



Existence and Uniqueness Results for a Mean-Field Game of Optimal Investment

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Abstract

We establish the existence and uniqueness of the equilibrium for a stochastic mean-field game of optimal investment. The analysis covers both finite and infinite time horizons, and the mean-field interaction of the representative company with a mass of identical and indistinguishable firms is modeled through the time-dependent price at which the produced good is sold. At equilibrium, this price is given in terms of a nonlinear function of the expected (optimally controlled) production capacity of the representative company at each time. The proof of the existence and uniqueness of the mean-field equilibrium relies on a priori estimates and the study of nonlinear integral equations, but employs different techniques for the finite and infinite horizon cases. Additionally, we investigate the deterministic counterpart of the mean-field game under study.

Keywords Mean-field games · Mean-field equilibrium · Forward-backward ODEs · Optimal investment · Price formation

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1 Introduction

In this paper, we provide existence, uniqueness, and characterization results for the equilibrium of a mean-field model of optimal investment with competition à la Cournot. In the absence of interventions, the production capacity of the representative company evolves stochastically over time as a geometric Brownian motion, and its level can be increased through investment, subject to quadratic costs. The representative company discounts profits and costs at a constant rate and aims to maximize total expected profits from production, net of investment costs. Instantaneous profits depend linearly on the company's production capacity (thus, production occurs at full capacity) and on the time-dependent price of the produced good. The mean-field equilibrium investment and average production processes $(\hat{\mathbf{u}}, \hat{q})$ are such that expected total net profits are maximized and, assuming an isoelastic inverse demand function, the price is given in terms of a nonlinear function of the average optimally controlled production at each time (see also Achdou et al. [2] and Sect. 2.1 below). We are able to prove the existence and uniqueness of the equilibrium pair $(\hat{\mathbf{u}}, \hat{q})$ when the problem's time horizon is finite or infinite. These equilibria are characterized as the unique solution to a nonstandard system of forward-backward ordinary differential equations. Furthermore, we demonstrate that the existence and uniqueness carry over to the deterministic counterpart of our model, essentially using the same techniques as in the stochastic case.

The investment problem under study falls under the category of mean-field games with scalar interaction. Mean-field games, independently introduced by Lasry and Lions [27], and Huang et al. [24], represent limit models of non-cooperative symmetric N -player games with mean-field interaction as the number of players N tends to infinity. An exhaustive review of mean-field models can be found in the two-volume book by Carmona and Delarue [11].

In our context, the consistency condition that the equilibrium price aligns at each time with a decreasing nonlinear function of the expected (optimally controlled) production capacity can be viewed as the limit, as the number N of identical and indistinguishable companies operating in the market diverges, of the requirement that price inversely depends on the aggregate production of the entire economy, scaled by a factor of $1/N$. As discussed in Huang et al. [24], this scaling can be justified by considering situations where “an increasing number of firms join together to serve an increasing number of consumers” (see the discussion in Huang et al. [24], after Eq. (2.4) therein).

The equilibrium construction for a given time horizon $T \in (0, +\infty]$ follows a three-step approach. Firstly, given a deterministic path of the average production $q := (q_s)_{s \in [0, T]}$, we solve the representative company's optimal investment problem. This is of linear-quadratic type and has an explicit solution, since the production

capacity evolves as a geometric Brownian motion and is linearly dependent on the investment process, and because the performance criterion has linear dependence on the production capacity and quadratic dependence on the investment cost. Moreover, the optimal control $\hat{\mathbf{u}}^{(T,q)}$ is deterministic, for any fixed average production trajectory q . In the second step, we calculate the expectation of the optimally controlled production capacity process, which is easily computable given the explicit representation of the state process. Finally, in the third step, we impose the consistency condition, requiring that this expectation must match q_s for each time $s \in [0, T]$. This condition leads to a nonlinear integral equation for the equilibrium average production trajectory $\hat{q} := (\hat{q}_s)_{s \in [0, T]}$. By construction the control $\hat{\mathbf{u}} := \hat{\mathbf{u}}^{(T, \hat{q})}$ and the function $\hat{q}$ form a mean-field equilibrium.

It is important to note that although the company's optimal control problem for a fixed average production trajectory q is linear-quadratic, the overall mean-field problem is *not* of linear-quadratic type, due to the nonlinear dependence of the net profit functional on q . In this regard, our results differ from those presented in Bensoussan et al. [6, 15, 29], among others, which focus on linear-quadratic mean-field games. It is also worth stressing that some of our results – particularly the existence and uniqueness results – extend to the case of an instantaneous profit function that is more general than the isoelastic demand function considered in this paper, as well as to more general mean-field interactions, provided that suitable monotonicity conditions are imposed on the problem's data. We refer to Sect. 2.1 for further details.

For our existence and uniqueness results, we do not rely on the study of the PDE system associated with the mean-field game (the Hamilton-Jacobi-Bellman equation and the Fokker-Planck equation), nor on the related master equation; we refer to the subsequent literature review for a discussion of related studies offering a PDE perspective on the problem. Instead, we provide an *ad hoc* analysis that hinges on the linear-quadratic optimization problem at hand and on the nonlinear mean-field interaction term under consideration.

This leads to a characterization of the unique equilibrium, both in the cases $T < +\infty$ and $T = +\infty$, through a nonlinear integral equation for the average production $\hat{q}$. However, different technical arguments are used to prove existence and uniqueness in the finite- and infinite-horizon settings. Indeed, the integral equation that uniquely characterizes the equilibrium average production $\hat{q}$ does not fall within the standard theory of integro-differential equations (see, e.g., Gripenberg et al. [23], Lakshmikantham and Rama Mohana Rao [26]), since we face an initial-value problem with a backward-in-time integral term, which requires a careful specific analysis.

More in detail, the proof of existence relies on a priori estimates and Schauder's fixed-point theorem when $T < +\infty$, whereas in the case $T = +\infty$ it relies on a priori estimates and the Fréchet–Kolmogorov compactness theorem in L^p . Uniqueness is established in both cases through suitable contradiction arguments that exploit structural properties of mean-field equilibria. Notably, when $T = +\infty$, we also prove that the time-dependent equilibrium $\hat{q}$ converges monotonically to the stationary average production level, which is obtained as the unique constant solution of the integral equation characterizing $\hat{q}$.

Given the geometric dynamics of the production capacity, the linear dependence of the profit functional on the controlled state variable, the quadratic costs of investment,

and the fact that the mean-field interaction is of scalar type and only involves the expected value of the latter, it is not surprising that the mean-field equilibrium is deterministic and does not depend on the volatility coefficient σ appearing in the production capacity's dynamics. Guided by this observation, we also consider the deterministic counterpart of the previously discussed mean-field game and show that a unique equilibrium exists in this setting as well. This equilibrium can indeed be constructed by following exactly the same arguments employed in the stochastic case.

Games of optimal investment into production capacity, both in stochastic and deterministic settings, are extensively studied in the economics literature, and a comprehensive overview of models and results can be found in Vives [30]. Specifically, mean-field problems with Cournot competition have attracted considerable attention in the recent literature; see Chan and Sircar [13] for an insightful overview. We now provide a non-exhaustive review of papers dealing with mean-field models of investment in capacity that are relevant for our setting.

Graber and Bensoussan [20] study the existence (and, under certain conditions, uniqueness) of solutions to a system of partial differential equations (PDEs)—namely, the Hamilton–Jacobi–Bellman (HJB) and Fokker–Planck equations—associated with a mean-field game involving Bertrand and Cournot competition among a continuum of players. In Graber and Sircar [21], existence and uniqueness of the master equation associated with a mean-field game of controls with absorption are established, while Chan and Sircar [12] analyze dynamic mean-field games with exhaustible capacities and interactions akin to Cournot and Bertrand competition. An optimal transport perspective on Cournot–Nash equilibria is explored in Acciaio et al. [1].

The existence of classical solutions to an extended mean-field game arising in the study of exhaustible resource production under market competition is addressed in Graber et al. [22], where uniqueness of the equilibrium is shown to hold if the Hamiltonian satisfies certain smallness and uniform convexity conditions. Economically, this corresponds to imposing an upper bound on the degree of substitutability among competing firms.

Camilli et al. [8] also consider Cournot mean-field games of controls for the production of an exhaustible resource by a continuum of producers. They establish uniqueness of the solution under general assumptions on the price function and prove convergence of a learning algorithm, which in turn yields existence of a solution to the mean-field game system.

In Escribe et al. [17], a stylized model for investment in renewable power plants under long-term uncertainty—arising from weather conditions, demand fluctuations, future fuel prices, and average national weather patterns—is considered. It is shown that the N -player formulation of the problem admits a Nash equilibrium and that any sequence of Nash equilibria of the N -player game converges to the unique solution of the corresponding mean-field game.

Investment in renewable energy production is also addressed in Alasseur et al. [5], which proposes a model with a large number of small competitive producers to study the effect of subsidies on the aggregate level of capacity, taking into account a cannibalization effect. The analysis is based on the master equation, and explicit formulas for the long-run equilibria are obtained.

Related price formation models, in the context of large systems of small agents trading a commodity or a security, are studied in Fujii and Takahashi [18], Gomes and Saúde [19] using mean-field techniques.

Regarding irreversible investment, we refer to Cao et al. [9], and Aïd et al. [3, 4]. Cao et al. [9] consider stationary discounted and ergodic mean-field games of singular controls motivated by irreversible investment and provide existence and uniqueness results, as well as connections between the two classes of problems. Applications to commodity markets and the role of regime switches in equilibrium formation are investigated in Aïd et al. [4], which studies a mean-field model of firms competing à la Cournot and irreversibly investing in production. Finally, Aïd et al. [3] studies the competition of firms investing in brown technologies to measure the impact on climate.

The rest of the paper is organized as follows. The stochastic mean-field game is introduced in Sect. 2, and in Sect. 2.1 we provide its economic foundation. In Sect. 3 the company's optimal investment problem is solved for both the cases $T < +\infty$ and $T = +\infty$, while existence and uniqueness of the equilibrium is shown in Sect. 4, again for the finite and infinite time horizon cases. The deterministic version of the mean-field problem is finally considered in Sect. 5.

1.1 Notation

In this section we collect the main notation used in this work.

- Throughout the paper the set $\mathbb{N}$ denotes the set of natural integers without the zero element, i.e., $\mathbb{N} = \{1, 2, \dots\}$, while $\mathbb{R}$ denotes the set of real numbers. Whenever $T = +\infty$, the notation $[0, T]$ indicates the interval $[0, +\infty)$.
- For any $p \geq 1$, any measure space $(E, \mathcal{E}, \mu)$, and any interval $I \subseteq \mathbb{R}$, we indicate by $L^p((E, \mathcal{E}, \mu); I)$ the set of all functions with values in I that are p -integrable with respect to μ . If $E \subseteq \mathbb{R}$ and it is Lebesgue-measurable, we take $\mathcal{E} = \mathcal{L}(E)$ as the Lebesgue σ -algebra and as μ the Lebesgue measure, which is denoted by Leb , and we simply write $L^p(E; I)$. When computing integrals with respect to this measure, we simply write dx instead of $\text{Leb}(dx)$.
- For any $\eta > 0$ and $E, I \subseteq \mathbb{R}$, the notation $L_\eta^p(E; I)$ indicates the set of all functions $f: E \rightarrow I$ such that $t \mapsto e^{-\eta t} f(t)$ is p -integrable with respect to the Lebesgue measure.
- Given two intervals $I, J \subseteq \mathbb{R}$, we denote by $C(I; J)$ the set of all continuous functions from I to J . If I is compact, we endow this set with the usual sup-norm, which makes it a Banach space. The notation $C^p(I; J)$, $p \in \mathbb{N} \cup \{+\infty\}$, denotes the set of functions from I to J that are continuously differentiable p times.

2 The Stochastic Model

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a complete filtered probability space, with $\mathbb{F}$ satisfying the usual assumptions, supporting an $\mathbb{F}$ -adapted standard Brownian motion B and an

independent $\mathcal{F}_0$ -measurable random variable ξ . Throughout the paper, $T \in (0, +\infty) \cup \{+\infty\}$ denotes a finite or infinite time horizon.

We consider a real-valued $\mathbb{F}$ -adapted process $X = (X_s)_{s \in [0, T]}$, satisfying the stochastic differential equation (SDE)

$$\begin{cases} dX_s = -\delta X_s ds + \sigma X_s dB_s + u_s ds, & s \in (0, T], \\ X_0 = \xi, \end{cases} \quad (2.1)$$

where $\delta, \sigma > 0$ are given coefficients, and $\mathbf{u} \in L^1(\Omega \times [0, T]; [0, +\infty))$, i.e., $\mathbf{u}$ is a non-negative, $(\mathcal{F}_s)_{s \in [0, T]}$ -progressively measurable process satisfying $\mathbb{E} \left[\int_0^t u_s ds \right] < +\infty$, for all $t \in [0, T]$. Process $\mathbf{u}$ is the control, which we will choose in a suitable class of admissible processes, that will be specified below. Whenever necessary, to stress the dependence of the solution to (2.1) on the initial condition ξ and on the control $\mathbf{u}$, we denote it by $X^{\xi, \mathbf{u}}$.

We observe that SDE (2.1) admits a unique strong solution, which can be written explicitly (cf. [25, Problem 5.6.15]), for each $\mathbf{u} \in L^1(\Omega \times [0, T]; [0, +\infty))$, as

$$X_s^{\xi, \mathbf{u}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} u_r dr, \quad s \in [0, T], \quad (2.2)$$

where

$$Y_s := e^{\sigma B_s - \left(\delta + \frac{\sigma^2}{2}\right)s}, \quad s \in [0, T]. \quad (2.3)$$

In the mean-field game studied in this paper, the process X describes the private state of a representative player. In particular, it models the evolution of her/his production capacity, which depreciates at a rate δ and can be increased by choosing the investment rate $\mathbf{u}$. Since we require process $\mathbf{u}$ to be non-negative, we are in a setting where investment actions are irreversible. From a mathematical point of view, this ensures useful properties of process X , which will be discussed below and used throughout the paper. Note also that the statistical distribution of the state of the other players in the economy does not influence the production capacity level of the representative player.

Throughout the paper, we work under the following assumption.

Assumption 2.1 *The initial condition ξ of SDE (2.1) is an $\mathcal{F}_0$ -measurable, positive, and integrable random variable. More precisely, $\xi > 0$, $\mathbb{P}$ -a.s., and $0 < \mathbb{E}[\xi] < +\infty$.*

The assumption above ensures that, for any $\mathbf{u} \in L^1(\Omega \times [0, T]; [0, +\infty))$, $X_s^{\xi, \mathbf{u}} > 0$, $\mathbb{P}$ -a.s., for all $s \in [0, T]$. Thus, the production capacity level of the representative agent is never negative. Moreover, it grants us the following result, which will be used later on.

