# Revisiting induced gravity in scalar-tensor thermodynamics

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Induced gravity, defined as a globally scale-invariant "first-generation" scalar-tensor theory, is investigated within the framework of the thermodynamics of modified gravity theories. The "temperature of gravity" and its evolution equation are derived for this model, and the resulting expressions are used to analyze general relativity equilibrium states and to investigate the possible existence of an attractor mechanism toward Einstein's theory with a cosmological constant.

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## I. INTRODUCTION

The generation of Einstein's gravity through processes of spontaneous symmetry breaking has been a topic of general interest since the early 1980s [1–5]. A very well-known approach along this line relies on the spontaneous breaking of scale invariance within the framework of scalar-tensor gravity [2–5]. Concretely, the most general "first-generation" globally scale-invariant scalar-tensor theory, originally proposed by Cooper and Venturi in Ref. [3] and commonly known as *induced gravity*, reads [6]

$$\mathcal{S}[g,\phi] = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\xi}{2}\phi^2 R - \frac{1}{2}\nabla_a\phi\nabla^a\phi - \frac{\lambda}{4}\phi^4\right] + \mathcal{S}^{(\mathrm{m})}, \tag{1}$$

where $\lambda$, $\xi > 0$ are dimensionless constants, $\phi$ is a scalar field, $g$ is the determinant of the spacetime metric $g_{ab}$ with Ricci scalar $R$, $\nabla$ is the Levi-Civita connection, and $\mathcal{S}^{(\mathrm{m})}$ denotes the action for matter fields (assumed to be independent of $\phi$).

The field equations for the action in Eq. (1) read

$$\Box\phi + \xi R\phi - \lambda\phi^3 = 0, \tag{2}$$

$$\xi\phi^2 G_{ab} = T^{(\mathrm{m})}_{ab} + \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi - \frac{\lambda}{4}\phi^4 g_{ab} + \xi\left(\nabla_a\nabla_b\phi^2 - g_{ab}\Box\phi^2\right), \tag{3}$$

with $\Box \coloneqq g^{ab}\nabla_a\nabla_b$ being the Laplace-Beltrami operator, $G_{ab} \coloneqq R_{ab} - (R/2)g_{ab}$ the Einstein tensor, $R_{ab}$ the Ricci tensor associated with $g_{ab}$, and $T^{(\mathrm{m})}_{ab}$ denoting the matter stress-energy tensor.

It is worth pointing out (see also Ref. [8]) that this model can be mapped onto the following realization of Brans-Dicke (BD) theory:

$$\mathcal{S}[g,\psi] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\psi R - \frac{1}{4\xi}\frac{\nabla_a\psi\nabla^a\psi}{\psi} - \frac{\lambda}{2\xi^2}\psi^2\right] + \mathcal{S}^{(\mathrm{m})}, \tag{4}$$

via the field redefinition

$$\psi = \xi\phi^2, \tag{5}$$

thus resulting in a BD model with constant coupling $\omega = 1/(4\xi)$ and potential $V(\psi) = \lambda\psi^2/(2\xi^2)$. Notably, the condition $\xi > 0$ allows the model to avoid pathological behaviors such as phantom phenomenology $(2\omega + 3 < 0)$ and a nondynamical BD scalar $(2\omega + 3 = 0)$.

Taking advantage of this mapping onto BD theory, we can briefly revisit the discussion in Ref. [3] concerning the spontaneous breaking of scale invariance in a cosmological setting.

Consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, with line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2),$$

where $t$ is cosmic time, $a(t)$ is the scale factor, $r$ denotes the comoving radial coordinate, and $\mathrm{d}\Omega^2$ is the line element on the unit 2-sphere. Restricting our discussion to the *vacuum* case, we find that the dynamics of the scale factor and of the scalar field $\psi = \psi(t)$ are determined by the scalar field equation and the Friedmann equation, (see Ref. [9]), i.e.

$$\ddot{\psi} + 3H\dot{\psi} = 0, \tag{6}$$

$$H^2 = \frac{\omega}{6}\frac{\dot{\psi}^2}{\psi^2} - H\frac{\dot{\psi}}{\psi} + \frac{V_0\psi}{6}, \tag{7}$$

with $\omega = 1/(4\xi)$, $V_0 = \lambda/(2\xi^2)$, $H \coloneqq \dot{a}/a$, and the dot denoting a derivative with respect to $t$.

