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Existence of Positive Eigenfunctions to an Anisotropic Elliptic Operator via Sub-Super Solutions Method

Simone Ciani, Giovany M. Figueiredo and Antonio Suárez

Abstract. Using the sub-supersolution method we study the existence of positive solutions for the anisotropic problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda u^{q-1} \tag{0.1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $q > 1$ and $\lambda > 0$.

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Keywords. Anisotropic p-Laplacian, Positive Solution, Sub-Supersolution, Eigenvalues.

1. Introduction

In this paper the main goal is to show the existence of positive solutions of the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $p_i > 1$, $i = 1, \ldots, N$, $q > 1$ and λ is a real parameter. We will assume without loss of generality that the p_i are ordered increasingly, that is, $p_1 < \ldots < p_N$.

There is a vast literature concerning to anisotropic elliptic problems. We mention here only those references most strongly related to (1.1). First, in [9] it was proved that for $q < p_N$ for any $\gamma > 0$ there exists $\lambda_\gamma > 0$ and u_γ with $\|u_\gamma\|_p = \gamma$ and u_γ solution of (1.1) with $\lambda = \lambda_\gamma$. As the authors themselves claim, from this result it can not be deduced the existence of solutions of (1.1) for a given λ. In [4], using mainly variational methods, it was proved that if $p_1 < q < p_N$ then there exist $0 < \lambda_* \leq \lambda^*$ such that:

- If $\lambda \leq \lambda_*$, (1.1) does not posses positive solution.
- If $\lambda > \lambda^*$, (1.1) possesses at least a positive solution.

Finally, for the general results of [14] (Corollary 1) we can deduce that for the case $1 < q < p_1$ there exist $0 < \lambda_* < \lambda_{**}$ such that (1.1) possesses at least a solution for $\lambda \in (0, \lambda_*) \cup (\lambda_{**}, \infty)$.
In this paper we complete and improve the above results. For that, we use the sub-supersolution method, see [1], [5] and [16], (see also [6], [7], [8] and references therein for the application of this method to problems with non-linear reaction function including singularities or critical exponent).
This method allows us not only to prove the existence of a solution, but also gives us lower and upper bounds of such solution. Specifically, our main result is the following.

Theorem 1.1.

1. *Assume that $1 < q < p_1$. There exists a positive solution of (1.1) if and only if $\lambda > 0$.*
2. *Assume that $p_1 \leq q < p_N$. There exists $\Lambda > 0$ such that (1.1) does not posses positive solutions for $\lambda < \Lambda$ and (1.1) possesses at least one positive solution for $\lambda > \Lambda$.*

An outline of the paper is as follows: in Section 2 we recall some definitions and some properties of the eigenvalues and eigenfunctions of the classical p-Laplacian. Next in Section 3 we enunciate the sub-supersolution method. Then in Section 4 we construct sub and super-solutions by multiplication of powers of p-Laplacian eigenfunctions to be applied in the existence theorem.

2. Preliminary Lemmas and Setting

Consider $h(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ a Caratheodory function, i.e. measurable in x and continuous in the second variable s. Consider the anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = h(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The natural framework to study (2.1) is the anisotropic Sobolev Space $W_0^{1,\mathbf{p}}(\Omega)$, that is, the closure of $C_0^\infty(\Omega)$ under the anisotropic norm

$$\|u\|_{W^{1,\mathbf{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$$

where $\frac{\partial u}{\partial x_i}$ denotes the $i-$th weak partial derivative of u.
Recall that if we denote

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1, \quad p_i > 1 \quad \forall i = 1, \dots, N, \quad p^* := \frac{N}{\sum \frac{1}{p_i} - 1}, \quad p_\infty := \max\{p^*, p_N\}, \tag{2.2}$$

then for every $r \in [1, p_\infty]$ the embedding

$$W_0^{1,\mathbf{p}}(\Omega) \subset L^r(\Omega)$$

is continuous, and it is compact if $r < p_\infty$. More precisely, it holds the following directional Poincaré-type inequality for any $u \in C_c^1(\Omega)$ (see for instance [9])

$$||u||_r \leq \frac{d^i\, r}{2}\left|\left|\frac{\partial u}{\partial x_i}\right|\right|_r, \quad \forall r \geq 1, \quad d^i = \sup_{x,y\in\Omega} \langle x-y, e_i\rangle, \tag{2.3}$$