Lemma 2.2 *Under Assumption 2.1 and for any $\mathbf{u} \in L^1(\Omega \times [0, T]; [0, +\infty))$, the unique solution to SDE (2.1) has finite first moment, given by*

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[\xi]e^{-\delta s} + \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} u_r dr \right], \quad s \in [0, T]. \quad (2.4)$$

Proof Fix ξ and $\mathbf{u}$ as above. Using the expression of $X_s^{\xi, \mathbf{u}}$ given in (2.2) we have that

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[Y_s \xi] + \mathbb{E} \left[\int_0^s \frac{Y_s}{Y_r} u_r \right] dr, \quad s \in [0, T].$$

Since the random variable $Y_s \xi$ is non-negative and Y_s is independent of $\mathcal{F}_0$, for all $s \in [0, T]$, we can directly compute the first summand

$$\mathbb{E}[Y_s \xi] = \mathbb{E}[\xi] \mathbb{E}[Y_s] = \mathbb{E}[\xi] e^{-\delta s}.$$

For the second summand, observe that the integrand is non-negative and that, for each fixed $s \in [0, T]$, the random variable $\frac{Y_s}{Y_r}$ is independent of $\mathcal{F}_r$, for all $r \in [0, s]$. Therefore, applying the Fubini-Tonelli theorem,

$$\mathbb{E} \left[\int_0^s \frac{Y_s}{Y_r} u_r dr \right] = \int_0^s \mathbb{E} \left[\frac{Y_s}{Y_r} u_r \right] dr = \int_0^s \mathbb{E} \left[\frac{Y_s}{Y_r} \right] \mathbb{E}[u_r] dr = \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} u_r dr \right],$$

which is finite thanks to the assumptions on $\mathbf{u}$. Therefore, we get (2.4). $\square$

In our mean-field game, the representative company's instantaneous profit function is derived from the isoelastic demand function of Spence-Dixit-Stiglitz preferences (see Achdou et al. [2, p. 7, Footnote 5] and Luttmer [28]), as explained in Sect. 2.1 below. Moreover, the company faces a quadratic cost of investment. This leads to the expected net profit functional

$$J_{T,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds \right]. \quad (2.5)$$

Here, $\rho > 0$ is a discount factor, $\beta > 0$ is a fixed parameter, and $q = (q_s)_{s \in [0, T]}$ is a given deterministic measurable function. The control process $\mathbf{u}$ is chosen in either of the following two classes of admissible controls: if $T < +\infty$,

$$\begin{aligned} \mathcal{U}_T := & \left\{ \mathbf{u} = (u_s)_{s \in [0, T]} \text{ s.t. } \mathbf{u}: \Omega \times [0, T] \rightarrow [0, +\infty) \right. \\ & \text{is } (\mathcal{F}_s)_{s \in [0, T]} \text{-progressively measurable and} \\ & \left. \mathbb{E} \left[\int_0^T u_s^2 ds \right] < +\infty \right\}; \end{aligned} \quad (2.6)$$

if, instead, $T = +\infty$,

$$\begin{aligned} \mathcal{U}_\infty := & \left\{ \mathbf{u} = (u_s)_{s \geq 0} \text{ s.t. } \mathbf{u}: \Omega \times [0, \infty) \rightarrow [0, +\infty) \text{ is } \mathbb{F}\text{-progressively measurable,} \right. \\ & \left. \mathbb{E} \left[\int_0^t u_s ds \right] < +\infty, \mathbb{P}\text{-a.s., } \forall t \geq 0, \text{ and } \mathbb{E} \left[\int_0^{+\infty} e^{-\rho s} u_s^2 ds \right] < +\infty \right\}. \end{aligned} \quad (2.7)$$

At equilibrium, the function q will identify with the average production capacity of the whole population of agents. This is formalized in the next definition.

Definition 2.3 Fix a random variable ξ under Assumption 2.1.

A pair $(\hat{\mathbf{u}}, \hat{q})$, where $\hat{\mathbf{u}} \in \mathcal{U}_T$ and $\hat{q}: [0, T] \rightarrow (0, +\infty)$ is a measurable function, is an *equilibrium* of the mean-field game if

- (i) $J_{T, \hat{q}}(\xi, \hat{\mathbf{u}}) \geq J_{T, \hat{q}}(\xi, \mathbf{u})$, for all $\mathbf{u} \in \mathcal{U}_T$;
- (ii) $\hat{q}_s = \mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}]$, for all $s \in [0, T]$.

From SDE (2.1), the expression of the expected net profit functional in (2.5), and the definition of equilibrium given above, we see that the mean-field interaction between firms appears only in the functional to be maximized, and not also in the state equation. We also emphasize that the interaction is only through the average production capacity, thus our problem is set in a Cournot-like competition framework and, as we are going to see, the optimal control for the representative firm is going to depend on this quantity.

The very specific nature of our problem allows us to tackle it with *ad hoc* arguments, as we outline below, rather than using either the standard analytic approach (see, e.g., [10]), where a system of coupled Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Plank equations is studied, or the standard probabilistic approach (see, e.g., [11]), where a system of Forward Backward Stochastic Differential Equations is considered.

In the next sections we show that there exists a unique equilibrium by adopting a classic fixed point approach. The first step is to solve an optimization problem for a given measurable function q , representing the evolution of the average production capacity level of the agents in the economy. We show that there exists a unique explicit and deterministic optimal control $\hat{\mathbf{u}} \in \mathcal{U}_T$, depending on q , which provides us with the best response of the representative agent to the distribution of the states of the other agents in the economy, summarized by the average q .

The second step is to determine the optimally controlled dynamics of the production capacity level of the representative agent. Thanks to (2.4), we can compute explicitly $\mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}]$, $s \in [0, T]$. Since the optimal control $\hat{\mathbf{u}}$ depends on q , the fixed point argument follows from condition (ii) in Definition 2.3. We show that proving that there exists a unique equilibrium reduces to finding the unique solution of an integro-differential equation.

2.1 The Economic Foundation

To provide an economic foundation of our model, let us introduce the N -player game corresponding to the mean-field game introduced above.

We consider a finite number of firms N , whose production capacities evolve stochastically over time according to the following dynamics

$$\begin{cases} dX_s^i = -\delta X_s^i ds + \sigma X_s^i dB_s^i + u_s^i ds, & s \in (0, T], \\ X_0^i = \xi^i, \end{cases} \quad (2.8)$$

where $B^1, \dots, B^N$ are independent Wiener processes and $\xi^1, \dots, \xi^N$ are i.i.d., $\mathcal{F}_0$ -measurable, positive, and integrable random variables, independent from the Brownian motions B^i , $i = 1, \dots, N$.

Each firm may increase its own production capacity, by investing at rate $\mathbf{u}^i = (u_s^i)_{s \in [0, T]} \in \mathcal{U}_T$, and aims at maximizing the expected net profit functional

$$\mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ X_s^i \left(\frac{1}{N} \sum_{j=1}^N X_s^j \right)^{-\beta} - \frac{1}{2} (u_s^i)^2 \right\} ds \right]. \quad (2.9)$$

Taking the limit as $N \rightarrow \infty$ and considering Nash equilibria, we expect that the dynamics (2.8) of each firm converges to the dynamics (2.1) of the representative player of the mean-field game, and that at equilibrium the functional (2.9) converges to $J_T, \hat{q}$ given in (2.5), with $\hat{q}$ satisfying the consistency condition of Definition 2.3-(ii).

We leave for future research the study of the N -player game and its convergence to the corresponding mean-field game, since this requires a detailed and careful analysis which is outside the scope of this paper, as we intend to focus on directly studying the mean-field game.

Now, instead, we want to focus on providing the precise derivation of the instantaneous profit function appearing in (2.9), i.e., the function

$$(x_1, \dots, x_n) \mapsto x_i \left(\frac{1}{N} \sum_{j=1}^N x_j \right)^{-\beta}, \quad (2.10)$$

for any fixed $i = 1, \dots, N$. This in turn justifies the form of the profit function $(x, p) \mapsto xp^{-\beta}$ appearing in (2.5). We follow an approach close to the one discussed in Achdou et al. [2, p. 7, Footnote 5].

Assume that each player $i = 1, \dots, N$ in our economy is a firm producing a single good and facing the demand function

$$\left(\frac{\pi_i}{P} \right)^{-\gamma}, \quad \text{with } \gamma > 1,$$

where γ is the elasticity of substitution, π_i is the price set by firm i and $P = \left(\frac{1}{N} \sum_{j=1}^N \pi_j^{1-\gamma} \right)^{1/(1-\gamma)}$ is a *price index*. This choice of the demand function is standard in economics and is justified by the Spence-Dixit-Stiglitz preferences (see, e.g., Dixit and Stiglitz [16]).

Each firm aims at maximizing the function

$$\pi_i \mapsto \left\{ \pi_i \left(\frac{\pi_i}{P} \right)^{-\gamma} - \frac{1}{x_i^\eta} \left(\frac{\pi_i}{P} \right)^{-\gamma} \right\},$$

where $x_i > 0$ is the productivity (or size) of firm i and, hence, costs decrease with respect to productivity, according to the factor $x_i^{-\eta}$, where $\eta > 0$ is a given parameter.

We can consider each firm production capacity as a proxy for x_i and, thus, this setting is coherent with the model introduced above.

It is easy to see that the maximum profit is

$$\frac{1}{\gamma - 1} \left(\frac{\gamma}{\gamma - 1} \right)^{-\gamma} x_i^{\eta(\gamma-1)} P^\gamma, \quad (2.11)$$

which is attained setting the price $\pi_i = \frac{\gamma}{(\gamma-1)} x_i^{-\eta}$.

This entails that $P = \left(\frac{\gamma}{(\gamma-1)} \left(\frac{1}{N} \sum_{j=1}^N x_j^{\eta(\gamma-1)} \right)^{-1/(\gamma-1)} \right)$, and hence, substituting this expression into (2.11), we get that the profit is

$$\frac{1}{\gamma - 1} x_i^{\eta(\gamma-1)} \left(\frac{1}{N} \sum_{j=1}^N x_j^{\eta(\gamma-1)} \right)^{-\frac{\gamma}{\gamma-1}}. \quad (2.12)$$

Therefore, setting $\beta = \frac{\gamma}{\gamma-1}$, $\eta = \frac{1}{\gamma-1}$, we get the profit function (2.10), except for the constant $\frac{1}{\gamma-1}$ which does not affect the optimization problem. We emphasize once more that (2.10) justifies the form of the profit function $(x, p) \mapsto xp^{-\beta}$ appearing in (2.5). Indeed, we formally substitute the production capacity x_i of firm i with the production capacity x of the representative firm, and we replace $\frac{1}{N} \sum_{j=1}^N x_j$ with variable p which, at equilibrium, will coincide with the average production capacity of the mean-field game.

The discussion above highlights that it is possible to consider more general forms of interaction between players in the profit function, which can be taken of the form

$$\Phi(x) F \left(\frac{1}{N} \sum_{j=1}^N f(x_j) \right), \quad (2.13)$$

for some $\Phi, f: \mathbb{R} \rightarrow [0, +\infty)$ strictly increasing and $F: [0, +\infty) \rightarrow [0, +\infty)$, see, e.g., [9]. However, with the aim of obtaining an explicit characterization of the equilibrium, in this paper we focus on the particular case in which Φ and f are the identity and $F(x) = x^{-\beta}$, as in (2.12). Nonetheless, it is worth noting that part of our subsequent results—in particular the existence ones—could be extended to the more general case of profit function (2.13), upon exploiting the monotonicity of functions Φ , F , and f .

3 The Optimization Problem

In this section we consider the optimization problem associated to the mean-field game described in Sect. 2. We show that, for each possible choice (in a suitable class of measurable functions) of the function q appearing in the discounted net profit

functional (2.5), the corresponding maximization problem has a unique solution and we compute explicitly the associated optimal control.

We divide our analysis into two subsections, the first devoted to the finite time horizon case, the second one to the infinite time horizon case. Since an additional technical condition is required to solve the infinite time horizon case, we discuss the two problems separately, for the sake of clarity.

3.1 The Finite Time Horizon Case

Let us consider the finite time horizon case, i.e., fix $T < +\infty$. We introduce the set

$$\mathcal{Q}_T := \{q: [0, T] \rightarrow (0, +\infty) \text{ s.t. } q^{-\beta} \in L^1((0, T]; (0, +\infty))\}. \quad (3.1)$$

For each fixed $q \in \mathcal{Q}_T$, we consider the problem of maximizing the functional

$$J_{T,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds \right], \quad (3.2)$$

where ξ satisfies Assumption 2.1, $\mathbf{u}$ is chosen in the class of admissible controls introduced in (2.6), and $X_s^{\xi, \mathbf{u}}$ is the unique solution of SDE (2.1). We also introduce the value function corresponding to the optimization problem, namely,

$$V_{T,q}(\xi) := \sup_{\mathbf{u} \in \mathcal{U}_T} J_{T,q}(\xi, \mathbf{u}), \quad \xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty)). \quad (3.3)$$

In the next Lemma we are going to show important properties of the functional $J_{T,q}$.

Lemma 3.1 *Fix a random variable ξ satisfying Assumption 2.1, $\mathbf{u} \in \mathcal{U}_T$, and $q \in \mathcal{Q}_T$. Then, the functional $J_{T,q}$ defined in (3.2) is finite and verifies*

$$J_{T,q}(\xi, \mathbf{u}) = \mathbb{E}[\xi] z_0^{(T,q)} + \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right], \quad (3.4)$$

where $z^{(T,q)}: [0, T] \rightarrow [0, +\infty)$ is the deterministic function given by

$$z_s^{(T,q)} := \int_s^T e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \in [0, T]. \quad (3.5)$$

Moreover, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $J_{T,q}(\xi; \mathbf{u}) = J_{T,q}(\xi'; \mathbf{u})$, for any $\mathbf{u} \in \mathcal{U}_T$.