It is possible to show (see Refs. [3,9]) that this system has no power-law solutions, except for the trivial one with

$$\psi = \psi_0 \quad \text{and} \quad H^2 = \frac{\lambda\psi_0}{12\xi^2} \tag{8}$$

for any $\psi_0 > 0$ constant. This solution of *vacuum* BD gravity leads to the well-known *de Sitter solution* discussed in Refs. [3,5], i.e.

$$\phi = \phi_0 \quad \text{and} \quad H = H_0 = \sqrt{\frac{\lambda\phi_0^2}{12\xi}} \tag{9}$$

in the induced gravity theory, by taking advantage of the field redefinition in Eq. (5). Notably, although the field equations of the theory are scale invariant, these *vacuum* solutions are not. Indeed, by selecting a value $\phi_0 \neq 0$, the scalar field acquires a nonvanishing *vacuum expectation value* $\langle\phi\rangle = \phi_0 \neq 0$, and hence the *vacuum* is not scale invariant. In other words, a spontaneous breaking of scale invariance occurs, leading to the generation of the Planck scale, $M_{\mathrm{P}}^2 = \xi\phi_0^2$.

Induced gravity coupled with dust and radiation has been shown to have general relativity (GR) plus a cosmological constant as a stable attractor [8] for a range of initial conditions. This scenario reminds us of the general attractor-to-GR mechanism in scalar-tensor gravity proposed by Damour and Nordvedt (see Refs. [10,11]).

In this work, we revisit the induced gravity paradigm by taking advantage of the recently developed framework of the thermodynamics of scalar-tensor gravity (see Refs. [12–16]).

## II. THERMODYNAMICS OF SCALAR-TENSOR GRAVITY: AN OVERVIEW

In this section, we recall some essential notions of the so-called thermodynamics of scalar-tensor theories. This discussion will be based on Refs. [12–16] (specifically, [16] offers an introductory review of the general approach), to which we refer the interested reader for further details on the formalism. We will also further specialize the discussion to cosmological spacetimes, thus recalling some ideas from Refs. [17,18] and from [19], the latter concerning the attractor-to-GR mechanism in scalar-tensor gravity.

Consider the action of *viable Horndeski gravity* coupled to matter:

$$\mathcal{S}[g_{ab}, \Phi, \zeta] = \mathcal{S}^{(\mathrm{g})}[g_{ab}, \Phi] + \mathcal{S}^{(\mathrm{m})}[g_{ab}, \zeta], \tag{10}$$

with

$$\mathcal{S}^{(\mathrm{g})} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}[G_4 R + G_2 - G_3\Box\Phi], \tag{11}$$

where $G_i$ are arbitrary functions of $\Phi$ and/or $X \coloneqq -(1/2)\nabla^a\Phi\nabla_a\Phi$ and with the requirement that $G_{4X} = 0$, adopting the convention $f_\Phi \equiv \partial f/\partial\Phi$ and $f_X \equiv \partial f/\partial X$.

The field equations for viable Horndeski gravity can be rewritten as (see Refs. [15,18,20])

$$G_{ab} = G_{\mathrm{eff}}(\Phi)T_{ab}^{(\mathrm{m})} + T_{ab}^{(\mathrm{eff})}, \tag{12}$$

$$\frac{\delta S^{(\mathrm{g})}}{\delta\Phi} = 0, \tag{13}$$

$$\frac{\delta S^{(\mathrm{m})}}{\delta\zeta} = 0; \tag{14}$$

i.e., they split into an effective Einstein equation, an equation for the scalar field $\Phi$, and the field equations for the matter fields $\zeta$. Note that the effective Einstein equation has two contributions on the right-hand side. The first contribution is due to the stress-energy tensor of matter which couples to (Einstein) gravity via an effective gravitational coupling, that for the case of viable Horndeski gravity is simply $G_{\mathrm{eff}}(\Phi) = 1/G_4(\Phi)$. The second term $T_{ab}^{(\mathrm{eff})}$ denotes an effective stress-energy tensor accounting for the rest of the contributions of $\Phi$ to the effective Einstein equation.