denoting by $\{e_1, \ldots, e_N\}$ the canonical basis of $\mathbb{R}^N$.
The theory of embeddings of this kind of anisotropic Sobolev spaces is vast and we refer to [9] for directional Poincaré-type inequality and to [12] for Sobolev and Morrey's embeddings of the whole $W^{1,\mathbf{p}}(\Omega)$ space, obtained with an important geometric condition on the domain Ω, namely that it must be semi-rectangular. It is not a case that this semi-rectangular condition reflects in our construction of the solution: the existence of traces for this kind of functions is heavily depending on the geometry of the domain as shown in [12]. Regularity theory for orthotropic operators as the one defined by equation (1.1) is still a challenging open problem, see for example [3].
We also recall the following definition.

Definition 2.1. *A function $u \in W^{1,\mathbf{p}}(\Omega)$ is defined to be a a sub-(super-) solution to the problem (2.1) if $u \leq (\geq)\, 0$ in $\partial\Omega$ and $\forall\, 0 \leq \phi \in W_0^{1,\mathbf{p}}(\Omega)$ it satisfies*

$$\int_\Omega \left[\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} - h(x, u(x))\phi\right] dx \leq (\geq) 0. \tag{2.4}$$

Finally, a solution $u \in W_0^{1,\mathbf{p}}(\Omega)$ to (2.1) has to satisfy

$$\int_\Omega \left[\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} - h(x, u(x))\phi\right] dx = 0 \quad \forall \phi \in W_0^{1,\mathbf{p}}(\Omega).$$

Now, we recall some well-known results concerning the eigenvalue problem for the p-Laplacian. Specifically, the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right)$$

The following result is well-known:

Lemma 2.1. *The eigenvalue problem (2.5) has a unique eigenvalue $\lambda = \lambda_1$ with the property of having a positive associated eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$,* *called principal eigenfunction. Moreover, λ_1 is simple, isolated and is defined by*

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p dx = 1\right\}.$$

Furthermore, $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and $\partial\varphi_1/\partial n < 0$ on $\partial\Omega$, where n is the outward unit normal on $\partial\Omega$. Finally, for $N = 1$ we have that

$$|\nabla\varphi_1|^{p-2}\nabla\varphi_1 \in W^{1,2}(\Omega), \tag{2.6}$$

and in fact

$$-\Delta_p\varphi_1(x) = \lambda_1\varphi_1(x) \quad a.e.\ x \in \Omega.$$

Remark 2.1. *The existence of λ_1 and main properties of φ_1 are well-known, see* [10], [13], [15]*. Property (2.6) holds in $N = 1$, see for instance* [11], *and for $N \geq 2$ is some specific domains, for example for Ω convex, see* [2].

3. An existence sub-supersolution theorem

We start by stating an important theorem, see [1] and [5], that assures the existence of a solution between a sub and a supersolution.

Theorem 3.1. *Suppose that $h : \mathbb{R} \mapsto \mathbb{R}$ a continuous function and that there exist $\underline{u}, \overline{u} \in W^{1,\mathbf{p}}(\Omega) \cap L^\infty(\Omega)$ subsolution and supersolution of (2.1) such that $\underline{u} \leq \overline{u}$. Then there exists $u \in W_0^{1,\mathbf{p}}(\Omega)$ solution to (2.1) such that*

$$\underline{u} \leq u \leq \overline{u}.$$

Proof. Since $\underline{u}$ and $\overline{u}$ belong to $L^\infty(\Omega)$, then h verifies condition (h_2) of [1]. This concludes the proof. ☐

4. Construction of sub and super-solutions: proof of the main result

In this section we prove Theorem 1.1. For that, we apply Theorem 3.1 to (1.1). Mainly, we construct the sub and the supersolution.

