On a class of globally analytic hypoelliptic sums of squares

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ON A CLASS OF GLOBALLY ANALYTIC
HYPOELLIPTIC SUMS OF SQUARES

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ABSTRACT. We consider sums of squares operators globally defined on the torus. We show that if some assumptions are satisfied the operators are globally analytic hypoelliptic. The purpose of the assumptions is to rule out the existence of a Hamilton leaf on the characteristic variety lying along the fiber of the cotangent bundle, i.e. the case of the (global) Métivier operator.

1. INTRODUCTION

The aim of this paper is to study a class of sums of squares operators defined globally on a torus $\mathbb{T}^n$. More precisely let P be a sum of squares operator of the form

$$(1.1) \qquad P(x, D) = \sum_{j=1}^{N} X_j(x, D)^2.$$

Here $X_j(x, D)$ denotes a vector field with real analytic coefficients defined on $\mathbb{T}^n$, and we shall always assume that Hörmander condition is satisfied:

(H) The vector fields and their iterated commutators generate a Lie algebra of the same dimension of the ambient space, i.e. n.

We are concerned with the regularity of the solutions to the equation $Pu = f$, where $f \in C^\omega(\mathbb{T}^n)$. It is known that in general the solutions u are not real analytic on $\mathbb{T}^n$. Actually let us consider the global Métivier operator

$$(1.2) \qquad P_M = D_x^2 + (\sin x)^2 D_y^2 + (\sin y \ D_y)^2.$$

Proposition 1.1 (see Treves, [27]). *The operator P_M in (1.2) is globally Gevrey 2 hypoelliptic on the torus $\mathbb{T}^2$ and not better.*

We recall the definition of the Gevrey classes:

Definition 1.1. *Let U be an open subset of $\mathbb{R}^n$. The space $G^s(U)$, $s \geq 1$, the class of Gevrey functions of order s in U, denotes the set of all $f \in C^\infty(U)$ such that for every compact set $K \Subset U$ there are two positive constants C_K and A such that for every $\alpha \in \mathbb{Z}_+^n$*

$$|D^\alpha f(x)| \leq AC_K^{|\alpha|}|\alpha|^{s|\alpha|}, \qquad \forall x \in K.$$

$G^s(\mathbb{T}^n)$, the class of the global Gevrey functions of order s in $\mathbb{T}^n$, denotes the set of all $f \in C^\infty(T^n)$ such that there exists a positive constant C, for which

$$|D^\alpha f(x)| \leq C^{|\alpha|+1}|\alpha|^{s|\alpha|}, \qquad \forall \alpha \in \mathbb{Z}_+^n, x \in \mathbb{T}^n.$$

Definition 1.2. *An operator P is said to be G^s-hypoelliptic, $s \geq 1$, in U, open subset of $\mathbb{R}^n$, if for any U' open subset of U, the conditions $u \in \mathscr{D}'(U)$ and $Pu \in G^s(U)$ imply that $u \in G^s(U')$.*

P is said to be globally G^s-hypoelliptic, $s \geq 1$, in $\mathbb{T}^n$ if the conditions $u \in \mathscr{D}'(\mathbb{T}^n)$ and $Pu \in G^s(\mathbb{T}^n)$ imply that $u \in G^s(\mathbb{T}^n)$.

Proof of Proposition 1.1. First of all we remark that the characteristic variety of P is $\mathrm{Char}(P) = \{(0,0;0,\eta) \mid \eta \neq 0\} \cup \{(\pi,\pi;0,\eta) \mid \eta \neq 0\}$. Let U_1 be a neighborhood of the origin in $\mathbb{T}^2$ and U_2 a neighborhood of the point (π,π) in $\mathbb{T}^2$. Moreover denote by V_j, $j = 1,2$, open neighborhoods of $(0,0)$, (π,π) respectively in $\mathbb{T}^2$, such that $V_j \Subset U_j$. Then $\{U_1, U_2, \mathbb{T}^2 \setminus (\bar{V}_1 \cup \bar{V}_2)\}$ is an open covering of $\mathbb{T}^2$.

Using [21], we know that any solution in U_j of the equation $P_M u_j = f \in C^\omega(U_j)$ belongs to $G^2(U_j)$ and is not better than that. Moreover $u_1 \in C^\omega(U_1 \setminus \{(0,0)\})$ and $u_2 \in C^\omega(U_2 \setminus \{(\pi,\pi)\})$, since P_M is analytic hypoelliptic on any open set not containing the origin and the points $(0,0)$ and (π,π).

Since the cohomology groups with coefficients in the sheaf of real analytic functions vanish in degree ≥ 1 (see [20], [15], [9]), we have that, on $U_1 \setminus \bar{V}_1$, $u_1 = g_1 - f_1$, where $g_1 \in C^\omega(U_1)$, $f_1 \in C^\omega(\mathbb{T}^2 \setminus (\bar{V}_1 \cup \bar{V}_2)$. Analogously, on $U_2 \setminus \bar{V}_2$, we have $u_2 = g_2 - f_1$, where $g_2 \in C^\omega(U_2)$.

Define now $v = u_1 - g_1$ on U_1, $v = u_2 - g_2$ on U_2 and $v = -f_1$ on $\mathbb{T}^2 \setminus (\bar{V}_1 \cup \bar{V}_2)$. We have that $P_M v \in C^\omega(\mathbb{T}^2)$ and sing supp$_a$ $v \subset \{(0,0)\} \cup \{(\pi,\pi)\}$ and is nonempty.

This proves the statement. $\square$

Remark 1.1. *An analogous proof can be made also for operators vanishing of higher order on the characteristic variety, e.g.*

$$(1.3) \qquad P_1 = D_x^2 + (\sin x)^{2(q-1)} D_y^2 + ((\sin y)^a \ D_y)^2,$$

*where $q > 2$, $a \geq 2$. The only difference is that Métivier's theorem [21]
has to be replaced by that proved in [7].*

We explicitly point out that both P_M and P_1 have a characteristic variety which is a non symplectic real analitic manifold, where the Hamilton leaves lie along the η fibers of the cotangent bundle.

On the other hand there are situations where, even though the characteristic variety is a symplectic real analitic manifold, still we have the occurrence of "strata" of Métivier type, i.e. whose Hamilton leaves lie along the fibers of the cotangent bundle. A class of models of this type is given by

$$(1.4) \qquad P_2 = D_x^2 + (\sin x)^{2(q-1)} D_y^2 + ((\sin x)^{p-1}(\sin y)^a \ D_y)^2,$$

where $1 < p < q$, $a \geq 1$. The analog of Métivier result [21] is not known in general, however, if $q = 2p$, Chinni in [12] has proved the optimality of the Gevrey regularity s_0, where

$$\frac{1}{s_0} = 1 - \frac{1}{2a}.$$

It seems reasonable to surmise that the only globally non analytic hypoelliptic operators are operators where the "Métivier situation" occurs, meaning that there is a Hamilton leaf lying along the fibers of the cotangent bundle.

Unfortunately this poses some major problem even in its formulation, since we do not know a good way to stratify the characteristic variety, let alone to define a Hamilton leaf, due to [1], at least in dimension greater or equal to 3.

Just in passing we mention that in special situations we actually are able to define the strata and we refer to [5] for this.

In the present paper we consider a class of sums of squares operators where we think that Métivier type situations do not occur.

