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# POHOZAEV-TYPE IDENTITIES FOR DIFFERENTIAL OPERATORS DRIVEN BY HOMOGENEOUS VECTOR FIELDS

STEFANO BIAGI, ANDREA PINAMONTI, AND EUGENIO VECCHI

ABSTRACT. We prove Pohozaev-type identities for smooth solutions of Euler-Lagrange equations of second and fourth order that arise from functional depending on homogeneous Hörmander vector fields. We then exploit such integral identities to prove non-existence results for the associated boundary value problems.

## 1. INTRODUCTION

In 1965 in [33] Pohozaev proved an integral identity for solutions of the following elliptic boundary value problem:

$$(1.1) \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$  is an open and bounded set with smooth boundary  $\partial\Omega$ . In particular, under the geometric assumption on the set  $\Omega$  of being star-shaped, he was able to show that (1.1) does not admit non-trivial solutions in  $C^2(\Omega) \cap C^1(\bar{\Omega})$ . In 1986 in [34]. Pucci and Serrin extended the approach of Pohozaev proving integral identities for solutions of a large class of variational PDEs coming from functionals possibly depending on the Hessian.

The starting idea of Pohozaev goes actually back to Rellich and Nehari [35, 30] and can be summarized as follows: given a sufficiently smooth solution of (1.1), it suffices to multiply the equation  $-\Delta u = f(u)$  by  $x \cdot \nabla u$ , integrate over  $\Omega$  and apply the Divergence Theorem. This will lead to the celebrated Pohozaev identity

$$(1.2) \quad \frac{n-2}{2} \int_{\Omega} \|\nabla u\|^2 dx - n \int_{\Omega} F(u) dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle x, \nu \rangle dx = 0,$$

where  $F$  is a primitive of  $f$  and  $\nu$  denotes the unit outward normal of  $\partial\Omega$ . From (1.2) it is pretty easy to get nonexistence results (of non-trivial solutions) under appropriate assumptions on the nonlinearity  $f$ , and therefore on  $F$  itself. Assume, e.g., that  $F(u) \leq 0$ . Then

$$0 \geq n \int_{\Omega} F(u) dx = \frac{n-2}{2} \int_{\Omega} \|\nabla u\|^2 dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle x, \nu \rangle dx.$$

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Since the first integral on the right hand side (r.h.s., in short) is non-negative, it is now clear to the entire sign of the r.h.s. depends on  $x \cdot \nu$ ; the latter is a purely geometric quantity, only depending on the set  $\Omega$  and not on the solution  $u$ . Therefore, assuming for example that  $\Omega$  is star-shaped, one can achieve the famous non-existence result of Pohozaev.

Since the paper of Pucci and Serrin [34], the intimate connection between integral identities of Pohozaev-type and non-existence results has been the object of study of many papers, who has extended the ideas previously recalled to cover an always wider class of PDEs. The streamline has been definitely interested in extending Pohozaev's results to more general equations, such as quasi-linear elliptic equations, polyharmonic equations and fractional differential equations. Without any attempt of completeness, we refer to [23, 15] for the case of the  $p$ -laplacian, to [29] for higher order differential operators and to [18, 36] for more recent contributions dealing with nonlocal operators. We must however remind that there has been a certain interest also in studying non-existence results in case of more general domains, see e.g. [14, 28, 13] and the references therein. We also refer to [37, 38] for slightly different approaches to the proof of the classical Pohozaev identity previously recalled.

Another interesting line of research moved to consider non-Euclidean ambient spaces, like Riemannian manifolds (see e.g. [11, 12]) and Carnot groups, which are the prototypical examples of sub-Riemannian manifolds. In this setting Pohozaev-type identities, and the related non-existence results for certain classes of semilinear subelliptic PDEs, have been established in [20, 21, 22, 31]. Among several technical issues to be faced in this setting, we want to stress that one has to understand how to replace the star-shape assumption that naturally appears in the Euclidean case. In this perspective, in [20] the authors introduced the notion of  $\delta_\lambda$ -star-shaped set, which is closely related to the anisotropic dilations defined on Carnot groups. Roughly speaking, this choice allows to *give a sign* to the quantity that naturally replaces the term  $x \cdot \nu$ . We address the interested reader to [16, 19, 17] for further comments on the notion of star-shaped and some applications.

The aim of this note is to continue along the line tracked in [24], where the authors proved similar results for a class of differential operators called  $\Delta_\lambda$  laplacians. In particular, we will focus on sufficiently regular solutions of second order variational PDEs that are Euler-Lagrange equations associated to functionals depending on homogeneous Hörmander vector fields (see Section 2 for the details). The setting of the homogeneous Hörmander vector fields has been studied in the series of papers [1, 2, 3, 4, 5, 6, 7], where several *global results* are established: in fact, this setting seems to allow the possibility of developing an interesting global theory, without assuming the existence of a group of translations. A similar context has been also exploited in [32] to study multiplicity results for solutions of possibly degenerate equation in divergence form and in [25, 26, 27] to study the converge of minimizers for integral functionals [25, 26, 27].

Following the spirit of [34] we will also deduce a Pohozaev-type identity for smooth solutions of variational higher-order PDEs. To be more precise, we will start by considering functionals depending on the intrinsic  $X$ -Hessian as well, and this choice naturally lead to fourth-order PDEs. As already mentioned, such kind of integral identities should be the good tool to study non-existence of non-trivial solutions for boundary value problems. Quite surprisingly, to conclude this kind of non-existence

results it seems necessary to require that the *full* Euclidean gradient vanishes on  $\partial\Omega$ , and not only the more intrinsic  $X$ -gradient (see Section 4 for a more detailed comment on this aspect). We believe that this feature is strictly related to the geometric assumption of  $\delta_\lambda$ -star-shaped set. As a part of a future research project, we shall investigate more general geometric assumptions on  $\Omega$ , possibly allowing us to deal with boundary conditions only involving the intrinsic  $X$ -gradient.

*Plan of the paper.* A short plan of the paper is now in order.

- In Section 2 we recall and discuss all the basic notions needed in the proof of our main theorem
- Section 3 is devoted to the proof of the Pohozaev identity for the second order case and its application to some non-existence results.
- Finally, in Section 4 we will comment on the case in which our operator depends also on the  $X$ -Hessian matrix.

## 2. MAIN ASSUMPTIONS

Throughout the sequel, we denote by  $\mathcal{X}(\mathbb{R}^n)$  the Lie algebra of the smooth vector fields in  $\mathbb{R}^n$ . Moreover, if  $A \subseteq \mathcal{X}(\mathbb{R}^n)$ , we let  $\text{Lie}(A)$  be the smallest Lie sub-algebra of  $\mathcal{X}(\mathbb{R}^n)$  containing  $A$ . Finally, if  $Y \in \mathcal{X}(\mathbb{R}^n)$  is of the form

$$Y = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} \quad (\text{for some } a_1, \dots, a_n \in C^\infty(\mathbb{R}^n)),$$

and if  $x \in \mathbb{R}^n$  is arbitrary, we define

$$Y(x) := \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix} \in \mathbb{R}^n.$$

**Assumptions.** Let  $X := \{X_1, \dots, X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$  be a family of *linearly independent* smooth vector fields in  $\mathbb{R}^n$  satisfying the following assumptions.

**(H.1):**  $X_1, \dots, X_m$  are homogeneous of degree 1 with respect to a family of non-isotropic dilations  $\{\delta_\lambda\}_{\lambda>0}$  in  $\mathbb{R}^n$  of the form

$$(2.1) \quad \delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n), \quad \begin{array}{l} \text{where } \sigma_1, \dots, \sigma_n \in \mathbb{N} \text{ and} \\ 1 = \sigma_1 \leq \dots \leq \sigma_n. \end{array}$$

We define the  $\delta_\lambda$ -homogeneous dimension of  $\mathbb{R}^n$  as

$$(2.2) \quad q := \sum_{k=1}^n \sigma_k \geq n$$

**(H.2):**  $X_1, \dots, X_m$  satisfy Hörmander's condition at  $x = 0$ , that is,

$$(2.3) \quad \dim \left\{ Y(0) : Y \in \text{Lie}(X) \right\} = n.$$

As regards assumption (H.1), we remind that a vector field  $Y \in \mathcal{X}(\mathbb{R}^n)$  is homogeneous of degree  $\alpha \in \mathbb{R}$  with respect to  $\{\delta_\lambda\}_{\lambda>0}$  if

$$(2.4) \quad Y(u \circ \delta_\lambda) = \lambda^\alpha (Y u) \circ \delta_\lambda \quad \text{for all } u \in C^\infty(\mathbb{R}^n) \text{ and } \lambda > 0.$$

Writing  $Y = \sum_{i=1}^n a_i(x) \partial_{x_i}$  (for suitable functions  $a_1, \dots, a_n \in C^\infty(\mathbb{R}^n)$ ), it is easy to check that (2.4) is equivalent to one of the following conditions:

(a)  $a_i$  is  $\delta_\lambda$ -homogeneous of degree  $\sigma_i - \alpha$ , that is,

$$(2.5) \quad a_i(\delta_\lambda(x)) = \lambda^{\sigma_i - \alpha} a_i(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \lambda > 0;$$

(b) for every  $x \in \mathbb{R}^n$  and every  $\lambda > 0$  one has the identity

$$(2.6) \quad \delta_\lambda(Y(x)) = \lambda^\alpha Y(\delta_\lambda(x)).$$

**Remark 2.1.** We list, for future reference, some easy consequences of assumptions (H.1)-(H.2) which shall be useful in the sequel.