4.1. Sub-solutions

Let us consider a rectangular bounded domain $U \subseteq \Omega$ i.e.

$$U := \prod_{i=1}^N U_i, \quad \text{where } U_i = (a_i, b_i), \quad a_i, b_i \in \mathbb{R} \quad \forall\, i = 1, .., N.$$

Denote by $v_i = v_i(x_i)$ a positive principal eigenfunction of $-\Delta_{p_i}$ in U_i, that is,

$$\begin{cases} -\Delta_{p_i} v_i = \eta_i |v_i|^{p_i-2} v_i & \text{in } U_i, \\ v_i = 0 & \text{on } \partial U_i. \end{cases} \tag{4.1}$$

From Lemma 2.1, recall that, if n_i is the outward normal derivative to ∂U_i, we have

$$\frac{\partial v_i}{\partial n_i} < 0 \quad \text{on } \partial U_i. \tag{4.2}$$

Let us consider the function

$$\underline{u}(x) = \begin{cases} \epsilon \prod_{i=1}^{N} v_i^{\alpha_i}(x_i) & x \in U, \\ 0 & x \in \Omega \setminus \overline{U}, \end{cases} \tag{4.3}$$

where $\alpha_i > 0$, $i = 1, \dots, N$, and $\epsilon > 0$ will be chosen later.

Remark 4.1. *We note that $\underline{u}(x) > 0$ in $\emptyset \neq U \subset \Omega$.*

As v_i are bounded, it is clear that $\underline{u} \in W^{1,\mathbf{p}}(\Omega)$ and that $\underline{u}_{|\partial\Omega} = 0$. Hence, $\underline{u}$ is subsolution of (1.1) provided that

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i}\, dx \leq \lambda \int_\Omega \underline{u}^{q-1} \phi\, dx \quad \forall\, \phi \in W_0^{1,\mathbf{p}}(\Omega), \quad \phi \geq 0.$$

Observe that

$$\lambda \int_\Omega \underline{u}^{q-1} \phi\, dx = \lambda \epsilon^{q-1} \int_U \prod_{i=1}^{N} v_i^{\alpha_i(q-1)} \phi\, dx. \tag{4.4}$$

On the other hand, observe that

$$\frac{\partial \underline{u}}{\partial x_i} = \epsilon\, \alpha_i \left(\prod_{j\neq i} v_j^{\alpha_j}\right) v_i^{\alpha-1} \frac{\partial v_i}{\partial x_i} \quad \text{in } U_i.$$

Then, taking into account the positivity of v_i, $\forall\, i = 1,..,N$,

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i}\, dx =$$

$$\sum_{i=1}^{N} \int_{\prod_{j\neq i} U_j} \left\{ \int_{U_i} \left[\epsilon \alpha_i \left(\prod_{j\neq i} v_j^{\alpha_j}\right) v_i^{\alpha_i - 1}\right]^{p_i-1} \left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \frac{\partial \phi}{\partial x_i}\, dx_i \right\} d\hat{x}^i$$

with the obvious notation for $d\hat{x}^i$. Next, by using an integration by parts argument and the Fubini-Tonelli theorem, we get

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i}\, dx =$$

$$-\sum_{i=1}^{N} \int_{\prod_{j\neq i} U_j} \left(\epsilon \alpha_i \prod_{j\neq i} v_j^{\alpha_j}\right)^{p_i-1} \int_{U_i} \frac{\partial}{\partial x_i}\left(v_i^{(\alpha_i-1)(p_i-1)} \left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2} \frac{\partial v_i}{\partial x_i}\right) \phi\, dx_i d\hat{x}^i$$

$$+\sum_{i=1}^{N} \int_{\prod_{j\neq i} U_j} \left(\epsilon \alpha_i \prod_{j\neq i} v_j^{\alpha_j}\right)^{p_i-1} \left\{ \int_{\partial U_i} v_i^{(\alpha_i-1)(p_i-1)} \left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2} \frac{\partial v_i}{\partial n_i} \phi\, dx_i \right\} d\hat{x}^i.$$

The second term on the right can be discarded as $\frac{\partial v_i}{\partial n_i} < 0$ in ∂U_i, see (4.2). Considering that

$$\frac{\partial}{\partial x_i}\left(v_i^{(\alpha_i-1)(p_i-1)}\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2}\frac{\partial v_i}{\partial x_i}\right) =$$

$$(\alpha_i-1)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i} + v_i^{(\alpha_i-1)(p_i-1)}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2}\frac{\partial v_i}{\partial x_i}\right)$$

and, from Lemma 2.1, that

$$\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2}\frac{\partial v_i}{\partial x_i}\right) = -\eta_i v_i^{p_i-1} \quad \text{in } U_i,$$

our sub-solution condition becomes

$$\sum_{i=1}^{N}\int_{\prod_{j\neq i}U_j}\left(\epsilon\alpha_i\prod_{j\neq i}v_j^{\alpha_j}\right)^{p_i-1}\int_{U_i}\left\{\left[(1-\alpha_i)(p_i-1)v_i^{(\alpha_i-1)(p_i-1)-1}\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i}+\right.\right.$$
$$\left.\left.+\,v_i^{(\alpha_i-1)(p_i-1)}\eta_i v_i^{p_i-1}\right] - \lambda\epsilon^{q-1}\left(\prod_{k=1}^{N}v_k^{\alpha_k(q-1)}\right)\right\}\phi\, dx_i d\hat{x}^i \leq 0. \tag{4.5}$$

