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Chapter 1

Regularity results for nonlocal minimal surfaces

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Abstract

In this note, we present some recent results in the study of nonlocal minimal surfaces. The notion of nonlocal minimal surface was introduced by Caffarelli, Roquejoffre, and Savin, they are boundaries of sets which minimize the nonlocal (or fractional) perimeter. In the last years, much interest has been devoted to the study of their regularity properties. Similarly to the classical local setting, a crucial ingredient in the study of regularity, is the classification of minimal cones. In the nonlocal setting, only partial results are available, dealing mainly with the low-dimensional case. We describe the main achievements in the field, focusing in particular on the difference with respect to the classical theory and in the difficulties which arise due to the nonlocal character of the problem.

1.1 Introduction

We describe some recent results in the study of regularity properties of *nonlocal*, or *fractional, minimal surfaces*. These geometric objects were defined by Caffarelli, Roquejoffre, and Savin in [5], as the boundaries of sets whose characteristic functions minimize a fractional Sobolev norm.

More precisely, in [5] the following notion of *fractional perimeter* was introduced.

Let $s \in (0,1)$. Given E a bounded subset of $\mathbb{R}^n$, the fractional s-perimeter of E is given by

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$$\mathrm{Per}_s(E) = c_s \int_E \int_{\mathbb{R}^n \setminus E} \frac{1}{|x-y|^{n+s}}\, dx\, dy = c_s [\chi_E]_{W^{s,1}(\mathbb{R}^n)}, \tag{1.1}$$

where χ_E denotes the characteristic function of the set E, $[\,\cdot\,]_{W^{s,1}(\mathbb{R}^n)}$ denotes the seminorm in the fractional Sobolev space $W^{s,1}$, and c_s is a constant depending on s which behaves like $(1-s)$ as $s \uparrow 1$. To be more precise, in [5] the definition of $\mathrm{Per_s}$ was given in terms of the squared $W^{s/2,2}-$seminorm of χ_E, but it is easily seen that their definition coincide with the one given above.

Written as in (1.1), one can better appreciate the analogy with the notion of classical perimeter in the sense of De Giorgi, defined as

$$\mathrm{Per}(E) = [\,\chi_E\,]_{BV(\mathbb{R}^n)},$$

where $[\,\cdot\,]_{BV(\mathbb{R}^n)}$ denotes the seminorm in the space BV. In (1.1) we are considering a fractional order derivative of the characteristic function of a set and the two notions are consistent in the sense that $\mathrm{Per_s} \to \mathrm{Per}$ as $s \uparrow 1$ (see e.g. [1, 7, 11]).

Roughly speaking, the s-perimeter captures the interactions between a set E and its complement, these interactions take place in the whole $\mathbb{R}^n$ and are weighted by a kernel with polynomial decay. Due to its nonlocal character, the s-perimeter has several applications, for example in image reconstruction and nonlocal capillarity models, see e.g. [3, 14].

A set E which is a minimizer for the fractional perimeter is called a *fractional (or nonlocal) minimal set*, and its boundary is referred to as a nonlocal minimal surface.

As it happens for the classical notion of area-minimizing surfaces, if the set E is not bounded, in order to give the notion of minimizer for the perimeter functional, one needs to introduce a localized version of perimeter, since the perimeter of an unbounded set E in the whole $\mathbb{R}^n$ could be infinite.

The localized notion of s-perimeter is the following: let Ω be a bounded domain in $\mathbb{R}^n$, we define the fractional s-perimeter of a measurable set $E \subset \mathbb{R}^n$ relative to Ω as

$$\mathrm{Per}_s(E,\Omega) := \int_{E\cap\Omega} \int_{E^c} \frac{1}{|x-y|^{n+s}}\, dx\, dy + \int_{E\setminus\Omega} \int_{\Omega\setminus E} \frac{1}{|x-y|^{n+s}}\, dx\, dy, \tag{1.2}$$

where E^c denotes the complement of E in $\mathbb{R}^n$.

The choice of the set of integration in the definition of the fractional perimeter is the natural one which does not change the variational structure of the functional, once we have fixed the set E outside of Ω. We can now give the definition of minimizer for $\mathrm{Per_s}$ in Ω.

Definition 1. We say that a set E is a *minimizer* for the s-perimeter in Ω if

$$\mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega), \qquad \text{for all } F \text{ such that } E \setminus \Omega = F \setminus \Omega.$$

Moreover, we say that E is a minimizer for the s-perimeter in $\mathbb{R}^n$, if E is a minimizer in a ball B_R, for all radii $R > 0$.