(1) It is not difficult to check that the combination of (H.1) and (H.2) implies the validity of Hörmander's condition at every point  $x \in \mathbb{R}^n$ , i.e.,

$$(2.7) \quad \dim\{Y(x) : Y \in \text{Lie}(X)\} = n \quad \text{for every } x \in \mathbb{R}^n.$$

(2) As a consequence of assumption (H.1), and since the  $\sigma_i$ 's are increasingly ordered (see (2.1)), we derive from [9, Rem. 1.3.7] that  $X_1, \dots, X_m$  are *pyramid-shaped*. More precisely, if we write (for  $i = 1, \dots, m$ )

$$X_i = \sum_{k=1}^n a_{k,i}(x) \frac{\partial}{\partial x_k},$$

we have that  $a_{k,i}(x)$  does not depend on the variables  $x_k, \dots, x_n$ .

(3) By crucially exploiting the pyramid-shape of the  $X_i$ 's, it is easy to see that the following integration-by-parts formula holds: if  $\Omega \subseteq \mathbb{R}^n$  is a bounded open set, regular for the Divergence Theorem, and if  $u, v \in C^1(\overline{\Omega})$ , then

$$(2.8) \quad \int_{\Omega} u X_i v \, dx = \int_{\partial\Omega} u v \langle X_i(x), \nu \rangle \, dH^{n-1} - \int_{\Omega} v X_i u \, dx,$$

where  $\nu$  is the outward normal to  $\Omega$ . In particular, choosing  $u \equiv 1$ , we get

$$(2.9) \quad \int_{\Omega} X_i v \, dx = \int_{\partial\Omega} v \langle X_i(x), \nu \rangle \, dH^{n-1}.$$

If  $\Omega \subseteq \mathbb{R}^n$  is a bounded open set and  $u \in C^1(\Omega)$ , we define the *X-gradient* of  $u$  as

$$\nabla_X u = (X_1 u, \dots, X_m u) \in C(\Omega; \mathbb{R}^m);$$

moreover, if  $F = (F_1, \dots, F_m) \in C^1(\Omega; \mathbb{R}^m)$ , we define the *X-divergence* of  $F$  as

$$\text{div}_X F = -\sum_{i=1}^m X_i F_i \in C(\Omega).$$

**Example 2.2.** Before proceeding, we present some concrete examples of vector fields  $X_1, \dots, X_m$  satisfying the 'structural' assumptions (H.1)-(H.2).

(1) In Euclidean space  $\mathbb{R}^n$ , let  $m = n$  and

$$X_i := \partial_{x_i} \quad (\text{for } i = 1, \dots, n).$$

Clearly,  $X_1, \dots, X_n$  are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = (\lambda x_1, \dots, \lambda x_n) = \lambda x,$$

so that assumption (H.1) is fulfilled. Moreover, since

$$[X_i, X_j] = 0 \quad \text{for all } i, j = 1, \dots, n,$$

we have that  $\text{Lie}(X) = \text{span}\{\partial_{x_1}, \dots, \partial_{x_n}\}$ , and assumption (H.2) trivially holds. We explicitly notice that, in this context, we have

- $\nabla_X u = \nabla u$  for every  $u \in C^1(\Omega)$ ;
- $\text{div}_X(F) = -\text{div}(F)$  for every  $F = (F_1, \dots, F_n) \in C^1(\Omega; \mathbb{R}^n)$ .

Moreover, the ‘sum of squares’ naturally associated with  $X$  is nothing but

$$\Delta_X = -\sum_{i=1}^n \partial_{x_i}^2 = -\Delta.$$

(2) Let  $k, n_1, n_2 \in \mathbb{N}$  be arbitrarily fixed, and let  $n := n_1 + n_2 \geq 2$ . We denote a generic point  $x \in \mathbb{R}^n \equiv \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  by

$$x = (y, t), \text{ where } y \in \mathbb{R}^{n_1} \text{ and } t \in \mathbb{R}^{n_2},$$

and we consider the vector fields on  $\mathbb{R}^n$  defined as follows:

$$\begin{aligned} Y_i &:= \partial_{y_i} & (i = 1, \dots, n_1); \\ T_{i,j} &:= y_i^k \partial_{t_j} & (i = 1, \dots, n_1 \text{ and } j = 1, \dots, n_2). \end{aligned}$$

Then, it is not difficult to recognize that the family

$$X := \{Y_i, T_{i,j} : i = 1, \dots, n_1 \text{ and } j = 1, \dots, n_2\}$$

satisfies both assumptions (H.1) and (H.2). In fact, an easy computation based on (2.6) shows that the elements of  $X$  are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = \delta_\lambda(y, t) := (\lambda y, \lambda^{k+1} t),$$

and thus assumption (H.1) is fulfilled. As regards (H.2), since

$$\underbrace{[Y_i, [Y_i, \dots [Y_i, T_{i,j}] \dots]]}_{k \text{ times}} = k! \partial_{t_j},$$

we derive that  $\text{Lie}(X) \supset \text{span}\{\partial_{y_i}, \partial_{t_{i,j}}\}$ ; using this inclusion, it is straightforward to conclude that (2.3) is satisfied. We explicitly notice that the ‘sum of squares’ naturally associated with  $X$  is the sub-elliptic operator

$$\Delta_X := -\sum_{i=1}^{n_1} Y_i^2 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} T_{i,j}^2 = -\Delta_y - (y_1^{2k} + \dots + y_{n_1}^{2k}) \Delta_t.$$

(3) In Euclidean space  $\mathbb{R}^n$ , with  $n \geq 2$ , we consider the vector fields

$$X_1 := \partial_{x_1}, \quad X_2 := x_1 \partial_{x_2} + \frac{x_1^2}{2!} \partial_{x_3} + \dots + \frac{x_1^{n-1}}{(n-1)!} \partial_{x_n}.$$

We claim that the family

$$X := \{X_1, X_2\}$$

satisfies both assumptions (H.1) and (H.2). In fact, a direct computation based on (2.6) shows that  $X_1, X_2$  are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) := (\lambda x_1, \lambda^2 x_2, \dots, \lambda^n x_n),$$

and thus assumption (H.1) is fulfilled. As for (H.2), since

$$Y_i := \underbrace{[X_1, [X_1, \dots [X_1, X_2] \dots]]}_{i \text{ times}} = \partial_{x_{i+1}} + \sum_{j=2}^{n-i} \frac{x_1^{j-1}}{(j-1)!} \partial_{x_{i+j}} \quad (1 \leq i \leq n-1),$$

we have that  $Y_1, \dots, Y_{n-1} \in \text{Lie}(X)$ ; as a consequence, we get

$$\{Y(0) : Y \in \text{Lie}(X)\} \supset \{e_1, \dots, e_n\},$$

and this readily proves that (2.3) is satisfied. We explicitly notice that the ‘sum of squares’ naturally associated with  $X$  is the sub-elliptic operator

$$\Delta_X := -X_1^2 - X_2^2 = -\partial_{x_1}^2 + \left(x_1 \partial_{x_2} + \dots + \frac{x_1^{n-1}}{(n-1)!} \partial_{x_n}\right)^2.$$

This operator was first introduced by Bony in his celebrated 1969 paper [10].

With all the above assumptions and notation at hand, we are ready to introduce the main object of our study. Let  $\mathcal{D} := \Omega \times \mathbb{R} \times \mathbb{R}^m$  and let

$$\mathcal{F} = \mathcal{F}(x, z, p) : \mathcal{D} \longrightarrow \mathbb{R}$$

satisfy the next two properties:

(P.1)  $\mathcal{F}$  is of class  $C^1$  on (an open neighborhood of)  $\overline{\mathcal{D}}$ ;

(P.2) for every  $i = 1, \dots, m$ , the function  $\partial_{p_i} \mathcal{F}$  is of class  $C^1$  on  $\overline{\mathcal{D}}$ .