Let us require a condition on the pointwise integrand

$$\lambda \geq$$

$$\sum_{i=1}^{N}\left(\epsilon\alpha_i\prod_{j\neq i}v_j^{\alpha_j}\right)^{p_i-q} v_i^{(\alpha_i-1)(p_i-1)-1-\alpha_i(q-1)}\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i}+\eta_i v_i^{p_i}\right]$$
$$= \sum_{i=1}^{N}\left(\epsilon\alpha_i\prod_{j\neq i}v_j^{\alpha_j}\right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i}\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i}+\eta_i v_i^{p_i}\right].$$

Now we consider various cases.

- If $1 < q < p_1$, then by choosing $\alpha_i > \frac{p_i}{(p_i-q)} > 1$, letting $\epsilon \to 0^+$ we obtain that $\underline{u}$ is a subsolution provided $\lambda > 0$.
- Assume that some $i_0 \in \{1, .., N\}$ we have $p_{i_0+1} > q \geq p_{i_0}$ taking $p_{N+1} = \infty$. Then $\underline{u}$ is a subsolution if

$$\lambda \geq \lambda_* := \max_U \mathcal{S}$$

where

$$\mathcal{S} = \sum_{i=1}^{N}\left(\epsilon\alpha_i\prod_{j\neq i}v_j^{\alpha_j}\right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i}\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i}+\eta_i v_i^{p_i}\right].$$

We show that λ_* is finite. Observe that $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$ where

$$\mathcal{S}_0 = \sum_{i=1}^{i_0}\left(\epsilon\alpha_i\prod_{j\neq i}v_j^{\alpha_j}\right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i}\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i}+\eta_i v_i^{p_i}\right].$$

and

$$\mathcal{S}_1 = \sum_{i=i_0+1}^{N} \left(\epsilon\alpha_i \prod_{j\neq i} v_j^{\alpha_j}\right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i}\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i} + \eta_i v_i^{p_i}\right].$$

It is clear that $\mathcal{S}_1$ is finite, taking $\alpha_i > p_i/(p_i-q)$.

On the other hand, observe that the behaviour next to ∂U is controlled: when $v_i \to 0^+$ then as $\frac{\partial}{\partial n_i} v_i < 0$ on ∂U_i we have that there exists $\delta > 0$ small enough such that the quantity

$$\left[(1-\alpha_i)(p_i-1)\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i} + \eta_i v_i^{p_i}\right] < 0 \quad \text{in } U_i^\delta, \text{ for all } i = 1, \dots, p_{i_0},$$

where

$$U_i^\delta := \{x_i \in U_i : \operatorname{dist}(x_i\,, \partial U_i) \geq \delta\}.$$

Moreover, $\mathcal{S}_0$ is bounded in $U \cap U_i^\delta$. Then, $\mathcal{S}_0$ is bounded in U and we can conclude that λ_* is finite.

4.2. Supersolutions

Since Ω is bounded, we can choose a domain U such that

$$\Omega \subset U = \prod_{i=1}^{N} U^i, \qquad U^i = (a_i, b_i), \quad a_i, b_i \quad \text{in} \quad \mathbb{R}.$$

Now for $M > 0$ we consider the function

$$\overline{u}(x) := M \prod_{i=1}^{N} v_i(x_i), \qquad x \in \Omega,$$

where v_i are the first eigenfunctions to the p_i-Laplacian in U^i, whose first eigenvalue we denote by η^i. Observe that

$$\overline{u}_{\partial\Omega} > 0.$$

Then, $\overline{u}$ is a supersolution to (1.1) for all $0 \leq \phi \in W_0^{1,\mathbf{p}}(\Omega)$ holds

$$\lambda \int_\Omega M^{q-1} \prod_{i=1}^{N} v_i^{q-1} \phi\, dx \leq \int_\Omega \sum_{i=1}^{N} \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\,\frac{\partial \phi}{\partial x_i}\, dx.$$