Proof Fix $\xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty))$, $\mathbf{u} \in \mathcal{U}_T$, and $q \in \mathcal{Q}_T$. Since the assumptions of Lemma 2.2 are verified, we can use (2.4) and apply the Fubini-Tonelli theorem (note that the integrand is non-negative) to get

$$\mathbb{E} \left[\int_0^T e^{-\rho s} X_s^{\xi, \mathbf{u}} q_s^{-\beta} ds \right]$$

$$= \mathbb{E}[\xi] \int_0^T e^{-\rho s} e^{-\delta s} q_s^{-\beta} ds + \mathbb{E} \left[\int_0^T \int_0^s e^{-\rho s} e^{-\delta(s-r)} q_s^{-\beta} u_r dr ds \right].$$

Clearly, the first summand of the last equality is finite, thanks to the assumptions on ξ and q . Moreover,

$$\mathbb{E}[\xi] \int_0^T e^{-\rho s} e^{-\delta s} q_s^{-\beta} ds = \mathbb{E}[\xi] z_0^{(T,q)}.$$

Also the second summand is finite, thanks to the assumptions on q and $\mathbf{u}$. Exchanging the order of the two time integrals we get

$$\begin{aligned} \mathbb{E} \left[\int_0^T \int_0^s e^{-\rho s} e^{-\delta(s-r)} q_s^{-\beta} u_r dr ds \right] &= \mathbb{E} \left[\int_0^T e^{\delta r} u_r \int_r^T e^{-(\rho+\delta)s} q_s^{-\beta} ds dr \right] \\ &= \mathbb{E} \left[\int_0^T e^{-\rho r} u_r \int_r^T e^{-(\rho+\delta)(s-r)} q_s^{-\beta} ds dr \right] = \mathbb{E} \left[\int_0^T e^{-\rho r} z_r^{(T,q)} u_r dr \right]. \end{aligned}$$

It easily follows that $J_{T,q}$ is finite and that Eq. (3.4) holds, which also entails the last statement of the Lemma. $\square$

Remark 3.2 The last statement in Lemma 3.1 implies that the value function $V_{T,q}$ depends on the initial condition ξ only through its average. More precisely, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $V_{T,q}(\xi) = V_{T,q}(\xi')$.

Thanks to the Lemma above we can find the optimal control for our optimization problem and this, as a byproduct, allows us to explicitly compute the value function.

Proposition 3.3 Fix a random variable ξ satisfying Assumption 2.1 and $q \in \mathcal{Q}_T$. Then, $\hat{\mathbf{u}}^{(T,q)} := z^{(T,q)} \in \mathcal{U}_T$, where $z^{(T,q)}$ is the function defined in (3.5), is an optimal control for problem (3.3), which is deterministic and independent of ξ .

Moreover, $\hat{\mathbf{u}}^{(T,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(T,q)} \in \mathcal{U}_T$ is an optimal control for problem (3.3) different from $\hat{\mathbf{u}}^{(T,q)}$, then

$$\bar{u}_s^{(T,q)} = \hat{u}_s^{(T,q)}, \quad \text{for } \mathbb{P} \otimes \text{Leb-a.e. } (\omega, s) \in \Omega \times [0, T].$$

Finally, the value of the optimization problem admits the explicit expression

$$V_{T,q}(\xi) = \mathbb{E}[\xi] z_0^{(T,q)} + \frac{1}{2} \int_0^T e^{-\rho s} (z_s^{(T,q)})^2 ds, \quad (3.6)$$

and the optimally controlled state process $X^{\xi, \hat{\mathbf{u}}^{(T,q)}}$ is given by

$$X_s^{\xi, \hat{\mathbf{u}}^{(T,q)}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} z_r^{(T,q)} dr, \quad s \in [0, T], \quad (3.7)$$

where Y is the process defined in (2.3).

Proof Fix $\xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty))$ $q \in \mathcal{Q}_T$. From (3.4) it immediately follows that

$$V_{T,q}(\xi) = \mathbb{E}[\xi]z_0^{(T,q)} + \sup_{\mathbf{u} \in \mathcal{U}_T} \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right]. \quad (3.8)$$

Hence, if we can find $\hat{\mathbf{u}}^{(T,q)} \in \mathcal{U}_T$ that maximizes the integrand $z_s^{(T,q)} u_s - \frac{1}{2} u_s^2$, for $\mathbb{P} \otimes \text{Leb}$ -a.e. $(\omega, s) \in \Omega \times [0, T]$, then $\hat{\mathbf{u}}^{(T,q)}$ it must be optimal. Clearly, the integrand is maximized in the sense above if we take $\hat{\mathbf{u}}^{(T,q)} = z^{(T,q)}$. Its admissibility is a consequence of the fact that $z^{(T,q)}$ is bounded. Therefore, $\hat{\mathbf{u}}^{(T,q)}$ is optimal, and clearly essentially unique in the sense specified above. Substituting its definition in (3.8) we immediately get (3.6) and (3.7). $\square$

3.2 The Infinite Time Horizon Case

We analyze now the infinite time horizon case, i.e., we set $T = +\infty$. Let us define, for any given measurable $q: [0, +\infty) \rightarrow (0, +\infty)$ the function

$$z_s^{(\infty,q)} := \int_s^{+\infty} e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \geq 0. \quad (3.9)$$

We introduce the set

$$\mathcal{Q}_\infty := \{q: [0, +\infty) \rightarrow (0, +\infty) \text{ s.t. } q^{-\beta} \in L^1((0, +\infty); (0, +\infty)) \text{ and } z^{(\infty,q)} \text{ is bounded}\}. \quad (3.10)$$

Note that in the definition of $\mathcal{Q}_\infty$ we require an additional technical condition with respect to the definition of $\mathcal{Q}_T$ in the finite time horizon case. This is needed to prove the results of this subsection, namely, Lemma 3.4 and Proposition 3.6.

For each fixed $q \in \mathcal{Q}_\infty$ we consider the problem of maximizing the functional

$$J_{\infty,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^{+\infty} e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds \right], \quad (3.11)$$

where ξ satisfies Assumption 2.1, $\mathbf{u}$ is chosen in the class of admissible controls defined in (2.7), and $X^{\xi, \mathbf{u}}$ is the unique solution of SDE (2.1). We also introduce the value function corresponding to the optimization problem, namely,

$$V_{\infty,q}(\xi) := \sup_{\mathbf{u} \in \mathcal{U}_\infty} J_{\infty,q}(\xi, \mathbf{u}), \quad \xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty)). \quad (3.12)$$

The next result is analogous to Lemma 3.1. The proof proceeds along the same lines and, thus, we omit it.

Lemma 3.4 Fix a random variable ξ satisfying Assumption 2.1, $\mathbf{u} \in \mathcal{U}_\infty$, and $q \in \mathcal{Q}_\infty$. Then, the functional $J_{\infty,q}$ defined in (3.11) is finite and verifies

$$J_{\infty,q}(\xi, \mathbf{u}) = \mathbb{E}[\xi]z_0^{(\infty,q)} + \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s} \left\{ z_s^{(\infty,q)} u_s - \frac{1}{2} u_s^2 \right\} ds\right]. \quad (3.13)$$

Moreover, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $J_{\infty,q}(\xi, \mathbf{u}) = J_{\infty,q}(\xi', \mathbf{u})$, for any $\mathbf{u} \in \mathcal{U}_\infty$.

Remark 3.5 Also in this case (cf. Remark 3.2) the last statement in Lemma 3.4 entails that the value function $V_{\infty,q}$ depends on the initial condition ξ only through its average.

Thanks to Lemma 3.4, also in the infinite time horizon case we can find the optimal control for problem (3.12) and the explicit form of the corresponding value function, as stated in the next Proposition. Its proof is omitted, being similar to that of Proposition 3.3.

Proposition 3.6 Fix a random variable ξ satisfying Assumption 2.1 and $q \in \mathcal{Q}_\infty$.

Then, $\widehat{\mathbf{u}}^{(\infty,q)} := z^{(\infty,q)} \in \mathcal{U}_\infty$ is an optimal control for problem (3.12), which is deterministic and independent of ξ .

Moreover, $\widehat{\mathbf{u}}^{(\infty,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(\infty,q)} \in \mathcal{U}_\infty$ is an optimal control for problem (3.12) different from $\widehat{\mathbf{u}}^{(\infty,q)}$, then

$$\bar{u}_s^{(\infty,q)} = \widehat{u}_s^{(\infty,q)}, \quad \text{for } \mathbb{P} \otimes \text{Leb-a.e. } (\omega, s) \in \Omega \times [0, +\infty).$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{\infty,q}(\xi) = \mathbb{E}[\xi]z_0^{(\infty,q)} + \frac{1}{2} \int_0^{+\infty} e^{-\rho s} (z_s^{(\infty,q)})^2 ds, \quad (3.14)$$

and the optimally controlled state process $X^{\xi, \widehat{\mathbf{u}}^{(\infty,q)}}$ is given by

$$X_s^{\xi, \widehat{\mathbf{u}}^{(\infty,q)}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} z_r^{(\infty,q)} dr, \quad s \geq 0, \quad (3.15)$$

where Y is the process defined in (2.3).

Remark 3.7 All the results of this section can be straightforwardly extended to the case where the optimization problem starts at a time $t > 0$. More precisely, setting the initial condition $X_t = x > 0$ in SDE (2.1), we get that (2.4) becomes

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[\xi]e^{-\delta(s-t)} + \mathbb{E}\left[\int_t^s e^{-\delta(s-r)} u_r dr\right], \quad s \in [t, T],$$

while the expressions of the functional $J_{T,q}$ and of the value function $V_{T,q}$ (now both dependent on t) given in (3.4) and (3.6), if $T < +\infty$, and in (3.13) and (3.14), if

$T = +\infty$, become

$$J_{T,q}(t, \xi, \mathbf{u}) = \mathbb{E}[\xi] e^{-\rho t} z_t^{(T,q)} + \mathbb{E} \left[\int_t^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right],$$

$$V_{T,q}(t, \xi) = \mathbb{E}[\xi] e^{-\rho t} z_t^{(T,q)} + \frac{1}{2} \int_t^T e^{-\rho s} (z_s^{(T,q)})^2 ds,$$

with optimal control $\widehat{\mathbf{u}}^{(T,q)} := z^{(T,q)}$, where

$$z_s^{(T,q)} := \int_s^T e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \in [t, T].$$

4 Existence and Uniqueness of Equilibria

In this section we discuss the existence and the uniqueness of an equilibrium for the mean-field game introduced in Sect. 2.

As usual in mean-field games theory, looking for an equilibrium boils down to finding a fixed point of a suitable map. According to Definition 2.3, in our case this map is $(q_s)_{s \in [0, T]} \mapsto \left(\mathbb{E}[X_s^{\xi, \widehat{\mathbf{u}}^{(T,q)}}] \right)_{s \in [0, T]}$, where $\widehat{\mathbf{u}}^{(T,q)}$ is the optimal control for the maximization problem studied in Sect. 3, whose expression is given either in Proposition 3.3, in the case $T < +\infty$, or in Proposition 3.6, in the case $T = +\infty$.

The next result formalizes this fact, showing also that the fixed point map is precisely the solution map of an integral equation.

Theorem 4.1 *Let us fix a random variable ξ satisfying Assumption 2.1.*

Consider the mean-field game introduced in Sect. 2 in the finite time horizon case, i.e., $T < +\infty$. Then,

- (i) *If there exists an equilibrium $(\widehat{\mathbf{u}}, \widehat{q})$ in the sense of Definition 2.3, such that $\widehat{q} \in \mathcal{Q}_T$, then $\widehat{q}$ is a solution to the integral equation*

$$y_s = e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T]. \quad (4.1)$$

- (ii) *Vice versa, if there exist a unique solution $\widehat{q} \in \mathcal{Q}_T$ to (4.1), then there exists a unique equilibrium $(\widehat{\mathbf{u}}, \widehat{q}) = (z^{(T, \widehat{q})}, \widehat{q})$ of the mean-field game among all equilibria $(\widetilde{\mathbf{u}}, \widetilde{q}) \in \mathcal{U}_T \times \mathcal{Q}_T$, where $z^{(T, \widehat{q})}$ is the function defined in (3.5).*

Consider, instead, the same mean-field game in the infinite time horizon case, i.e., $T = +\infty$. Then,

- (iii) *If there exists an equilibrium $(\widehat{\mathbf{u}}, \widehat{q})$ in the sense of Definition 2.3, such that $\widehat{q} \in \mathcal{Q}_\infty$, then $\widehat{q}$ is a solution to the integral equation*

$$y_s = e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^{+\infty} e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \geq 0. \quad (4.2)$$

- (iv) Vice versa, if there exist a unique solution $\widehat{q} \in \mathcal{Q}_\infty$ to (4.2), then there exists a unique equilibrium $(\widehat{\mathbf{u}}, \widehat{q}) = (z^{(\infty, \widehat{q})}, \widehat{q})$ of the mean-field game among all equilibria $(\widetilde{\mathbf{u}}, \widetilde{q}) \in \mathcal{U}_\infty \times \mathcal{Q}_\infty$, where $z^{(\infty, q)}$ is the function defined in (3.9).

Proof We prove only the first two statements in the finite time horizon case, as the case $T = +\infty$ is analogous.

(i). Let $(\widehat{\mathbf{u}}, \widehat{q})$ be an equilibrium of the mean-field game, with $\widehat{\mathbf{u}} \in \mathcal{U}_T$ and $\widehat{q}: [0, T] \rightarrow (0, +\infty)$, such that $\widehat{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$. Since point (i) of Definition 2.3 holds, from Proposition 3.3 we have that $\widehat{\mathbf{u}} = z^{(T, \widehat{q})}$, for $\mathbb{P} \otimes \text{Leb}$ -a.e. $(\omega, s) \in \Omega \times [0, T]$. Combining the expression of $z^{(T, q)}$, given in (3.5), and (2.4), we get

$$\begin{aligned} \mathbb{E}[X_s^{\xi, \widehat{\mathbf{u}}}] &= \mathbb{E}[\xi]e^{-\delta s} + \mathbb{E}\left[\int_0^s e^{-\delta(s-r)} \widehat{u}_r \, dr\right] \\ &= \mathbb{E}[\xi]e^{-\delta s} + \mathbb{E}\left[\int_0^s e^{-\delta(s-r)} z_r^{(T, \widehat{q})} \, dr\right] \\ &= e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} \widehat{q}_u^{-\beta} \, du \, dr, \quad s \in [0, T]. \end{aligned} \quad (4.3)$$

Therefore, by point (ii) of Definition 2.3, we get that $\widehat{q}$ is a solution to (4.1).