If the 4-gradient of $\Phi$ is timelike, we can construct a 4-velocity field

$$u^a \coloneqq \epsilon\frac{\nabla^c\Phi}{\sqrt{2X}}, \qquad 2X = -\nabla^e\Phi\nabla_e\Phi,$$

where $\epsilon = \pm 1$ has to be chosen so that $u^a$ is future-oriented by definition. The stress-energy tensor $T_{ab}^{(\mathrm{eff})}$ in Eq. (12) then describes an effective dissipative fluid with 4-velocity $u^a$. This "fluid" always admits (see Ref. [21]) an imperfect fluid decomposition [12,22]

$$T_{ab}^{(\mathrm{eff})} = \rho u_a u_b + P h_{ab} + \pi_{ab} + 2q_{(a}u_{b)}, \tag{15}$$

where $\rho$, $P$, $\pi_{ab}$, and $q_a$ are, respectively, the effective energy density, isotropic pressure, anisotropic stress tensor, and heat flux density. $h_{ab} \coloneqq g_{ab} + u_a u_b$ is the induced metric onto the 3-space orthogonal to $u^a$, and indices enclosed in parentheses are understood to be symmetrized. Explicit expressions for the fluid quantities can be obtained from the expression for $T_{ab}^{(\mathrm{eff})}$ in Eq. (12) via orthogonal projections (for details, see, e.g., Refs. [15,18]).

Taking advantage of this representation for $T_{ab}^{(\mathrm{eff})}$ and comparing the expressions for the fluid quantities with kinematic quantities associated with the fluid lines, one can infer some constitutive laws for the effective fluid (see Refs. [15,18]). In particular, these relations allow for an analogy with Eckart's first-order thermodynamics [23], which yields a notion of temperature (multiplied by the thermal conductivity of the fluid) of modified gravity that reads

$$\mathcal{K}\mathcal{T} = \epsilon \frac{\sqrt{2X}(G_{4\Phi} - XG_{3X})}{G_4}. \tag{16}$$

This quantity measures the departure of Hondeski gravity from GR (see Refs. [13–16], and Refs. [15,18] for the specific case of Hondeski).

Hondeski gravity reduces to GR for $\Phi =$ constant, which implies $\mathcal{K}\mathcal{T} = 0$. However, note that for GR, the imperfect fluid representation of the effective fluid is not defined, hence this should be understood as a definition of the temperature of GR within the formalism. A remark at the end of this section will make this point more precise, showing the self-consistency of the approach.

It is worth noting that there exist some alternative, yet physically equivalent, definitions of such a temperature of gravity [20], but they will not be discussed further in this work.

By combining Eq. (16) with the field equations of viable Horndeski gravity, it is also possible to derive an evolution equation for the temperature of gravity. This equation reads [18,20]

$$\frac{\mathrm{d}(\mathcal{K}\mathcal{T})}{\mathrm{d}\tau} = \left(\epsilon \frac{\Box\Phi}{\sqrt{2X}} - \Theta\right)\mathcal{K}\mathcal{T} + \nabla^c\Phi\nabla_c\left(\frac{G_{4\Phi} - XG_{3X}}{G_4}\right), \tag{17}$$

where $\tau$ stands for the proper time along the fluid lines (i.e., $\mathrm{d}/\mathrm{d}\tau \coloneqq u^c\nabla_c$) and $\Theta \coloneqq \nabla_c u^c$ denotes the expansion scalar. Concretely, Eq. (17) determines whether an exact solution of a specified (viable) Horndeski model approaches GR over time.