More precisely let n, m be two positive integers and P be as in (1.1). We assume the following:

  (1) The X_j are real analytic vector fields defined on the torus $\mathbb{T}^{n+m}$. We denote the variable as (t, x) where $t \in \mathbb{T}^n$, $x \in \mathbb{T}^m$.
  (2) Let $n' < n$ and consider $X_j(t, x, D_t, D_x)$ for $1 \leq j \leq n'$. We assume that

$$(1.5) \qquad\qquad X_j = \sum_{i=1}^{n'} a_{ji}(t')D_{t_i},$$

where $t = (t', t'')$ with $t' \in \mathbb{T}^{n'}$, $t'' \in \mathbb{T}^{n-n'}$. Furthermore we assume that the vector fields X_j, $1 \le j \le n'$, are linearly independent for every $t' \in \mathbb{T}^{n'}$.

(3) Consider now X_j for $n' + 1 \le j \le N$. We assume that $N \ge n$ and that X_j has the form

$$(1.6) \qquad X_j = a_j(t') D_{t_{q(j)}} + \sum_{k=1}^{m} b_{jk}(t) D_{x_k},$$

where a_j, b_{jk} are real analytic funtions defined in $\mathbb{T}^{n'}$, $\mathbb{T}^n$ respectively and q is a surjective map from $\{n'+1, \dots, N\}$ onto $\{n'+1, \dots, n\}$. Hence $q^{-1}(\{j\})$ is a partition of $\{n'+1, \dots, N\}$ with non empty subsets.

Furthermore we assume that for each $j = n'+1, \dots, n$, there exists $\lambda_j \in q^{-1}(\{j\})$, such that

$$(1.7) \qquad \sum_{r=n'+1}^{N} \sum_{k=1}^{m} |b_{rk}(t)| \le C |a_\ell(t')|,$$

for every $\ell \in \{\lambda_j \mid j \in \{n'+1, \dots, n\}\}$.
We also assume that

$$(1.8) \qquad \sum_{r=n'+1}^{N} |a_r(t')| \le C |a_\ell(t')|,$$

for every $\ell \in \{\lambda_j \mid j \in \{n'+1, \dots, n\}\}$.

(4) The vector fields X_j, $1 \le j \le N$, satisfy Hörmander condition.

Remark 1.2. *We could also consider the vector fields above in the case when $n = n'$ with no condition* (1.7). *Then the corresponding operator is in a subclass of that considered by Cordaro and Himonas in* [14].

The vector fields described above can be used to produce the global analog of some well known examples.

Example 1. Take $n' = 1$, $n = 2$, $m = 1$. Then $X_1 = D_{t_1}$, $X_2 = D_{t_2}$ and $X_3 = a(t_1) D_x$, where a denotes a non identically vanishing real analytic function defined on $\mathbb{T}^1$, give the globally defined version of the (possibly generalized) Baouendi-Goulaouic operator:

$$D_{t_1}^2 + D_{t_2}^2 + a^2(t_1) D_x^2,$$

see also [14]. We recall that the local version of the Baouendi-Goulaouic operator is given by $D_{t_1}^2 + D_{t_2}^2 + t_1^{2k} D_x^2$, $k \in \mathbb{Z}_+$.

Example 2. Take $n' = 1$, $n = 2$, $m = 1$. Then $X_1 = D_{t_1}$, $X_2 = a(t_1) D_{t_2}$ and $X_3 = b(t_1) D_x$, where a, b are non identically vanishing

real analytic functions defined on $\mathbb{T}^1$. Condition (1.7) becomes $|b(t_1)| \leq C|a(t_1)|$. The corresponding operator

$$D_{t_1}^2 + a^2(t_1)D_{t_2}^2 + b^2(t_1)D_x^2,$$

is a globally defined version of

> *i)* the Oleĭnik–Radkeviç operator if b vanishes only where a vanishes. We recall that the local version of the Oleĭnik–Radkeviç operator is given by $D_{t_1}^2 + t_1^{2(p-1)}D_{t_2}^2 + t_1^{2(q-1)}D_x^2$, $p, q \in \mathbb{Z}_+$, $1 < p \leq q$.
> *ii)* the Baouendi-Goulaouic operator if b vanishes and a does not.
> *iii)* an elliptic operator if neither a nor b vanish.

For a generalization of this example we refer to the paper [10].

Example 3. Let $n' = n = 2$, $m = 2$. Then $X_1 = D_{t_1}$, $X_2 = D_{t_2}$, $X_3 = a(t_1)D_{x_1}$, $X_4 = a(t_1)D_{x_2}$, $X_5 = b(t_2)D_{x_1}$, $X_6 = c(t_2)D_{x_2}$, a, b, c real analytic functions.

Consider the corresponding operator

$$(1.9) \qquad D_{t_1}^2 + D_{t_2}^2 + a^2(t_1)\left(D_{x_1}^2 + D_{x_2}^2\right) + b^2(t_2)D_{x_1}^2 + c^2(t_2)D_{x_2}^2.$$

Assume that a, b, c vanish at the origin of order $r - 1$, $p - 1$ and $q - 1$ respectively, with $r < p < q$. Then the operator above is the global version of the operator

$$D_{t_1}^2 + D_{t_2}^2 + t_1^{2(r-1)}\left(D_{x_1}^2 + D_{x_2}^2\right) + t_2^{2(p-1)}D_{x_1}^2 + t_2^{2(q-1)}D_{x_2}^2,$$

which has been proved to violate Treves conjecture in [1]. On the other hand Chinni in [13] has proved that the above globally defined operator is analytic hypoelliptic. The operator in (1.9) is also in the class studied by Cordaro and Himonas, [14].

If we choose $p < q < r$, then the corresponding operator in (1.9) is again globally analytic hypoelliptic and satisfies Treves conjecture.

Example 4. Let $n' = 1$, $n = 2$ and $m = 2$. Then $X_1 = D_{t_1}$, $X_2 = a(t_1)D_{t_2}$, $X_3 = b(t_1)D_{x_1}$, $X_4 = b(t_1)D_{x_2}$, $X_5 = c(t_1, t_2)D_{x_1}$, $X_6 = d(t_1, t_2)D_{x_2}$, a, b, c and d are real analytic functions.

Consider the corresponding operator

(1.10)

$$D_{t_1}^2 + a^2(t_1)D_{t_2}^2 + b^2(t_1)\left(D_{x_1}^2 + D_{x_2}^2\right) + c^2(t_1, t_2)D_{x_1}^2 + d^2(t_1, t_2)D_{x_2}^2.$$

Assume that a does not vanish and also that the operator in (1.10) is not elliptic. Then condition (1.7) implies that the operator in (1.10) is a slight generalization of that in (1.9). In particular it belongs to the class studied in [14].

Assume then that a vanishes at the origin. Then $a(t_1) = t_1^\ell \tilde{a}(t_1)$, where $\ell \in \mathbb{N}$, $\tilde{a}(0) \neq 0$. Hence condition (1.7) implies that $b, c, d = \mathcal{O}(t_1^\ell)$.

Assume that $b = \mathcal{O}(t_1^{\ell+r-1})$ for a certain $r \in \mathbb{N}$, $r > 1$. Moreover assume that $t_1^{-\ell} c(t) = \mathcal{O}(t_2^{p-1})$, and that $t_1^{-\ell} d(t) = \mathcal{O}(t_2^{q-1})$, for certain $p, q \in \mathbb{N}$, $1 < r < p < q$.

Then the operator in (1.10) is the global analog of the operator

$$D_{t_1}^2 + t_1^{2\ell} D_{t_2}^2 + t_1^{2(\ell+r-1)} \left(D_{x_1}^2 + D_{x_2}^2 \right) + t_1^{2\ell} \left(t_2^{2(p-1)} D_{x_1}^2 + t_2^{2(q-1)} D_{x_2}^2 \right);$$

it has a symplectic characteristic real analytic manifold having a symplectic stratification according to Treves. It has been proved to violate Treves conjecture in [4].