(ii). Let $\widehat{q}$ be the unique solution to (4.1) such that $\widehat{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$. Define $\widehat{\mathbf{u}} := z^{(T, \widehat{q})}$ according to (3.5). Then, by Proposition 3.3, we know that $\widehat{\mathbf{u}}$ is optimal for the problem of maximizing $J_{T, \widehat{q}}$, i.e., that point (i) of Definition 2.3 holds. We also have that Eq. (4.3) above is verified, and hence, since we assumed that $\widehat{q}$ solves (4.1), we get that $\widehat{q}_s = \mathbb{E}[X_s^{\xi, \widehat{\mathbf{u}}}]$, for all $s \in [0, T]$, i.e., that point (ii) of Definition 2.3 also holds. This means that $(\widehat{\mathbf{u}}, \widehat{q})$ is an equilibrium of the mean-field game.

If $(\widetilde{\mathbf{u}}, \widetilde{q})$ is any other equilibrium of the mean-field game, such that $\widetilde{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$, then necessarily $\widetilde{\mathbf{u}} = z^{(T, \widetilde{q})}$, for $\mathbb{P} \otimes \text{Leb}$ -a.e. $(\omega, s) \in \Omega \times [0, T]$, by Proposition 3.3. Therefore, we can repeat the same argument of the proof of point (i) to get that $\widetilde{q}$ is a solution to (4.1), and hence that $(\widetilde{\mathbf{u}}, \widetilde{q}) = (\widehat{\mathbf{u}}, \widehat{q})$, since we assumed that (4.1) has a unique solution. $\square$

Thanks to Theorem 4.1 we can reduce our search for a unique equilibrium of our mean-field game to proving existence and uniqueness of a solution to the integral Eq. (4.1), in the finite time horizon case, or (4.2), in the infinite time horizon case. This, in turn, is equivalent to showing existence and uniqueness of a solution $y: [0, T] \rightarrow (0, +\infty)$ to the integro-differential equation

$$\begin{cases} \frac{d}{ds} y_s = -\delta y_s + \int_s^T e^{-(\rho+\delta)(u-s)} y_u^{-\beta} \, du, & s \in [0, T], \\ y_0 = x. \end{cases} \quad (4.4)$$

for any initial datum $x > 0$.

Indeed, this equivalence follows from the next definition and from Lemma 4.3 below, whose proof is omitted, being a standard result, see, e.g., [26, Chap. 1].

Definition 4.2

- (i) A function $y \in C^1([0, T]; (0, +\infty))$ that satisfies (4.4) for all $s \in [0, T]$ is called a *classical solution* to (4.4).
- (ii) A function $y \in C([0, T]; (0, +\infty))$ that satisfies the integral equation

$$y_s = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T],$$

is called a *mild solution* to (4.4).

Lemma 4.3 Any classical solution $y \in C^1([0, T]; (0, +\infty))$ to (4.4) is also a mild solution. Vice versa, any mild solution $y \in C([0, T]; (0, +\infty))$ to (4.4) is continuously differentiable in $[0, T]$ and also a classical solution.

Remark 4.4 From now on, we will just speak of a solution (in either sense) to (4.4), unless otherwise specified.

It is convenient to introduce a further integro-differential equation, namely

$$\begin{cases} \frac{d}{ds} z_s = e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du, & s \in [0, T], \\ z_0 = x, \end{cases} \quad (4.5)$$

and the corresponding integral equation

$$z_s = x + \int_0^s e^{(\rho+2\delta)r} \int_r^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du dr, \quad s \in [0, T], \quad (4.6)$$

which we consider for any initial datum $x > 0$.

The notions of classical and mild solution to (4.5) are analogous to those given in Definition 4.2. Also in this case, a result similar to Lemma 4.3 shows that these two solution concepts are equivalent.

The following result, whose proof is omitted being based on routine computations, shows that solutions to (4.4) can be expressed in terms of those of (4.5) and vice versa. Thus, we can study either of these two equations, depending on convenience.

Lemma 4.5 If $y: [0, T] \rightarrow (0, +\infty)$ is a solution to (4.4), then $z_s := e^{\delta s} y_s$, $s \in [0, T]$ is a solution to (4.5). Vice versa, if $z: [0, T] \rightarrow (0, +\infty)$ is a solution to (4.5), then $y_s := e^{-\delta s} z_s$, $s \in [0, T]$ is a solution to (4.4).

Remark 4.6 Note that equations (4.4) and (4.5) do not fall into the standard theory of integro-differential equations (see, e.g. [23, 26]), as we have an initial-value problem with the integral being backward in time.

Now, we proceed to study existence and uniqueness of a solution to (4.4) or, equivalently, (4.5). As in the previous section, we are going to divide our analysis into two subsections, the first devoted to the finite time horizon case, the second one to the

infinite time horizon case. In particular, to prove the well-posedness of the integro-differential equation (4.4) in the finite time horizon case, we are going to rely on a fixed point argument to show existence of a solution, and on an *ad-hoc* argument based on the structure of the equation to show its uniqueness. This result will be used in the infinite time horizon case to show existence of a solution to (4.4), by constructing a candidate one as a sequence of solutions to the integro-differential equation with finite time horizon. Then, we prove uniqueness of the solution and its convergence to a steady state using arguments that rely on the analysis of its properties.

4.1 The Finite Time Horizon Case

Let us fix $T < +\infty$ and $x > 0$. We observe, first, that solutions of (4.5) satisfy suitable a priori estimates.

Proposition 4.7 *Let $z \in C([0, T]; (0, +\infty))$ be a solution to (4.5). Then, for all $s \in [0, T]$,*

$$z_s \geq \underline{\alpha}_s^x, \quad \text{and} \quad z_s \leq \bar{\alpha}_s^x, \quad (4.7)$$

where

$$\underline{\alpha}_s^x := x, \quad s \in [0, T], \quad (4.8)$$

and, for any $s \in [0, T]$,

$$\bar{\alpha}_s^x := \begin{cases} x - \frac{x^{-\beta}}{\rho + \delta - \delta\beta} \left\{ \frac{1 - e^{\delta(1+\beta)s}}{\delta(1+\beta)} - e^{-(\rho+\delta-\delta\beta)T} \frac{1 - e^{(\rho+2\delta)s}}{\rho + 2\delta} \right\}, & \text{if } \beta \neq 1 + \frac{\rho}{\delta}, \\ x - \frac{x^{-\beta}}{(\rho + 2\delta)^2} \left\{ [1 + (\rho + 2\delta)(T - s)]e^{(\rho+2\delta)s} - [1 + (\rho + 2\delta)T] \right\}, & \text{if } \beta = 1 + \frac{\rho}{\delta}. \end{cases} \quad (4.9)$$

Proof Since $z_s > 0$, for all $s \in [0, T]$, the right hand side of (4.6) is positive, and hence, again for all $s \in [0, T]$,

$$z_s \geq x, \quad \text{and} \quad z_s^{-\beta} \leq x^{-\beta}, \quad (4.10)$$

which provides us with the lower bound. To get the upper bound, it is enough to substitute the second inequality in (4.10) into the right hand side of (4.6) and compute the integrals. $\square$

Given the result above our goal will be to show that (4.5) admits a unique solution in the (nonempty) set

$$\mathcal{C}_x := \{f \in C([0, T]; (0, +\infty)) : \underline{\alpha}_s^x \leq f_s \leq \bar{\alpha}_s^x, \forall s \in [0, T]\}. \quad (4.11)$$

Theorem 4.8 *For each fixed $x > 0$, there exists a unique solution to the integro-differential Eq. (4.5) in the set $\mathcal{C}_x$ defined in (4.11), which is also twice continuously differentiable. Consequently, the integro-differential Eq. (4.4) admits a unique solution $y \in C^2([0, T]; (0, +\infty))$ satisfying the bounds*

$$e^{-\delta s} x \leq y_s \leq e^{-\delta s} \bar{\alpha}_s^x, \quad s \in [0, T]. \quad (4.12)$$

Proof Fix $x > 0$. Showing that (4.5) admits a unique solution is equivalent to proving that the operator Γ , defined on $C([0, T]; (0, +\infty))$ and given by

$$\Gamma(f)_s := x + \int_0^s e^{(\rho+2\delta)r} \int_r^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du dr, \quad s \in [0, T], \quad (4.13)$$

admits a unique fixed point in $\mathcal{C}_x$.

It is easy to verify that $\mathcal{C}_x$ is a closed, bounded, and convex subset of $C([0, T]; (0, +\infty))$ (which is endowed with the usual sup-norm). Moreover, Γ maps $\mathcal{C}_x$ into itself. Indeed, from (4.13) we readily verify that, for all $f \in \mathcal{C}_x$ and all $s \in [0, T]$, $\Gamma(f)_s \geq \underline{\alpha}_s^x$, since f is a positive function; moreover, noting that for all $f, g \in \mathcal{C}_x$

$$f_s \geq g_s \implies \Gamma(f)_s \leq \Gamma(g)_s, \quad s \in [0, T],$$

we immediately check that $\Gamma(f)_s \leq \Gamma(\underline{\alpha}^x)_s = \bar{\alpha}_s^x$. Finally, Γ is compact in $\mathcal{C}_x$ as a consequence of the Ascoli-Arzelà theorem. Indeed, any sequence in $\mathcal{C}_x$ is equibounded, thanks to the definition of $\mathcal{C}_x$, and equi-continuous since, for all $f \in \mathcal{C}_x$, $\Gamma(f)$ is differentiable and

$$\begin{aligned} \sup_{f \in \mathcal{C}} \sup_{s \in [0, T]} |\Gamma(f)'_s| &= \sup_{f \in \mathcal{C}} \sup_{s \in [0, T]} \left\{ e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du \right\} \\ &\leq \sup_{s \in [0, T]} \left| \frac{d}{ds} \bar{\alpha}_s^x \right| < +\infty. \end{aligned}$$

In fact, since $f_s^{-\beta} \leq (\underline{\alpha}_s^x)^{-\beta} = x^{-\beta}$, for all $s \in [0, T]$, we have that, if $\beta \neq 1 + \frac{\rho}{\delta}$,

$$\begin{aligned} 0 &\leq e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du \\ &\leq \frac{x^{-\beta}}{\rho + \delta - \delta\beta} \left\{ e^{\delta(1+\beta)s} - e^{-(\rho+\delta-\delta\beta)T} e^{(\rho+2\delta)s} \right\} = \frac{d}{ds} \bar{\alpha}_s^x, \quad s \in [0, T], \end{aligned}$$

while if $\beta = 1 + \frac{\rho}{\delta}$,

$$0 \leq e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du \leq x^{-\beta} (T-s) e^{(\rho+2\delta)s} = \left| \frac{d}{ds} \bar{\alpha}_s^x \right|, \quad s \in [0, T].$$

Therefore, by Schauder's fixed point theorem the map Γ has a fixed point $z \in \mathcal{C}_x$, i.e., z verifies (4.6). Hence, z is a solution of Eq. (4.5). The fact that $z \in C^2([0, T]; (0, +\infty))$ simply follows by differentiating twice (4.6).

To show uniqueness, assume that two different solutions $z, \tilde{z} \in \mathcal{C}_x$ of (4.5) exist. It is convenient to write the integro-differential equation satisfied by $w_s := z_{T-s}$ and

$$\tilde{w}_s := \tilde{z}_{T-s}, s \in [0, T],$$

$$\begin{cases} \frac{d}{ds} w_s = -e^{\delta(1+\beta)(T-s)} \int_0^s e^{-(\rho+\delta-\delta\beta)(s-r)} w_r^{-\beta} dr, & s \in [0, T], \\ w_T = x. \end{cases} \quad (4.14)$$

From this equation we deduce that

$$w'_s - \tilde{w}'_s = -e^{\delta(1+\beta)(T-s)} \int_0^s e^{-(\rho+\delta-\delta\beta)(s-r)} [w_r^{-\beta} - \tilde{w}_r^{-\beta}] dr, \quad s \in [0, T]. \quad (4.15)$$

The following two cases may occur:

Case 1. Consider $w_0 \neq \tilde{w}_0$ and assume, without loss of generality, that $w_0 > \tilde{w}_0$. Let us define

$$s^* := \inf\{u > 0: w_u = \tilde{w}_u\}.$$

For each fixed $s \in [0, s^*)$, we have that $w_s > \tilde{w}_s$ and, hence, $w_r^{-\beta} - \tilde{w}_r^{-\beta} < 0$, for all $r \in [0, s]$; thus, we deduce from (4.15) that $w'_s - \tilde{w}'_s > 0$, i.e., that the distance between the two solutions increases in time. Therefore, $s^* = +\infty$, but this contradicts the fact that $w_T = \tilde{w}_T$.

Case 2. Consider $w_0 = \tilde{w}_0$ and note that both w and $\tilde{w}$ are bounded below by the constant x , since $z, \tilde{z} \in \mathcal{C}_x$. Observe, also, that the function $w \mapsto w^{-\beta}$ is Lipschitz in $[x, +\infty)$, with Lipschitz constant $L := x^{-\beta-1}$. Therefore, integrating (4.15) on $[0, s]$ we get

$$w_s - \tilde{w}_s = - \int_0^s e^{\delta(1+\beta)(T-u)} \int_0^u e^{-(\rho+\delta-\delta\beta)(u-r)} [w_r^{-\beta} - \tilde{w}_r^{-\beta}] dr du, \quad s \in [0, T],$$

whence

$$|w_s - \tilde{w}_s| \leq L \int_0^s \left\{ e^{\delta(1+\beta)(T-u)} \int_0^u e^{-(\rho+\delta-\delta\beta)(u-r)} dr \right\} \sup_{r \in [0, u]} |w_r - \tilde{w}_r| du, \quad s \in [0, T].$$

Since $s \mapsto \sup_{r \in [0, s]} |w_r - \tilde{w}_r|$ is bounded on $[0, T]$, applying Gronwall's lemma we get that $w_s = \tilde{w}_s$ for all $s \in [0, T]$, which contradicts the assumption that w and $\tilde{w}$ are different.

Therefore, Eq. (4.5) admits a unique solution in $\mathcal{C}_x$. Finally, from Lemma 4.5, we immediately deduce that also (4.4) has a unique solution in $C^2([0, T]; (0 + \infty))$ satisfying the bounds in (4.12). $\square$

Thanks to the result above, we can establish the following.

Theorem 4.9 *For each fixed random variable ξ satisfying Assumption 2.1, there exists a unique equilibrium $(\hat{u}, \hat{q})$ for the mean-field game with finite time horizon, among all equilibria $(\tilde{u}, \tilde{q}) \in \mathcal{U}_T \times \mathcal{Q}_T$.*

More precisely, $\widehat{u} = z^{(T, \widehat{q})}$, where $z^{(T, \widehat{q})}$ is the function defined in (3.5), and $\widehat{q}$ is the unique solution to (4.4), with initial condition $x = \mathbb{E}[\xi]$.