In this paper, we shall be concerned with the functional

$$(2.10) \quad F : C^2(\Omega) \longrightarrow \mathbb{R}, \quad F(u) := \int_{\Omega} \mathcal{F}(x, u(x), \nabla_X u(x)) \, dx.$$

**Remark 2.3.** We explicitly notice that the functional  $F$  is intimately related with the following *non-linear sub-elliptic* PDE

$$(2.11) \quad \operatorname{div}_X \left( \mathcal{F}_p(x, u(x), \nabla_X u(x)) \right) + \partial_z \mathcal{F}(x, u(x), \nabla_X u(x)) = 0,$$

where we have used the obvious notation

$$\mathcal{F}_p = (\partial_{p_1} \mathcal{F}, \dots, \partial_{p_m} \mathcal{F}).$$

In fact, if  $u \in C^2(\Omega)$  is a minimum/maximum point of  $F$ , we have

$$\left. \frac{d}{dt} \right|_{t=0} F(u + t\varphi) = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

From this, taking into account properties (P.1)-(P.2) of  $\mathcal{F}$ , an application of Lebesgue's Dominated Convergence Theorem shows that, for every  $\varphi \in C_0^\infty(\Omega)$ ,

$$\begin{aligned} 0 &= \int_{\Omega} \left[ \partial_z \mathcal{F}(x, u(x), \nabla_X u(x)) \varphi + \langle \mathcal{F}_p(x, u(x), \nabla_X u(x)), \nabla_X \varphi \rangle \right] dx \\ &\quad \text{(using (2.8), and since } \varphi \in C_0^\infty(\Omega)) \\ &= \int_{\Omega} \left[ \partial_z \mathcal{F}(x, u(x), \nabla_X u(x)) + \operatorname{div}_X \left( \mathcal{F}_p(x, u(x), \nabla_X u(x)) \right) \right] \varphi \, dx, \end{aligned}$$

and thus  $u$  satisfies (2.11) (point-wise on  $\Omega$ ).

### 3. RELICH-POHOZAEV IDENTITY FOR $F$

Throughout the sequel,  $X = \{X_1, \dots, X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$  is a fixed family satisfying assumptions (H.1)-(H.2) and  $\Omega$  is a bounded open set, regular for the Divergence Theorem. Moreover,  $F$  is as in (2.10) with  $\mathcal{F}$  fulfilling (P.1)-(P.2).

Our main aim in this section is to prove some integral identities related to  $F$ . To this end, if  $\sigma_1, \dots, \sigma_n$  are as in (2.1), we introduce the vector field

$$(3.1) \quad \mathbb{T} := \sum_{i=1}^n \sigma_i x_i \frac{\partial}{\partial x_i},$$

which shall be referred to as the *infinitesimal generator* of  $\{\delta_\lambda\}_{\lambda>0}$ .

**Remark 3.1.** We explicitly observe, for a future reference, that the vector field  $\mathbb{T}$  is related with the family  $\{\delta_\lambda\}_{\lambda>0}$  via the following 'Euler-type' theorem: *a function  $u \in C^\infty(\mathbb{R}^n)$  is  $\delta_\lambda$ -homogeneous of degree  $m \geq 0$  if and only if*

$$\mathbb{T}u = mu.$$



As a consequence, taking into account (2.5), we deduce that a smooth vector field  $Y \in \mathcal{X}(\mathbb{R}^n)$  is  $\delta_\lambda$ -homogeneous of degree  $\alpha \in \mathbb{R}$  if and only if

$$[Y, \mathbb{T}] = \alpha Y.$$

We are now ready to prove the following Pohozaev identity for  $F$ . In order to keep our results as clear as possible, in the sequel we shall write

$$\mathcal{F}, \quad \mathcal{F}_x := (\partial_{x_1}\mathcal{F}, \dots, \partial_{x_n}\mathcal{F}), \quad \partial_z\mathcal{F}, \quad \mathcal{F}_p = (\partial_{p_1}\mathcal{F}, \dots, \partial_{p_m}\mathcal{F})$$

and, unless otherwise specified, we shall understand that the above functions are evaluated at points  $(x, u(x), \nabla_X u(x))$  (with  $x \in \Omega$ ).

**Proposition 3.2.** *For every  $u \in C^2(\bar{\Omega})$ , we have the identity*

$$\begin{aligned} (3.2) \quad & \int_{\Omega} (q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) dx \\ & + \int_{\Omega} \mathbb{T}u (\operatorname{div}_X(\mathcal{F}_p) + \partial_z\mathcal{F}) dx \\ & = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1} \end{aligned}$$

where  $H^{n-1}$  is the standard  $(n-1)$ -dimensional Hausdorff measure in  $\mathbb{R}^n$ ,  $\nu$  is the outward normal to  $\Omega$  and  $\nu_X$  is given by

$$(3.3) \quad \nu_X := (\langle X_1(x), \nu \rangle, \dots, \langle X_m(x), \nu \rangle).$$

Finally,  $q$  is the  $\delta_\lambda$ -homogeneous dimension of  $\mathbb{R}^n$  defined in (2.2).

*Proof.* Since  $\Omega$  is regular for the Divergence Theorem, we have

$$\int_{\Omega} \operatorname{div}(\mathcal{F} \cdot \mathbb{T}(x)) dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1}.$$

From this, observing that  $\operatorname{div}(\mathbb{T}(x)) = \sum_{i=1}^n \sigma_i = q$  (see (2.2)), we obtain

$$(3.4) \quad q \int_{\Omega} \mathcal{F} dx + \int_{\Omega} \langle \nabla \mathcal{F}, \mathbb{T}(x) \rangle dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1},$$

where we have used the notation

$$\nabla \mathcal{F} = \nabla(x \mapsto \mathcal{F}(x, u(x), \nabla_X u(x))).$$

Now, according to this notation, a direct calculation gives

$$\begin{aligned} (3.5) \quad & \langle \nabla \mathcal{F}, \mathbb{T}(x) \rangle = \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u(x), \nabla_X u(x)) \\ & + \mathcal{F}_u \cdot \mathbb{T}u(x) + \sum_{i=1}^m \frac{\partial \mathcal{F}}{\partial p_i} \cdot \mathbb{T}(X_i u). \end{aligned}$$

On the other hand, since the  $X_i$ 's are  $\delta_\lambda$ -homogeneous of degree 1, from Remark 3.1 we derive that  $[X_i, \mathbb{T}] = X_i$  (for every  $i = 1, \dots, m$ ); hence, we can write

$$\begin{aligned} (3.6) \quad & \sum_{i=1}^m \partial_{p_i} \mathcal{F} \cdot \mathbb{T}(X_i u) = \sum_{i=1}^m \partial_{p_i} \mathcal{F} \cdot X_i(\mathbb{T}u) + \sum_{i=1}^m \partial_{p_i} \mathcal{F} \cdot [\mathbb{T}, X_i]u \\ & = \langle \mathcal{F}_p, \nabla_X(\mathbb{T}u) \rangle - \sum_{i=1}^m \partial_{p_i} \mathcal{F} \cdot X_i u \\ & = \langle \mathcal{F}_p, \nabla_X(\mathbb{T}u) \rangle - \langle \mathcal{F}_p, \nabla_X u \rangle. \end{aligned}$$

By combining (3.4), (3.5) and (3.6), we then get

$$(3.7) \quad \int_{\Omega} (q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) dx + \int_{\Omega} [\mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) + \partial_z \mathcal{F} \cdot \mathbb{T}u] dx \\ + \int_{\Omega} \langle \mathcal{F}_p, \nabla_X (\mathbb{T}u) \rangle dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1}.$$

To proceed further, we integrate by parts in the last integral in the left-hand side of (3.7): using formula (2.8) in Remark 2.1-(3), we get

$$(3.8) \quad \int_{\Omega} \partial_{p_i} \mathcal{F} \cdot X_i(\mathbb{T}u) dx = \int_{\partial\Omega} \partial_{p_i} \mathcal{F} \cdot \mathbb{T}u \cdot \langle X_i(x), \nu \rangle dH^{n-1} - \int_{\Omega} \mathbb{T}u X_i(\mathcal{F}_{p_i}) dx.$$

Gathering together (3.7), (3.8) and the definition of  $\nu_X$  in (3.3), we then obtain

$$\int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) dx \\ + \int_{\Omega} \mathbb{T}u (\operatorname{div}_X(\mathcal{F}_p) + \partial_z \mathcal{F}) dx \\ = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \cdot \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1},$$

which is exactly the desired (3.2).  $\square$

Just about to illustrate the applicability of Proposition 3.2, we write down the integral identity (3.2) for a general  $\mathcal{F}$  in two concrete examples.