This operator does not belong to the class studied in [14].

Example 5. Let $n' = 1$, $n = 3$ and $m = 1$. Then $X_1 = D_{t_1}$, $X_2 = a_2(t_1) D_{t_2}$, $X_3 = a_3(t_1) D_{t_3} + b(t) D_x$, where a_2, a_3, b are real analytic functions in $\mathbb{T}^3$.

Assume that $a_2 = \mathcal{O}(t_1^{p-1})$ for $t_1 \to 0$, then condition (1.8) implies that $a_3 = \mathcal{O}(t_1^{p-1})$ for $t_1 \to 0$, and condition (1.7) inplies that $b = \mathcal{O}(t_1^{p-1})$ for $t_1 \to 0$, i.e. $b(t) = \mathcal{O}(t_1^{p-1}) \tilde{b}(t)$.

For this example, when condition (1.8) is not satisfied, we do not know a general analytic hypoellipticity result.

We state now the main result of the paper

Theorem 1.1. *Let P be a sum of squares operator as in* (1.1) *with real analytic coefficients globally defined on the torus $\mathbb{T}^{n+m}$. Assume that conditions* **(2), (3), (4)** *above are satisfied. Then P is globally analytic hypoelliptic.*

We point out explicitly that even though P is globally analytic hypoelliptic, it is not in general locally analytic hypoelliptic, as we can see from the above examples.

2. PROOF OF THEOREM 1.1

In order to prove the theorem we use the maximal hypoellipticity global L^2 estimate for P.

$$(2.1) \qquad \sum_{j=1}^{N} \|X_j u\|_0^2 \leq C \Big(\langle Pu, u \rangle + \|u\|_0^2 \Big),$$

where $u \in C^\infty(\mathbb{T}^{n+m})$. We explicitly remark that the subelliptic term $\|u\|_\varepsilon^2$—Sobolev norm of order ε, which is present in the estimates proved by Hörmander and Rothschild and Stein in [16], [24], is not needed,

since we cannot use it to prove analytic regularity, but only to prove a Gevrey regularity.

Since the characteristic variety of P is contained in

$$\{(t', t'', x; 0, \tau'', \xi) \mid |\tau''| + |\xi| > 0\},$$

we may use t''- and x-derivatives to establish real analyticity.

Actually we want to bind the quantity

$$\Big(\sum_{j=n'+1}^{n} D_{t_j}^p + \sum_{k=1}^{m} D_{x_k}^p \Big) u,$$

where p is a large integer and u is a smooth solution of $Pu = f$, with $f \in C^\omega(\mathbb{T}^{n+m})$ and P is given by

$$(2.2) \quad P(t, D_t, D_x) = \sum_{j=1}^{n'} \Big(\sum_{i=1}^{n'} a_{ji}(t') D_{t_i} \Big)^2$$

$$+ \sum_{j=n'+1}^{N} \Big(a_j(t') D_{t_{q(j)}} + \sum_{k=1}^{m} b_{jk}(t) D_{x_k} \Big)^2 .$$

2.1. **The x-derivatives.** Let $k \in \{1, \ldots, m\}$. We want to estimate $\|X D_{x_k}^p u\|_0$, where X denotes one of the vector fields X_j, $j = 1, \ldots, N$.

To treat the error term on the right hand side of (2.1) we use the subelliptic estimate with a generic subelliptic term. We know that there exists a positive number, ε, such that the Lie algebra is generated by brackets of length $\leq \varepsilon^{-1}$ so that we have

$$(2.3) \qquad \|u\|_\varepsilon^2 + \sum_{j=1}^{N} \|X_j u\|_0^2 \leq C\Big(\langle Pu, u \rangle + \|u\|_0^2 \Big).$$

Let us start by considering $\|D_{x_k}^p u\|_0$, i.e. the error term on the right hand side of (2.3) where u has been replaced by $D_{x_k}^p u$.

We start off by showing that the error term $\|D_{x_k}^p u\|_0^2$ can actually be absorbed in the l.h.s. of (2.3). To this end, denote by χ a smooth cutoff function such that $\chi(t) = 1$ if $|t| \geq 2$ and $\chi(t) = 0$ if $|t| \leq 1$. It turns out that $\chi(p^{-1}\langle D \rangle) \in OPS^0$, where $\langle D \rangle = (1 + |D|^2)^{\frac{1}{2}}$. (see Def. A.1 in Appendix) and then

$$(2.4) \qquad \|D_{x_k}^p u\|_0 \leq \|(1 - \chi(p^{-1}\langle D \rangle)) D_{x_k}^p u\|_0 + \|\chi(p^{-1}\langle D \rangle) D_{x_k}^p u\|_0.$$

The first summand can be easily estimated, using Proposition A.1, because the support of the cutoff $1 - \chi$ is contained in $\{\xi \in \mathbb{Z}^m \mid |\xi| \leq 2p\}$.

Whence we obtain

$$\|(1 - \chi(p^{-1}\langle D \rangle))D_{x_k}^p u\|_0 \le C^{p+1}p^p,$$

which is an analytic growth estimate.

Thus we are left with the estimate of the second summand in the r.h.s. of (2.4). We have that

$$\|\chi(p^{-1}\langle D \rangle)D_{x_k}^p u\|_0 = p^{-\varepsilon}\|p^{\varepsilon}\chi(p^{-1}\langle D \rangle)\langle D \rangle^{-\varepsilon}\langle D \rangle^{\varepsilon}D_{x_k}^p u\|_0.$$

Due to the support of the cutoff χ, we see that

$$\sigma\big(p^{\varepsilon}\chi(p^{-1}\langle D \rangle)\langle D \rangle^{-\varepsilon}\big) = p^{\varepsilon}\chi(p^{-1}\langle \xi \rangle)\langle \xi \rangle^{-\varepsilon} \in S^0$$

with the S^0-semi-norms uniformly bounded with respect to p; thus the L^2 continuity theorem A.1 it follows that

$$\|p^{\varepsilon}\chi(p^{-1}\langle D \rangle)\langle D \rangle^{-\varepsilon}\|_{\mathcal{L}(L^2,L^2)} \le C,$$

C being a positive constant independent of p. Summarizing we obtained the following inequality

$$(2.5) \qquad \|D_{x_k}^p u\|_0 \le C^{p+1}p^p + Cp^{-\varepsilon}\|D_{x_k}^p u\|_{\varepsilon}.$$

Hence, plugging $D_{x_k}^p u$ into (2.3), because of (2.5), we obtain

$$(2.6) \qquad \|D_{x_k}^p u\|_{\varepsilon}^2 + \sum_{j=1}^N \|X_j D_{x_k}^p u\|_0^2 \le C\Big(\langle PD_{x_k}^p u, D_{x_k}^p u \rangle + C^{2p}p^{2p}\Big),$$

with possibly a larger constant C. Here the term $Cp^{-\varepsilon}\|D_{x_k}^p u\|_{\varepsilon}$ has been absorbed on the left hand side of (2.6).