Proof Let $\widehat{q}$ be the unique solution to (4.4), with $x = \mathbb{E}[\xi]$. Since $\widehat{q}$ satisfies the bounds (4.12) and since the functions $s \mapsto e^{-\delta s}x$ and $s \mapsto e^{-\delta s}\overline{\alpha}_s^x$ are continuous on $[0, T]$, hence therein bounded, we deduce that also $\widehat{q}$ is bounded on $[0, T]$. More precisely,

$$0 < m \leq \widehat{q}_s \leq M, \quad \forall s \in [0, T], \quad \text{with } m := e^{-\delta T}x \text{ and } M := \sup_{s \in [0, T]} e^{-\delta s}\overline{\alpha}_s^x. \quad (4.16)$$

Therefore, $\widehat{q} \in \mathcal{Q}_T$ and applying Theorem 4.1-(ii) we get the result. $\square$

4.2 The Infinite Time Horizon Case

We discuss now the infinite time horizon case, i.e., we fix $T = +\infty$. Here we work under the assumption $\beta < 1 + \frac{\rho}{\delta}$, which will be used to guarantee convergence of the integral appearing in (4.5).

A reasoning similar to the one used to prove Proposition 4.7 shows that the following holds.

Proposition 4.10 *Let $z \in C([0, +\infty); (0, +\infty))$ be a solution to (4.5), with $T = +\infty$, and assume that $\beta < 1 + \frac{\rho}{\delta}$. Then, for all $s \geq 0$,*

$$z_s \geq \underline{\alpha}_s^{x, \infty}, \quad \text{and} \quad z_s \leq \overline{\alpha}_s^{x, \infty}. \quad (4.17)$$

where

$$\underline{\alpha}_s^{x, \infty} := x, \quad s \geq 0, \quad (4.18)$$

and

$$\overline{\alpha}_s^{x, \infty} := x - \frac{x^{-\beta}}{\rho + \delta - \delta\beta} \left\{ \frac{1 - e^{\delta(1+\beta)s}}{\delta(1+\beta)} \right\}, \quad s \geq 0. \quad (4.19)$$

As the notation suggests, the function $\overline{\alpha}^{x, \infty}$ given in (4.19) is the pointwise limit, as $T \rightarrow +\infty$, of the sequence $\{\overline{\alpha}^{x, T}\}_{T>0}$, where for each $T > 0$ the function $\overline{\alpha}^{x, T}$ has the expression given in (4.9).

We aim at showing that (4.5) admits a unique solution in the (nonempty) set

$$\mathcal{C}_{x, \infty} := \{f \in C([0, +\infty); (0, +\infty)) : \underline{\alpha}_s^{x, \infty} \leq f_s \leq \overline{\alpha}_s^{x, \infty}, \forall s \geq 0\}. \quad (4.20)$$

Note, in particular, that for any $f \in \mathcal{C}_{x, \infty}$ we have that

$$\begin{aligned} \int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du &\leq \int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} x^{-\beta} du \\ &= \frac{x^{-\beta}}{\rho + \delta - \delta\beta} e^{-(\rho+\delta-\delta\beta)s} < +\infty, \quad \forall s \geq 0, \end{aligned}$$

as f is bounded below by the constant x and we assumed $\beta < 1 + \frac{\rho}{\delta}$.

Theorem 4.11 Let $T = +\infty$ and assume that $\beta < 1 + \frac{\rho}{\delta}$. Then, for each fixed $x > 0$, there exists a unique solution to the integro-differential Eq. (4.5) in the set $\mathcal{C}_{x,\infty}$ defined in (4.20), which is also twice continuously differentiable. Moreover, the integro-differential Eq. (4.4) admits a unique solution $y \in C^2([0, +\infty); (0, +\infty))$ satisfying the bounds

$$e^{-\delta s}x \leq y_s \leq e^{-\delta s} \left[x - \frac{x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)} \right] + \frac{e^{\delta\beta s}x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)}, \quad s \geq 0. \quad (4.21)$$

Finally, if $x = y_\infty := (\delta(\rho+\delta))^{-\frac{1}{\beta+1}}$, then the function $y_s = y_\infty$, for all $s \geq 0$, is the unique constant solution to (4.4). If, instead, $x \neq y_\infty$, then the corresponding solution y to (4.4) is strictly monotone and such that

$$\lim_{s \rightarrow +\infty} y_s = y_\infty.$$

Proof We divide the proof as follows. First, we show existence and uniqueness of a solution to (4.5) in the set $\mathcal{C}_{x,\infty}$, for each fixed $x > 0$. As in the finite time horizon case, the fact that such solution is twice continuously differentiable follows by differentiation of (4.6). Then, it immediately follows from Lemma 4.5 that (4.4) admits a unique solution $y \in C^2([0, +\infty); (0, +\infty))$ satisfying the bounds given in (4.21). Finally, we will show that the constant function equal to y_∞ is the unique constant solution to (4.4), and that the solutions starting at $x \neq y_\infty$ are monotone in time and converge to y_∞ .

Existence. Fix $x > 0$, and consider the family of solutions $\mathcal{Z} := \{z^{(n)}\}_{n \geq 1}$, where $z^{(n)}$ is the solution to the finite horizon problems with terminal time $T = n$, i.e.,

$$\begin{cases} \frac{d}{ds} z_s^{(n)} = e^{(\rho+2\delta)s} \int_s^n e^{-(\rho+\delta-\delta\beta)u} (z_u^{(n)})^{-\beta} du, & s \in [0, n], \\ z_0 = x. \end{cases} \quad (4.22)$$

We extend each of them by continuity taking them to be constant over $(n, +\infty)$, so that $z^{(n)} \in \mathcal{C}_{x,\infty}$, as $\bar{\alpha}_s^x \leq \bar{\alpha}_s^{x,\infty}$, for all $s \in [0, n]$. Differently from the finite horizon case, here we cannot directly apply the Ascoli-Arzelà theorem, as the infinite horizon does not allow to get the same equicontinuity estimates as in the proof of Theorem 4.8. Hence, use a weaker compactness criterion based on the Fréchet-Kolmogorov theorem.

Let us define the set

$$L_{\rho+2\delta}^1([0, \infty); \mathbb{R}) := \left\{ f : [0, \infty) \rightarrow \mathbb{R} : \int_0^\infty e^{-(\rho+2\delta)s} |f_s| ds < +\infty \right\},$$

endowed with the usual weighted norm $\|\cdot\|_{1,\rho+2\delta}$. Clearly $\mathcal{Z} \subset L_{\rho+2\delta}^1([0, \infty); \mathbb{R})$.

We observe that, given the bounds in (4.20), the family $\mathcal{Z}$ is equibounded. We want to show that it is also equi-integrable in $L^1_{\rho+2\delta}$. For any $s \geq 0$ and $h > 0$ we have that

$$z_{s+h}^{(n)} - z_s^{(n)} = \mathbf{1}_{[0, n-h]}(s) \int_s^{s+h} (z_r^{(n)})' dr + \mathbf{1}_{[n-h, n]}(s) \int_s^n (z_r^{(n)})' dr.$$

Let $K := \frac{x^{-\beta}}{\rho+\delta-\delta\beta}$. Using the fact that $0 < (z_s^{(n)})' \leq K e^{\delta(1+\beta)s}$, for all $s \geq 0$, and all $n \geq 1$, we get that, for any $h > 0$ small enough,

$$\begin{aligned} \|z_{\cdot+h}^{(n)} - z_{\cdot}^{(n)}\|_{1, \rho+2\delta} &= \int_0^{+\infty} e^{-(\rho+2\delta)s} |z_{s+h}^{(n)} - z_s^{(n)}| ds \\ &\leq \int_0^{n-h} e^{-(\rho+2\delta)s} \int_s^{s+h} (z_r^{(n)})' dr ds + \int_{n-h}^n e^{-(\rho+2\delta)s} \int_s^n (z_r^{(n)})' dr ds \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)} \int_0^{n-h} e^{-(\rho+\delta-\delta\beta)s} ds \\ &\quad + \underbrace{\frac{K e^{\delta(1+\beta)n}}{\delta(1+\beta)} \int_{n-h}^n e^{-(\rho+2\delta)s} ds - \frac{K}{\delta(1+\beta)} \int_{n-h}^n e^{-(\rho+\delta-\delta\beta)s} ds}_{\leq 0} \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)(\rho+\delta-\delta\beta)} \underbrace{[1 - e^{-(\rho+\delta-\delta\beta)(n-h)}]}_{\leq 0} + \frac{K e^{-(\rho+\delta-\delta\beta)n}}{\delta(1+\beta)(\rho+2\delta)} [e^{(\rho+2\delta)h} - 1] \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)(\rho+\delta-\delta\beta)} + \frac{K}{\delta(1+\beta)(\rho+2\delta)} [e^{(\rho+2\delta)h} - 1], \quad \forall n \geq 1, \end{aligned}$$

where in the last inequality we used the condition $\beta < 1 + \frac{\rho}{\delta}$. Therefore, we obtain that

$$\lim_{h \rightarrow 0^+} \sup_{n \geq 1} \|z_{\cdot+h}^{(n)} - z_{\cdot}^{(n)}\|_{1, \rho+2\delta} = 0,$$

i.e., that the family $\mathcal{Z}$ is equi-integrable in $L^1_{\rho+2\delta}$.

By the Frechet-Kolmogorov theorem, it follows that $\mathcal{Z}$ is relatively compact, so we can extract a subsequence $\{z^{(n)}\}_{n \geq 1} \subset \mathcal{Z}$, still labeled (with an abuse of notation) by $\{z^{(n)}\}_{n \geq 1}$, such that $z^{(n)} \rightarrow z^{(\infty)}$ with respect to $\|\cdot\|_{1, \rho+2\delta}$, as $n \rightarrow \infty$. Then, we can extract a sub-subsequence, still labeled (with an abuse of notation) by $\{z^{(n)}\}_{n \geq 1}$, such that $z_s^{(n)} \rightarrow z_s^{(\infty)}$, for almost every $s \geq 0$, as $n \rightarrow \infty$. Clearly, the limit $z^{(\infty)}$ belongs to $L^1_{\rho+2\delta}$ and, since $z^{(n)} \in \mathcal{C}_{x, \infty}$, for any $n \geq 1$, we have that $\underline{\alpha}_s^{x, \infty} \leq z_s^{(\infty)} \leq \bar{\alpha}_s^{x, \infty}$, for almost every $s \geq 0$, where $\underline{\alpha}^{x, \infty}$ and $\bar{\alpha}^{x, \infty}$ are the functions defined in (4.18) and (4.19). Now we use (4.5) to get that also the derivative $\{(z^{(n)})'\}$ of the last subsequence $\{z^{(n)}\}$ converges a.e. to the function

$$w_s^{(\infty)} := e^{(\rho+2\delta)s} \int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} (z_u^{(\infty)})^{-\beta} du, \quad s \geq 0,$$

which belongs to $L^1_{\rho+2\delta}$. By standard computations, it follows that $\{(z^{(n)})'\} \subset L^1_{\rho+2\delta}$ and that

$$0 < (z_s^{(n)})' \leq \frac{x^{-\beta}}{\rho + \delta - \delta\beta} e^{\delta(1+\beta)s} =: g_s, \quad \forall s \geq 0, \forall n \geq 1,$$

with $g \in L^1_{\rho+2\delta}$. Therefore, the sequence of derivatives $\{(z^{(n)})'\}$ converges in $L^1_{\rho+2\delta}$ to $w^{(\infty)}$. We deduce that the subsequence $\{z^{(n)}\}$ converges in the weighted Sobolev space $W^{1,1}_{\rho+2\delta}$ (cf. [7, Chap. 8, Remark 4]). By completeness of this space, we get that $w_s^{(\infty)} = (z_s^{(\infty)})'$, for a.e. $s \geq 0$. Moreover, using the same argument as in the proof of [7, Theorem 8.2], we get that there exists a function $\bar{z}^{(\infty)} \in C([0, +\infty); (0, +\infty))$ such that $\bar{z}_s^{(\infty)} = z_s^{(\infty)}$, for almost every $s \geq 0$, and

$$\bar{z}_s^{(\infty)} = x + \int_0^s (z_u^{(\infty)})' du = x + \int_0^s w_u^{(\infty)} du, \quad \forall s \geq 0.$$

Substituting the expression of $w^{(\infty)}$ in the previous equality we get that

$$\bar{z}_s^{(\infty)} = x + \int_0^s e^{(\rho+2\delta)r} \int_r^{+\infty} e^{-(\rho+\delta-\delta\beta)u} (\bar{z}_u^{(\infty)})^{-\beta} du dr, \quad s \geq 0,$$

that is, $\bar{z}^{(\infty)} \in C_{x,\infty}$ and solves (4.5).

Uniqueness. As we observed at the beginning of the proof, any solution z to (4.5) is (at least) twice differentiable. Differentiating (4.5) with respect to the time variable and denoting by $'$ and $''$ first- and second-order derivatives with respect to this variable, we have that z verifies

$$\begin{cases} z_s'' = (\rho + 2\delta)z_s' - e^{\delta(1+\beta)s} z_s^{-\beta}, & s \geq 0, \\ z_0 = x. \end{cases} \quad (4.23)$$

The equation above is a second-order ODE with locally Lipschitz coefficients over $(s, z) \in [0, +\infty) \times (0, +\infty)$, to which we can associate the following initial value problem

$$\begin{cases} z_s'' = (\rho + 2\delta)z_s' - e^{\delta(1+\beta)s} z_s^{-\beta}, & s \geq 0, \\ z_0 = x, \\ z_0' = \zeta > 0. \end{cases} \quad (4.24)$$

Note that we only need to consider $\zeta > 0$, since solutions to (4.5) are such that their first derivative is strictly positive for all $s \geq 0$. For each fixed $x > 0$ and $\zeta > 0$, (4.24) has a unique solution $z^{x,\zeta}$ on the maximal interval $[0, \tau^*(x, \zeta))$, with $\tau^*(x, \zeta) \leq +\infty$.

Since any solution z to (4.5) also verifies (4.23), it also satisfies (4.24) for some $\bar{\zeta} > 0$. Therefore, if we show that for each given $x > 0$, there exists a unique $\bar{\zeta} > 0$ such that (4.24) has a unique global solution $z^{x,\bar{\zeta}} \in C_{x,\infty}$, then necessarily $\bar{\zeta} = \bar{\zeta}$ and

$z^{x,\zeta}$ must also be the unique solution to (4.5). The idea is, thus, to study, for each fixed $x > 0$, the dependence of the solution $z^{x,\zeta}$ to (4.24) on $\zeta > 0$.