**Example 3.3.** In Euclidean space  $\mathbb{R}^n$ , let  $m = n$  and

$$X_1 = \partial_{x_1}, \dots, X_n = \partial_{x_n}.$$

Moreover, let  $\Omega \subseteq \mathbb{R}^n$  be a fixed open set, regular for the Divergence Theorem, and let  $k \in (1, n]$  be a fixed real number. We then define

$$\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathcal{F}(x, z, p) := \frac{|p|^k}{k} - G(z),$$

where  $G \in C^1(\mathbb{R})$  is a suitable function satisfying  $G(0) = 0$ .

We have already recognized in Example 2.2-(1) that  $X = \{X_1, \dots, X_n\}$  satisfies assumption (H.1) and (H.2); in particular, we have

$$\nabla_X u = \nabla u \quad \text{for all } u \in C^1(\Omega).$$

Moreover, since the  $X_i$ 's are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = (\lambda x_1, \dots, \lambda x_n),$$

from (2.2) and (3.1) we derive that

$$q = n \quad \text{and} \quad \mathbb{T} = \sum_{i=1}^n x_i \partial_{x_i}.$$

Taking into account all these facts, identity (3.2) boils down to

$$(3.9) \quad \left(\frac{n}{k} - 1\right) \int_{\Omega} |\nabla u|^k dx - n \int_{\Omega} G(u) dx + \int_{\Omega} \langle x, \nabla u \rangle (\Delta_k u + G'(u)) dx \\ = \frac{1}{k} \int_{\partial\Omega} \langle x, \nu \rangle |\nabla u|^k dH^{n-1} - \int_{\partial\Omega} |\nabla u|^{k-2} \langle x, \nabla u \rangle \frac{\partial u}{\partial \nu} dH^{n-1},$$

where  $\Delta_k u = -\operatorname{div}(|\nabla u|^{k-2} \nabla u)$  is the usual  $k$ -Laplacian of  $u$ .

**Example 3.4.** In Euclidean space  $\mathbb{R}^n$ , let  $m < n$  and let  $X = \{X_1, \dots, X_m\}$  satisfy assumptions (H.1)-(H.2). Moreover, let  $\Omega \subseteq \mathbb{R}^n$  be a fixed open set, regular for the Divergence Theorem, and let  $k \in (1, n]$ . As in Example 3.3, we then define

$$\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathcal{F}(x, z, p) := \frac{|p|^k}{k} - G(u),$$

where  $G \in C^1(\mathbb{R})$  and  $G(0) = 0$ . In this scenario, identity (3.2) boils down to

$$(3.10) \quad \begin{aligned} & \left(\frac{q}{k} - 1\right) \int_{\Omega} |\nabla_X u|^k dx - q \int_{\Omega} G(u) dx + \int_{\Omega} \mathbb{T}u(\Delta_{X,k}u + G'(u)) dx \\ &= \frac{1}{k} \int_{\partial\Omega} \langle \mathbb{T}(x), \nu \rangle |\nabla_X u|^k dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u |\nabla u|^{k-2} \langle \nabla_X u, \nu_X \rangle dH^{n-1}, \end{aligned}$$

where  $\Delta_{X,k}u$  is the so-called *horizontal  $k$ -Laplacian* of  $u$ , that is,

$$\Delta_{X,k}u = \operatorname{div}_X(|\nabla_X u|^{k-2} \nabla_X u),$$

and  $q, \mathbb{T}, \nu_X$  are given, respectively, by (2.2), (3.1), (3.3).

**3.1. Applications.** As it happens in the ‘classical Euclidean case’, Proposition 3.2 is a key tool in the study of the PDE (2.11) naturally associated with  $F$ , i.e.,

$$(3.11) \quad \operatorname{div}_X(\mathcal{F}_p(x, u(x), \nabla_X u(x))) + \partial_z \mathcal{F}(x, u(x), \nabla_X u(x)) = 0.$$

More precisely, we have the following key result.

**Theorem 3.5.** *Let  $u \in C^2(\overline{\Omega})$  be a solution of (3.11). Then*

$$(3.12) \quad \begin{aligned} & \int_{\Omega} (q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) dx \\ &= \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1}. \end{aligned}$$

If, in addition,  $u \equiv 0$  on  $\partial\Omega$ , for every  $a \in \mathbb{R}$  we have

$$(3.13) \quad \begin{aligned} & \int_{\Omega} (q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) dx \\ & - a \int_{\Omega} u \partial_z \mathcal{F} dx = \int_{\partial\Omega} (\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) \cdot \langle \mathbb{T}(x), \nu \rangle dH^{n-1}. \end{aligned}$$

*Proof.* As regards identity (3.12), it is an immediate consequence of (3.2) and of the fact that,  $u$  being a solution of (3.11), one has

$$\int_{\Omega} \mathbb{T}u (\operatorname{div}_X(\mathcal{F}_p) + \partial_z \mathcal{F}) dx = 0.$$

We then turn to prove identity (3.13) under the additional assumption that  $u \equiv 0$  on  $\partial\Omega$ . To this end, we first claim that

$$(3.14) \quad \int_{\Omega} (\langle \mathcal{F}_p, \nabla_X u \rangle + u \partial_z \mathcal{F}) dx = 0.$$

Indeed, since  $u$  is a solution of (3.11), we have

$$\sum_{i=1}^m X_i(u \partial_{p_i} \mathcal{F}) = \langle \nabla_X u, \mathcal{F}_p \rangle - u \operatorname{div}_X(\mathcal{F}_p) = \langle \mathcal{F}_p, \nabla_X u \rangle + u \partial_z \mathcal{F};$$

from this, using Divergence's Theorem and the fact that  $u \equiv 0$  on  $\partial\Omega$ , we get

$$\begin{aligned} \int_{\Omega} (\langle \mathcal{F}_p, \nabla_X u \rangle + u \partial_z \mathcal{F}) \, dx &= \sum_{i=1}^m \int_{\Omega} X_i (u \partial_{p_i} \mathcal{F}) \, dx \\ &\text{(writing, as usual, } X_i = \sum_{k=1}^n a_{k,i}(x) \partial_{x_k} \text{)} \\ &= \sum_{i=1}^m \sum_{k=1}^n \int_{\Omega} a_{k,i}(x) \partial_{x_k} (u \partial_{p_i} \mathcal{F}) \, dx \\ &= \sum_{i=1}^m \sum_{k=1}^n \left( \int_{\partial\Omega} a_{k,i}(x) u \partial_{p_i} \mathcal{F} \cdot \nu_k - \int_{\Omega} u \partial_{p_i} \mathcal{F} \cdot \partial_{x_k} a_{k,i} \, dx \right) = 0, \end{aligned}$$

which is precisely the claimed (3.14) (see also Remark 2.1-(2)). By combining this last identity with (3.12) we then obtain, for every choice of  $a \in \mathbb{R}$ ,

$$(3.15) \quad \begin{aligned} \int_{\Omega} (q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u \rangle) \, dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) \, dx \\ - a \int_{\Omega} u \partial_z \mathcal{F} \, dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle \, dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle \, dH^{n-1}. \end{aligned}$$

To complete the demonstration, we now perform some algebraic computations on the term  $\mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle$ . Using the very definition of  $\mathbb{T}$  and  $\nu_X$ , and writing

$$X_k = \sum_{j=1}^n a_{j,k}(x) \partial_{x_j} \quad (i = 1, \dots, m),$$

we derive the following chain of equality:

$$\begin{aligned} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle &= \sum_{i=1}^n \sum_{k=1}^m \sigma_i x_i \partial_{x_i} u \cdot \partial_{p_k} \mathcal{F} \cdot \langle X_k(x), \nu \rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^m \sigma_i x_i \partial_{x_i} u \cdot \partial_{p_k} \mathcal{F} \cdot a_{j,k}(x) \nu_j =: (\star). \end{aligned}$$

On the other hand, since  $u \equiv 0$  on  $\partial\Omega$ , for every  $i = 1, \dots, n$  we have

$$\partial_{x_i} u = \frac{\partial u}{\partial \nu} \cdot \nu_i;$$

as a consequence, we obtain

$$\begin{aligned} (\star) &= \sum_{i,j=1}^n \sum_{k=1}^m \partial_{p_k} \mathcal{F} \cdot (\sigma_i x_i \nu_i) \cdot a_{j,k}(x) \left( \frac{\partial u}{\partial \nu} \cdot \nu_j \right) \\ &= \left( \sum_{i=1}^n \sigma_i x_i \nu_i \right) \cdot \sum_{k=1}^m \partial_{p_k} \mathcal{F} \cdot \left( \sum_{j=1}^n a_{j,k}(x) \partial_{x_j} u \right) \\ &= \langle \mathbb{T}(x), \nu \rangle \cdot \sum_{k=1}^m \partial_{p_k} \mathcal{F} \cdot X_k u = \langle \mathbb{T}(x), \nu \rangle \cdot \langle \mathcal{F}_p, \nabla_X u \rangle. \end{aligned}$$

By inserting this last identity into (3.15), we finally get

$$\begin{aligned} \int_{\Omega} (q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u \rangle) \, dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) \, dx \\ - a \int_{\Omega} u \partial_z \mathcal{F} \, dx = \int_{\partial\Omega} \langle \mathbb{T}(x), \nu \rangle (\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) \, dH^{n-1}, \end{aligned}$$

which is exactly the desired (3.13). This ends the proof.  $\square$

With Theorem 3.5 at hand, we are ready to prove our non-existence result for the Dirichlet problem associated with (3.11). Before doing this, we fix a definition.