Due to conditions $(1) - (3)$, D_{x_k} commutes with P, so that

$$(2.7) \quad \|D_{x_k}^p u\|_{\varepsilon}^2 + \sum_{j=1}^N \|X_j D_{x_k}^p u\|_0^2 \le C\Big(\|D_{x_k}^p Pu\|_0^2 + \|D_{x_k}^p u\|_0^2 + C^{2p}p^{2p}\Big).$$

Hence, applying (2.5) to the next to last term above and keeping into account that Pu is an analytic function, we obtain

$$(2.8) \qquad \|D_{x_k}^p u\|_{\varepsilon} \le C^{p+1}p^p,$$

for every $k = 1, \ldots, m$, i.e. an analytic growth rate with respect to the variable x.

2.2. **The t''-derivatives.** Next we consider $D_{t_j}^q$, $j \in \{n'+1,\ldots,n\}$ and q is a large integer. As above from (2.3) we deduce

$$(2.9) \quad \|D_{t_j}^q u\|_\varepsilon^2 + \sum_{r=1}^N \|X_r D_{t_j}^q u\|_0^2 \leq C\Big(\langle PD_{t_j}^q u, D_{t_j}^q u\rangle + \|D_{t_j}^q u\|_0^2\Big).$$

Applying (2.5) to the second term on the right hand side of the above inequality we obtain, with a possibly larger constant,

$$(2.10) \quad \|D_{t_j}^q u\|_\varepsilon^2 + \sum_{r=1}^N \|X_r D_{t_j}^q u\|_0^2 \leq C\Big(\langle PD_{t_j}^q u, D_{t_j}^q u\rangle + C^{2q}q^{2q}\Big).$$

Consider now $\langle PD_{t_j}^q u, D_{t_j}^q u\rangle$. We have

$$\langle PD_{t_j}^q u, D_{t_j}^q u\rangle = \sum_{r=1}^N \langle X_r^2 D_{t_j}^q u, D_{t_j}^q u\rangle$$

$$= \sum_{r=1}^N \Big(\langle D_{t_j}^q X_r^2 u, D_{t_j}^q u\rangle + \langle [X_r^2, D_{t_j}^q]u, D_{t_j}^q u\rangle\Big).$$

Now

$$[X_r^2, D_{t_j}^q] = 2X_r[X_r, D_{t_j}^q] - [[D_{t_j}^q, X_r], X_r].$$

Let us examine first $\langle X_r[X_r, D_{t_j}^q]u, D_{t_j}^q u\rangle$, for $r = 1,\ldots,N$. If $r \in \{1,\ldots,n'\}$ then the commutator is zero, so that we have to consider the case $r \in \{n'+1,\ldots,N\}$. Since in this case X_r is a self adjoint vector field, we have

$$\langle X_r[X_r, D_{t_j}^q]u, D_{t_j}^q u\rangle = \langle [X_r, D_{t_j}^q]u, X_r D_{t_j}^q u\rangle.$$

By Cauchy-Schwartz we get

$$\Big|\langle X_r[X_r, D_{t_j}^q]u, D_{t_j}^q u\rangle\Big| \leq \delta\|X_r D_{t_j}^q u\|_0^2 + C_\delta\|[X_r, D_{t_j}^q]u\|_0^2.$$

Choosing δ in a convenient way allows us to absorb the first norm on the right hand side above on the left hand side of (2.10).

Thus

$$[D_{t_j}^q, X_r] = \sum_{k=1}^m [D_{t_j}^q, b_{rk}(t)]D_{x_k} = \sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}(\text{ad } D_{t_j})^\ell(b_{rk})D_{t_j}^{q-\ell}D_{x_k}.$$

Hence

$$(2.11) \quad \|[X_r, D_{t_j}^q]u\|_0 \leq \sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}\|(D_{t_j}^\ell b_{rk})D_{t_j}^{q-\ell}D_{x_k}u\|_0.$$

By condition (1.7) each coefficient $|b_{rk}| \leq C|a_\ell|$, for every $\ell \in \{\lambda_j \mid j \in \{n'+1,\ldots,n\}\}$ and for every $k \in \{1,\ldots,m\}$, $r \in \{n'+1,\ldots,N\}$.

If the a_ℓ do not vanish on $\mathbb{T}^{n'}$ for $\ell \in \{\lambda_j \mid j \in \{n' + 1, \ldots, n\}\}$, we do nothing.

Assume that $a_\ell^{-1}(0) \neq \varnothing$. For this we need a partition of unity in $\mathbb{T}^{n'}$. Let U_h, $h = 1, \ldots, M$, denote an open covering of $\mathbb{T}^{n'}$ and let $\varphi_h(t') \in C_0^\infty(U_h)$, $\sum_{h=1}^M \varphi_h(t') = 1$, be a partition of unity subordinated to the covering U_h.

We are going to estimate $\|\varphi_h[X_r, D_{t_j}^q]u\|_0$, $h = 1, \ldots, M$.

We want to discuss the restriction of b_{rk} to U_h. For the sake of simplicity we argue for U_h when the origin belongs to U_h and we assume that $a_\ell^{-1}(0) \cap U_h \neq \varnothing$. We may always assume that $a_\ell(0) = 0$, since the estimate we obtain is independent of h.

Without loss of generality we may also assume that $a_\ell(t_1, 0)$ has an isolated zero of multiplicity p at zero. Then by the Weierstraß preparation theorem, see e.g. [28], we may write

$$a_\ell(t') = e_\ell(t')\mathfrak{p}_\ell(t_1, \tilde{t}),$$

where

$$\mathfrak{p}_\ell(t_1, \tilde{t}) = t_1^p + \sum_{j=1}^p a_{\ell j}(\tilde{t})t_1^{p-j},$$

where $\tilde{t} = (t_2, \ldots, t_{n'})$, $e_\ell(t') \in C^\omega(U_h)$ nowhere vanishing and $a_{\ell j}(\tilde{t})$ are real analytic functions vanishing at $\tilde{t} = 0$.

Consider now $b_{rk}(t_1, 0, t'')$. It vanishes when $t_1 = 0$, for every $t'' \in \mathbb{T}^{n''}$ due to our assumptions. We may always perform a small linear change of the variables t' in such a way that both $a_\ell(t_1, 0)$ and $b_{rk}(t_1, 0, t'')$ have an isolated zero at $t_1 = 0$, even though of different multiplicities.

Then by the Weierstraß preparation theorem we may write

$$b_{rk}(t', t'') = \varepsilon_{rk}(t', t'')\mathfrak{q}_{rk}(t_1, \tilde{t}, t''),$$

where

$$\mathfrak{q}_{rk}(t_1, \tilde{t}, t'') = t_1^q + \sum_{j=1}^q b_{krj}(\tilde{t}, t'')t_1^{q-j}.$$

Our assumption that $|b_{rk}| \leq C|a_\ell|$ imply that $|\mathfrak{q}_{rk}| \leq C_1|\mathfrak{p}_\ell|$. This implies that $q \geq p$ and that $\mathfrak{p}_\ell$ divides $\mathfrak{q}_{rk}$, i.e.

(2.12) $$b_{rk}(t) = \varepsilon_{rk}(t)\mathfrak{p}_\ell(t_1, \tilde{t})\mathfrak{q}'(t_1, \tilde{t}, t''),$$

for every $r \in \{n' + 1, \ldots, N\}$, $k \in \{1, \ldots, m\}$ and $\ell \in \{\lambda_j \mid j \in \{n' + 1, \ldots, n\}\}$. Here $\mathfrak{q}'$ denotes a Weierstraß polynomial of degree $q - p$.

We note explicitly that the dependence on t'' of the functions b_{rk} is confined to the unity ε_{rk} and the coefficients of $\mathfrak{q}'$.