Let us fix $x > 0$. Standard results (see, e.g., [14, Theorem 7.5, Chapter 1]) ensure that the solution $z^{x,\zeta}$ to (4.24) and its first- and second-order derivatives with respect to the time variable are differentiable with respect to ζ and that the order of differentiation can be exchanged.

Therefore, defining $\widehat{z}_s(\zeta) := \frac{\partial}{\partial \zeta} z_s^{x,\zeta}$, for all $s \geq 0$, we get from (4.24) that $\widehat{z}(\zeta)$ solves the initial value problem

$$\begin{cases} \widehat{z}_s''(\zeta) = (\rho + 2\delta)\widehat{z}_s'(\zeta) + \beta e^{\delta(1+\beta)s} z_s(\zeta)^{-\beta-1} \widehat{z}_s(\zeta), & s \geq 0, \\ \widehat{z}_0(\zeta) = 0, \\ \widehat{z}_0'(\zeta) = 1, \end{cases} \quad (4.25)$$

where $z_s(\zeta) := z_s^{x,\zeta}$. Let

$$s^*(\zeta) := \inf\{s \geq 0 : \widehat{z}_s'(\zeta) \leq 0\} > 0.$$

By definition $\widehat{z}_s'(\zeta) > 0$, for all $s \in [0, s^*(\zeta))$, which implies

$$\widehat{z}_s(\zeta) > 0, \quad \forall s \in [0, s^*(\zeta)).$$

Therefore, using (4.25) we get that $\widehat{z}_s''(\zeta) > 0$, for all $s \in [0, s^*(\zeta))$. It follows that

$$\widehat{z}_s'(\zeta) \geq 1, \quad \forall s \in [0, s^*(\zeta)),$$

which implies that $s^*(\zeta) = +\infty$ and, in turn, that $\widehat{z}_s(\zeta) \geq s$, for all $s \geq 0$.

Using again (4.25), we then get

$$\begin{cases} \widehat{z}_s''(\zeta) \geq (\rho + 2\delta)\widehat{z}_s'(\zeta), & s \geq 0, \\ \widehat{z}_0'(\zeta) = 1, \end{cases} \quad (4.26)$$

which, implies, by comparison,

$$\widehat{z}_s'(\zeta) \geq e^{(\rho+2\delta)s}, \quad \forall s \geq 0,$$

and, consequently,

$$\widehat{z}_s(\zeta) \geq \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

Thus, for every $a < b$, we have

$$z_s(b) - z_s(a) \geq \int_a^b \widehat{z}_s(\zeta) d\zeta \geq (b-a) \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

We know that, for some $\bar{\zeta} > 0$, there exists a global solution $z(\bar{\zeta}) \in \mathcal{C}_{x,\infty}$ to (4.24), since any solution z to (4.5) also verifies this equation. Using the estimate above and (4.20), we get, for all $b > a := \bar{\zeta}$,

$$z_s(b) \geq z_s(\bar{\zeta}) + (b - \bar{\zeta}) \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1] \geq x + (b - \bar{\zeta}) \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

This implies that $z(b)$ would grow exponentially at a rate at least $\rho + 2\delta > \delta(1 + \beta)$, which contradicts the fact that solutions in (4.20) need to grow at a rate at most $\delta(1 + \beta)$.

Similarly, we get, for all $a < \bar{\zeta} =: b$,

$$z_s(a) \leq z_s(\bar{\zeta}) - (\bar{\zeta} - a) \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1] \leq \bar{\alpha}_s^{x,\infty} - (\bar{\zeta} - a) \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

This implies that $z(a)$ becomes negative in finite time, which contradicts the fact that solutions in (4.20) are positive.

By arbitrariness of $b > \bar{\zeta}$ and $a < \bar{\zeta}$, we conclude.

Convergence. As recalled at the beginning of the proof, uniqueness of the solution to (4.5) implies uniqueness of the solution to (4.4), which is also twice continuously differentiable. Differentiating (4.4), we get the second-order ODE

$$y''_s = \rho y'_s + (\rho + \delta)\delta y_s - y_s^{-\beta}, \quad s \geq 0. \quad (4.27)$$

Hence, any solution to (4.4) is also a solution to (4.27). We observe that y_∞ is the unique solution to the algebraic equation

$$(\rho + \delta)\delta y - y^{-\beta} = 0. \quad (4.28)$$

Hence, the function $y_s = y_\infty$, for all $s \geq 0$, is the unique stationary solution to (4.4).

Now, we show monotonicity and convergence of the solution to (4.4) when $x \neq y_\infty$. We prove the case $x > y_\infty$, as the other one can be established by similar arguments.

Let y be the unique solution to (4.4) and fix $x > y_\infty$. We prove, first, that $y_s > y_\infty$, for all $s \geq 0$. Let us define

$$s_0 := \inf\{s \geq 0 : y_s = y_\infty\},$$

and assume, by contradiction, that $s_0 < +\infty$. We have two cases:

- (i) $y'_{s_0} < 0$;
- (ii) $y'_{s_0} = 0$.

Consider the first case. By the flow property, the function $\tilde{y}_s := y_{s-s_0}$, $s \geq s_0$, solves (4.4) with initial condition equal to y_∞ and it satisfies the a priori bounds given in (4.21) (with $x = y_\infty$). However, this solution is different from the constant one, which is the unique solution to (4.4) with initial condition y_∞ and satisfying the aforementioned bounds. Hence we have a contradiction.

The second case leads to a contradiction as well. Indeed, we would have two different solutions in the interval $[0, s_0]$ to the Cauchy problem (4.27), with terminal conditions $y_{s_0} = y_\infty$ and $y'_{s_0} = 0$, which clearly admits unique solution, that is, $y_s = y_\infty$, for all $s \geq 0$.

Hence, we have proved that $y_s > y_\infty$, for all $s \geq 0$. This implies, by strict monotonicity of the map $y \mapsto (\rho + \delta)\delta y - y^{-\beta}$ and (4.28), that

$$(\rho + \delta)\delta y_s - y_s^{-\beta} > (\rho + \delta)\delta y_\infty - y_\infty^{-\beta} = 0, \quad \forall s \geq 0, \quad (4.29)$$

and hence, from (4.27), that

$$y''_s > \rho y'_s, \quad \forall s \geq 0. \quad (4.30)$$

Now we show that $y'_s < 0$, for all $s \geq 0$. Let

$$s_1 := \inf\{s \geq 0 : y'_s \geq 0\}.$$

Assume, by contradiction, that $s_1 < +\infty$. Then, from (4.30) we get that $y''_{s_1} > 0$. It follows that there exist $\eta, \varepsilon > 0$ such that $y'_{s_1+\varepsilon} = \eta > 0$. Using (4.30) again, we get

$$\begin{cases} y''_s > \rho y'_s, & \forall s \geq s_1 + \varepsilon, \\ y'_{s_1+\varepsilon} = \eta > 0. \end{cases} \quad (4.31)$$

Therefore, solving the corresponding Cauchy problem and using the comparison, we get

$$y'_s \geq \eta e^{\rho[s-(s_1+\varepsilon)]} > 0, \quad s \geq s_1 + \varepsilon. \quad (4.32)$$

Next, differentiating (4.27), we get that $y \in C^3([0, +\infty); (0, +\infty))$ and that it satisfies

$$y'''_s = \rho y''_s + (\rho + \delta)\delta y'_s + \beta y_s^{-\beta-1} y'_s, \quad s \geq 0. \quad (4.33)$$

Since, by (4.32), we know that $y'_s > 0$ for all $s \geq s_1 + \varepsilon$, we deduce that

$$\begin{cases} y'''_s > \rho y''_s + (\rho + \delta)\delta y'_s, & \forall s \geq s_1 + \varepsilon, \\ y'_{s_1+\varepsilon} = \eta, \\ y''_{s_1+\varepsilon} = \kappa, \end{cases}$$

with $\eta > 0$ and $\kappa > \eta\rho > 0$, by (4.31). Solving the corresponding Cauchy problem and using the comparison, we get

$$y'_s > \frac{\eta\delta + \kappa}{\rho + 2\delta} e^{(\rho+\delta)[s-(s_1+\varepsilon)]} + \frac{\eta(\rho + \delta) - \kappa}{\rho + 2\delta} e^{-\delta[s-(s_1+\varepsilon)]}, \quad \forall s \geq s_1 + \varepsilon.$$

This implies that y grows, in the long run, at least with rate $\rho + \delta$ which is strictly bigger than $\delta\beta$, as prescribed by the admissibility condition in (4.21). The contradiction follows, and hence we proved that $y'_s < 0$, for all $s \geq 0$.

We are now going to show that $y''_s > 0$, for all $s \geq 0$. Since we proved that $y'_s < 0$, for all $s \geq 0$, we deduce from (4.33) that

$$y'''_s < \rho y''_s, \quad \forall s \geq 0. \quad (4.34)$$

Assume by contradiction that $y''_{s_2} \leq 0$ for some $s_2 \geq 0$. Then, from (4.34) we get that $y'''_{s_2} < 0$. It follows that there exist $\theta < 0$, $\tau > 0$ such that $y''_{s_2+\tau} = \theta < 0$. Using again (4.34), we get

$$\begin{cases} y'''_s < \rho y''_s, & \forall s \geq s_2 + \tau, \\ y''_{s_2+\tau} = \theta. \end{cases}$$

Therefore, $y''_s \leq \theta e^{\rho s} < 0$, for all $s \geq s_2 + \tau$. Considering that $y'_s < 0$, for all $s \geq 0$, this implies that the graph of $s \mapsto y_s$ lies below a straight line with strictly negative slope, contradicting the fact that $y_s > y_\infty$, for all $s \geq 0$, as previously established.

So, we have proved that, for all $s \geq 0$,

$$y_s > y_\infty, \quad y'_s < 0, \quad y''_s > 0.$$

This implies that there exists $\bar{y} := \lim_{s \rightarrow \infty} y_s \geq y_\infty$ and that $\lim_{s \rightarrow \infty} y'_s = 0$. However, it cannot be that $\bar{y} > y_\infty$, as it would imply, using (4.27), that $\lim_{s \rightarrow \infty} y''_s > 0$, contradicting the fact that $y'_s \rightarrow 0$, as $s \rightarrow +\infty$. Therefore, $\bar{y} = y_\infty$. $\square$

We are now ready to establish existence and uniqueness of an equilibrium for the mean-field game.

Theorem 4.12 *For each fixed random variable ξ satisfying Assumption 2.1, there exists a unique equilibrium $(\hat{u}, \hat{q})$ for the mean-field game with infinite time horizon, among all equilibria $(\tilde{u}, \tilde{q}) \in \mathcal{U}_\infty \times \mathcal{Q}_\infty$.*

More precisely, $\hat{u} = z^{(\infty, \hat{q})}$, where $z^{(\infty, q)}$ is the function defined in (3.9), and $\hat{q}$ is the unique solution to (4.4), with initial condition $x = \mathbb{E}[\xi]$.

Proof Let $\hat{q}$ be the unique solution to (4.4), with $x = \mathbb{E}[\xi]$. We need to check that $\hat{q} \in \mathcal{Q}_\infty$. We have the following three cases.

Case $x = y_\infty$. From Theorem 4.11, we know that $\hat{q}_s = y_\infty$, for all $s \geq 0$. Therefore,

$$z^{(\infty, \hat{q})}_s = K := \frac{y_\infty^{-\beta}}{\rho + \delta}, \quad \forall s \geq 0, \quad (4.35)$$

and hence $\hat{q} \in \mathcal{Q}_\infty$.

Case $x > y_\infty$. From Theorem 4.11, we know that $\hat{q}$ is monotone decreasing and converges to y_∞ . This implies that, $0 < y_\infty < \hat{q}_s \leq x$, for all $s \geq 0$, and that $z^{(\infty, \hat{q})}$ is positive and bounded above by the constant K defined in (4.35). Therefore, $\hat{q} \in \mathcal{Q}_\infty$.

Case $x < y_\infty$. From Theorem 4.11, we know that $\hat{q}$ is monotone increasing and converges to y_∞ . This implies that, $0 < x \leq \hat{q}_s < y_\infty$, for all $s \geq 0$, and that $z^{(\infty, \hat{q})}$ is positive and bounded above by $\frac{x-\beta}{\rho+\delta}$. Therefore, $\hat{q} \in \mathcal{Q}_\infty$.

Applying Theorem 4.1-(iv) we get the result. $\square$

A crucial question regarding the robustness of the mean-field equilibrium is whether the aggregate behavior of the agents remains consistent when passing from a large but finite horizon to an infinite one. Thanks to the characterization of the equilibrium provided by Theorem 4.1 and the analytical results obtained in Theorem 4.11, we can formally establish that the finite-horizon equilibrium $\hat{q}^{(T)}$ converges to the unique infinite-horizon equilibrium $\hat{q}^{(\infty)}$. This result, summarized in the following corollary, ensures the stability of the model's predictions with respect to the time horizon T and justifies the use of the infinite-horizon case as a structural limit for long-term strategic interactions.

Corollary 4.13 (Convergence of the Mean-Field Equilibrium) *For each $T > 0$, let $(\hat{u}^{(T)}, \hat{q}^{(T)})$ be the unique equilibrium of the mean-field game with finite horizon T , where $\hat{q}^{(T)}$ is the unique solution to (4.4). Under the assumptions of Theorem 4.11, as $T \rightarrow +\infty$, the sequence of equilibrium trajectories $\{\hat{q}^{(T)}\}_{T>0}$ converges to the unique infinite-horizon equilibrium $\hat{q}^{(\infty)}$ in the weighted space $L^1_{\rho+2\delta}([0, \infty); \mathbb{R})$. Consequently, the optimal strategies $\hat{u}^{(T)}$ converge to the infinite-horizon optimal strategy $\hat{u}^{(\infty)}$ in the same sense.*

Proof The proof of Theorem 4.11 shows that the family of transformed solutions $\mathcal{Z} := \{z^{(T)}\}_{T>0}$ is relatively compact in $L^1_{\rho+2\delta}$ by means of the Fréchet-Kolmogorov theorem. Let $t_n \rightarrow \infty$ be an arbitrary sequence, consider the corresponding sequence $\{z^{(t_n)}\}$, and take any subsequence $\{z^{(t_{n_k})}\}$. By the existence part of the proof of Theorem 4.11, its limit must solve the infinite-horizon integro-differential equation. Since the uniqueness part of the same Theorem guarantees that such a solution is unique within the admissible set $\mathcal{C}_{x,\infty}$, we conclude that $z^{(t_{n_k})} \rightarrow z^{(\infty)}$. Since this holds for an arbitrary sequence $t_n \rightarrow \infty$, we conclude that the entire family $\mathcal{Z}$ converges to $z^{(\infty)}$ as $T \rightarrow \infty$. By the continuous mapping between the auxiliary function z and the equilibrium trajectory $\hat{q}$ (and the corresponding optimal control $\hat{u}$), the convergence of the equilibrium follows. $\square$

Remark 4.14 Also in this case the results of this section can be easily extended to the case where we consider an initial time $t > 0$.

