{+}(\Sigma, \Pi):=\nabla_{\mathtt{M}}^{A}(\Sigma, \Pi)$.

Proposition 8 *There exists an effective translation* $(-)^{\oslash} : \mathtt{M}(A) \to \mathtt{Pos}_B(\mathtt{M}(A))$ *such that a sentence* $\varphi \in \mathtt{M}(A)$ *is monotone in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^{\oslash}$.

Proof To define the translation, by Fact 2, we may assume without loss of generality that φ is in the normal form $\bigvee \nabla_{\mathtt{M}}(\Sigma)$. We define the translation as

$$\left(\bigvee \nabla_{\mathtt{M}}(\Sigma) \right)^{\oslash} := \bigvee \nabla_{\mathtt{M}}^{B}(\Sigma).$$

From the construction it is clear that $\varphi^{\oslash} \in \mathtt{Pos}_B(\mathtt{M}(A))$. Then, the right-to-left direction of the proposition is immediate by Proposition 6. For the left-to-right direction, assume that φ is monotone in B. It suffices to prove that $(D, V) \models \varphi$ if and only if $(D, V) \models \varphi^{\oslash}$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D, V) \models \varphi^{\oslash}$ and let Σ be such that $(D, V) \models \nabla_{\mathtt{M}}^{B}(\Sigma)$. If $D = \varnothing$, then $\Sigma = \varnothing$ and $\nabla_{\mathtt{M}}^{B}(\Sigma) = \nabla_{\mathtt{M}}(\Sigma)$. Hence, assume $D \neq \varnothing$, and clearly $\Sigma \neq \varnothing$.

We claim the existence of a surjective map $T : D \to \Sigma$ such that $(D, V) \models \tau_{T_d}^{B}(d)$, for every d in D. To see this, first note that, because of the existential part of $\nabla_{\mathtt{M}}^{B}(\Sigma)$, every type $S \in \Sigma$ has a 'B-witness' in $\mathbb{D}$, that is, an element $d_S \in D$ such that $(D, V) \models \tau_{S}^{B}(d_S)$. It is in fact safe to assume that all these witnesses are *distinct* (this is because (D, V) can be proved to be $\mathtt{M}$-equivalent to such a model, cf. Proposition 18). This means that we may define $T : d_S \mapsto S$, which will ensure the surjectivity of T. It remains to extend the definition of T to the elements of D that are not of the form d_S for some $S \in \Sigma$. But this is easy: because of the universal part of $\nabla_{\mathtt{M}}^{B}(\Sigma)$, we may find for every element d in D some type S_d in Σ such that $(D, V) \models \tau_{S_d}^{B}(d)$. Putting these observations together, it should be clear that the map $T : D \to \wp(A)$ given by $T(d):=S$ if $d = d_S$ for some $S \in \Sigma$, and $T(d):=S_d$ otherwise, satisfies the requirements.

Now let $U : A \to \wp(D)$ be the valuation of which T is the associated colouring, cf. Definition 2. That is, we put $U(a):=\{d \in D \mid a \in T_d\}$. The definition of U is tailored towards the claim that

$$(D, U) \models \nabla_{\mathtt{M}}(\Sigma). \tag{6}$$

To see why this is the case, first take an arbitrary $d \in D$; it is immediate by the definitions that $(D, U) \models \tau_{T_d}(d)$, and since $T_d \in \Sigma$, this takes care of the universal conjunct of the formula $\nabla_{\mathtt{M}}(\Sigma)$. Now take an arbitrary $S \in \Sigma$. It follows by the surjectivity of T that there is a $d \in D$ such that $\Sigma = T_d$; and since we saw that $(D, U) \models \tau_{T_d}(d)$, this takes care of the existential part.

Clearly it follows from (6) that $(D, U) \models \varphi$. But then by monotonicity of φ, we are done if we can show that

$$U \leq_B V. \tag{7}$$

To see this, observe that for $a \in A \setminus B$ we have the following equivalences:

$$d \in U(a) \iff a \in T_d \iff (D, V) \models a(d) \iff d \in V(a),$$

while for $b \in B$ we can prove

$$d \in U(b) \iff b \in T_d \implies (D, V) \models b(d) \iff d \in V(b).$$

This suffices to prove (7), and finishes the proof of the Proposition. □

A careful analysis of the translation gives us the following corollary, providing normal forms for the monotone fragment of M.

Corollary 1 *For any sentence $\varphi \in M(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla_M^B(\Sigma)$ for some types $\Sigma \subseteq \wp(A)$.*
2. *The formula φ is monotone in every $a \in A$ iff φ is equivalent to a formula $\bigvee \nabla_M^+(\Sigma)$ for some types $\Sigma \subseteq \wp(A)$.*

In both cases the normal forms are effective.

4.2 Monotone fragment of ME

In order to prove the ME-variant of Proposition 7, we need to introduce some new notation.

Definition 20 Let $B \subseteq A$ be a finite set of names, $\Pi \subseteq \wp(A)$ be some types, and $\overline{T} \in \wp(A)^k$ some list of types. The B-monotone variant of $\nabla_{ME}(\overline{T}, \Pi)$ is given as follows:

$$\nabla_{ME}^B(\overline{T}, \Pi) := \exists \overline{x}.\left(\text{diff}(\overline{x}) \wedge \bigwedge_i \tau_{T_i}^B(x_i) \wedge \forall z.(\text{diff}(\overline{x}, z) \to \bigvee_{S \in \Pi} \tau_S^B(z))\right).$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant $\nabla_{ME}^+(\overline{T}, \Pi)$ of $\nabla_{ME}(\overline{T}, \Pi)$ is defined as above but with $+$ in place of B.

Proposition 9 *There exists an effective translation $(-)^{\oslash} : ME(A) \to \text{Pos}_B(ME(A))$ such that a sentence $\varphi \in ME(A)$ is monotone in B if and only if $\varphi \equiv \varphi^{\oslash}$.*

Proof In proposition 10 this result is proved for ME^{∞} (i.e., ME extended with generalised quantifiers). It is not difficult to adapt the proof for ME. The translation is defined as follows. By Theorem 1 we may assume without loss of generality that φ is in basic normal form $\bigvee \nabla_{ME}(\overline{T}, \Pi)$. Then $\varphi^{\oslash} := \bigvee \nabla_{ME}^B(\overline{T}, \Pi)$. □

Combining the normal form for ME and the proof of the above proposition, we obtain a normal form for the monotone fragment of ME.

Corollary 2 *For any sentence $\varphi \in \text{ME}(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^B_{\text{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*
2. *The formula φ is monotone in all $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\text{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*

In both cases, normal forms are effective.

4.3 Monotone fragment of ME^∞

First, in this case too we introduce some notation for the positive variant of a sentence in normal form.