As a consequence of the above argument we have that

$$(2.13) \qquad D_{t_j}^{\ell_1} b_{rk}(t) = D_{t_j}^{\ell_1}(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))\mathfrak{p}_\ell(t_1, \tilde{t}).$$

Going back to (2.11), we rewrite (2.13) as

$$(2.14) \quad D_{t_j}^{\ell} b_{rk}(t) = D_{t_j}^{\ell}(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))\mathfrak{p}_{\lambda_j}(t_1, \tilde{t})$$
$$= D_{t_j}^{\ell}(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))e_{\lambda_j}(t')^{-1}a_{\lambda_j}(t').$$

Then

$$\|\varphi_h[X_r, D_{t_j}^q]u\|_0$$
$$\leq \sum_{k=1}^{m}\sum_{\ell=1}^{q}\binom{q}{\ell}\|\varphi_h D_{t_j}^{\ell}(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))e_{\lambda_j}(t')^{-1}a_{\lambda_j}(t')D_{t_j}^{q-\ell}D_{x_k}u\|_0.$$

There are two cases: the first is $\ell = q$ and the second is $\ell < q$. In the first case we obtain

$$(2.15) \quad \sum_{k=1}^{m}\|\varphi_h D_{t_j}^q(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))e_{\lambda_j}(t')^{-1}a_{\lambda_j}(t')D_{x_k}u\|_0$$
$$\leq \sum_{k=1}^{m}C_1^{q+1}q!\|D_{x_k}u\|_0 \leq C_2^{q+1}q!\|u\|_0.$$

Consider the second case

$$(2.16)$$
$$\sum_{k=1}^{m}\sum_{\ell=1}^{q-1}\binom{q}{\ell}\|\varphi_h D_{t_j}^{\ell}(\varepsilon_{rk}(t)\mathfrak{q}'(t_1, \tilde{t}, t''))e_{\lambda_j}(t')^{-1}a_{\lambda_j}(t')D_{t_j}^{q-\ell}D_{x_k}u\|_0$$
$$\leq \sum_{k=1}^{m}\sum_{\ell=1}^{q-1}C_2^{\ell+1}q^{\ell}\|a_{\lambda_j}(t')D_{t_j}D_{t_j}^{q-\ell-1}D_{x_k}u\|_0$$
$$\leq \sum_{k=1}^{m}\sum_{\ell=1}^{q-1}C_2^{\ell+1}q^{\ell}\|X_{\lambda_j}D_{t_j}^{q-\ell-1}D_{x_k}u\|_0$$
$$+ \sum_{k=1}^{m}\sum_{\ell=1}^{q-1}\sum_{k_1=1}^{m}C_2^{\ell+1}q^{\ell}\|b_{\lambda_j k_1}D_{t_j}^{q-\ell-1}D_{x_{k_1}}D_{x_k}u\|_0.$$

The first term is ready for an induction and hence we have to consider the last term above: by assumption (1.7) we may write

$$\sum_{k=1}^{m}\sum_{\ell=1}^{q-1}\sum_{k_1=1}^{m}C_2^{\ell+1}q^{\ell}\|b_{\lambda_j k_1}D_{t_j}^{q-\ell-1}D_{x_{k_1}}D_{x_k}u\|_0$$

$$\leq C \sum_{k=1}^{m} \sum_{k_1=1}^{m} C_2^q q^{q-1} \| D_{x_{k_1}} D_{x_k} u \|_0$$

$$+ C \sum_{k=1}^{m} \sum_{\ell=1}^{q-2} \sum_{k_1=1}^{m} C_2^{\ell+1} q^{\ell} \| a_{\lambda_j} D_{t_j} D_{t_j}^{q-\ell-2} D_{x_{k_1}} D_{x_k} u \|_0$$

$$\leq C \sum_{k=1}^{m} \sum_{k_1=1}^{m} C_2^q q^{q-1} \| D_{x_{k_1}} D_{x_k} u \|_0$$

$$+ C \sum_{k=1}^{m} \sum_{\ell=1}^{q-2} \sum_{k_1=1}^{m} C_2^{\ell+1} q^{\ell} \| X_{\lambda_j} D_{t_j}^{q-\ell-2} D_{x_{k_1}} D_{x_k} u \|_0$$

$$+ C \sum_{k=1}^{m} \sum_{\ell=1}^{q-2} \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} C_2^{\ell+1} q^{\ell} \| b_{\lambda_j k_2} D_{t_j}^{q-\ell-2} D_{x_{k_2}} D_{x_{k_1}} D_{x_k} u \|_0.$$

Again the first term yields an analytic growth rate, the second is ready for an induction. For the last term above we may iterate $q - 1$ times the procedure obtaining

$$\sum_{k=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} \| \varphi_h D_{t_j}^{\ell} (\varepsilon_{rk}(t) \mathfrak{q}'(t_1, \tilde{t}, t'')) e_{\lambda_j}(t')^{-1} a_{\lambda_j}(t') D_{t_j}^{q-\ell} D_{x_k} u \|_0$$

$$\leq \sum_{i=1}^{q-1} \sum_{k=1}^{m} \sum_{k_1=1}^{m} \cdots \sum_{k_{i-1}=1}^{m} C_3^{q+1} q^{q-i+1} \| D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{i-1}}} u \|_0$$

$$+ \sum_{k=1}^{m} \sum_{k_1=1}^{m} \cdots \sum_{k_{q-2}=1}^{m} C_3^{q+1} q \| X_{\lambda_j} D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{q-2}}} u \|_0$$

$$+ \sum_{k=1}^{m} \sum_{k_1=1}^{m} \cdots \sum_{k_{q-1}=1}^{m} C_3^{q+1} q \| D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{q-1}}} u \|_0.$$

Consider now the second summation on the right hand side above. Using (2.7), (2.8), and remarking that in those inequalities one may swap the derivative $D_{x_k}^p$ with D_x^α, where α is a multliindex with $|\alpha| = p$, we have that

$$(2.17) \quad \sum_{k=1}^{m} \sum_{k_1=1}^{m} \cdots \sum_{k_{q-2}=1}^{m} C_3^{q+1} q \| X_{\lambda_j} D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{q-2}}} u \|_0 \leq C_4^{q+1} q^q.$$

The first sum and the third can be rewritten as

$$\sum_{i=1}^{q} \sum_{k=1}^{m} \sum_{k_1=1}^{m} \cdots \sum_{k_{i-1}=1}^{m} C_3^{q+1} q^{q-i+1} \| D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{i-1}}} u \|_0.$$

Now, by (2.5), we have, for a multiindex α,

$$\|D_x^\alpha u\|_0 \leq C^{|\alpha|+1}|\alpha|^{|\alpha|} + C|\alpha|^{-\varepsilon}\|D_x^\alpha u\|_\varepsilon.$$

Hence, using (2.8), we get
(2.18)
$$\sum_{i=1}^q \sum_{k=1}^m \sum_{k_1=1}^m \cdots \sum_{k_{i-1}=1}^m C_3^{q+1} q^{q-i+1} \|D_{x_k} D_{x_{k_1}} \cdots D_{x_{k_{i-1}}} u\|_0 \leq C_4^{q+1} q^q,$$

for a suitable positive constant C_4.

Summing over h the above estimate implies that

(2.19)
$$\|[X_r, D_{t_j}^q]u\|_0 \leq C_4^{q+1} q^q,$$

with a slightly larger constant.