Clearly, we need to adapt (in an obvious way) the definition of equilibrium given in Definition 2.3 to include the initial time t . The statement of Theorem 4.1 is also adapted accordingly.

The integro-differential Eq. (4.4) becomes

$$\begin{cases} \frac{d}{ds} y_s = -\delta y_s + \int_s^T e^{-(\rho+\delta)(u-s)} y_u^{-\beta} du, & s \in [t, T], \\ y_t = x, \end{cases} \quad (4.36)$$

while Eq. (4.5) becomes

$$\begin{cases} \frac{d}{ds} z_s = e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du, & s \in [t, T], \\ z_t = e^{\delta t} x. \end{cases} \quad (4.37)$$

The sets in which we look for solutions to (4.37) (now dependent on t) are

$$\mathcal{C}_{t,x} := \{f \in C([t, T]; (0, +\infty)) : e^{\delta t} \underline{\alpha}_s^x \leq f_s \leq e^{\delta t} \overline{\alpha}_s^x, \forall s \in [t, T]\}, \quad (4.38)$$

in the finite time horizon case, and

$$\mathcal{C}_{t,x,\infty} := \{f \in C([t, +\infty); (0, +\infty)) : e^{\delta t} \underline{\alpha}_s^{x,\infty} \leq f_s \leq e^{\delta t} \overline{\alpha}_s^{x,\infty}, \forall s \geq t\}, \quad (4.39)$$

in the infinite time horizon case.

The statements of Theorems 4.8, 4.9, 4.11, and 4.12, remain the same (except for minor modifications). The bounds given in (4.12) become

$$e^{-\delta(s-t)} x \leq y_s \leq e^{-\delta(s-t)} \overline{\alpha}_s^x, \quad s \in [t, T], \quad (4.40)$$

while those given in (4.21) become

$$\begin{aligned} e^{-\delta(s-t)} x \leq y_s \leq e^{-\delta(s-t)} & \left[x - \frac{x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)} \right] \\ & + \frac{e^{\delta\beta(s-t)} x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)}, \quad s \geq t. \end{aligned} \quad (4.41)$$

4.2.1 Economic Interpretation of the Stationary Equilibrium

The explicit nature of the stationary solution in the infinite horizon problem allows for a clear economic characterization of the long-term market equilibrium. In this setting, the unique stationary average production level y_∞ is the positive solution to the algebraic equation $(\rho + \delta)\delta y - y^{-\beta} = 0$ (cf. (4.28)), given by:

$$y_\infty = \left[\frac{1}{\delta(\rho + \delta)} \right]^{\frac{1}{1+\beta}}.$$

Some comments on the economics of this result are in place.

Sensitivity to parameters. The term $\delta(\rho + \delta)$ can be interpreted as the *effective user cost of capital*. It combines the physical depreciation rate δ with the opportunity cost of capital ρ . As either the discount rate ρ or the depreciation rate δ increases, the stationary production capacity y_∞ decreases. This reflects the intuition that firms are less willing to maintain high production levels when the future is heavily discounted or when maintaining capacity requires more intensive reinvestment. Furthermore, the

parameter β represents the inverse of the price elasticity of demand. A higher β (less elastic demand) leads to a lower y_∞ , as firms strategically limit capacity to avoid the sharp price decreases associated with high aggregate supply in inelastic markets.

Market stability and convergence. The monotone convergence result of Theorem 4.11 provides further economic insight. If the initial production level exceeds y_∞ , the equilibrium path decreases monotonically toward the steady state; if it starts below y_∞ , it increases monotonically. Thus, the stationary equilibrium is globally stable and dynamically attractive. Deviations from the steady state are gradually absorbed through investment adjustments without oscillations or overshooting, suggesting that competitive investment incentives dampen imbalances rather than amplify them. Notably, the stationary level y_∞ and the trajectory of the average production $\hat{q}$ are independent of the volatility parameter σ . This suggests that while individual firms face stochastic fluctuations, the "mean-field" or aggregate economy follows a deterministic and stable path toward equilibrium. We refer also to Sect. 5 below on this.

5 The Deterministic Model

In this section we discuss the deterministic version of the mean-field game problem introduced in Sect. 2. The structure of the problem will allow us to exploit the results given in Sects. 3 and 4, to get the unique equilibrium of the mean-field game in an explicit form. Indeed, the results obtained for the stochastic model reveal that the average production capacity and the optimal control at equilibrium do not depend on the parameter σ appearing in SDE (2.1), which describes the sensitivity of the production capacity with respect to random shocks. Thus, by formulating a deterministic counterpart of the problem studied in the previous sections, i.e., formally setting $\sigma = 0$ in the dynamics of the production capacity, we are able to show that results similar to those proved in Sects. 3 and 4 are valid also in this context.

We emphasize that no absolute continuity assumption on the initial distribution is needed in the deterministic model. The evolution of the population distribution will be described directly through the pushforward of the initial law by the deterministic flow. This formulation naturally includes, in particular, Dirac initial distributions.

Let us consider the ordinary differential equation (ODE)

$$\begin{cases} \frac{d}{ds} X_s = -\delta X_s + u_s, & s \in (0, T], \\ X_0 = x, \end{cases} \quad (5.1)$$

where $\delta > 0$ is a given coefficient and $\mathbf{u} := (u_s)_{s \in [0, T]}$ is the control, chosen in a suitable class of admissible non-negative functions that will be specified below.

Note that, for any $x \in \mathbb{R}$ and any $\mathbf{u} \in L^1_{\text{loc}}([0, T]; [0, +\infty))$, Eq. (5.1) has a unique solution $X^{x, \mathbf{u}}$, given by

$$X^{x, \mathbf{u}}_s = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} u_r \, dr, \quad s \in [0, T]. \quad (5.2)$$

As in the stochastic case, X describes the evolution of the production capacity of a representative agent, which depreciates at a rate δ and can be increased by choosing the investment rate $\mathbf{u}$. Also in the deterministic case we are in an irreversible investment decisions setting, given the non-negativity of the control.

Next, we consider the discounted net profit functional

$$J_{T,q}(x, \mathbf{u}) := \int_0^T e^{-\rho s} \left\{ X_s^{x,\mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds, \quad (5.3)$$

where $\rho > 0$ is a discount factor, $\beta > 0$ is a fixed parameter, and $q = (q_s)_{s \in [0, T]}$ is a given deterministic measurable function. The control $\mathbf{u}$ is chosen in either of the following two classes of admissible controls: if $T < +\infty$,

$$\mathcal{U}_T := \left\{ \mathbf{u}: [0, T] \rightarrow [0, +\infty) \text{ measurable and such that } \int_0^T u_s^2 ds < +\infty \right\}; \quad (5.4)$$

if, instead, $T = +\infty$,

$$\mathcal{U}_\infty := \left\{ \mathbf{u}: [0, \infty) \rightarrow [0, +\infty) \text{ measurable and such that } \int_0^s u_r dr < +\infty, \forall s \geq 0, \right. \\ \left. \text{and } \int_0^{+\infty} e^{-\rho s} u_s^2 ds < +\infty \right\}. \quad (5.5)$$

From an economic point of view, the deterministic model captures an economy in which the production capacity of the representative firm is not affected by random shocks. Its evolution over time is, thus, uniquely determined by the investment efforts to increase it.

We introduce the following assumption.

Assumption 5.1 *The initial distribution v_0 of the agents in the economy belongs to $\mathcal{P}_1((0, +\infty))$, namely,*

$$v_0((0, +\infty)) = 1, \quad \int_0^{+\infty} x v_0(dx) < +\infty.$$

Here and in what follows, $\mathcal{P}(E)$ denotes the set of probability measures on a Borel set $E \subseteq \mathbb{R}$, and $\mathcal{P}_1(E)$ denotes the subset of probability measures with finite first moment. We identify $v_0 \in \mathcal{P}_1((0, +\infty))$ with its extension to $\mathbb{R}$ concentrated on $(0, +\infty)$. If $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ is Borel-measurable and $\nu \in \mathcal{P}(\mathbb{R})$, we denote by $\Phi_\# \nu$ the pushforward of ν through Φ , i.e., the measure given, for any Borel set $E \subseteq \mathbb{R}$, by

$$\Phi_\# \nu(E) := \int_{\mathbb{R}} \mathbf{1}_E(\Phi(x)) \nu(dx) = \nu(\Phi^{-1}(E)).$$

Since the representative agent is chosen randomly by picking her/his initial production capacity level x according to the initial distribution ν_0 , Assumption 5.1 ensures that, for any $\mathbf{u} \in \mathcal{U}_T$, $X_s^{x,\mathbf{u}} > 0$, for all $s \in [0, T]$ and for ν_0 -a.e. $x > 0$. Thus, the production capacity level of the representative agent is never negative.

To describe the evolution of the population distribution, we consider the flow associated with (5.1) on the whole real line. This avoids introducing any boundary condition at zero; since the initial distribution is concentrated on $(0, +\infty)$ and the control is non-negative, the resulting law remains concentrated on $(0, +\infty)$. For any $\mathbf{u} \in \mathcal{U}_T$, set

$$\Phi_s^{\mathbf{u}}(x) := e^{-\delta s}x + \int_0^s e^{-\delta(s-r)}u_r \, dr, \quad (s, x) \in [0, T] \times \mathbb{R}. \quad (5.6)$$

Then the population distribution at time s is defined by

$$\nu_s^{\nu_0, \mathbf{u}} := (\Phi_s^{\mathbf{u}})_\# \nu_0, \quad s \in [0, T]. \quad (5.7)$$

Equivalently, for every bounded Borel-measurable function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(x) \nu_s^{\nu_0, \mathbf{u}}(dx) = \int_{\mathbb{R}} \varphi(\Phi_s^{\mathbf{u}}(x)) \nu_0(dx). \quad (5.8)$$

5.1 The Continuity Equation

Formally, the equation associated to the flow of measures above, i.e. the so called *continuity equation* associated with (5.1), is the measure valued PDE

$$\partial_t \nu_t + \frac{\partial}{\partial x} \left([-\delta x + u_s] \nu_s \right) = 0. \quad (5.9)$$

Definition 5.2 A narrowly continuous family $\nu = (\nu_s)_{s \in [0, T]}$ in $\mathcal{P}_1(\mathbb{R})$ is called a *weak measure-valued solution* to the continuity Eq. (5.9) with initial condition ν_0 if

$$\begin{aligned} & \int_{\mathbb{R}} \phi(t, x) \nu_t(dx) \\ &= \int_{\mathbb{R}} \phi(0, x) \nu_0(dx) + \int_0^t \int_{\mathbb{R}} \left\{ \frac{\partial}{\partial s} \phi(s, x) + [-\delta x + u_s] \frac{\partial}{\partial x} \phi(s, x) \right\} \nu_s(dx) \, ds, \end{aligned} \quad (5.10)$$

for every $\phi \in C_c^\infty([0, T] \times \mathbb{R})$ and every $t \in [0, T]$.

The family $\nu^{\nu_0, \mathbf{u}} = (\nu_s^{\nu_0, \mathbf{u}})_{s \in [0, T]}$ is the measure-valued solution to the continuity equation associated with (5.1). The next lemma records this fact and the corresponding first-moment identity.

Lemma 5.3 For any $\mathbf{u} \in \mathcal{U}_T$ and any $\nu_0 \in \mathcal{P}_1((0, +\infty))$, the family $\nu^{\nu_0, \mathbf{u}}$ defined by (5.7) is the unique weak measure-valued solution to the continuity equation

associated with (5.1), in the sense of Definition 5.2. More precisely, for every $\phi \in C_c^\infty([0, T] \times \mathbb{R})$ and every $t \in [0, T]$,

$$\begin{aligned} & \int_{\mathbb{R}} \phi(t, x) \nu_t^{v_0, \mathbf{u}}(dx) \\ &= \int_{\mathbb{R}} \phi(0, x) \nu_0(dx) + \int_0^t \int_{\mathbb{R}} \left\{ \frac{\partial}{\partial s} \phi(s, x) + [-\delta x + u_s] \frac{\partial}{\partial x} \phi(s, x) \right\} \nu_s^{v_0, \mathbf{u}}(dx) ds. \end{aligned} \quad (5.11)$$

Moreover, $\nu_s^{v_0, \mathbf{u}}((0, +\infty)) = 1$ for every $s \in [0, T]$, and $\nu_s^{v_0, \mathbf{u}}$ has finite first moment, given by

$$\int_0^{+\infty} x \nu_s^{v_0, \mathbf{u}}(dx) = e^{-\delta s} \int_0^{+\infty} x \nu_0(dx) + \int_0^s e^{-\delta(s-r)} u_r dr, \quad s \in [0, T]. \quad (5.12)$$

Proof Fix $\mathbf{u} \in \mathcal{U}_T$ and $\nu_0 \in \mathcal{P}_1((0, +\infty))$. For every $x \in \mathbb{R}$, the map $s \mapsto \Phi_s^{\mathbf{u}}(x)$ is absolutely continuous and solves (5.1). Hence, for every $\phi \in C_c^\infty([0, T] \times \mathbb{R})$ and every $x \in \mathbb{R}$, the chain rule gives, for Lebesgue-a.e. $s \in [0, T]$,

$$\frac{d}{ds} \phi(s, \Phi_s^{\mathbf{u}}(x)) = \partial_s \phi(s, \Phi_s^{\mathbf{u}}(x)) + [-\delta \Phi_s^{\mathbf{u}}(x) + u_s] \partial_x \phi(s, \Phi_s^{\mathbf{u}}(x)).$$

The narrow continuity of $s \mapsto \nu_s^{v_0, \mathbf{u}}$ follows from (5.8), the continuity of $s \mapsto \Phi_s^{\mathbf{u}}(x)$, and dominated convergence. Integrating the previous identity over $[0, t]$ and then with respect to $\nu_0(dx)$, and using the definition of pushforward, yields (5.10), hence (5.11). Since $\Phi_s^{\mathbf{u}}(x) > 0$ whenever $x > 0$, we also have $\nu_s^{v_0, \mathbf{u}}((0, +\infty)) = 1$.