Definition 21 Let $B \subseteq A$ be a finite set of names, $\Sigma, \Pi \subseteq \wp(A)$ be some types, and $\overline{\mathbf{T}} \in \wp(A)^k$ some list of types. The B-positive variant of $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is given as follows:

$$\nabla^B_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla^B_{\text{ME}}(\overline{\mathbf{T}}, \Pi) \wedge \nabla^B_\infty(\Sigma)$$
$$\nabla^B_\infty(\Sigma) := \bigwedge_{S \in \Sigma} \exists^\infty y . \tau^B_S(y) \wedge \forall^\infty y . \bigvee_{S \in \Sigma} \tau^B_S(y).$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant of $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is defined as $\nabla^+_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla^A_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

We are now ready to proceed with the proof of the ME^∞-variant of Proposition 7 and to give the translation.

Proposition 10 *There is an effective translation $(-)^\varnothing : \text{ME}^\infty(A) \to \text{Pos}_B(\text{ME}^\infty(A))$ such that a sentence $\varphi \in \text{ME}^\infty(A)$ is monotone in B if and only if $\varphi \equiv \varphi^\varnothing$.*

Proof By Theorem 2, we assume that φ is in the normal form $\bigvee \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = \nabla_{\text{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$ for some sets and list of types $\Pi, \Sigma, \overline{\mathbf{T}} \subseteq \wp(A)$ with $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. For the translation we define

$$\left(\bigvee \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) \right)^\varnothing := \bigvee \nabla^B_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma).$$

From the construction it is clear that $\varphi^\varnothing \in \text{Pos}_B(\text{ME}^\infty(A))$. Then, the right-to-left direction of the proposition is immediate by Proposition 6. For the left-to-right direction, assume that φ is monotone in B, we have to prove that $(D, V) \models \varphi$ if and only if $(D, V) \models \varphi^\varnothing$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D, V) \models \varphi^\varnothing$, and in particular that $(D, V) \models \nabla^B_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. If $D = \varnothing$, then $\Sigma = \Pi = \overline{\mathbf{T}} = \varnothing$ and $\nabla^B_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Hence, assume $D \neq \varnothing$. Observe that the elements of D can be partitioned in the following way:

(a) distinct elements $t_i \in D$ such that each t_i satisfies $\tau^B_{T_i}(x)$,
(b) for every $S \in \Sigma$ an infinite set D_S, such that every $d \in D_S$ satisfies τ^B_S,
(c) a *finite* set D_Π of elements, each satisfying one of the B-positive types τ^B_S with $S \in \Pi \setminus \Sigma$.

Following this partition, with every element $d \in D$ we may associate a type S_d in, respectively, (a) $\overline{\mathbf{T}}$, (b) Σ, or (c) $\Pi \setminus \Sigma$, such that d satisfies $\tau^B_{S_d}$. As in the proof of Proposition 8, we now consider the valuation U defined as $U^\flat(d):=S_d$, and as before we can show that $U \leq_B V$. Finally, it easily follows from the definitions that $(D, U) \models \nabla_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, implying that $(D, U) \models \varphi$. But then by the assumed B-monotonicity of φ it is immediate that $(D, V) \models \varphi$, as required. □

As with the previous two cases, the translation provides normal forms for the monotone fragment of ME^∞.

Corollary 3 *For any sentence $\varphi \in \mathrm{ME}^\infty(A)$, the following hold:*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula $\bigvee \nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for $\Sigma \subseteq \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ for some k.*
2. *The formula φ is monotone in every $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for types $\Sigma \subseteq \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ for some k.*

In both cases, normal forms are effective.

5 Continuity

In this section we study the sentences that are *continuous* in some set B of monadic predicate symbols.

Definition 22 Let U and V be two A-valuations on the same domain D. For a set $B \subseteq A$, we write $U \leq^\omega_B V$ if $U \leq_B V$ and $U(b)$ is finite, for every $b \in B$.

Given a monadic logic L and a formula $\varphi \in \mathrm{L}(A)$ we say that φ is *continuous in* $B \subseteq A$ if φ is monotone in B and satisfies the following:

$$\text{if } (D, V), g \models \varphi \text{ then } (D, U), g \models \varphi \text{ for some } U \leq^\omega_B V. \tag{8}$$

for every monadic model (D, V) and every assignment $g : \mathsf{iVar} \to D$.

Remark 5 As for monotonicity it is straightforward to show that a formula φ is continuous in a set B iff it is continuous in every $b \in B$.

What explains both the name and the importance of this property is its equivalence to so-called *Scott continuity*. To understand it, we may formalise the dependence of the meaning of a monadic sentence φ with m free variables $\overline{\mathbf{x}}$ in a monadic model $\mathbb{D} = (D, V)$ on a fixed name $b \in A$ as a map $\varphi^\mathbb{D}_b : \wp(D) \to \wp(D^m)$ defined by

$$X \subseteq D \mapsto \{\overline{\mathbf{d}} \in D^m \mid (D, V[b \mapsto X]) \models \varphi(\overline{\mathbf{d}})\}.$$

One can then verify that a sentence φ is continuous in b if and only if the operation $\varphi_b^{\mathbb{D}}$ is continuous with respect to the Scott topology on the powerset algebras.[6] Scott continuity is of key importance in many areas of theoretical computer sciences where ordered structures play a role, such as domain theory (see e.g. [1]).

Similarly as for monotonicity, the semantic property of continuity can be provided with a syntactic characterisation.

Definition 23 Let $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$ The fragment of $\mathtt{L}(A)$-formulas that are *syntactically continuous* in a subset $B \subseteq A$ is defined by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{L}(A \setminus B)$. In both cases, we let $\mathtt{Con}_B(\mathtt{L}(A))$ denote the set of B-continuous sentences.

To define the syntactically continuous fragment of $\mathtt{ME}^\infty$, we first introduce the following binary generalised quantifier $\mathbf{W}$: given two formulas $\varphi(x)$ and $\psi(x)$, we set

$$\mathbf{W}x.(\varphi, \psi) := \forall x.(\varphi(x) \vee \psi(x)) \wedge \forall^\infty x.\psi(x).$$

The intuition behind $\mathbf{W}$ is the following. If $(D, V), g \models \mathbf{W}x.(\varphi, \psi)$, then because of the second conjunct there are only finitely many $d \in D$ refuting ψ. The point is that this weakens the universal quantification of the first conjunct to the effect that only the finitely many mentioned elements refuting ψ need to satisfy φ.

Definition 24 The fragment of $\mathtt{ME}^\infty(A)$-formulas that are *syntactically continuous* in a subset $B \subseteq A$ is given by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x.\varphi \mid \mathbf{W}x.(\varphi, \psi),$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \setminus B)$. We let $\mathtt{Con}_B(\mathtt{ME}^\infty(A))$ denote the set of B-continuous $\mathtt{ME}^\infty$-sentences.

For $\mathtt{M}$ and $\mathtt{ME}$, the equivalence between the semantic and syntactic properties of continuity was established by van Benthem in [5]. To keep this paper self-contained, we give a sketch of this proof, which is based on a compactness argument.