The "generalized heat equation" (17) is the basis for the thermodynamic interpretation of the attractor-to-GR mechanism for "first-generation" scalar-tensor theories (see Ref. [19]). Indeed, if we consider the simplified scenario of BD gravity with conformal matter [$T^{(\mathrm{m})} = 0$], constant BD coupling $\omega$, and a quadratic potential, then Eq. (17) reduces to

$$\frac{\mathrm{d}(\mathcal{K}\mathcal{T})}{\mathrm{d}\tau} = \mathcal{K}\mathcal{T}(\mathcal{K}\mathcal{T} - \Theta), \tag{18}$$

which implies that, if $\mathcal{K}\mathcal{T} > 0$ and $\Theta > 0$, then

(1) If a solution of the BD field equations begins above the line $\mathcal{K}\mathcal{T} = \Theta$, then $\mathcal{K}\mathcal{T} \to +\infty$; this means that the model diverges away from GR.
(2) If a solution of the BD field equations begins below the line $\mathcal{K}\mathcal{T} = \Theta$, then $\mathcal{K}\mathcal{T} \to 0$; thus, the model converges toward the GR equilibrium state.

This discussion has been made explicit for the "first quadrant" of the $(\Theta, \mathcal{K}\mathcal{T})$ plane, but it can easily be extended to the other quadrants. For further details, we refer the reader to Ref. [19]. Note that, in contrast to earlier interpretations [19], the line $\mathcal{K}\mathcal{T} = \Theta$ can be dynamically crossed, provided that the condition $\mathrm{d}(\mathcal{K}\mathcal{T})/\mathrm{d}\Theta = 0$ holds at the crossing point [24].

*Remark*. Note that the condition $\Phi =$ constant seems to be inconsistent with the proposed formalism, because in said configuration, it is not possible to define a comoving fluid velocity $u^a$, since $X \equiv 0$. However, it is easy to observe that for $\Phi =$ constant, the effective stress-energy tensor becomes that of a perfect fluid—specifically, of the cosmological-constant type—i.e., $T_{ab}^{(\mathrm{eff})} = g_{ab}[G_2/(2G_4)]$; see Ref. [15]. Thus, any normalized timelike vector field is an eigenvector of $T^{(\mathrm{eff})a}{}_b$, corresponding to the same eigenvalue, and there is no preferred notion of velocity of the fluid. Notably, however, if we go back to the general framework of the fluid formalism, together with the temperature $\mathcal{K}\mathcal{T}$ in Eq. (16), one can also compute the shear and bulk viscosities for the effective fluid of viable Horndeski gravity, which read $\eta_{\mathrm{bulk}} = 0$ and $\eta_{\mathrm{shear}} \propto \sqrt{2X}$ (see Ref. [15]). If we now take these quantities as formal expressions, irrespective of their connection with the effective fluid representation, we observe that they vanish identically if we set $\Phi =$ constant. In other words, the dissipation coefficients vanish as we approach the GR configuration. Moreover, local thermal equilibrium in Eckart's theory requires a vanishing bulk pressure, $q^a \equiv 0$ and $\pi_{ab} \equiv 0$, which is exactly what happens to the effective fluid for $\Phi =$ constant.

Since the effective fluid for $\Phi =$ constant does not uniquely fix the fluid velocity $u^a$, for any choice of a normalized timelike vector field we can assume that the constitutive laws of Eckart's hydrodynamics still hold—i.e.,

$$0 \equiv q^a\big|_{\Phi=\mathrm{const.}} = -\mathcal{K}(h_{ab}\nabla^b\mathcal{T} + \mathcal{T}u^c\nabla_c u^a),$$
$$0 \equiv \pi_{ab}\big|_{\Phi=\mathrm{const.}} = -2\eta_{\mathrm{shear}}\sigma_{ab},$$