Said in other words, a nonlocal minimal surface in Ω is the boundary of a set E, whose characteristic function minimize the $W^{s,1}$-seminorm, among all sets which coincide with E in the complement of Ω.

In [5] the Euler-Lagrange equation for this functional has been derived: similarly to the classical case, a nonlocal minimal set E must have vanishing *fractional mean curvature* H_s, where H_s is given by the following expression

$$H_s(x) = c_s \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y)}{|x-y|^{n+s}} dy. \tag{1.3}$$

Here c_s denotes again a constant depending on s which behaves like $(1-s)$ as $s \uparrow 1$.

The first example of a surface with zero nonlocal mean curvature is a half-space. Other examples of sets with vanishing nonlocal mean curvature have been studied in the recent contributions [8, 12]. In [12], the nonlocal analogue of catenoids are constructed, but they differ from the standard catenoids since they approach a singular cone at infinity instead of having a logarithmic growth. These surfaces are constructed using perturbative methods, by performing small perturbation along the normal vector to ∂E. Instead in [8] it is proven, just by an easy symmetry argument, that the standard helicoids are surfaces with zero nonlocal mean curvature.

In [5], the study of regularity of nonlocal minimal surfaces has been started. More precisely, Caffarelli, Roquejoffre, and Savin established density estimates, the improvement of flatness for minimizers, a monotonicity formula, a blow-up and a dimension reduction argument. Nevertheless, the regularity theory for minimizers of the fractional perimeter is still widely open. In the following sections we describe the main results and the main open questions in the field.

1.2 Classification of s-minimal cones in low dimensions

We start by recalling the following well-known results in the regularity theory for classical area-minimizing surfaces.

Every minimal cone in $\mathbb{R}^n$ is a hyperplane, whenever $n < 8$. The condition on the dimension is optimal, indeed in $\mathbb{R}^8$ the Simons cone defined as

$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$

is a minimizer for the perimeter functional.

The classification of minimal cones is one of the main ingredients in both the classification of entire minimal surfaces (that is surfaces that are minimizer of the perimeter functional in the whole $\mathbb{R}^n$) and in the study of regularity for minimizers of the perimeter in a bounded set Ω. Indeed, the classification of minimal cones leads, on the one hand, to the classification of *any* entire area minimizing surfaces via a blow-down argument. On the other hand the nonexistence of singular minimal cones in space dimension $n \leq 7$ implies, via blow-up and a dimension reduction argument, that any minimal surface is smooth outside a singular set of Hausdorff dimension $n-8$. Moreover, again the classification of minimal cones leads to the classification of entire minimal graphs (the so called Bernstein problem): If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a half-space, whenever $n < 9$. Note that the critical dimension for a graph to be flat is one more than the one for a general set. The main ingredients in the proof of these results are given by density estimates, perimeter estimates, improvement of flatness for minimizers and a monotonicity formula.

As already mentioned in the Introduction, many of these ingredients in the nonlocal setting were established in [5]. With these tools, Caffarelli, Roquejoffre and Savin could reduce the study of regularity for nonlocal minimal surfaces to the classification of nonlocal minimal cones. More precisely they proved that, if the blow-up, around the origin, of an s-minimal set E is flat, then ∂E is $C^{1,\alpha}$ in a neighborhood of the origin (see [5, Theorem 9.4]). As a consequence of a dimension reduction argument, they proved $C^{1,\alpha}$ regularity outside a singular set of Hausdorff dimension at most $n-2$ (see [5, Theorem 10.4]). The bound $n-2$ on the dimension of the singular set was not optimal due to the fact that in [5] the classification of nonlocal minimal cones was not known, not even in $\mathbb{R}^2$.

Later, in [16] Savin and Valdinoci proved that in $\mathbb{R}^2$ an s-minimal cone is necessarily a half-plane. As a consequence they could improve the bound on the Hausdorff dimension of the singular set from $n-2$ to $n-3$ and via a blow-down argument they obtained the classification of any s-minimal surface in $\mathbb{R}^2$.

Moreover, in [2] Barrios, Figalli, and Valdinoci showed that if E is an s-minimal set such that $\partial E \in C^{1,\alpha}$, then ∂E is in fact C^∞ (such a result holds in every dimension). This is a consequence of a more general regularity result for solutions to integro-differential equations via a bootstrap argument. In [13], Figalli and Valdinoci addressed the fractional version of the Bernstein problem and proved that, if there are not s-minimal singular cones in $\mathbb{R}^n$, then the only entire s-minimal graphs in $\mathbb{R}^{n+1}$ are the hyperplanes.