Proposition 11 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$. Then φ is continuous in a set $B \subseteq A$ if and only if there is an equivalent sentence $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{L}(A))$.*

Proof The direction from right to left is covered by Proposition 12 below, so we immediately turn to the completeness part of the statement. The case of $\mathtt{M}$ being treated in Sect. 5.1, we only discuss the statement for $\mathtt{ME}$. Hence, let $\varphi \in \mathtt{ME}(A)$ be continuous in B. For simplicity in the exposition, we assume $B = \{b\}$; the case of an arbitrary B can easily be generalised from what follows.

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[6] The interested reader is referred to [15, Sec. 8] for a more precise discussion of the connection.

Let $y_0, y_1, \ldots$ be an infinite list of variables not occurring in φ. For $k \in \omega$, consider the formula

$$\varphi_k := \exists y_0 \cdots \exists y_{k-1} \left(\bigwedge_{\ell < k} b(y_\ell) \wedge \varphi([\overline{y}/b]) \right),$$

where $\varphi([\overline{y}/b])$ is obtained from φ by substituting each occurrence of an atomic formula of the form $b(x)$ with the formula $\bigvee_{\ell < k} x \approx y_\ell$. Intuitively, φ_k expresses that φ holds if we reduce the current interpretation of b to some subset of size k. Define $\Phi := \{\varphi_k \mid k \in \omega\} \cup \{\varphi_{\mathbb{D}_\varnothing}\}$, where $\varphi_{\mathbb{D}_\varnothing} := \forall x.\bot$ if $\mathbb{D}_\varnothing \models \varphi$ and $\varphi_{\mathbb{D}_\varnothing} := \exists x.\bot$ otherwise. Then by construction $\Phi \subset \mathrm{Con}_B(\mathrm{ME}(A))$. Now by continuity of φ we find that

$$\varphi \models \bigvee \Phi,$$

that is, any non-empty monadic model that validates φ must validate one of the φ_k. But then by compactness of first-order logic, there is an $n \in \omega$ such that $\varphi \models \bigvee_{k<n} \varphi_k \vee \varphi_{\mathbb{D}_\varnothing}$. By monotonicity, $\varphi_k \models \varphi$, for every $k \in \omega$, and by definition $\varphi_{\mathbb{D}_\varnothing} \models \varphi$. We therefore conclude that $\varphi \equiv \bigvee_{k<n} \varphi_k \vee \varphi_{\mathbb{D}_\varnothing}$. As $\mathrm{Con}_B(\mathrm{ME}(A))$ is closed under disjunctions, this ends the proof of the statement. $\qquad\square$

In this paper, we extend such a characterisation to ME^∞. Moreover, analogously to what we did in the previous section, for M and ME^∞ we provide both an explicit translation and a decidability result. The corresponding results in the case of ME remain open.

Theorem 4 *Let φ be a sentence of the monadic logic $\mathrm{L}(A)$, where $\mathrm{L} \in \{\mathrm{M}, \mathrm{ME}^\infty\}$. Then φ is continuous in a set $B \subseteq A$ if and only if there is an equivalent sentence $\varphi^\ominus \in \mathrm{Con}_B(\mathrm{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \mathrm{L}(A)$ has this property or not.*

Analogously to the previous case of monotonicity, the proof of the theorem is composed of two parts. We start with the right-left implication of the first claim (the preservation statement), which also holds for ME.

Proposition 12 *Every sentence $\varphi \in \mathrm{Con}_B(\mathrm{L}(A))$ is continuous in B, where $\mathrm{L} \in \{\mathrm{M}, \mathrm{ME}, \mathrm{ME}^\infty\}$.*

Proof First observe that φ is monotone in B by Proposition 6. The case for $D = \varnothing$ being clear, we assume $D \neq \varnothing$. We show, by induction, that any first-order formula φ in the fragment satisfies (8), for every non-empty monadic model (D, V) and assignment $g : \mathrm{iVar} \to D$.

- If $\varphi = \psi \in \mathrm{L}(A \setminus B)$, changes in the B part of the valuation will not affect the truth value of φ and hence the condition is trivial.
- Case $\varphi = b(x)$ for some $b \in B$: if $(D, V), g \models b(x)$ then $g(x) \in V(b)$. Let U be the valuation given by $U(b) := \{g(x)\}$, $U(a) := \varnothing$ for $a \in B \setminus \{b\}$ and $U(a) := V(a)$ for $a \in A \setminus B$. Then it is obvious that $(D, U), g \models b(x)$, while it is immediate by the definitions that $U \leq_B^\omega V$.

- Case $\varphi = \varphi_1 \vee \varphi_2$: assume $(D, V), g \models \varphi$. Without loss of generality we can assume that $(D, V), g \models \varphi_1$ and hence by induction hypothesis there is $U \leq_B^\omega V$ such that $(D, U), g \models \varphi_1$ which clearly implies $(D, U), g \models \varphi$.
- Case $\varphi = \varphi_1 \wedge \varphi_2$: assume $(D, V), g \models \varphi$. By induction hypothesis we have $U_1, U_2 \leq_B^\omega V$ such that $(D, U_1), g \models \varphi_1$ and $(D, U_2), g \models \varphi_2$. Let U be the valuation defined by putting $U(a) := U_1(a) \cup U_2(a)$; then clearly we have $U \leq_B^\omega V$, while it follows by monotonicity that $(D, U), g \models \varphi_1$ and $(D, U), g \models \varphi_2$. Clearly then $(D, U), g \models \varphi$.
- Case $\varphi = \exists x. \varphi'(x)$ and $(D, V), g \models \varphi$. By definition there exists $d \in D$ such that $(D, V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis there is a valuation $U \leq_B^\omega V$ such that $(D, U), g[x \mapsto d] \models \varphi'(x)$ and hence $(D, U), g \models \exists x. \varphi'(x)$.
- Case $\varphi = \mathbf{W}x.(\varphi', \psi) \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$ and $(D, V), g \models \varphi$. Define the formulas $\alpha(x)$ and β as follows:

$$\varphi = \forall x. \underbrace{(\varphi'(x) \vee \psi(x))}_{\alpha(x)} \wedge \underbrace{\forall^\infty x. \psi(x)}_{\beta} .$$

Suppose that $(D, V), g \models \varphi$. By the induction hypothesis, for every $d \in D$ which satisfies $(D, V), g_d \models \alpha(x)$ (where we write $g_d := g[x \mapsto d]$) there is a valuation $U_d \leq_B^\omega V$ such that $(D, U_d), g_d \models \alpha(x)$. The crucial observation is that because of β, only finitely many elements of d refute $\psi(x)$. Let U be the valuation defined by putting $U(a) := \bigcup \{U_d(a) \mid (D, V), g_d \not\models \psi(x)\}$. Note that for each $b \in B$, the set $U(b)$ is a finite union of finite sets, and hence finite itself; it follows that $U \leq_B^\omega V$. We claim that

$$(D, U), g \models \varphi. \tag{9}$$

It is clear that $(D, U), g \models \beta$ because ψ (and hence β) is B-free. To prove that $(D, U), g \models \forall x\, \alpha(x)$, we have to show that $(D, U), g_d \models \varphi'(x) \vee \psi(x)$ for any $d \in D$. We consider two cases: If $(D, V), g_d \models \psi(x)$ we are done, again because ψ is B-free. On the other hand, if $(D, V), g_d \not\models \psi(x)$, then $(D, U_d), g_d \models \alpha(x)$ by assumption on U_d, while it is obvious that $U_d \leq_B U$; but then it follows by monotonicity of α that $(D, U), g_d \models \alpha(x)$.