It is clear that

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\,\frac{\partial \phi}{\partial x_i}\, dx =$$

$$\int_\Omega \sum_{i=1}^{N} \left[M\left(\prod_{j\neq i} v_j\right)\right]^{p_i-1} \left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2} \frac{\partial v_i}{\partial x_i}\,\frac{\partial \phi}{\partial x_i}\, dx =$$

$$-\sum_{i=1}^{N} \int_\Omega \left(M\prod_{j\neq i} v_j\right)^{p_i-1} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v_i}{\partial x_i}\right|^{p_i-2} \frac{\partial v_i}{\partial x_i}\right)\phi\, dx =$$

$$\sum_{i=1}^{N} \eta^i \int_\Omega \left(M\prod_{j=1}^{N} v_j\right)^{p_i-1} \phi\, dx$$

Thus we may ask for the strong condition

$$\lambda^* := \sum_{i=1}^{N} \eta^i \left(M \prod_{j=1}^{N} v_j \right)^{p_i - q} \geq \lambda. \tag{4.6}$$

Hence, if $1 < q < p_N$ by letting $M \to \infty$ we have that $\overline{u}$ is a super solution $\forall \lambda > 0$.

Proof of Theorem 1.1.

1. Assume $1 < q < p_1$. Fix $\lambda > 0$. Then, we can choose $\epsilon > 0$ small and M large enough such that $\underline{u}$, $\overline{u}$ are sub-supersolution of (1.1) and $\underline{u} \leq \overline{u}$ in Ω. Theorem 3.1 assures the existence of a solution u of (1.1) such that $\underline{u} \leq u \leq \overline{u}$. This completes this case.
2. Assume $p_1 \leq q < p_N$. In this case, taking for example $\epsilon = 1$, we have that $\underline{u}$ is subsolution provided that $\lambda \geq \lambda_*$ for some λ_*. On the other hand, we can take M large such that $\overline{u}$ is supersolution and $\underline{u} \leq \overline{u}$. Thus, there exists a positive solution for $\lambda \geq \lambda_*$.
   Now, we define

   $$\Lambda := \inf\{\lambda : (1.1) \text{ possesses at least a positive solution}\}.$$

   We have proved that $\Lambda < \infty$. For $p_1 < q < p_N$, in [4] it was proved that $0 < \Lambda$. We show now that this is also true for $q = p_1$. Indeed, let now consider $q = p_1$ and let us multiply the equation (1.1) by u and integrate it on Ω to obtain

   $$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^{N} \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_{p_i}^{p_i} = \lambda ||u||_{p_1}^{p_1} = \lambda \int_\Omega |u|^{p_1} dx$$

   Now we use the embedding (2.3) on $r = p_1$ to get

   $$\left(\frac{d^1 p_1}{2} \right)^{-p_1} ||u||_{p_1}^{p_1} \leq \left|\left| \frac{\partial u}{\partial x_1} \right|\right|_{p_1}^{p_1} + \sum_{i=2}^{N} \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_{p_i}^{p_i} = \lambda ||u||_{p_1}^{p_1}$$

   and thus

   $$||u||_{p_1}^{p_1} \left[\lambda - \left(\frac{2}{d^1 p_1} \right)^{p_1} \right] \geq 0.$$

   But if $\lambda < \left(\frac{2}{d^1 p_1} \right)^{p_1}$ this quantity is negative and we arrive to the absurd of declaring $||u||_{p_1} \neq 0$.
   We prove now that for all $\lambda > \Lambda$ we have the existence of positive solution. Indeed, fix $\lambda_0 > \Lambda$. Then, by definition of Λ, there exists $\mu \in (\Lambda, \lambda_0)$ and a positive solution, denoted by u_μ, of (1.1) for $\lambda = \mu$. Since $\mu < \lambda_0$, it is clear that u_μ is subsolution of (1.1) for $\lambda = \lambda_0$. On the other hand, for M large, there exists $\overline{u}$ supersolution of (1.1) for $\lambda = \lambda_0$. Finally, thanks to regularity results, see for instance Proposition 4.1 in [1] or Lemma 2.4 in [5], we have that $u_\mu \in L^\infty(\Omega)$. Hence, for M large $u_\mu \leq \overline{u}$, and we can conclude the existence of positive solution for $\lambda = \lambda_0$. This completes the proof.