with $\sigma_{ab}$ being the shear tensor of $u^a$. This implies that the shear viscosity must vanish and that one of the following applies: (i) $\mathcal{K} \equiv 0$; (ii) $h_{ab}\nabla^b\mathcal{T} + \mathcal{T}u^c\nabla_c u^a \equiv 0$; (iii) $h_{ab}\nabla^b\mathcal{T} = 0$ and $\mathcal{K}\mathcal{T} \equiv 0$. The latter case is exactly what happens to our dissipation coefficients ($\eta_{\mathrm{shear/bulk}}$ and $\mathcal{K}\mathcal{T}$) for viable Horndeski gravity for $\Phi =$ constant. Hence, we can consistently extend the validity of the expressions for these coefficients to $\Phi =$ constant. Furthermore,

$\Phi = \text{constant}$ also implies that $\nabla_a(\mathcal{KT}) = 0$, and hence $\mathcal{KT}$ is both constant and vanishing, as one would expect from a zero-temperature equilibrium state. In other words, the "zero-temperature equilibrium state" corresponding to GR is embedded into the formalism as the *singular limit case* (in the sense that Eckart's $\eta_{\text{shear/bulk}}$ and $\mathcal{KT}$ are formally defined and well-behaved at $\Phi = \text{constant}$, although one cannot construct a unique normalized comoving velocity of the fluid), in which the temperature is stuck at $\mathcal{KT} = 0$ and the effective dissipative fluid becomes that of a cosmological constant or completely disappears from the field equations.

Conversely, let us consider a solution of "first-generation" scalar-tensor gravity with a nonconstant scalar field, satisfying the conditions leading to Eq. (18), and initialized below the critical line $\mathcal{KT} = \Theta$. Then, we have the observation that $\mathcal{KT}(\tau = \tau_0) \neq 0$ and $\mathcal{KT}(\tau)$ evolves toward zero, which is reached only asymptotically (as $\tau \to \infty$). This means that throughout the evolution of the system, we are never exactly in GR [i.e., $X(\tau) \neq 0,\ \forall\ \tau \geq \tau_0$], and the imperfect fluid components (and the associated thermodynamic quantities) are always well defined.

## III. THE CASE OF INDUCED GRAVITY

Induced gravity [Eq. (1)] can easily be understood as a subclass of viable Horndeski gravity [Eq. (11)] with $\Phi = \phi$ and

$$G_4 = \xi\phi^2, \qquad G_2 = 2X - \frac{\lambda\phi^4}{2}, \qquad G_3 = 0, \tag{19}$$

with $2X = -\nabla_a\phi\nabla^a\phi$. Then, the temperature of (induced) gravity reads

$$\mathcal{KT} = 2\epsilon\frac{\sqrt{-\nabla_a\phi\nabla^a\phi}}{\phi}, \tag{20}$$

provided that $\phi$ has a timelike gradient.

Similarly, we can compute the evolution equation for the temperature from that of viable Horndeski [Eq. (17)], which yields

$$\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} = \frac{\mathcal{KT}}{2}(\mathcal{KT} - 2\Theta) + 2\frac{\Box\phi}{\phi}. \tag{21}$$

Indeed, from Eq. (20), it suffices to observe that

$$\sqrt{2X} = \frac{\phi\mathcal{KT}}{2\epsilon} \quad \text{and} \quad \frac{(\mathcal{KT})^2}{2} = \frac{4X}{\phi^2},$$

hence

$$\begin{aligned}
\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} &= \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\mathcal{KT} + \nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right) \\
&= \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\mathcal{KT} + \nabla^c\phi\nabla_c\left(\frac{2}{\phi}\right) \\
&= \left(\frac{2\epsilon^2\Box\phi}{\phi\mathcal{KT}} - \Theta\right)\mathcal{KT} - \frac{2}{\phi^2}\nabla^c\phi\nabla_c\phi \\
&= \left(2\frac{\Box\phi}{\phi\mathcal{KT}} - \Theta\right)\mathcal{KT} + \frac{4X}{\phi^2} \\
&= 2\frac{\Box\phi}{\phi} - \Theta\mathcal{KT} + \frac{(\mathcal{KT})^2}{2} \\
&= \frac{\mathcal{KT}}{2}(\mathcal{KT} - 2\Theta) + 2\frac{\Box\phi}{\phi},
\end{aligned}$$

which concludes the derivation of Eq. (21).