Consider next the term, with $r \in \{n'+1, \ldots, N\}$, $j = n'+1, \ldots, n$,

$$\langle [[D_{t_j}^q, X_r], X_r]u, D_{t_j}^q u\rangle.$$

We have

$$[[D_{t_j}^q, X_r], X_r] = \sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}[(\operatorname{ad} D_{t_j})^\ell(b_{rk})D_{t_j}^{q-\ell}, X_r]D_{x_k}$$

$$= \sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}[(D_{t_j}^\ell b_{rk}), X_r]D_{t_j}^{q-\ell}D_{x_k}$$

$$+ \sum_{k=1}^m \sum_{\ell=1}^q \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \binom{q}{\ell}\binom{q-\ell}{\ell_1}(D_{t_j}^\ell b_{rk})(\operatorname{ad} D_{t_j})^{\ell_1}(b_{rk_1})D_{t_j}^{q-\ell-\ell_1}D_{x_k}D_{x_{k_1}}$$

$$= -\sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}a_r(t')(D_{t_{q(r)}}D_{t_j}^\ell b_{rk})D_{t_j}^{q-\ell}D_{x_k}$$

$$+ \sum_{k=1}^m \sum_{\ell=1}^q \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \binom{q}{\ell}\binom{q-\ell}{\ell_1}(D_{t_j}^\ell b_{rk})(D_{t_j}^{\ell_1}b_{rk_1})D_{t_j}^{q-\ell-\ell_1}D_{x_k}D_{x_{k_1}}$$

$$= J_1 + J_2.$$

Let us examine J_1 first. We have

$$\langle J_1 u, D_{t_j}^q u\rangle = -\sum_{k=1}^m \sum_{\ell=1}^q \binom{q}{\ell}\langle (D_{t_{q(r)}}D_{t_j}^\ell b_{rk})D_{t_j}^{q-\ell}D_{x_k}u, a_r(t')D_{t_j}^q u\rangle$$

$$= -\sum_{k=1}^m \langle (D_{t_{q(r)}}D_{t_j}^q b_{rk})D_{x_k}u, a_r(t')D_{t_j}^q u\rangle$$

$$- \sum_{k=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} \langle (D_{t_{q(r)}} D_{t_j}^{\ell} b_{rk}) D_{t_j}^{q-\ell-1} D_{x_k} u, a_r(t') D_{t_j}^{q+1} u \rangle$$

$$+ \sum_{k=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} \langle (D_{t_{q(r)}} D_{t_j}^{\ell+1} b_{rk}) D_{t_j}^{q-\ell-1} D_{x_k} u, a_r(t') D_{t_j}^{q} u \rangle$$

$$= J_{11} + J_{12} + J_{13}.$$

Consider J_{11}. We have that

$$|J_{11}| \leq C_0^{q+1} q! \sup_k \|D_{x_k} u\|_0 \|D_{t_j}^q u\|_0,$$

which gives an analytic growth rate by using an analog of (2.5). Consider then J_{12}. For the left hand side factor of the scalar product we apply assumption (1.7) and proceed analogously to (2.16). For the right hand side factor of the scalar product we apply assumption (1.8). We obtain

$$(2.20) \quad |J_{12}| \leq \sum_{k=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} C_b^{\ell+2} (\ell+1)! \|X_{\lambda_j} D_{t_j}^{q-\ell-2} D_{x_k} u\|_0 \|X_{\lambda_j} D_{t_j}^q u\|_0$$

$$+ \sum_{k=1}^{m} \sum_{k_1=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} C_b^{\ell+2} (\ell+1)! \|b_{\lambda_j k_1} D_{t_j}^{q-\ell-2} D_{x_k} D_{x_{k_1}} u\|_0 \|X_{\lambda_j} D_{t_j}^q u\|_0$$

$$+ \sum_{k=1}^{m} \sum_{k_1=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} C_b^{\ell+2} (\ell+1)! \|X_{\lambda_j} D_{t_j}^{q-\ell-2} D_{x_k} u\|_0 \|b_{\lambda_j k_1} D_{t_j}^q D_{x_{k_1}} u\|_0$$

$$+ \sum_{k=1}^{m} \sum_{k_1=1}^{m} \sum_{k_1'=1}^{m} \sum_{\ell=1}^{q-1} \binom{q}{\ell} C_b^{\ell+2} (\ell+1)! \|b_{\lambda_j k_1'} D_{t_j}^{q-\ell-2} D_{x_k} D_{x_{k_1'}} u\|_0$$

$$\cdot \|b_{\lambda_j k_1} D_{t_j}^q D_{x_{k_1}} u\|_0.$$

As for the first summand we observe that

$$\binom{q}{\ell} (\ell+1)! \leq q^{\ell+1},$$

so that the norm $q^{\ell+1} \|X_{\lambda_j} D_{t_j}^{q-\ell-2} D_{x_k} u\|_0$ yields an analytic growth rate, since the norm $\|X_{\lambda_j} D_{t_j}^q u\|_0$ can be absorbed on the left hand side of the a priori estimate.

Let us consider the other summands in the above inequality. Observe that the norms appearing in the sums are of the same type as those on the right hand side of (2.16). Hence arguing in the same way we can conclude as we did for the case of the simple commutator (see the argument preceding (2.18).)

Consider then J_{13}. It is a lower order term due to the fact that one derivative landed onto b_{rk}. Thus its treatment is completely analogous to that of J_{12}, but simpler once (2.5) is used to absorb on the left hand side $q^{-\varepsilon}\|D_{t_j}^q u\|_\varepsilon$.

Finally we have to examine J_2:

$$\langle J_2 u, D_{t_j}^q u\rangle = \sum_{k=1}^m \sum_{\ell=1}^{q-1} \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \binom{q}{\ell}\binom{q-\ell}{\ell_1}$$

$$\cdot \langle (D_{t_j}^\ell b_{rk})(D_{t_j}^{\ell_1} b_{rk_1})D_{t_j}^{q-\ell-\ell_1} D_{x_k} D_{x_{k_1}} u, D_{t_j}^q u\rangle + \left(C_0^{q+1}q!\right)^2 + \delta\|D_{t_j}^q u\|_\varepsilon^2,$$

where δ is a small constant.

Next we are going to forget about the last two summands in the above relation.

We start off by bringing a x-derivative to the right hand side of the scalar product and ℓ_1 t_j-derivatives to the left, so that

$$\langle J_2 u, D_{t_j}^q u\rangle = \sum_{k=1}^m \sum_{\ell=1}^{q-1} \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \binom{q}{\ell}\binom{q-\ell}{\ell_1}$$

$$\cdot \langle (D_{t_j}^\ell b_{rk})D_{t_j}^{q-\ell-\ell_1} D_{x_k} u, (D_{t_j}^{\ell_1} b_{rk_1})D_{t_j}^q D_{x_{k_1}} u\rangle$$

$$= \sum_{k=1}^m \sum_{\ell=1}^{q-1} \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \sum_{s=0}^{\ell_1} \sum_{\sigma=0}^{\ell_1-s}(-1)^s \binom{q}{\ell}\binom{q-\ell}{\ell_1}\binom{\ell_1}{s}\binom{\ell_1-s}{\sigma}$$

$$\cdot \langle (D_{t_j}^{\ell+\sigma} b_{rk})D_{t_j}^{q-\ell-s-\sigma} D_{x_k} u, (D_{t_j}^{\ell_1+s} b_{rk_1})D_{t_j}^{q-\ell_1} D_{x_{k_1}} u\rangle,$$

where the adjoint Newton binomial formula has been used. Hence

$$|\langle J_2 u, D_{t_j}^q u\rangle| \le$$

$$\sum_{k=1}^m \sum_{\ell=1}^{q-1} \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \sum_{s=0}^{\ell_1} \sum_{\sigma=0}^{\ell_1-s} \frac{q!}{\ell!}\frac{1}{(q-\ell-\ell_1)!}\frac{1}{s!}\frac{1}{\sigma!(\ell_1-s-\sigma)!}$$

$$\cdot \|(D_{t_j}^{\ell+\sigma} b_{rk})D_{t_j}^{q-\ell-s-\sigma} D_{x_k} u\|_0 \cdot \|(D_{t_j}^{\ell_1+s} b_{rk_1})D_{t_j}^{q-\ell_1} D_{x_{k_1}} u\|_0.$$