This finishes the proof.   □

The second part of the proof of the theorem, is constituted by the following stronger version of the expressive completeness result that provides, as a corollary, normal forms for the syntactically continuous fragments.

Proposition 13 *Let* $\mathtt{L}$ *be one of the logics* $\{\mathtt{M}, \mathtt{ME}^\infty\}$. *There exists an effective translation* $(-)^\ominus : \mathtt{L}(A) \to \mathtt{Con}_B(\mathtt{L}(A))$ *such that a sentence* $\varphi \in \mathtt{L}(A)$ *is continuous in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^\ominus$.

We prove the two manifestations of Proposition 13 separately, in two respective subsections.

By putting together the two propositions above, we are able to conclude.

Proof of Theorem 4 The first claim follows from Proposition 13. Hence, by applying Fact 1 to Proposition 13, the problem of checking whether a sentence $\varphi \in \mathtt{L}(A)$ is continuous in $B \subseteq A$ or not, is decidable. □

We conjecture that Proposition 13, and therefore Theorem 4, holds also for $\mathtt{L} = \mathtt{ME}$.

5.1 Continuous fragment of M

Since continuity implies monotonicity, by Theorem 3, in order to verify the $\mathtt{M}$-variant of Proposition 13, it is enough to prove the following result.

Proposition 14 *There is an effective translation* $(-)^\ominus : \mathtt{Pos}_B(\mathtt{M}(A)) \to \mathtt{Con}_B(\mathtt{M}(A))$ *such that a sentence* $\varphi \in \mathtt{Pos}_B(\mathtt{M}(A))$ *is continuous in $B \subseteq A$ if and only if* $\varphi \equiv \varphi^\ominus$.

Proof By Corollary 1, to define the translation we may assume without loss of generality that φ is in the basic form $\bigvee \nabla_\mathtt{M}^B(\Sigma)$. For the translation, let

$$\left(\bigvee \nabla_\mathtt{M}^B(\Sigma) \right)^\ominus := \bigvee \nabla_\mathtt{M}^B(\Sigma, \Sigma_B^-)$$

where $\Sigma_B^- := \{ S \in \Sigma \mid B \cap S = \varnothing \}$. From the construction, it is clear that $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{M}(A))$. Then the right-to-left direction of the proposition is immediate by Proposition 12.

For the left-to-right direction, assume that φ is continuous in B. We have to prove that $(D, V) \models \varphi$ iff $(D, V) \models \varphi^\ominus$, for every monadic model (D, V). Our proof strategy consists of proving the same equivalence for the model $(D \times \omega, V_\pi)$, where $D \times \omega$ consists of ω many copies of each element in D and V_π is the valuation given by $V_\pi(a) := \{ (d, k) \mid d \in V(a), k \in \omega \}$. It is easy to see that $(D, V) \equiv^\mathtt{M} (D \times \omega, V_\pi)$ (see Proposition 18) and so it suffices indeed to prove that

$$(D \times \omega, V_\pi) \models \varphi \text{ iff } (D \times \omega, V_\pi) \models \varphi^\ominus.$$

Consider first the case where $D = \varnothing$. Then $(D \times \omega, V_\pi) = \mathbb{D}_\varnothing$, and then the claim is true since $\nabla_\mathtt{M}^B(\varnothing) = \nabla_\mathtt{M}^B(\varnothing, \varnothing_B^-)$ and $\mathbb{D}_\varnothing \models \nabla_\mathtt{M}^B(\Sigma)$ iff $\Sigma = \varnothing$.
In the remainder of the proof we focus on the case where $D \neq \varnothing$.
$\boxed{\Rightarrow}$ Let $(D \times \omega, V_\pi) \models \varphi$. As φ is continuous in B there is a valuation $U \leq_B^\omega V_\pi$ satisfying $(D \times \omega, U) \models \varphi$. This means that $(D \times \omega, U) \models \nabla_\mathtt{M}^B(\Sigma)$ for some disjunct $\nabla_\mathtt{M}^B(\Sigma)$ of φ. Below we will use the following fact (which can easily be verified):

$$(D \times \omega), U \models \tau_S^B(d, k) \text{ iff } U^\flat(d, k) \setminus B = S \setminus B \text{ and } U^\flat(d, k) \subseteq S \cap B. \quad (10)$$

Our claim is now that $(D \times \omega, U) \models \nabla_\mathtt{M}^B(\Sigma, \Sigma_B^-)$.
The existential part of $\nabla_\mathtt{M}^B(\Sigma, \Sigma_B^-)$ is trivially true. To cover the universal part, it remains to show that every element of $(D \times \omega, U)$ realizes a B-positive type in Σ_B^-. Take an arbitrary pair $(d, k) \in D \times \omega$ and let T be the (full) type of (d, k), that is, let $T := U^\flat(d, k)$. If $B \cap T = \varnothing$ then trivially $T \in \Sigma_B^-$ and we are done. So suppose $B \cap T \neq \varnothing$. Observe that in $D \times \omega$ we have infinitely many copies of $d \in D$. Hence,

as $U(b)$ is finite for every $b \in B$, there must be some (d, k') with type $U^\flat(d, k') = V^\flat_\pi(d, k') \setminus B = V^\flat_\pi(d, k) \setminus B = T \setminus B$. It follows from $(D \times \omega, U) \models \nabla^B_M(\Sigma)$ and (10) that there is some $S \in \Sigma$ such that $S \setminus B = U^\flat(d, k') \setminus B = U^\flat(d, k')$ and $S \cap B \subseteq U^\flat(d, k') \cap B = \varnothing$. From this we easily derive that $S = U^\flat(d, k')$ and $S \in \Sigma^-_B$. Finally, we observe that $S \setminus B = U^\flat(d, k') \setminus B = U^\flat(d, k) \setminus B$ and $S \cap B = \varnothing \subseteq U^\flat(d, k)$, so that by (10) we find that $D \times \omega, U) \models \tau^B_S(d, k)$ indeed.

Finally, by monotonicity it directly follows from $(D \times \omega, U) \models \nabla^B_M(\Sigma, \Sigma^-_B)$ that $(D \times \omega, V_\pi) \models \nabla^B_M(\Sigma, \Sigma^-_B)$, and from this it is immediate that $(D \times \omega, V_\pi) \models \varphi^\ominus$. $\fbox{$\Leftarrow$}$ Let $(D \times \omega, V_\pi) \models \nabla^B_M(\Sigma, \Sigma^-_B)$. To show that $(D \times \omega, V_\pi) \models \nabla^B_M(\Sigma)$, the existential part is trivial. For the universal part just observe that $\Sigma^-_B \subseteq \Sigma$. $\square$

A careful analysis of the translation provides us with normal forms for the continuous fragment of M. We also formulate a version of this result which holds when we restrict to the positive fragment of M; this version, which can be proved in the same manner as the main result, will be needed in our companion paper.