Due to the field equations (2), we also have that

$$\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} = \frac{\mathcal{KT}}{2}(\mathcal{KT} - 2\Theta) + 2\lambda\phi^2 - 2\xi R. \tag{22}$$

If we require GR to be a stable equilibrium point in the far future, this implies that $\mathcal{KT} \to 0$ and $\mathrm{d}(\mathcal{KT})/\mathrm{d}\tau \to 0$ must hold together as $\tau \to +\infty$ for a solution of Eqs. (2) and (3) that relaxes to this state. In other words, the following limit must hold:

$$\lim_{\tau\to+\infty}(\lambda\phi^2 - \xi R) = 0, \tag{23}$$

provided that the limit exists.

Let us now specialize the discussion to cosmological solutions, and specifically to flat FLRW spacetimes, for which the temperature associated with induced gravity reads

$$\mathcal{KT} = 2\epsilon\frac{|\dot\phi|}{\phi} \tag{24}$$

for a scalar field $\phi$ which is not identically zero. Furthermore, using the fact that $\Theta = 3H$ and $R = 6(\dot H + 2H^2)$, we can rewrite Eq. (22) as

$$\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} = \frac{\mathcal{KT}}{2}(\mathcal{KT} - 6H) + 2\lambda\phi^2 - 12\xi(\dot H + 2H^2), \tag{25}$$

where in turn $H$ is related to the contribution of matter through the effective Friedmann equations for induced gravity.

A natural observation that can be made concerns the de Sitter *vacuum* solution found in Ref. [3] and discussed in Sec. I. Indeed, from Eq. (9), one immediately finds $\mathcal{KT} = 0$ and $\mathrm{d}(\mathcal{KT})/\mathrm{d}\tau = 0$; i.e., we are in GR and this configuration is a stable equilibrium state. This is consistent with the original analysis of Refs. [3,5], where it was noted that for such a solution, the action in Eq. (1) reduces to the Einstein-Hilbert action with gravitational coupling $M_P^2 = \xi\phi_0^2$ plus a cosmological constant $\Lambda = \lambda\phi_0^4/4$.

## IV. MORE ON THE ATTRACTOR TO GR

It turns out that the effective heat equation for induced gravity [Eq. (21)] can be further simplified if we make explicit its dependence on the stress-energy tensor of matter.

Consider the trace of Eq. (3), which yields

$$-\xi\phi^2 R = T^{(\mathrm{m})} + 2X - \lambda\phi^4 - 3\xi\Box\phi^2. \tag{26}$$

It is easy to see that

$$\begin{aligned}\Box\phi^2 &= \nabla_c\nabla^c\phi^2 = 2(\nabla_c\phi\nabla^c\phi + \phi\Box\phi)\\ &= 2(-2X + \phi\Box\phi).\end{aligned} \tag{27}$$

Inserting these results into Eq. (2), one finds

$$\begin{aligned}\Box\phi &= -\xi\phi R + \lambda\phi^3\\ &= \frac{T^{(\mathrm{m})}}{\phi} + \frac{2X}{\phi} - \frac{6\xi}{\phi}(-2X + \phi\Box\phi)\\ &= \frac{1}{\phi}[T^{(\mathrm{m})} + (1+6\xi)2X - 6\xi\phi\Box\phi],\end{aligned} \tag{28}$$

which implies

$$(1+6\xi)[\phi\Box\phi - 2X] = T^{(\mathrm{m})}, \tag{29}$$

which, taking advantage of Eq. (27), can be rewritten as

$$(1+6\xi)\Box\phi^2 = 2T^{(\mathrm{m})}. \tag{30}$$

Dividing Eq. (27) by $2\phi^2$, one finds

$$\frac{\Box\phi}{\phi} = \frac{2X}{\phi^2} + \frac{\Box\phi^2}{2\phi^2}, \tag{31}$$

and recalling that $(\mathcal{KT})^2/4 = 2X/\phi^2$ and Eq. (30), then Eq. (31) implies

$$\frac{\Box\phi}{\phi} = \frac{(\mathcal{KT})^2}{4} + \frac{T^{(\mathrm{m})}}{(1+6\xi)\phi^2}. \tag{32}$$