By (2.14) we obtain

$$|\langle J_2 u, D_{t_j}^q u\rangle| \le$$

$$\sum_{k=1}^m \sum_{\ell=1}^{q-1} \sum_{k_1=1}^m \sum_{\ell_1=1}^{q-\ell} \sum_{s=0}^{\ell_1} \sum_{\sigma=0}^{\ell_1-s} C_b^{\ell+\ell_1+s+\sigma+1} \frac{q!}{\ell!}\frac{1}{(q-\ell-\ell_1)!}\frac{1}{s!}\frac{1}{\sigma!(\ell_1-s-\sigma)!}$$

$$\cdot (\ell+\sigma)!(\ell_1+s)!\|a_{\lambda_j}(t')D_{t_j}^{q-\ell-s-\sigma} D_{x_k} u\|_0 \cdot \|a_{\lambda_j}(t')D_{t_j}^{q-\ell_1} D_{x_{k_1}} u\|_0$$

$$\leq \sum_{k=1}^{m} \sum_{\ell=1}^{q-1} \sum_{k_1=1}^{m} \sum_{\ell_1=1}^{q-\ell} \sum_{s=0}^{\ell_1} \sum_{\sigma=0}^{\ell_1-s} C_1^{\ell+\ell_1+s+\sigma+1} q^{\ell+s+\sigma}$$

$$\cdot \|a_{\lambda_j}(t') D_{t_j} D_{t_j}^{q-\ell-s-\sigma-1} D_{x_k} u\|_0 \cdot q^{\ell_1} \|a_{\lambda_j}(t') D_{t_j} D_{t_j}^{q-\ell_1-1} D_{x_{k_1}} u\|_0.$$

Then we may argue for each factor above as in (2.16) to get analytic growth rate.

This ends the estimate of the left hand side of (2.10) for $j \in \{n' + 1, \ldots, n\}$. This implies that

$$(2.21) \qquad |\Delta_{(t,x)}^q u| \leq C^{2q+1} q!^2, \quad \text{on } \mathbb{T}^{n+m},$$

which proves our statement.

3. Further thoughts on Example 5

Let us consider Example 5 without assuming condition (1.8):

$$(3.1) \qquad D_{t_1}^2 + a_2(t_1)^2 D_{t_2}^2 + (a_3(t_1) D_{t_3} + b(t) D_x)^2,$$

where

$$a_2(t_1) = \mathcal{O}(t_1^{p-1}) \text{ for } t_1 \to 0,$$

and assuming that

$$b(t) = \mathcal{O}(t_1^k) \tilde{b}(t) \text{ for } t_1 \to 0, \text{ where } k \geq 2(p-1).$$

The proof of the global analytic hypoellipticity of (3.1) differs from the above proof only in the estimate of the double commutator term.

More precisely the estimate of the right hand side factor of the scalar product in (2.20) can be obtained by using the above assumption on the function b instead that condition (1.8).

A. Appendix

For the sake of completeness we recall here some well-known facts about pseudodifferential operators on the torus $\mathbb{T}^n$.

Definition A.1. *For any $m \in \mathbb{R}$, we denote by $S^m(\mathbb{T}^n)$ the set of all the functions $p(x, \xi) \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ such that for every multi-index α, there exits a positive constant C_α for which*

$$|\partial_x^\alpha p(x, \xi)| \leq C_\alpha \langle \xi \rangle^m,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

We denote by OPS^m the class of the corresponding pseudodifferential operators $P = p(x, D)$ whose action on smooth functions on the torus is defined as

$$p(x, D)u(x) = \sum_{\xi \in \mathbb{Z}^n} p(x, \xi) \hat{u}_\xi e^{ix \cdot \xi},$$

where $u(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{u}_\xi e^{ix\cdot\xi}$ *is the Fourier series of* u.

It is trivial to see that the symbol class $S^m(\mathbb{T}^n)$ equipped with the semi-norms

$$|p|_\ell^{(m)} = \max_{|\alpha|\leq\ell} \sup_{x\in\mathbb{T}^n} \{|\partial_x^\alpha p(x,\xi)|\langle\xi\rangle^{-m}\}, \quad \ell \in \mathbb{N},$$

is a Fréchet space.

We need the L^2-continuity of the pseudodifferential operators in the above classes. We state below a formulation of such a theorem for pseudodifferential operators of order σ.

Proposition A.1 (Chinni and Cordaro, [11]). *Let* Q *be a pseudodifferential operator on* $\mathbb{T}^n$ *of order* $\sigma \in \mathbb{R}$. *Then there is a constant* $M > 0$ *such that, for every* $k \in \mathbb{Z}_+$,

$$\|Qu\|_k \leq M|q|_{k+[|\sigma|]+n+1}^{(\sigma)} (\|u\|_{k+\sigma} + \|u\|_\sigma), \quad u \in C^\infty(\mathbb{T}^n).$$

Here we denoted by $[|\sigma|]$ *the integral part of* $|\sigma|$ *plus 1, if* $|\sigma|$ *is not an integer, and* $|\sigma|$ *if* $|\sigma|$ *is an integer.*

Proof. Let $\tilde{q}(x,\eta) = e_{-\eta}(x)Q(e_\eta)(x)$ be the discrete symbol of Q. Here $e_\eta(x) = e^{ix\cdot\eta}$. By the definition of a pseudodifferential operator of order σ, there is a constant $C > 0$ such that

$$|D_x^\alpha \tilde{q}(x,\eta)| \leq C_\alpha(1 + |\eta|)^\sigma, \quad x \in \mathbb{T}^n, \alpha \in \mathbb{Z}_+^n.$$

Set $q(\xi,\eta) = \langle Q(e_\eta), e_\xi\rangle_0$. Then the following inequality holds:

(A.1) $\qquad |q(\xi,\eta)| \leq C_N(1 + |\xi - \eta|)^{-N}(1 + |\eta|)^\sigma, \quad \xi, \eta \in \mathbb{Z}^n,$

for every $N \in \mathbb{N}$ and for some constant $C_N > 0$. Moreover we can write

$$\widehat{(Qu)}(\xi) = \sum_{\eta\in\mathbb{Z}^n} q(\xi,\eta)\hat{u}(\eta), \quad \xi \in \mathbb{Z}^n, u \in C^\infty(\mathbb{T}^N).$$

Now we estimate

$$\|Qu\|_k = \left(\sum_{\xi\in\mathbb{Z}^n}(1 + |\xi|)^{2k}|\widehat{Qu}(\xi)|^2\right)^{1/2}$$

$$= \left(\sum_{\xi\in\mathbb{Z}^n}\left(\sum_{\eta\in\mathbb{Z}^n}(1 + |\xi|)^k|q(\xi,\eta)||\widehat{u}(\eta)|\right)^2\right)^{1/2}$$

$$\leq \left(\sum_{\xi\in\mathbb{Z}^n}\left(\sum_{\ell=0}^k \binom{k}{\ell}\sum_{\eta\in\mathbb{Z}^n}|\xi-\eta|^\ell|q(\xi,\eta)|(1+|\eta|)^{k-\ell}|\widehat{u}(\eta)|\right)^2\right)^{1/2}.$$