Corollary 4 *For any sentence $\varphi \in \mathrm{M}(A)$, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff it is equivalent to a formula $\bigvee \nabla^B_M(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp(A)$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*
2. *If φ positive in A (i.e., $\varphi \in \mathrm{M}^+(A)$) then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_M(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp(A)$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*

5.2 Continuous fragment of ME$^\infty$

As for the previous case, the ME$^\infty$-variant of Proposition 13 is an immediate consequence of Theorem 3 and the following proposition.

Proposition 15 *There is an effective translation $(-)^\ominus : \mathrm{Pos}_B(\mathrm{ME}^\infty(A)) \to \mathrm{Con}_B(\mathrm{ME}^\infty(A))$ such that a sentence $\varphi \in \mathrm{Pos}_B(\mathrm{ME}^\infty(A))$ is continuous in B if and only if $\varphi \equiv \varphi^\ominus$.*

Proof By Corollary 3, we may assume that φ is in basic normal form, i.e., $\varphi = \bigvee \nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, with $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. For the translation let $\left(\bigvee \nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^\ominus := \bigvee \nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus$ where

$$\nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus := \begin{cases} \nabla^B_{\mathrm{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) & \text{if } B \cap \bigcup \Sigma = \varnothing \\ \bot & \text{if } B \cap \bigcup \Sigma \neq \varnothing. \end{cases}$$

First we prove the right-to-left direction of the proposition. By Proposition 12 it is enough to show that $\varphi^\ominus \in \mathrm{Con}_B(\mathrm{ME}^\infty(A))$. We focus on the disjuncts of $\varphi^\ominus$. The interesting case is where $B \cap \bigcup \Sigma = \varnothing$. Define the formulas $\varphi'(\overline{\mathbf{x}}, z)$ and $\psi(z)$ as follows:

$$\varphi'(\overline{\mathbf{x}}, z) := \neg\mathrm{diff}(\overline{\mathbf{x}}, z) \vee \bigvee_{S \in \Pi \setminus \Sigma} \tau^B_S(z)$$
$$\psi(z) \quad := \bigvee_{S \in \Sigma} \tau^B_S(y).$$

Then we may rearrange the internal structure of the formula $\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ somewhat, arriving at the following:

$$\exists \overline{\mathbf{x}}.\Big(\text{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \forall z.(\underbrace{\neg\text{diff}(\overline{\mathbf{x}}, z) \vee \bigvee_{S \in \Pi \setminus \Sigma} \tau^B_S(z)}_{\varphi'(\overline{\mathbf{x}}, z)} \vee \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(z)}_{\psi(z)})$$

$$\wedge \forall^\infty y. \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(y)}_{\psi(y)}\Big) \wedge \bigwedge_{S \in \Sigma} \exists^\infty y.\tau^B_S(y),$$

so that we find

$$\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) \equiv \exists \overline{\mathbf{x}}.\Big(\text{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \mathbf{W}z.(\varphi'(\overline{\mathbf{x}}, z), \psi(z))\Big) \wedge \bigwedge_{S \in \Sigma} \exists^\infty y.\tau^B_S(y),$$

which belongs to the required fragment because $B \cap \bigcup \Sigma = \varnothing$.

For the left-to-right direction of the proposition, we have to prove that $\varphi \equiv \varphi^\ominus$.

$\boxed{\Rightarrow}$ Let $(D, V) \models \varphi$. Because φ is continuous in B we may assume that $V(b)$ is finite, for all $b \in B$. Let $\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ be a disjunct of φ such that $(D, V) \models \nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. If $D = \varnothing$, then $\overline{\mathbf{T}} = \Pi = \Sigma = \varnothing$, and $\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = (\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\ominus$. Hence, let $D \neq \varnothing$. Suppose for contradiction that $B \cap \bigcup \Sigma \neq \varnothing$, then there must be some $S \in \Sigma$ with $B \cap S \neq \varnothing$. Because $(D, V) \models \nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ we have, in particular, that $(D, V) \models \exists^\infty x.\tau^B_S(x)$ and hence $V(b)$ must be infinite, for any $b \in B \cap S$, which is absurd. It follows that $B \cap \bigcup \Sigma = \varnothing$, but then we trivially conclude that $(D, V) \models \varphi^\ominus$ because the disjunct remains unchanged.

$\boxed{\Leftarrow}$ Let $(D, V) \models \varphi^\ominus$. This direction is trivial, because the only difference between φ and $\varphi^\ominus$ is that some disjuncts may have been replaced by $\bot$. $\qquad\square$

We conclude the section by stating the following corollary, providing normal forms for the continuous fragment of ME^∞. As in the case of M we formulate, for future reference, a variation of this result which applies to the positive fragment of ME^∞.

Corollary 5 *For any sentence $\varphi \in ME^\infty(A)$, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff φ is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\nabla^B_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where $\Sigma, \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*
2. *If φ is positive (i.e., $\varphi \in ME^{\infty+}(A)$) then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\bigvee \nabla^+_{ME^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $\Sigma, \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*

6 Submodels and quotients

There are various natural notions of morphism between monadic models; the one that we will be interested here is that of a (strong) homomorphism.

Definition 25 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. A map $f : D \to D'$ is a *homomorphism* from $\mathbb{D}$ to $\mathbb{D}'$, notation: $f : \mathbb{D} \to \mathbb{D}'$, if we have $d \in V(a)$ iff $f(d) \in V'(a)$, for all $a \in A$ and $d \in D$.

In this section we will be interested in the sentences of M, ME and ME$^\infty$ that are preserved under taking submodels and the ones that are invariant under quotients.

Definition 26 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. We call $\mathbb{D}$ a *submodel* of $\mathbb{D}'$ if $D \subseteq D'$ and the inclusion map $\iota_{DD'} : D \hookrightarrow D'$ is a homomorphism, and we say that $\mathbb{D}'$ is a *quotient* of $\mathbb{D}$ if there is a surjective homomorphism $f : \mathbb{D} \to \mathbb{D}'$.

Now let φ be an L-sentence, where L $\in$ {M, ME, ME$^\infty$}. We say that φ is *preserved under taking submodels* if $\mathbb{D} \models \varphi$ implies $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a submodel of $\mathbb{D}$. Similarly, φ is *invariant under taking quotients* if we have $\mathbb{D} \models \varphi$ iff $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a quotient of $\mathbb{D}$.

The first of these properties (preservation under taking submodels) is well known from classical model theory—it is for instance the topic of the Łos-Tarski Theorem. When it comes to quotients, in model theory one is usually more interested in the formulas that are *preserved* under surjective homomorphisms (and the definition of homomorphism may also differ from ours). For instance, this is the topic of Lyndon's Theorem [23] which characterises the formulas that are preserved under a weaker notion of homomorphism as the ones that are *positive* in all predicates occurring in the formula. Our preference for the notion of *invariance* under quotients stems from the fact that the property of invariance under quotients plays a key role in characterising the *bisimulation-invariant fragments* of various monadic second-order logics, as is explained in our companion paper [10].