**Definition 3.6.** Let  $\Omega \subseteq \mathbb{R}^n$  be a non-void open set with  $C^1$  boundary. We say that  $\Omega$  is  $\delta_\lambda$ -star shaped (with respect to the origin) if

$$(3.16) \quad \langle \mathbb{T}(x), \nu \rangle \geq 0 \quad \text{for every } x \in \partial\Omega,$$

where  $\mathbb{T}$  is as in (3.1) and  $\nu$  is the outward normal to  $\partial\Omega$ .

**Theorem 3.7.** Let  $\Omega \subseteq \mathbb{R}^n$  be a connected open set, regular for the Divergence Theorem and  $\delta_\lambda$ -star shaped with respect to the origin. We assume that

- (i)  $\mathcal{F}(x, 0, p) - \langle \mathcal{F}_p(x, 0, p), p \rangle \leq 0$  for all  $x \in \partial\Omega$  and  $p \in \mathbb{R}^m$ ;
- (ii) there exists a number  $a_0 \in \mathbb{R}$  such that

$$(3.17) \quad \begin{aligned} & q\mathcal{F}(x, z, p) - (a_0 + 1)\langle \mathcal{F}_p(x, z, p), p \rangle \\ & + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p)) - a_0 z \partial_z \mathcal{F}(x, z, p) \geq 0 \end{aligned}$$

for every  $x \in \Omega$  and every  $(z, p) \in \mathbb{R} \times \mathbb{R}^m$ ;

- (iii) either  $z = 0$  or  $p = 0$  when (3.17) holds.

Then, the boundary-value problem

$$(3.18) \quad \begin{cases} \operatorname{div}_X(\mathcal{F}_p(x, u(x), \nabla_X u(x))) + \partial_z \mathcal{F}(x, u(x), \nabla_X u(x)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has no non-trivial solutions  $u \in C^2(\overline{\Omega})$ .

*Proof.* Let  $u \in C^2(\overline{\Omega})$  be a solution of (3.18). We need to prove that

$$(3.19) \quad u \equiv 0 \text{ in } \Omega.$$

First of all, since  $u \equiv 0$  on  $\partial\Omega$  and since  $\Omega$  is  $\delta_\lambda$ -star shaped (with respect to the origin), by combining assumption (i) and identity (3.15) we derive that

$$\begin{aligned} & \int_{\Omega} (q\mathcal{F} - (a_0 + 1)\langle \mathcal{F}_p, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) dx \\ & - a_0 \int_{\Omega} u \partial_z \mathcal{F} dx \leq 0 \quad (\text{for every } a \in \mathbb{R}); \end{aligned}$$

from this, using assumption (ii) we obtain

$$q\mathcal{F} - (a_0 + 1)\langle \mathcal{F}_p, \nabla_X u \rangle + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p))(x, u, \nabla_X u) - a_0 u \partial_z \mathcal{F} = 0$$

for every  $x \in \Omega$ . Now, according to assumption (iii), only two cases can occur:

- (a)  $u \equiv 0$  on  $\Omega$ . In this case, (3.19) is satisfied and the proof is complete.

(b)  $\nabla_X u \equiv 0$  on  $\Omega$ . In this case, since  $X_i u \equiv 0$  on  $\Omega$  for every  $i = 1, \dots, m$ , we see that  $u$  is constant along any integral curve of  $\pm X_1, \dots, \pm X_m$  contained in  $\Omega$ . On the other hand, since  $\Omega$  is connected and the  $X_i$ 's satisfy Hörmander's rank condition, the Chow-Rashevsky Connectivity Theorem holds (see, e.g., [8, Thm. 6.22]): for every  $x, y \in \Omega$  there exists a continuous path  $\gamma : [0, 1] \rightarrow \Omega$ , which is piecewise an integral curve of the vector fields  $\pm X_1, \dots, \pm X_m$ , such that

$$\gamma(0) = x \quad \text{and} \quad \gamma(1) = y.$$

As a consequence,  $u$  is constant along  $\gamma$  and, in particular,

$$u(x) = u(y).$$

Due to the arbitrariness of  $x, y \in \Omega$ , we then conclude that  $u$  is constant throughout on  $\Omega$ . From this, since  $u \equiv 0$  on  $\partial\Omega$ , we derive that  $u \equiv 0$  on  $\Omega$ , and the needed (3.19) is again satisfied. This ends the proof.  $\square$

We end this section by ‘specializing’ the assumptions of Theorem 3.7 in the concrete cases discussed in Examples 3.3 and 3.4.

**Example 3.8.** In Euclidean space  $\mathbb{R}^n$ , let  $m = n$  and let  $X = \{X_1, \dots, X_n\}$ ,  $\mathcal{F}$  be as in Example 3.3. For every  $x \in \mathbb{R}^n$  and every  $p \in \mathbb{R}^n$ , we have

$$\mathcal{F}(x, 0, p) - \langle \mathcal{F}_p(x, 0, p), p \rangle = \frac{|p|^k}{k} - G(0) - |p|^{k-2} \langle p, p \rangle = |p|^k \left( \frac{1}{k} - 1 \right) \leq 0$$

(as  $k > 1$  and  $G(0) = 0$ ); hence, assumption (i) of Theorem 3.7 is satisfied. Moreover, since in this case  $q = n$  and  $\mathcal{F}$  is independent of  $x$ , one also has

$$\begin{aligned} q\mathcal{F}(x, z, p) - (a+1)\langle \mathcal{F}_p(x, z, p), p \rangle + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p)) - a z \partial_z \mathcal{F}(x, z, p) \\ = \frac{n}{k} |p|^k - nG(z) - (a+1)|p|^k + a z G'(z). \end{aligned}$$

Let us now suppose that  $G$  enjoys the following additional properties:

- (a)  $G'(0) = 0$ ;
- (b) for every  $z \in \mathbb{R} \setminus \{0\}$  we have the growth condition

$$(3.20) \quad G(z) < \varrho z G'(z), \quad \text{where } \varrho := \frac{1}{k} - \frac{1}{n}.$$

Then, it is immediate to check that assumptions (ii)-(iii) of Theorem 3.7 are satisfied with the choice  $a_0 := \varrho$ . As a consequence, the Dirichlet problem

$$\begin{cases} \Delta_k u + G'(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

possesses no non-trivial solutions  $u \in C^2(\overline{\Omega})$  if (3.20) holds.

**Example 3.9.** In Euclidean space  $\mathbb{R}^n$ , let  $m < n$  and let  $X = \{X_1, \dots, X_m\}$ ,  $\mathcal{F}$  be as in Example 3.4. By arguing exactly as in Example 3.8, we see that assumption (i) of Theorem 3.7 is satisfied; moreover, since  $\mathcal{F}$  is independent of  $x$ , we have

$$\begin{aligned} q\mathcal{F}(x, z, p) - (a+1)\langle \mathcal{F}_p(x, z, p), p \rangle + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p)) - a z \partial_z \mathcal{F}(x, z, p) \\ = \frac{q}{k} |p|^k - qG(z) - (a+1)|p|^k + a z G'(z), \end{aligned}$$

where  $q \geq n$  is as in (2.2). Again as in Example 3.8, let us assume that

- (a)  $G'(0) = 0$ ;
- (b) for every  $z \in \mathbb{R} \setminus \{0\}$  we have the growth condition

$$(3.21) \quad G(z) < \varrho_q z G'(z), \quad \text{where } \varrho_q := \frac{1}{k} - \frac{1}{q}.$$

Then, it is immediate to check that assumptions (ii)-(iii) of Theorem 3.7 are satisfied with the choice  $a_0 := \varrho_q$ . As a consequence, the Dirichlet problem

$$\begin{cases} \Delta_{X,k}u + G'(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

possesses no non-trivial solutions  $u \in C^2(\overline{\Omega})$  if (3.21) holds.