Applying the Minkowsky's inequality we can estimate the right hand side of the above inequality:

$$\|Qu\|_k \le \sum_{\ell=0}^k \binom{k}{\ell} \left(\sum_{\xi\in\mathbb{Z}^n} \left(\sum_{\eta\in\mathbb{Z}^n} |\xi-\eta|^\ell |q(\xi,\eta)|(1+|\eta|)^{k-\ell}|\widehat{u}(\eta)| \right)^2 \right)^{1/2}$$

$$\le \sum_{\ell=0}^k \binom{k}{\ell} \left(\sum_{\xi\in\mathbb{Z}^n} \left(\sum_{\eta\in\mathbb{Z}^n} |\xi-\eta|^{2\ell}|q(\xi,\eta)| \right) \right.$$

$$\left. \left(\sum_{\eta\in\mathbb{Z}^n} |q(\xi,\eta)|(1+|\eta|)^{2(k-\ell)}|\widehat{u}(\eta)|^2 \right) \right)^{1/2} = I.$$

Applying (A.1) and Peetre inequality

$$(1+|\eta|)^\sigma \le (1+|\xi-\eta|)^{|\sigma|}(1+|\xi|)^\sigma, \quad \xi,\eta \in \mathbb{Z}^n,$$

we have

$$I \le C_k \sum_{\ell=0}^k \binom{k}{\ell} \left(\sum_{\xi\in\mathbb{Z}^n} \left(\sum_{\eta\in\mathbb{Z}^n} (1+|\eta|)^\sigma (1+|\xi-\eta|)^{-(n+1)} \right) \right.$$

$$\left. \cdot \left(\sum_{\eta\in\mathbb{Z}^n} |q(\xi,\eta)|(1+|\eta|)^{2(k-\ell)}|\widehat{u}(\eta)|^2 \right) \right)^{1/2}$$

$$\le C_k \sum_{\ell=0}^k \binom{k}{\ell} \left(\sum_{\xi\in\mathbb{Z}^n} (1+|\xi|)^\sigma \sum_{\eta\in\mathbb{Z}^n} |q(\xi,\eta)|(1+|\eta|)^{2(k-\ell)}|\widehat{u}(\eta)|^2 \right)^{1/2}$$

$$\le C_k \sum_{\ell=0}^k \binom{k}{\ell} \left(\sum_{\xi,\eta\in\mathbb{Z}^n} (1+|\eta|)^{2(k-\ell)+2\sigma}(1+|\xi-\eta|)^{|\sigma|-[|\sigma|]-1}|\widehat{u}(\eta)|^2 \right)^{1/2}$$

$$\le \tilde{C}_k \left(\|u\|_{k+\sigma} + \|u\|_\sigma \right),$$

where $C_k, \tilde{C}_k$ are suitable positive constants.

We explicitly point out that $C_k \le M|q|_{k+[|\sigma|]+n+1}^{(\sigma)}$, for a positive constant M. □

References

[1] P. Albano, A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture*, J. Funct. Anal. **274** (2018), no. 10, 2725–2753.

[2] M. S. Baouendi and Ch. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate op- erators*, Bull. A. M. S., **78** (1972), 483-486.

[3] A. Bove and M. Mughetti, *Analytic hypoellipticity for sums of squares in the presence of symplectic non Treves strata*, J. Inst. Math. Jussieu **19**(6) (2020), 1877–1888.

[4] A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture, II*, Analysis and PDE **10** (7) (2017), 1613–1635.

[5] A. Bove and M. Mughetti, *Gevrey Regularity for a Class of Sums of Squares of Monomial Vector Fields*, Advances in Math. **373**(2020) 107323, 35pp.

[6] A. Bove and M. Mughetti, *Analytic regularity for solutions to sums of squares: an assessment*, Complex Analysis and its synergies **6**(2020), no. 2. https://doi.org/10.1007/s40627-020-00055-8.

[7] A. Bove and M. Mughetti, *Optimal Gevrey Regularity for Certain Sums of Squares in Two Variables*, preprint, 2021.

[8] A. Bove and F. Treves, *On the Gevrey hypo-ellipticity of sums of squares of vector fields*, Ann. Inst. Fourier (Grenoble) **54**(2004), 1443-1475.

[9] H. Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bulletin de la Société Mathématique de France **85** (1957), 77–99.

[10] G. Chinni, *Gevrey regularity for a generalization of the Oleǐnik–Radkeviç operator*, J. Math. Anal. Appl. **415** (2014), 948–962.

[11] G. Chinni and P. D. Cordaro, *On global analytic and Gevrey hypoellipticity on the torus and the Métivier inequality*, Communications in P.D.E. **42**(1) (2017), 121–141.

[12] G. Chinni, *On the sharp Gevrey regularity for a generalization of the Métivier operator*, preprint, 2021.

[13] G. Chinni, *(Semi)-global analytic hypoellipticity for a class of sums of squares which fail to be locally analytic hypoelliptic*, Proc. A.M.S., to appear, DOI: https://doi.org/10.1090/proc/14464.

[14] D. Cordaro and A. A. Himonas, *Global analytic hypoellipticity for a class of degenerate elliptic operators on the torus*, Math. Res. Letters **1** (1994), 501–510.

[15] H. Grauert, *On Levi's Problem and the Imbedding of Real-Analytic Manifolds*, Annals of Mathematics **68** (1958), 460–472.

[16] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[17] L. Hörmander, *Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential Equations with Analytic Coefficients*, Communications Pure Appl. Math. **24** (1971), 671–704.

[18] L. Hörmander, *The Analysis of Partial Differential Operators, I*, Springer Verlag, 1985.

[19] L. Hörmander, *The Analysis of Partial Differential Operators, III*, Springer Verlag, 1985.

[20] S. G. Krantz, *Function theory of several complex variables*, 2nd ed., AMS Chelsea Publishing, 2001, Providence, R.I.

[21] G. Métivier, *Non-hypoellipticité Analytique pour $D_x^2 + (x^2 + y^2)D_y^2$*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 7, 401–404.

[22] O. A. Oleǐnik, *On the analyticity of solutions of partial differential equations and systems*, Colloque International CNRS sur les Équations aux Dérivées Partielles Linéaires (Univ. Paris- Sud, Orsay, 1972), 272–285. Astérisque, 2 et 3. Societé Mathématique de France, Paris, 1973.

[23] O. A. Oleĭnik and E. V. Radkevič, *The analyticity of the solutions of linear partial differential equations*, (Russian) Mat. Sb. (N.S.), **90(132)** (1973), 592–606.

[24] L. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (3 - 4) (1976), 247–320.

[25] J-P. Serre, *Applications de la théorie générale à divers problèmes globaux*, Séminaire Henri Cartan, 20, **4** (1951-1952).

[26] F. Treves, *Symplectic geometry and analytic hypo-ellipticity, in Differential equations*, La Pietra 1996 (Florence), Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999, 201-219.

[27] F. Treves, *On the analyticity of solutions of sums of squares of vector fields*, Phase space analysis of partial differential equations, Bove, Colombini, Del Santo ed.'s, 315-329, Progr. Nonlinear Differential Equations Appl., **69**, Birkhäuser Boston, Boston, MA, 2006.

[28] F. Treves, *Aspects of Analytic PDE*, book to appear.

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