6.1 Preservation under submodels

In this subsection we characterise the fragments of our predicate logics consisting of the sentences that are preserved under taking submodels. That is, the main result of this subsection is a Łos-Tarksi Theorem for ME$^\infty$.

Definition 27 The *universal fragment* of the set ME$^\infty(A)$ is the collection Univ(ME$^\infty(A)$) of formulas given by the following grammar:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \forall x.\varphi \mid \forall^\infty x.\varphi$$

where $x, y \in$ iVar and $a \in A$. The universal fragment Univ(ME(A)) is obtained by deleting the clause for $\forall^\infty$ from this grammar, and we obtain the universal fragment Univ(M(A)) by further deleting both clauses involving the equality symbol.

Theorem 5 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is preserved under taking submodels if and only if there is an equivalent formula $\varphi^\otimes \in \mathtt{Univ}(\mathtt{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \mathtt{L}(A)$ has this property or not.*

We start by verifying that universal formulas satisfy the property.

Proposition 16 *Let $\varphi \in \mathtt{Univ}(\mathtt{L}(A))$ be a universal sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is preserved under taking submodels.*

Proof It is enough to directly consider the case $\mathtt{L} = \mathtt{ME}^\infty$. Let (D', V') be a submodel of the monadic model (D, V). The case for $D = \varnothing$ being immediate, let us assume $D \neq \varnothing$. By induction on the complexity of a formula $\varphi \in \mathtt{Univ}(\mathtt{ME}^\infty(A))$ we will show that for any assignment $g : \mathsf{iVar} \to D'$ we have

$$(D, V), g' \models \varphi \text{ implies } (D', V'), g \models \varphi,$$

where $g' := g \circ \iota_{D'D}$. We will only consider the inductive step of the proof where φ is of the form $\forall^\infty x.\psi$. Define $X_{D,V} := \{d \in D \mid (D, V), g'[x \mapsto d] \models \psi\}$, and similarly, $X_{D',V'} := \{d \in D' \mid (D', V'), g[x \mapsto d] \models \psi\}$. By the inductive hypothesis we have that $X_{D,V} \cap D' \subseteq X_{D',V'}$, implying that $D' \setminus X_{D',V'} \subseteq D \setminus X_{D,V}$. But from this we immediately obtain that

$$|D \setminus X_{D,V}| < \omega \text{ implies } |D' \setminus X_{D',V'}| < \omega,$$

which means that $(D, V), g' \models \varphi$ implies $(D', V'), g \models \varphi$, as required. □

Turning to the much harder verification of the opposite implication of the theorem, we first define the appropriate translations from each monadic logic into its universal fragment.

Definition 28 We start by defining the translations for sentences in basic normal forms. Let $\Sigma, \Pi \subseteq \wp(A)$ be some types and $\overline{\mathbf{T}} \in \wp(A)^k$ some list of types. For $\mathtt{M}$-sentences in basic form we first set

$$\left(\nabla_{\mathtt{M}}(\Sigma)\right)^\otimes := \forall z \bigvee_{S \in \Sigma} \tau_S(z),$$

in the case of $\mathtt{ME}$ we define

$$\left(\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\right)^\otimes := \forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi} \tau_S(z),$$

while for basic formulas of ME^∞, the translation $(-)^\otimes$ is given as follows:

$$(\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\otimes := \forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z).$$

Second, in each case we define $(\bigvee_i \varphi_i)^\otimes := \bigvee \varphi_i^\otimes$.

Finally, for each $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^\infty\}$, we extend the translation $(-)^\otimes$ to the collection of all sentences by defining $\varphi^\otimes := (\varphi^*)^\otimes$, where φ^* is the basic normal form of φ as given by Fact 2 (in the case of M), by Theorem 1 (in the case of ME), and by Theorem 2 (in the case of ME^∞).

The missing part in the proof of the theorem is covered by the following result.

Proposition 17 *For any monadic logic $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^\infty\}$ there is an effective transla-tion $(-)^\otimes : \text{L}(A) \to \text{Univ}(\text{L}(A))$ such that a sentence $\varphi \in \text{L}(A)$ is preserved under taking submodels if and only if $\varphi \equiv \varphi^\otimes$.*

Proof We only consider the case where $\text{L} = \text{ME}^\infty$, leaving the other cases to the reader.

It is easy to see that $\varphi^\otimes \in \text{Univ}(\text{ME}^\infty(A))$, for every sentence $\varphi \in \text{ME}^\infty(A)$; but then it is immediate by Proposition 16 that φ is preserved under taking submodels if $\varphi \equiv \varphi^\otimes$.

For the left-to-right direction, assume that φ is preserved under taking submodels. It is easy to see that φ implies $\varphi^\otimes$, so we focus on proving the opposite. That is, we suppose that $(D, V) \models \varphi^\otimes$, and aim to show that $(D, V) \models \varphi$.

By Theorem 2 we may assume without loss of generality that φ is a disjunction of sentences of the form $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. It follows that (D, V) satisfies some disjunct $\forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z)$ of $\left(\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^\otimes$. Expand D with finitely many elements $\overline{\mathbf{d}}$, in one-one correspondence with $\overline{\mathbf{T}}$, and ensure that the type of each d_i is T_i. In addition, add, for each $S \in \Sigma$, infinitely many elements $\{e_n^S \mid n \in \omega\}$, each of type S. Call the resulting monadic model $\mathbb{D}' = (D', V')$.

This construction is tailored to ensure that $(D', V') \models \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, and so we obtain $(D', V') \models \varphi$. But obviously, $\mathbb{D}$ is a submodel of $\mathbb{D}'$. This implies that $(D, V) \models \varphi$, by our assumption on φ.   □

Proof of Theorem 5 The first part of the theorem is an immediate consequence of Proposition 17. By applying Fact 1 to Proposition 17 we finally obtain that for the three concerned formalisms the problem of deciding whether a sentence is preserved under taking submodels is decidable.   □

As an immediate consequence of the proof of the previous Proposition 17, we get effective normal forms for the universal fragments.

Corollary 6 *The following hold:*

1. *A sentence $\varphi \in \text{ME}(A)$ is preserved under taking submodels iff it is equivalent to a formula $\bigvee \left(\forall z \bigvee_{S \in \Sigma} \tau_S(z)\right)$, for types $\Sigma \subseteq \wp(A)$.*

2. *A sentence $\varphi \in \text{ME}(A)$ is preserved under taking submodels iff it is equivalent to a formula $\bigvee \left(\forall z \bigvee_{S \in \overline{T} \cup \Pi} \tau_S(z) \right)$, for types $\Pi \subseteq \wp(A)$ and $\overline{T} \in \wp(A)^k$ for some k.*

3. *A sentence $\varphi \in \text{ME}^\infty(A)$ is preserved under taking submodels iff it is equivalent to a formula $\bigvee \left(\forall z \bigvee_{S \in \overline{T} \cup \Pi} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z) \right)$ for types $\Sigma \subseteq \Pi \subseteq \wp(A)$ and $\overline{T} \in \wp(A)^k$ for some k.*

In all three cases, normal forms are effective.