#### 4. TOWARD A NON-EXISTENCE RESULT FOR FUNCTIONALS WITH $X$ -HESSIAN DEPENDENCE

In this last section, we discuss a possible extension of the non-existence result in Theorem 3.7 to functionals  $F$  with  $X$ -Hessian dependence. In order to clearly describe this setting, we fix a family  $X = \{X_1, \dots, X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$  satisfying assumptions (H.1)-(H.2), and we inherit all the notation introduced so far. In addition, if  $\Omega \subseteq \mathbb{R}^n$  is an open set and if  $u \in C^2(\Omega)$ , we define

$$(4.1) \quad \mathcal{H}_X u = (X_j(X_i u))_{i,j=1}^m = \begin{pmatrix} X_1^2 u & X_2(X_1 u) & \cdots & X_m(X_1 u) \\ X_1(X_2 u) & X_2^2 u & \cdots & X_m(X_2 u) \\ \vdots & \vdots & \ddots & \vdots \\ X_1(X_m u) & X_2(X_m u) & \cdots & X_m^2 u \end{pmatrix},$$

and we call the matrix  $\mathcal{H}_X u$  the  $X$ -Hessian of  $u$ .

**Remark 4.1.** It should be explicitly noticed that, since the  $X_i$ 's do not commute (in general), the matrix  $\mathcal{H}_X u$  is not symmetric, i.e.,  $X_i(X_j u) \neq X_j(X_i u)$ .

Let now  $\mathcal{O} := \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m^2}$  and let

$$\mathcal{F} : \mathcal{O} \longrightarrow \mathbb{R}, \quad \mathcal{F} = \mathcal{F}(x, z, p, r) = \mathcal{F}(x, z, p, \{r_{ij}\}_{i,j=1}^m),$$

satisfy the next three properties:

- (P.1)  $\mathcal{F}$  is of class  $C^1$  on (an open neighborhood of)  $\overline{\mathcal{O}}$ ;
- (P.2) for every  $i = 1, \dots, m$ , the function  $\mathcal{F}_{p_i}$  is of class  $C^1$  on  $\overline{\mathcal{O}}$ ;
- (P.3) for every  $i, j = 1, \dots, m$ , the function  $\mathcal{F}_{r_{ij}}$  is of class  $C^2$  on  $\overline{\mathcal{O}}$ .

We then consider the functional  $F : C^4(\Omega) \rightarrow \mathbb{R}$  defined as follows:

$$(4.2) \quad \begin{aligned} F(u) &:= \int_{\Omega} \mathcal{F}(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) \, dx \\ &= \int_{\Omega} \mathcal{F}(x, u(x), \nabla_X u(x), \{X_j(X_i u)\}_{i,j=1}^m) \, dx. \end{aligned}$$

**Remark 4.2.** Analogously to what described in Remark 2.3, the functional  $F$  defined in (4.2) is deeply related with the following non-linear PDE

$$(4.3) \quad \begin{aligned} \sum_{i,j=1}^m X_i X_j \left( \mathcal{F}_{r_{ij}}(x, u, \nabla_X u, \mathcal{H}_X u) \right) + \operatorname{div}_X \left( \mathcal{F}_p(x, u(x), \nabla_X u(x), \mathcal{H}_X u) \right) \\ + \mathcal{F}_z(x, u, \nabla_X u, \mathcal{H}_X u) = 0. \end{aligned}$$

In fact, if  $u \in C^4(\Omega)$  is a minimum/maximum point of  $F$ , we have

$$\left. \frac{d}{dt} \right|_{t=0} F(u + t\varphi) = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

From this, taking into account properties (P.1)-to-(P.3) of  $\mathcal{F}$ , an application of Lebesgue's Dominated Convergence Theorem shows that, for every  $\varphi \in C_0^\infty(\Omega)$ ,

$$\begin{aligned} 0 &= \int_{\Omega} \left[ \partial_z \mathcal{F}(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) \varphi + \langle \mathcal{F}_p(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)), \nabla_X \varphi \rangle \right. \\ &\quad \left. + \sum_{i,j=1}^m \partial_{r_{ij}} \mathcal{F}(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) \cdot X_j(X_i \varphi) \right] dx \\ &\quad \text{(using formula (2.8), and since } \varphi \in C_0^\infty(\Omega)\text{)} \\ &= \int_{\Omega} \left[ \partial_z \mathcal{F}(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) + \operatorname{div}_X \left( \mathcal{F}_p(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) \right) \right. \\ &\quad \left. + \sum_{i,j=1}^m X_i X_j \left( \mathcal{F}_{r_{ij}}(x, u, \nabla_X u, \mathcal{H}_X u) \right) \right] \varphi dx, \end{aligned}$$

and thus  $u$  satisfies (4.3) (point-wise on  $\Omega$ ).

With all these preliminaries at hand, we are ready to prove the analog of Proposition 3.2 in this context. For notational simplicity, in what follows we write

$$\begin{aligned} \mathcal{F}, \quad \mathcal{F}_x &= (\partial_{x_1} \mathcal{F}, \dots, \partial_{x_n} \mathcal{F}), \quad \partial_z \mathcal{F}, \\ \mathcal{F}_p &= (\partial_{p_1} \mathcal{F}, \dots, \partial_{p_m} \mathcal{F}), \quad \mathcal{F}_{r_{ij}} := \partial_{r_{ij}} \mathcal{F} \end{aligned}$$

and, unless otherwise specified, we understand that all the above functions are evaluated at points  $(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x))$  (for  $x \in \Omega$ ). Moreover, we tacitly assume that the open set  $\Omega$  is *bounded and regular for the Divergence Theorem*.

**Proposition 4.3.** *For every  $u \in C^4(\bar{\Omega})$ , we have the identity*

$$\begin{aligned} &\int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\ &\quad + \int_{\Omega} \mathbb{T}u (\operatorname{div}_X(\mathcal{F}_p) + \sum_{i,j=1}^m X_i(X_j(\mathcal{F}_{r_{ij}})) + \mathcal{F}_u) dx \\ &\quad - 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx \\ (4.4) \quad &= \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1} \\ &\quad + \sum_{i,j=1}^m \int_{\partial\Omega} (X_i(\mathcal{F}_{r_{ij}}) \mathbb{T}u - \mathcal{F}_{r_{ij}} X_i(\mathbb{T}u)) \langle X_j(x), \nu \rangle dH^{n-1}, \end{aligned}$$

where  $\nu$  is the outward normal to  $\Omega$  and  $\nu_X$  is as in (3.3).

*Proof.* We argue essentially as in the proof of Proposition 3.2. First of all, since  $\Omega$  is regular for the Divergence Theorem and  $\operatorname{div}(\mathbb{T}(x)) = q$ , we can write

$$(4.5) \quad q \int_{\Omega} \mathcal{F} dx + \int_{\Omega} \langle \nabla \mathcal{F}, \mathbb{T}(x) \rangle dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1},$$

where we have used the notation

$$\nabla \mathcal{F} = \nabla(x \mapsto \mathcal{F}(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x))).$$



On the other hand, using identity (3.6) in the proof of Proposition 3.2, we get

$$\begin{aligned}
 \langle \nabla \mathcal{F}, \mathbb{T}(x) \rangle &= \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)) \\
 (4.6) \quad &+ \mathcal{F}_u \cdot \mathbb{T}u(x) + \langle \mathcal{F}_p, \nabla_X(\mathbb{T}u) \rangle - \langle \mathcal{F}_p, \nabla_X u \rangle \\
 &+ \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} \cdot \mathbb{T}(X_j(X_i u)).
 \end{aligned}$$

Now, reminding that  $[X_i, \mathbb{T}] = X_i$  for every  $i = 1, \dots, m$  (as  $X_1, \dots, X_m$  are  $\delta_\lambda$ -homogeneous of degree 1, see Remark 3.1), we have

$$\begin{aligned}
 \mathcal{F}_{r_{ij}} \cdot \mathbb{T}(X_j(X_i u)) &= \mathcal{F}_{r_{ij}} \cdot \left( X_j(\mathbb{T}(X_i u)) + [\mathbb{T}, X_j](X_i u) \right) \\
 (4.7) \quad &= \mathcal{F}_{r_{ij}} \cdot \left( X_j(X_i(\mathbb{T}u)) + [\mathbb{T}, X_i]u - X_j(X_i u) \right) \\
 &= \mathcal{F}_{r_{ij}} \cdot \left( X_j(X_i(\mathbb{T}u)) - 2X_j(X_i u) \right);
 \end{aligned}$$

as a consequence, by combining (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned}
 (4.8) \quad &\int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\
 &+ \int_{\Omega} \mathbb{T}u \cdot \mathcal{F}_u dx + \int_{\Omega} [\langle \mathcal{F}_p, \nabla_X(\mathbb{T}u) \rangle + \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} \cdot X_j(X_i(\mathbb{T}u))] dx \\
 &- 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx = \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1}.
 \end{aligned}$$