If we now replace $\Box\phi/\phi$ in Eq. (21) with the expression in Eq. (32), then the effective heat equation for induced gravity becomes

$$\begin{aligned}\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} &= \frac{\mathcal{KT}}{2}(\mathcal{KT} - 2\Theta) + 2\frac{\Box\phi}{\phi}\\ &= \frac{(\mathcal{KT})^2}{2} - \Theta\mathcal{KT} + 2\left[\frac{(\mathcal{KT})^2}{4} + \frac{T^{(\mathrm{m})}}{(1+6\xi)\phi^2}\right]\\ &= (\mathcal{KT})^2 - \Theta\mathcal{KT} + \frac{2T^{(\mathrm{m})}}{(1+6\xi)\phi^2};\end{aligned}$$

namely,

$$\frac{\mathrm{d}(\mathcal{KT})}{\mathrm{d}\tau} = \mathcal{KT}(\mathcal{KT} - \Theta) + \frac{2T^{(\mathrm{m})}}{(1+6\xi)\phi^2}. \tag{33}$$

This equation is just a rewriting of Eq. (21), obtained by taking advantage of the field equations of induced gravity, and it holds for generic solutions of Eqs. (2) and (3) and generic matter sources.

Notably, if we now consider conformal matter, for which $T^{(\mathrm{m})} = 0$, the evolution equation for the temperature of induced gravity reduces to Eq. (18), and hence the existence of an attractor mechanism to GR can be determined in the same way as in Ref. [19], and as described in Sec. II. This is, after all, not surprising, since induced gravity maps onto a subclass of BD gravity analyzed in detail in Ref. [19] ($\omega = $ const. and $\psi V' - 2V = 0$).

From a cosmological perspective, if we now consider an expanding Universe with ordinary matter (dust or radiation), then $T^{(\mathrm{m})} \to 0$ as $t \to +\infty$, since $T^{(\mathrm{m})}$ is either identically zero (radiation), or matter dilutes as the Universe expands (dust, $T^{(\mathrm{m})} = -\rho^{(\mathrm{m})} \propto -a^{-3}$). This is consistent with the necessary condition (23) discussed in Sec. III for the existence of the GR attractor in the far future.

## V. CONCLUSIONS

In this work, we have revisited the induced gravity model based on the thermodynamics of scalar-tensor gravity. Specifically, we have taken advantage of the fact that the action proposed by Cooper and Venturi [3] belongs to a subclass of viable Horndeski theories, for which the thermodynamic formalism was developed in general in Ref. [15], and specialized to cosmology in Ref. [18]. In particular, we found an explicit expression for both the temperature of gravity [Eq. (20)] and its evolution equation (33). These results allowed us to reinterpret the scale-invariance-breaking de Sitter *vacuum* solution of induced gravity as a stable zero-temperature equilibrium state of the effective dissipative fluid associated with the scalar field $\phi$. Furthermore, taking advantage of the analysis in Ref. [19], we identified a necessary condition [Eq. (23)] for the existence of a stable GR equilibrium state at late times—i.e., an attractor mechanism to GR (with cosmological constant). Notably, in the case of conformal matter, the analysis of the attractor-to-GR mechanism reduces exactly to that of Ref. [19], therefore leading to the same conclusions (summarized in Sec. II).

Finding exact solutions upon which one could test this general analysis is rather difficult due to the complexity of the field equations. However, some solutions (both *vacuum* and with matter) have been investigated numerically [8] or for anisotropic cosmologies [25]. These solutions will be the focus of future investigations.

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## DATA AVAILABILITY

No data were created or analyzed in this study.

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