6.2 Invariance under quotients

The following theorem states that monadic first-order logic *without* equality (M) provides the quotient-invariant fragment of both monadic first-order logic with equality (ME), and of infinite-monadic predicate logic (ME$^\infty$). Recall that a formula φ is invariant under taking quotients if it satisfies the condition that $\mathbb{D} \models \varphi$ iff $\mathbb{D}' \models \varphi$, for any monadic model $\mathbb{D}$ and any quotient $\mathbb{D}'$ of $\mathbb{D}$.

Theorem 6 *Let φ be a sentence of the monadic logic $\text{L}(A)$, where $\text{L} \in \{\text{ME}, \text{ME}^\infty\}$. Then φ is invariant under taking quotients if and only if there is an equivalent sentence in M. Furthermore, it is decidable whether a sentence $\varphi \in \text{L}(A)$ has this property or not.*

We first state the 'easy' part of the first claim of the theorem. Note that in fact, we have already been using this observation in earlier parts of the paper.

Proposition 18 *Every sentence in M is invariant under taking quotients.*

Proof Let $f : D \to D'$ provide a surjective homomorphism between the models (D, V) and (D', V'), and observe that for any assignment $g : \text{iVar} \to D$ on D, the composition $f \circ g : \text{iVar} \to D'$ is an assignment on D'.

In order to prove the proposition one may show that, for an arbitrary M-formula φ and an arbitrary assignment $g : \text{iVar} \to D$, we have

$$(D, V), g \models \varphi \text{ iff } (D', V'), f \circ g \models \varphi. \tag{11}$$

We leave the proof of (11), which proceeds by a straightforward induction on the complexity of φ, as an exercise to the reader. □

To prove the remaining part of Theorem 6, we start with providing translations from ME and from ME$^\infty$, respectively, to M.

Definition 29 For ME-sentences in basic form we first define

$$\left(\nabla_{\text{ME}}(\overline{T}, \Pi) \right)^\circ := \bigwedge_i \exists x_i . \tau_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S(x),$$

whereas for ME$^\infty$-sentences in basic form we start with defining

$$\left(\nabla_{\text{ME}^\infty}(\overline{T}, \Pi, \Sigma) \right)^\bullet := \bigwedge_i \exists x_i . \tau_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Sigma} \tau_S(x).$$

In both cases, the translation is then extended to the full language as in Definition 28.

Note that the two maps may give *different* translations for ME-sentences. Also observe that the Π 'disappears' in the translation of the formula $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

The key property of these translations is the following.

Proposition 19 *1. For every monadic model (D, V) and every $\varphi \in \text{ME}(A)$ we have*

$$(D, V) \models \varphi^\circ \text{ iff } (D \times \omega, V_\pi) \models \varphi. \tag{12}$$

2. For every monadic model (D, V) and every $\varphi \in \text{ME}^\infty(A)$ we have

$$(D, V) \models \varphi^\bullet \text{ iff } (D \times \omega, V_\pi) \models \varphi. \tag{13}$$

Here V_π is the induced valuation given by $V_\pi(a) := \{(d, k) \mid d \in V(a), k \in \omega\}$.

Proof We only prove the claim for ME^∞ (i.e., the second part of the proposition), the case for ME being similar. Clearly it suffices to prove (13) for formulas of the form $\varphi = \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

First of all, if $\mathbb{D}$ is the empty model, we find $\overline{\mathbf{T}} = \Pi = \Sigma = \varnothing$, $(D, V) = (D \times \omega, V_\pi)$, and $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = (\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\bullet$. In other words, in this case there is nothing to prove.

In the sequel we assume that $D \neq \varnothing$.

$\boxed{\Rightarrow}$ Assume $(D, V) \models \varphi^\bullet$, we will show that $(D \times \omega, V_\pi) \models \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Let d_i be such that $V^\flat(d_i) = T_i$ in (D, V). It is clear that the (d_i, i) provide *distinct* elements, with each (d_i, i) satisfying τ_{T_i} in $(D \times \omega, V_\pi)$. Thus, the first-order existential part of φ is satisfied. With a similar argument it is straightforward to verify that the $\exists^\infty$-part of φ is also satisfied—here we critically use the observation that $\Sigma \subseteq \overline{\mathbf{T}}$, so that every type in Σ is witnessed in the model (D, V), and hence witnessed infinitely many times in $(D \times \omega, V_\pi)$.

For the universal parts of $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ it is enough to observe that, because of the universal part of $\varphi^\bullet$, *every* $d \in D$ realizes a type in Σ. By construction, the same applies to $(D \times \omega, V_\pi)$. This takes care of both universal quantifiers.

$\boxed{\Leftarrow}$ Assuming that $(D \times \omega, V_\pi) \models \nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, we will show that $(D, V) \models \varphi^\bullet$. The existential part of $\varphi^\bullet$ is trivial. For the universal part we have to show that every element of D realizes a type in Σ. Suppose not, and let $d \in D$ be such that $\neg\tau_S(d)$ for all $S \in \Sigma$. Then we have $(D \times \omega, V_\pi) \not\models \tau_S(d, k)$ for all k. That is, there are infinitely many elements not realising any type in Σ. Hence we have $(D \times \omega, V_\pi) \not\models \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y)$. Absurd, because this formula is a conjunct of $\nabla_{\text{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. $\square$

We will now show how the theorem follows from this. First of all we verify that in both cases M is expressively complete for the property of being invariant under taking quotients.

Proposition 20 *For any monadic logic $\text{L} \in \{\text{ME}, \text{ME}^\infty\}$ there is an effective translation $(-)^\circledast : \text{L}(A) \to \text{M}$ such that a sentence $\varphi \in \text{L}(A)$ is invariant under taking quotients if and only if $\varphi \equiv \varphi^\circledast$.*

Proof Let φ be a sentence of $\mathtt{ME}^\infty$, and let $\varphi^\circledast := \varphi^\bullet$ (we only cover the case of $\mathtt{L} = \mathtt{ME}^\infty$, the case for $\mathtt{L} = \mathtt{ME}$ is similar, just take $\varphi^\circledast := \varphi^\circ$) We will show that

$$\varphi \equiv \varphi^\circledast \text{ iff } \varphi \text{ is invariant under taking quotients.} \tag{14}$$

The direction from right to left is immediate by Proposition 18. For the other direction it suffices to observe that any model (D, V) is a quotient of its 'ω-product' $(D \times \omega, V_\pi)$, and to reason as follows:

$$
\begin{aligned}
(D, V) &\models \varphi \text{ iff } (D \times \omega, V_\pi) \models \varphi &\quad \text{(assumption on } \varphi) \\
&\text{iff } (D, V) \models \varphi^\bullet &\quad \text{(Proposition 19)}
\end{aligned}
$$

$\square$

Hence we can conclude.