We summarize all these results in the following Theorem.

Theorem 1. *The following facts hold:*

1. *Every s-minimal cone in $\mathbb{R}^2$ is a hyperplane ([16]);*
2. *If E is a minimizer of the s-perimeter in the whole $\mathbb{R}^2$, then E is a half-plane ([16]);*
3. *If E is a minimizer of the s-perimeter in $\mathbb{R}^n$ and ∂E is a graph, then E is a half-space, whenever $n \leq 3$ ([13]);*

4. If E is a minimizer of the s-perimeter, then ∂E is C^∞ outside a singular set Σ of Hausdorff dimension $n-3$ ([2, 5, 16]).

In addition, when s is close to 1 Caffarelli and Valdinoci proved that all the regularity results that hold in the classical setting are inherited, by a compactness argument, by s-nonlocal minimal surfaces (see [6, 7]).

Theorem 2 (Theorem 5 in [7]). *There exists $\epsilon_0 \in (0,1)$ such that if $s \geq 1-\epsilon_0$, then any s-minimal surfaces is C^∞ outside a singular set Σ of Hausdorff dimension $n-8$.*

Finally, in the very recent contribution [4], Cabré, Serra and the author proved flatness for nonlocal s-minimal cones in $\mathbb{R}^3$ for s close to 1. We emphasize that in [4], differently from [7], the proof is not based on a compactness argument and it permits to quantify how much s must be close to 1. This last result holds not only for cones that are minimizers for the s-perimeter, but for the more general class of *stable* cones. Stability here has to be understood in the variational sense, i.e. it corresponds to the fact that the second variation of the s-perimeter is nonnegative (we will comment on the notion of stability in the next section). The following is the main result in [4].

Theorem 3 (Theorem 1.2 in [4]). *There exists $s_* \in (0,1)$ such that for every $s \in (s_*, 1)$ the following statement holds.*

Let $\Sigma \subset \mathbb{R}^3$ be a cone with nonempty boundary of class C^2 away from 0. Assume that Σ is a stable set for the s-perimeter. Then, Σ is a half-space.

The proof of this result uses two crucial ingredients: the fractional Hardy inequality in $\mathbb{R}^2$ (with the precise behavior of its sharp constant as $s \uparrow 1$) and the perimeter estimates for stable sets contained in [9] and that we describe in the next Section.

1.3 Quantitative flatness results and perimeter estimates for stable sets

We now focus on the two-dimensional result proven by Savin and Valdinoci in [16] (see Theorem 1, point *1.*). The proof of this result relies on the following idea: given a minimal cone E in the whole $\mathbb{R}^n$ (i.e. a cone which is a minimizer in B_R for any $R > 0$), one considers perturbations E_R^+ that are small translations, in some direction, of E inside the half ball $B_{R/2}$ (and which coincide with E outside of B_R). A computation shows that the difference between the s-perimeter of E_R^+ and the s-perimeter of E is controlled in the following way:

$$\mathrm{Per}_s(E_R^+, B_R) - \mathrm{Per}_s(E, B_R) \leq C R^{n-2-s}.$$

Hence, when $n = 2$, this difference can be made arbitrarily small as $R \to \infty$. On the other hand, if E was not a half-plane, it could be modified in such a way to decrease its s-perimeter by a small but fixed amount and this leads to a contradiction. It is clear that this argument works only in dimension $n = 2$ (we are using that R^{n-s-2} goes to 0 as $R \to \infty$). We emphasize that the factor R^{n-s} comes from an optimal bound for the s-perimeter of minimizers. Indeed, by a comparison argument one can show that if E is an s-minimal set in B_R, then

$$\mathrm{Per}_s(E, B_R) \leq CR^{n-s},$$

and this bound is optimal.

These ideas were recently used in [9] to prove a quantitative version of this 2-dimensional flatness result, where *quantitative* has to be understood in the following sense.

Suppose that E is a minimizer for Per_s in a ball B_R for *some* R large enough (and not for *all* R). Is it true that E is "close" to be a half-plane in B_1? Moreover, can we give an estimate on this closeness depending on R? The following result, contained in [9], gives an answer to these questions.