To proceed further, we integrate by parts in the fourth integral in the left-hand side of (4.8). First, using identity (3.8) in the proof of Proposition 3.2, we get

$$(4.9) \quad \int_{\Omega} \langle \mathcal{F}_p, \nabla_X(\mathbb{T}u) \rangle dx = \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1} + \int_{\Omega} \mathbb{T}u \cdot \operatorname{div}_X(\mathcal{F}_p) dx;$$

moreover, by repeatedly exploiting formula (2.8) in Remark 2.1-(3), we have

$$\begin{aligned}
 (4.10) \quad &\int_{\Omega} \mathcal{F}_{r_{ij}} \cdot X_j(X_i(\mathbb{T}u)) dx \\
 &= \int_{\partial\Omega} \mathcal{F}_{r_{ij}} \cdot X_i(\mathbb{T}u) \langle X_j(x), \nu \rangle dH^{n-1} - \int_{\Omega} X_j(\mathcal{F}_{r_{ij}}) X_i(\mathbb{T}u) dx \\
 &= \int_{\partial\Omega} \mathcal{F}_{r_{ij}} \cdot X_i(\mathbb{T}u) \langle X_j(x), \nu \rangle dH^{n-1} \\
 &- \left( \int_{\partial\Omega} X_j(\mathcal{F}_{r_{ij}}) \mathbb{T}u \langle X_i(x), \nu \rangle dH^{n-1} - \int_{\Omega} \mathbb{T}u \cdot X_i(X_j(\mathcal{F}_{r_{ij}})) dx \right).
 \end{aligned}$$

By inserting (4.9) and (4.10) into (4.8), we finally obtain (4.4).  $\square$

**Example 4.4.** In Euclidean space  $\mathbb{R}^n$ , let  $m < n$  and let  $X = \{X_1, \dots, X_m\}$  satisfy assumptions (H.1)-(H.2). Moreover, let  $\Omega \subseteq \mathbb{R}^n$  be a fixed open set, regular for the Divergence Theorem. We then consider the function

$$\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m^2} \rightarrow \mathbb{R}, \quad \mathcal{F}(x, z, p, r) := \frac{1}{2} \left( \sum_{i=1}^m r_{ii} \right)^2 - G(z),$$

where  $G \in C^1(\mathbb{R})$  and  $G(0) = 0$ . In this context, identity (4.4) boils down to

$$\begin{aligned} & \left(\frac{q}{2} - 2\right) \int_{\Omega} (\Delta_X u)^2 dx - q \int_{\Omega} G(u) dx + \int_{\Omega} \mathbb{T}u (\Delta_X^2 u - G'(u)) dx \\ &= \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} \\ & \quad + \sum_{i=1}^m \int_{\partial\Omega} (X_i(\Delta_X u) \mathbb{T}u - \Delta_X u X_i(\mathbb{T}u)) \langle X_i(x), \nu \rangle dH^{n-1}, \end{aligned}$$

where  $\Delta_X u$  is the *horizontal Laplacian* of  $u$ , that is,

$$\Delta_X u = - \sum_{i=1}^m X_i^2 u,$$

and  $q, \mathbb{T}, \nu_X$  are given, respectively, by (2.2), (3.1), (3.3).

With Proposition 4.3 at hand, we could try to rerun the arguments of the previous section in order to establish a non-existence result for the PDE (4.3), i.e.,

$$(4.11) \quad \begin{aligned} & \sum_{i,j=1}^m X_i X_j \left( \mathcal{F}_{r_{ij}}(x, u, \nabla_X u, \mathcal{H}_X u) \right) + \operatorname{div}_X \left( \mathcal{F}_p(x, u(x), \nabla_X u(x), \mathcal{H}_X u) \right) \\ & \quad + \mathcal{F}_z(x, u, \nabla_X u, \mathcal{H}_X u) = 0. \end{aligned}$$

In fact, if  $u \in C^4(\overline{\Omega})$  solves (4.11), from identity (4.4) we get

$$\begin{aligned} & \int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\ & \quad - 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx \\ &= \int_{\partial\Omega} \mathcal{F} \langle \mathbb{T}(x), \nu \rangle dH^{n-1} - \int_{\partial\Omega} \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle dH^{n-1} \\ & \quad + \sum_{i,j=1}^m \int_{\partial\Omega} (X_i(\mathcal{F}_{r_{ji}}) \mathbb{T}u - \mathcal{F}_{r_{ij}} X_i(\mathbb{T}u)) \langle X_j(x), \nu \rangle dH^{n-1}. \end{aligned}$$

Moreover, if we further assume that  $u \equiv 0$  on  $\partial\Omega$ , we already know from the proof of Theorem 3.5 that, for every  $x \in \partial\Omega$ , the following equality holds:

$$(4.12) \quad \mathbb{T}u \langle \mathcal{F}_p, \nu_X \rangle = \langle \mathbb{T}(x), \nu \rangle \cdot \langle \mathcal{F}_p, \nabla_X u \rangle.$$

Gathering together these facts, we then obtain the identity

$$(4.13) \quad \begin{aligned} & \int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\ & \quad - 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx \\ &= \int_{\partial\Omega} \langle \mathbb{T}(x), \nu \rangle (\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle) dH^{n-1} \\ & \quad + \sum_{i,j=1}^m \int_{\partial\Omega} (X_i(\mathcal{F}_{r_{ji}}) \mathbb{T}u - \mathcal{F}_{r_{ij}} X_i(\mathbb{T}u)) \langle X_i(x), \nu \rangle dH^{n-1}. \end{aligned}$$

Now, following the arguments in the previous section, it is clear that identity (4.13) leads to a non-existence result for (4.11) (which a fourth-order PDE) if we couple the PDE with *two* boundary conditions, namely

$$(4.14) \quad u \equiv 0 \text{ on } \partial\Omega \quad \text{and} \quad \mathcal{B}u \equiv 0 \text{ on } \partial\Omega,$$

where  $\mathcal{B}$  is a suitable first-order operator. On the other hand, for the ‘geometrical’ assumption that  $\Omega$  is  $\delta_\lambda$ -star shaped to play a rôle, we need to choose  $\mathcal{B}$  in such a way that, for every solution  $u \in C^4(\bar{\Omega})$  satisfying (4.14), one has

$$(4.15) \quad \sum_{i,j=1}^m (X_i(\mathcal{F}_{r_{ij}}) \mathbb{T}u - \mathcal{F}_{r_{ij}} X_i(\mathbb{T}u)) \langle X_i(x), \nu \rangle \\ = \langle \mathbb{T}(x), \nu \rangle f(x, u, \nabla_X u, \mathcal{H}_X u),$$

where  $f = f(x, z, p, r)$  is a suitable function only depending on  $\mathcal{F}$  and on its derivatives. We explicitly point out that, when  $\mathcal{F}$  does not depend on  $r$ , the analog of (4.15) is identity (4.12) (holding true under the condition  $u \equiv 0$  on  $\partial\Omega$ ), where

$$f(x, z, p) = \langle p, \mathcal{F}_p(x, z, p) \rangle.$$

Quite unnaturally, we are able to prove an identity like (4.15) if we assume that  $\mathcal{B} = \nabla$ , that is, if we consider the boundary condition

$$(4.16) \quad \nabla u \equiv 0 \text{ on } \partial\Omega \iff \partial_{x_i} u \equiv 0 \text{ on } \partial\Omega \text{ for all } i = 1, \dots, n.$$

The reason why such a boundary condition is ‘unnatural’ is that it is not intrinsically defined through the vector fields  $X_1, \dots, X_m$  associated with  $\mathcal{F}$ ; in this spirit, a more ‘natural’ condition should involve the  $X$ -gradient of  $u$ , that is,

$$\nabla_X u \equiv 0 \text{ on } \partial\Omega.$$

We believe that the problem of finding adequate *intrinsic* boundary conditions which lead to non-existence results involving the same notion of  $\delta_\lambda$ -star shapedness used in the previous section could be an interesting challenge.

For the sake of completeness, we end this section by proving the non-existence result arising from Proposition 4.3 under the ‘non-intrinsic’ condition (4.16).