Proof of Theorem 6 The theorem is an immediate consequence of Proposition 20. Finally, the effectiveness of translation $(\cdot)^\bullet$, decidability of $\mathtt{ME}^\infty$ (Fact 1) and (14) yield that it is decidable whether a given $\mathtt{ME}^\infty$-sentence φ is invariant under taking quotients or not. $\square$

As a corollary, we obtain:

Corollary 7 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is invariant under taking quotients if and only if there is an equivalent sentence $\nabla_{\mathtt{M}}(\Sigma)$ for types $\Sigma \subseteq \wp(A)$. Moreover, such a normal form is effective.*

In our companion paper [10] on automata, we need versions of these results for the monotone and the continuous fragment. For this purpose we define some slight modifications of the translations $(\cdot)^\circ$ and $(\cdot)^\bullet$ which restricts to positive and syntactically continuous sentences.

Theorem 7 *There are effective translations $(\cdot)^\circ : \mathtt{ME}^+ \to \mathtt{M}^+$ and $(\cdot)^\bullet : \mathtt{ME}^{\infty+} \to \mathtt{M}^+$ such that $\varphi \equiv \varphi^\circ$ (respectively, $\varphi \equiv \varphi^\bullet$) iff φ is invariant under quotients. Moreover, we may assume that $(\cdot)^\bullet : \mathrm{Con}_B(\mathtt{ME}^\infty(A)) \cap \mathtt{ME}^{\infty+} \to \mathrm{Con}_B(\mathtt{M}(A)) \cap \mathtt{M}^+$, for any $B \subseteq A$.*

Proof We define translations $(\cdot)^\circ : \mathtt{ME}^+ \to \mathtt{M}^+$ and $(\cdot)^\bullet : \mathtt{ME}^{\infty+} \to \mathtt{M}^+$ as follows. For $\mathtt{ME}^+, \mathtt{ME}^{\infty+}$-sentences in simple basic form we define

$$
\begin{aligned}
\left(\nabla_{\mathtt{ME}}^+(\overline{\mathbf{T}}, \Pi) \right)^\circ &:= \bigwedge_i \exists x_i . \tau_{T_i}^+(x_i) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S^+(x), \\
\left(\nabla_{\mathtt{ME}^\infty}^+(\overline{\mathbf{T}}, \Pi, \Sigma) \right)^\bullet &:= \bigwedge_i \exists x_i . \tau_{T_i}^+(x_i) \wedge \forall x. \bigvee_{S \in \Sigma} \tau_S^+(x),
\end{aligned}
$$

and then we use, respectively, the Corollaries 2 and 3 to extend these translations to the full positive fragments $\mathtt{ME}^+$ and $\mathtt{ME}^{\infty+}$, as we did in Definition 29 for the full language.

We leave it as an exercise for the reader to prove the analogue of Proposition 19 for these translations, and to show how the first statements of the theorem follows from this.

Finally, to see why we may assume that $(\cdot)^\bullet$ restricts to a map from the syntactically B-continuous fragment of $\text{ME}^{\infty+}(A)$ to the syntactically B-continuous fragment of $\text{M}^+(A)$, assume that $\varphi \in \text{ME}^{\infty}(A)$ is continuous in $B \subseteq A$. By Corollary 5 we may assume that φ is a disjunction of formulas of the form $\nabla^+_{\text{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $B \cap \bigcup \Sigma = \varnothing$. This implies that in the formula $\varphi^\bullet$ no predicate symbol $b \in B$ occurs in the scope of a universal quantifier, and so $\varphi^\bullet$ is syntactically continuous in B indeed. □

7 Conclusions

In this paper we established some model-theoretic results about the logic ME^{∞}, a variation of monadic first-order logic that features the generalised quantifier $\exists^{\infty}$ ('there are infinitely many'), and about its classical fragments ME and M consisting of, respectively, monadid first-order logic with and without equality. For each logic $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^{\infty}\}$ we used the method of Ehrenfeucht–Fraïssé games to show that arbitrary sentences can be effectively rewritten into some normal form. We subsequently used these normal forms to prove a number of characterisation theorems, covering some well-known semantic properties, viz., monotonicity and preservation under submodels, but also some properties of more specific interest, viz., continuity and invariance under quotients. In all cases we actually proved a stronger result than a mere characterisation theorem: we provided a map, effectively translating arbitrary sentences into sentences of the required syntactic shape, and we showed that an arbitrary sentence in L has the semantic property under scrutiny iff it is equivalent to its translation. As a consequence of this result and the fact that each $\text{L} \in \{\text{M}, \text{ME}, \text{ME}^{\infty}\}$ has a decidable satisfiability problem, we showed that each of the mentioned properties is decidable for monadic first-order sentences.

Our main interest concerned the language ME^{∞} with the infinity quantifer. Since this operator does not make sense in finite models, we did not explicitly investigate which of our results on the other languages, M and ME, hold as well in the setting of finite model theory. We claim, however, that all of our results on normal forms, and on characterisations of monotonicity, preservation under submodels, and invariance under quotients, hold in this setting as well, with only minor adaptations of the proofs. (The remaining property of continuity is obviously not of interest in the setting of finite models.) For instance, some of our proofs use a model-theoretic copying construction that turns an arbitrary monadic model (D, V) into its ω-fold copy $(D \times \omega, V_\pi)$. In the setting of finite model theory, this construction needs to be replaced with a more fine-grained k-fold copying construction, with k a finite number depending on the sentence under investigation.

We finish with mentioning some suggestions for further research. First, given that many semantic properties of monadic predicate logics turn out to be decidable, a natural follow-up question would be to investigate the computational *complexity* of these problems. Second, by van Benthem's result (cf. Proposition 11), a sentence

$\varphi \in \mathrm{ME}(A)$ is continuous in a set $B \subseteq A$ if and only if it is equivalent to some $\varphi^{\ominus}$ in the syntactic fragment $\mathrm{Con}_B(\mathrm{L}(A))$. Intrigueingly we did not manage to prove this result using the normal form method; we conjecture, however, that the obvious analogues of Proposition 13 and Theorem 4 do hold for $\mathrm{L} = \mathrm{ME}$.

Finally, one perspective on our work is that it bears further witness to the fact that failure of compactness is in itself not an obstacle for the development of *model theory*—as is well known, of course, from the area of finite model theory that we just mentioned. It would be interesting to see which of our characterisation results still hold if we drop the restriction to *monadic* predicate logic, and investigate the full language FOE^{∞} of first-order logic (with equality) extended with the infinity quantifier $\exists^{\infty}$, or fragments of FOE^{∞} that are more expressive than ME^{∞}. A first step in this direction was taken by Ignacio Bellas Acosta, who wrote, under the supervision of the third author, an MSc thesis [3] on the *modal fragment* of FOE^{∞} establishing, among other things, a van Benthem-style bisimulation invariance result.

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