□

Remark 4.2. *Since our subsolution $\underline{u}$ is strictly positive in U, we have by Theorem 3.1 that $u \geq \underline{u} > 0$ in a non empty open set contained in Ω. In the case $p_1 \geq 2$ by the result of Corollary 4.4* [4] *we have $u > 0$ in Ω.*

Remark 4.3. *We comment a possible further generalization.*
Let $x = (x_1, ..., x_N)$, where $x_i \in \Omega_i \subset \mathbb{R}^{N_i}$, being Ω_i an open, bounded and convex domain. Denote with ∇_{x_i} the gradient along the vector x_i and div_{x_i} its divergence, and let

$$\Delta_{p_i} u = div_{x_i}(|\nabla_{x_i} u|^{p_i-2}\nabla_{x_i} u) = \sum_{j=1}^{N_i} \frac{\partial}{\partial x_{ij}} \left(\left| \frac{\partial u}{\partial x_{ij}} \right|^{p_i-2} \frac{\partial u}{\partial x_{ij}} \right)$$

be the p_i-Laplacian acting on the x_i vector. Problems of the kind of

$$\begin{cases} -\sum_{i=1}^{N} \Delta_{p_i} u = \lambda u^{q-1} & in\ \Omega = \prod \Omega_i, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4.7}$$

can be faced with the same technique, using properties of the p_i-Laplacian principal eigenfuctions, and owing integrability condition as (2.6) *to recent regularity results obtained in* [2].

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References

[1] C. O. Alves, A. El Hamidi, *Existence of solution for a anisotropic equation with critical exponent*, Differential Integral Equations, 21 (1-2), (2008), 25-40.

[2] A. Cianchi, V. Mazýa, *Second-Order Two-Sided Estimates in Nonlinear Elliptic Problems*, Arch. Rational Mech. Anal., 229, (2018), 569-599.

[3] E. DiBenedetto, U. Gianazza, V. Vespri, *Remarks on Local Boundedness and Local Holder Continuity of Local Weak Solutions to Anisotropic p-Laplacian Type Equations*, J. Elliptic Parabol. Equ., 2, (2016), 157–169.

[4] A. Di Castro, E. Montefusco, *Nonlinear eigenvalues for anisotropic quasilinear degenerate elliptic equations*, Nonlinear Anal., 70(11), (2009), 4093-4105.

[5] G. C. G. Dos Santos, G. M. Figueiredo, J. R. Silva, *Multiplicity of positive solutions for a anisotropic problem via Sub-supersolution method and Mountain Pass Theorem*, preprint 2020.

[6] G. M. Figueiredo, G. C. G. Santos, L. Tavares, *Existence results for some anisotropic singular problems via sub-supersolutions*, Milan Journal of Mathematics, 87, (2019), 249-272.

[7] G. M. Figueiredo, J. R. Silva, *A critical anisotropic problem with discontinuous nonlinearities*, Nonlinear Analysis, Real World Applications, 47, (2019), 364-372.

[8] G. M. Figueiredo, J. R. Silva, *Solutions to an anisotropic system via sub-supersolution method and Mountain Pass Theorem*, Electronic Journal Quality Theory in Differential Equations, 46, (2019), 1-13.

[9] I. Fragalà, F. Gazzola, B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), 715-734.

[10] E. Giusti, *Direct Methods in the Calcuus of Variations*, World Scientific Publishing, 2003.

[11] M. Guedda and L. Veron, *Bifurcation phenomena associated to the p-Laplace operator*, Trans. Amer. Math. Soc., 310 (1), (1988), 419-431.

[12] J. Haskovec, C. Schmiser, *A note on the anisotropic generalizations of the Sobolev and Morrey embedding theorems*, Monatshefte für Mathematik, 158 (1), (2009), 71–79.

[13] P. Lindqvist, *Notes on the Stationary p-Laplace Equation*, Springer Briefs in Mathematics, IX, 68, 2016.

[14] M. Mihailescu, P. Pucci, V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. 340 (2008) 687-698.

[15] I. Peral, *Multiplicity of Solutions for p-Laplacian*, International Center for Theorethical Physics Trieste, School of Nonlinear Functional Analysis and Applications to Differential Equations, 1997.

[16] M. Struwe, *Variational Methods*, Vol. 34. Springer.

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