**Theorem 4.5.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a connected open set, regular for the Divergence Theorem and  $\delta_\lambda$ -star shaped with respect to the origin. We assume that*

(i) *for every  $x \in \partial\Omega$  and every  $r \in \mathbb{R}^{m^2}$  we have*

$$(4.17) \quad \mathcal{F}(x, 0, 0, r) - \sum_{i,j=1}^m r_{ij} \mathcal{F}_{r_{ij}}(x, 0, 0, r) \leq 0;$$

(ii) *for every  $x \in \Omega$  and every  $(z, p, r) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m^2}$  we have*

$$(4.18) \quad q\mathcal{F}(x, z, p, r) - \langle p, \mathcal{F}_p(x, z, p, r) \rangle \\ + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r)) - 2 \sum_{i,j=1}^m r_{ij} \mathcal{F}_{r_{ij}}(x, z, p, r) \geq 0$$

(iii) *either  $z = 0$  or  $p = 0$  or  $r = 0$  when (4.18) holds.*

*Then, the boundary-value problem*

$$(4.19) \quad \begin{cases} \left( \sum_{i,j=1}^m X_i X_j(\mathcal{F}_{r_{ij}}) + \operatorname{div}_X(\mathcal{F}_p) + \mathcal{F}_z \right) (x, u, \nabla_X u, \mathcal{H}_X u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \nabla u|_{\partial\Omega} = 0, \end{cases}$$

has no non-trivial solutions  $u \in C^4(\overline{\Omega})$ .

*Proof.* Let  $u \in C^4(\overline{\Omega})$  be a solution of (4.19). We need to prove that

$$(4.20) \quad u \equiv 0 \text{ on } \Omega.$$

We first observe that, since  $\nabla u \equiv 0$  on  $\partial\Omega$ , we obviously have

$$\nabla_X u \equiv 0 \quad \text{and} \quad \mathbb{T}u = \langle \mathbb{T}(x), \nabla u \rangle \equiv 0 \quad \text{on } \partial\Omega.$$

As a consequence, since  $u \equiv 0$  on  $\partial\Omega$ , from identity (4.13) we get

$$(4.21) \quad \begin{aligned} & \int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\ & - 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx \\ & = \int_{\partial\Omega} \langle \mathbb{T}(x), \nu \rangle \mathcal{F} dH^{n-1} - \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} X_i(\mathbb{T}u) \langle X_j(x), \nu \rangle dH^{n-1}. \end{aligned}$$

We now claim that, for every  $x \in \partial\Omega$  and every  $i, j = 1, \dots, m$ , one has

$$(4.22) \quad X_i(\mathbb{T}u) \langle X_j(x), \nu \rangle = \langle \mathbb{T}(x), \nu \rangle \cdot X_j(X_i u).$$

In fact, using the very definition of  $\mathbb{T}$ , and writing

$$X_t = \sum_{h=1}^n a_{h,t}(x) \partial_{x_h} \quad (\text{for all } t = 1, \dots, m),$$

we obtain the following chain of equality (holding true for all  $x \in \partial\Omega$ ):

$$\begin{aligned} X_i(\mathbb{T}u) \langle X_j(x), \nu \rangle &= \sum_{h,k=1}^n a_{h,i}(x) \partial_{x_h}(\mathbb{T}u) a_{k,j}(x) \cdot \nu_k \\ &= \sum_{h,k,l=1}^n a_{h,i}(x) \partial_{x_h}(\sigma_l x_l \partial_{x_l} u) a_{k,j}(x) \cdot \nu_k \\ &= \sum_{h,k=1}^n \sigma_h a_{h,i}(x) (\partial_{x_h} u) a_{k,j}(x) \cdot \nu_k + \sum_{h,k,l=1}^n \sigma_l x_l a_{h,i}(x) \partial_{x_h x_l}^2 u a_{k,j}(x) \cdot \nu_k \\ & \text{(since } \partial_{x_h} u \equiv 0 \text{ on } \partial\Omega \text{ for every } h = 1, \dots, n) \\ &= \sum_{h,k,l=1}^n \sigma_l x_l a_{h,i}(x) \partial_{x_h x_l}^2 u a_{k,j}(x) \cdot \nu_k =: (\star). \end{aligned}$$

On the other hand, since  $u \equiv \nabla u \equiv 0$  on  $\partial\Omega$ , it is very easy to see that

$$\frac{\partial^2 u}{\partial x_h \partial x_l} u = \frac{\partial^2 u}{\partial \nu^2} \nu_h \nu_l \quad \text{on } \partial\Omega;$$

thus, by crucially exploiting this information, we get

$$\begin{aligned}
 (\star) &= \left( \sum_{l=1}^n \sigma_l x_l \nu_l \right) \cdot \sum_{h,k=1}^n a_{h,i}(x) a_{k,j}(x) \left( \frac{\partial^2 u}{\partial \nu^2} \nu_h \nu_k \right) \\
 &= \langle \mathbb{T}(x), \nu \rangle \cdot \sum_{h,k=1}^n a_{h,i}(x) a_{k,j}(x) \partial_{x_h x_k} u \\
 &\quad (\text{since } \partial_{x_h} u \equiv 0 \text{ on } \partial\Omega \text{ for all } h = 1, \dots, n) \\
 &= \langle \mathbb{T}(x), \nu \rangle \cdot \sum_{k=1}^n a_{k,j}(x) \partial_{x_k} \left( \sum_{h=1}^n a_{h,i}(x) \partial_{x_h} u \right) \\
 &= \langle \mathbb{T}(x), \nu \rangle \cdot X_j(X_i u),
 \end{aligned}$$

which is precisely the claimed (4.22).

With (4.22) at hand, we can proceed with the proof of the theorem. Indeed, by combining the cited (4.22) with (4.21), we get

$$\begin{aligned}
 &\int_{\Omega} (q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle) dx + \int_{\Omega} \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X u, \mathcal{H}_X u) dx \\
 &\quad - 2 \sum_{i,j=1}^m \int_{\Omega} \mathcal{F}_{r_{ij}} X_j(X_i u) dx \\
 &= \int_{\partial\Omega} \langle \mathbb{T}(x), \nu \rangle (\mathcal{F} - \sum_{i,j=1}^m X_j(X_i u) \mathcal{F}_{r_{ij}}) dH^{n-1};
 \end{aligned}$$

from this, using assumptions (i)-(ii) and reminding that  $\Omega$  is  $\delta_\lambda$ -star shaped with respect to the origin (see Definition 3.6), we obtain

$$\begin{aligned}
 q\mathcal{F} - \langle \nabla_X, \mathcal{F}_p \rangle + \mathbb{T}(x \mapsto \mathcal{F}(x, z, p, r))(x, u, \nabla_X, \mathcal{H}_X u) \\
 - 2 \sum_{i,j=1}^m X_j(X_i u) \mathcal{F}_{r_{ij}} = 0 \quad \text{for all } x \in \Omega.
 \end{aligned}$$

Now, according to assumption (iii), only three cases can occur:

(a)  $u \equiv 0$  on  $\Omega$ . In this case, (4.20) is satisfied and the proof is complete.

(b)  $\nabla_X u \equiv 0$  on  $\Omega$ . In this case, we have already recognized in the demonstration of Theorem 3.7 that  $u$  must be constant in  $\Omega$ . Since, by assumption,  $u \equiv 0$  on  $\partial\Omega$ , we conclude that  $u \equiv 0$  throughout  $\Omega$ , and (4.20) is again satisfied.

(c)  $\mathcal{H}_X u \equiv 0$  on  $\Omega$ . In this case, bearing in mind the definition of  $\mathcal{H}_X$ , we get

$$\nabla_X(X_i u) \equiv 0 \text{ on } \Omega \text{ for every } i = 1, \dots, n;$$

from this, by arguing exactly as in the proof of Theorem 3.7, we deduce that  $\nabla_X u$  must be constant on  $\Omega$ . On the other hand, since (by assumption)  $\nabla u \equiv 0$  on  $\partial\Omega$ , we necessarily have that  $\nabla_X u \equiv 0$  on  $\partial\Omega$  as well; as a consequence,

$$\nabla_X u \equiv 0 \quad \text{on } \partial\Omega.$$

From (b) we then conclude that  $u \equiv 0$  on  $\Omega$ , and (4.20) is again satisfied.  $\square$

**Remark 4.6.** With reference to discussion before the statement of Theorem 4.5, we point out that identity (4.22) leads to the needed (4.15): in fact, we have

$$\begin{aligned} & \sum_{i,j=1}^m (X_i(\mathcal{F}_{r_{ji}}) Tu - \mathcal{F}_{r_{ij}} X_j(Tu)) \langle X_j(x), \nu \rangle \\ &= - \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} X_j(Tu) \langle X_j(x), \nu \rangle = - \langle T(x), \nu \rangle \cdot \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} X_j(X_i u), \end{aligned}$$

and this is precisely identity (4.15) with the choice

$$f(x, z, p, r) := - \sum_{i,j=1}^m r_{ij} \mathcal{F}_{r_{ij}}.$$

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