Theorem 4 (Theorem 1.3 in [9]). *Let $n = 2$. Let $R \geq 2$ and E be a minimizer for the s-perimeter in the ball $B_R \subset \mathbb{R}^2$.*

Then, there exists a half-plane $\mathfrak{h}$ such that

$$|(E \triangle \mathfrak{h}) \cap B_1| \leq CR^{-s/2}. \tag{1.4}$$

Moreover, after a rotation, we have that $E \cap B_1$ is the subgraph of a measurable function $g : (-1, 1) \to (-1, 1)$ with oscillation $\operatorname{osc} g \leq CR^{-s/2}$ outside a "bad" set $\mathcal{B} \subset (-1, 1)$ with measure $CR^{-s/2}$.

As mentioned above, the proof of this result is based on the technique developed in [16] which uses perturbations given by small translations of the minimizer E (inside the ball B_R) and introducing quantitative elements which allow to keep track of the dependence on the radius R.

The ideas developed in [9] to prove Theorem 4 above have also been used to prove an optimal estimate for the classical perimeter of an s-minimal set E. Of course, such an estimate cannot be deduced just by a comparison argument (indeed, it is a genuine regularity estimate which improves the order of differentiability of χ_E) and needs a more sophisticated argument. More interestingly, this estimate holds true in the more general class of *stable sets*. Here stability has to be understood in the variational sense, that is we require the set to be a minimizer among *small* perturbations, which corresponds, for smooth objects, to the fact that the second variation of the s-perimeter is nonnegative. For the precise notion of stability that we use, we refer to [9, Definition 1.6] and [4, Section 2]. Once one has an estimate for the classical perimeter of E, by a standard interpolation, one can deduce an estimate for its s-perimeter. As already explained, for minimizers the upper bound on the s-perimeter comes easily by comparison, but for stable sets is highly nontrivial.

In order to explain the interest in perimeter estimates for stable objects, we recall some known facts in the classical local setting.

Stable minimal cones (for the classical perimeter) are completely classified: they are hyperplanes in space dimensions $n \leq 7$. In $\mathbb{R}^8$, the Simons cone is an example of stable cone which is singular (i.e., the classification that we have presented in the previous section for classical minimal surfaces holds true for *stable cones*). Once one has a complete classification of stable cones, using a blow-down technique, one can obtain the classification of *any* stable surface in the whole $\mathbb{R}^n$. A crucial tool needed for this argument is an optimal estimate for the perimeter of stable sets. It is well known that any minimizer of the classical perimeter in a ball B_R satisfies the estimate

$$\mathrm{Per}(E, B_R) \leq C R^{n-1}. \tag{1.5}$$

Unfortunately, an estimate like (1.5) is not known to hold for stable sets, unless we are in dimension $n = 3$ and we require some topological assumptions on the set E (see [10, 15]). The difficulty in proving perimeter estimates for stable sets relies on the fact that, when using a comparison argument, we are allowed to consider only competitors which are *small perturbations* of the given set E.

In dimension $n > 3$ the search for a perimeter estimate for stable sets is still completely open. As explained above, having a universal bound for the classical perimeter of embedded minimal surfaces in every dimension $n > 3$ would be a decisive step towards proving the following well-known and long standing conjecture: *The only stable embedded minimal (hyper)surfaces in $\mathbb{R}^n$ are hyperplanes as long as the dimension of the ambient space is less than or equal to 7.*

Surprisingly, in the fractional setting, the nonlocal character of the perimeter functional gives somehow more rigidity and allows to obtain the following result (which holds in every dimension):

Theorem 5 (Theorem 1.1 in [9]). *Let $s \in (0,1)$, $R > 0$ and E be a stable set in the ball $B_{2R} \subset \mathbb{R}^n$ for the nonlocal s-perimeter functional. Then,*

$$\mathrm{Per}(E, B_R) \leq C\, R^{n-1},$$

and

$$\mathrm{Per}_s(E, B_R) \leq C R^{n-s}.$$

As a consequence of Theorem 5, in [9] the quantitative flatness result in $\mathbb{R}^2$ was proven to hold also for stable set (and not only for minimizers) .

In a similar way to what described for the classical case, once one has a complete classification for s-minimal stable cones, the s-perimeter estimates of Theorem 5 would allow to classify any stable s-minimal surface. In this respect, the difficulties in the nonlocal setting are, in some way, dual to the ones in the local setting: in the first case, we have perimeter estimates in any dimensions but only the classification of stable cones in low dimensions is known; in the second the situation is reversed, since stable cones are completely classified but perimeter estimates are still missing in dimension $n > 3$.

Having in mind this picture, an interesting motivation in the study of nonlocal minimal surfaces is whether nonlocal techniques and nonlocal results could lead to give an answer to some important open questions in the local setting, such as, for example, the complete classification of stable surfaces.

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