Conversely, uniqueness follows from the uniqueness of the characteristics associated with the vector field $(s, x) \mapsto -\delta x + u_s$ on $\mathbb{R}$ and the standard characterization of measure-valued solutions to the continuity equation by pushforward along the flow. In the present one-dimensional affine case this characterization follows directly from the explicit formula (5.6).

Finally, applying (5.8) with $\varphi(x) = x$ and using (5.6), we obtain

$$\begin{aligned} \int_{\mathbb{R}} x \nu_s^{v_0, \mathbf{u}}(dx) &= \int_{\mathbb{R}} \Phi_s^{\mathbf{u}}(x) \nu_0(dx) \\ &= e^{-\delta s} \int_{\mathbb{R}} x \nu_0(dx) + \int_0^s e^{-\delta(s-r)} u_r dr, \end{aligned}$$

which is finite thanks to Assumption 5.1 and the admissibility of $\mathbf{u}$. $\square$

5.2 Mean Filed Equilibrium

We are now ready to state the definition of equilibrium for our deterministic mean-field game.

Definition 5.4 Fix an initial distribution ν_0 under Assumption 5.1.

A pair $(\hat{\mathbf{u}}, \hat{q})$, where $\hat{\mathbf{u}} \in \mathcal{U}_T$ and $\hat{q}: [0, T] \rightarrow (0, +\infty)$ is a measurable function, is an *equilibrium* of the deterministic mean-field game if

- (i) $J_{T,\hat{q}}(x, \hat{\mathbf{u}}) \geq J_{T,\hat{q}}(x, \mathbf{u})$, for all $\mathbf{u} \in \mathcal{U}_T$ and ν_0 -a.e. $x > 0$;
- (ii)

$$\hat{q}_s = \int_0^{+\infty} x \nu_s^{\nu_0, \hat{\mathbf{u}}}(\mathrm{d}x), \quad s \in [0, T],$$

where $\nu^{\nu_0, \hat{\mathbf{u}}}$ is the pushforward flow of measures defined by (5.7).

Also in this case, we can adopt a fixed point argument to show that there exists a unique equilibrium, by solving, first, the problem of maximizing (5.3) for a given measurable function q , then determining the optimally controlled dynamics of the production capacity level of the representative agent and, finally, setting the fixed point argument from condition (ii) in Definition 5.4.

5.2.1 The Optimization Problem

Let us consider the optimization problem

$$V_{T,q}(x) := \sup_{\mathbf{u} \in \mathcal{U}_T} J_{T,q}(x, \mathbf{u}), \quad x > 0, \quad (5.13)$$

where $J_{T,q}$ is the discounted net profit functional defined in (5.3), for any given and fixed measurable deterministic function q .

In the finite time horizon case, i.e. $T < +\infty$, the following result holds, which is analogous to Proposition 3.3.

Proposition 5.5 Fix $x > 0$ and $q \in \mathcal{Q}_T$, where $\mathcal{Q}_T$ is the set defined in (3.1). Then, $\hat{\mathbf{u}}^{(T,q)} := z^{(T,q)} \in \mathcal{U}_T$, where $z^{(T,q)}$ is the function defined in (3.5), is an optimal control for problem (5.13).

Moreover, $\hat{\mathbf{u}}^{(T,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(T,q)} \in \mathcal{U}_T$ is an optimal control for problem (5.13) different from $\hat{\mathbf{u}}^{(T,q)}$, then

$$\bar{u}_s^{(T,q)} = \hat{u}_s^{(T,q)}, \quad \text{for Leb-a.e. } s \in [0, T].$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{T,q}(x) = x z_0^{(T,q)} + \frac{1}{2} \int_0^T e^{-\rho s} (z_s^{(T,q)})^2 \mathrm{d}s, \quad (5.14)$$

and the optimally controlled state $X^{x, \hat{\mathbf{u}}^{(T,q)}}$ is given by

$$X_s^{x, \hat{\mathbf{u}}^{(T,q)}} = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} z_r^{(T,q)} \mathrm{d}r, \quad s \in [0, T]. \quad (5.15)$$

Proof Replicating the argument of the proof of Lemma 3.1, we get that

$$J_{T,q}(x, \mathbf{u}) = xz_0^{(T,q)} + \int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds.$$

Then, the result follows using the same reasoning as in the proof of Proposition 3.3. $\square$

In the infinite time horizon case, i.e. $T = +\infty$, we have the following statement, which is the deterministic counterpart of Proposition 3.6. We state it without proof.

Proposition 5.6 Fix $x > 0$ and $q \in \mathcal{Q}_\infty$, where $\mathcal{Q}_\infty$ is the set defined in (3.10). Then, $\widehat{\mathbf{u}}^{(\infty,q)} := z^{(\infty,q)} \in \mathcal{U}_\infty$, where $z^{(\infty,q)}$ is the function defined in (3.9), is an optimal control for problem (5.13). Moreover, $\widehat{\mathbf{u}}^{(\infty,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(\infty,q)} \in \mathcal{U}_\infty$ is an optimal control for problem (5.13) different from $\widehat{\mathbf{u}}^{(\infty,q)}$, then

$$\bar{u}_s^{(\infty,q)} = \widehat{u}_s^{(\infty,q)}, \quad \text{for Leb-a.e. } s \geq 0.$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{\infty,q}(x) = xz_0^{(\infty,q)} + \frac{1}{2} \int_0^{+\infty} e^{-\rho s} (z_s^{(\infty,q)})^2 ds, \quad (5.16)$$

and the optimally controlled state $X^{x,\widehat{\mathbf{u}}^{(\infty,q)}}$ is given by

$$X_s^{x,\widehat{\mathbf{u}}^{(\infty,q)}} = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} z_r^{(\infty,q)} dr, \quad s \geq 0. \quad (5.17)$$

5.2.2 Existence and Uniqueness of Equilibria

Also in the deterministic mean-field game previously introduced, the search for an equilibrium boils down to finding a fixed point of a suitable map. According to Definition 5.4, this map is

$$(q_s)_{s \in [0,T]} \mapsto \left(\int_0^{+\infty} x \nu_s^{v_0, \widehat{\mathbf{u}}^{(T,q)}}(dx) \right)_{s \in [0,T]},$$

where $\widehat{\mathbf{u}}^{(T,q)}$ is the optimal control for the optimization problem (5.13) – whose expression is given either in Proposition 5.5, in the case $T < +\infty$, or in Proposition 5.6, in the case $T = +\infty$ – and $\nu^{v_0, \widehat{\mathbf{u}}^{(T,q)}}$ is the pushforward flow of measures defined by (5.7).

The next result, which is analogous to Theorem 4.1, shows that also in the deterministic case the fixed point map is the solution map of the same integral equation studied in Sect. 4. We omit its proof, which is a straightforward adaptation of the proof of Theorem 4.1; the only change is that the first moment of the population law is now computed through (5.12).

Theorem 5.7 *Let us fix an initial distribution v_0 satisfying Assumption 5.1, and set*

$$\bar{x}_0 := \int_0^{+\infty} x v_0(dx).$$

Consider the deterministic mean-field game in the finite time horizon case, i.e., $T < +\infty$. Then,

- (i) *If there exists an equilibrium $(\hat{u}, \hat{q})$ in the sense of Definition 5.4, such that $\hat{q} \in \mathcal{Q}_T$, then $\hat{q}$ is a solution to the integral equation*

$$y_s = e^{-\delta s} \bar{x}_0 + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T]. \quad (5.18)$$

- (ii) *Vice versa, if there exists a unique solution $\hat{q} \in \mathcal{Q}_T$ to (5.18), then there exists a unique equilibrium $(\hat{u}, \hat{q}) = (z^{(T, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\tilde{u}, \tilde{q}) \in \mathcal{U}_T \times \mathcal{Q}_T$, where $z^{(T, q)}$ is the function defined in (3.5).*

Consider, instead, the deterministic mean-field game in the infinite time horizon case, i.e., $T = +\infty$. Then,

- (iii) *If there exists an equilibrium $(\hat{u}, \hat{q})$ in the sense of Definition 5.4, such that $\hat{q} \in \mathcal{Q}_\infty$, then $\hat{q}$ is a solution to the integral equation*

$$y_s = e^{-\delta s} \bar{x}_0 + \int_0^s e^{-\delta(s-r)} \int_r^{+\infty} e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \geq 0. \quad (5.19)$$

- (iv) *Vice versa, if there exists a unique solution $\hat{q} \in \mathcal{Q}_\infty$ to (5.19), then there exists a unique equilibrium $(\hat{u}, \hat{q}) = (z^{(\infty, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\tilde{u}, \tilde{q}) \in \mathcal{U}_\infty \times \mathcal{Q}_\infty$, where $z^{(\infty, q)}$ is the function defined in (3.9).*

Finally, we get the following result, which establishes existence and uniqueness of an equilibrium also for the deterministic mean-field game. We omit its proof, since it follows the same lines of the proofs of Theorems 4.9 and 4.12.

Theorem 5.8 *For each fixed v_0 satisfying Assumption 5.1, there exists a unique equilibrium $(\hat{u}, \hat{q})$ for the deterministic mean-field game with finite time horizon, respectively infinite time horizon, among all equilibria $(\tilde{u}, \tilde{q}) \in \mathcal{U}_T \times \mathcal{Q}_T$, respectively $(\tilde{u}, \tilde{q}) \in \mathcal{U}_\infty \times \mathcal{Q}_\infty$.*

More precisely, in the finite time horizon case, $\hat{u} = z^{(T, \hat{q})}$, where $z^{(T, \hat{q})}$ is the function defined in (3.5); in the infinite time horizon case, $\hat{u} = z^{(\infty, \hat{q})}$, where $z^{(\infty, \hat{q})}$ is the function defined in (3.9). In both cases, $\hat{q}$ is the unique solution to (4.4), with initial condition

$$x = \int_0^{+\infty} y v_0(dy).$$

In particular, the result applies to Dirac initial distributions $v_0 = \delta_x$, for every $x > 0$.

Remark 5.9 We observe that the equilibrium structure derived in the stochastic setting is invariant with respect to the diffusion parameter σ . This ensures a consistent transition to the deterministic case discussed in the present section. More precisely, for a fixed initial condition and along the same equilibrium control, the explicit representation of the optimally controlled state yields $\widehat{X}^\sigma \rightarrow \widehat{X}^0$ locally uniformly in time, in probability, as $\sigma \downarrow 0$; in fact, the convergence holds pathwise on compact time intervals along Brownian sample paths. The equilibrium mean production capacity $\widehat{q}$ and the equilibrium investment rate $\widehat{u}$ are therefore the same objects obtained in the deterministic formulation.

6 Numerical Experiments

In this section we discuss some numerical simulations based on our results, to visualize the behaviour of the equilibrium average production capacity and of the optimal investment strategy at equilibrium, and at the same time highlighting the differences between the finite and infinite time horizon cases.

We consider the following values for the model parameters: $\beta = 2$, $\delta = 0.01$, $\rho = 0.03$. As stressed in the previous sections, the equilibrium average production capacity and the optimal investment strategy do not depend on volatility σ appearing in SDE (2.1). Nonetheless, we are going to discuss the effect of such parameter on the equilibrium production capacity.

6.1 The Finite Time Horizon Case

We consider three different time horizons, namely, $T = 3, 30, 300$. Thus, we compare the dynamics at equilibrium of the average production capacity $\widehat{q}$, given in (4.1), and of the optimal investment strategy $\widehat{u}$, which coincides with $z^{T, \widehat{q}}$ in (3.5), on a short, medium, and long time horizon.

We anticipate here that the longer horizon $T = 300$ is the same as the one used in the infinite-time horizon case, where it is used to truncate the time domain; this will allow us to compare the behaviour of $\widehat{q}$ and of $\widehat{u}$ between the two cases. The specific choice of $T = 300$ will be motivated in Sect. 6.2 below.

First, for each of the three time horizons we solve numerically the integro-differential Eq. (4.5), whose solution z is used to compute the equilibrium average production capacity $\widehat{q}$, via the transformation $\widehat{q}_s = e^{-\delta s} z_s$, $s \in [0, T]$ (cf. Lemma 4.5); finally, we integrate numerically the expression appearing in (3.5) to compute the equilibrium optimal investment strategy $\widehat{u}$.

In Fig. 1, we plot the three functions computed above for each of the three time horizons, choosing as initial average production capacity $x = 10$; the graphs on each column show a comparison between the short, medium, and long time horizons.

We observe that in the short time horizon case, the capital deterioration effect dominates the dynamics of $\widehat{q}$ and the investment level is relatively small. In the medium time horizon case, the investment effect predominates in the first half of the time interval, but then, as the investment effort decreases, the capital deterioration effect

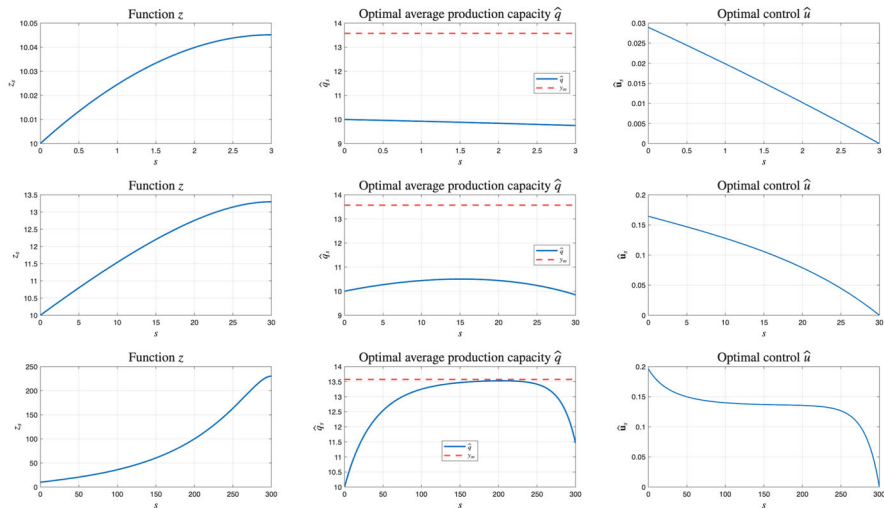


Fig. 1 The solution z to the integro-differential Eq. (4.5), the equilibrium average production capacity $\hat{q}$, and the equilibrium optimal investment strategy $\hat{u}$, in the short, medium, and long time horizon cases. Initial condition $x = 10$

takes over. In the long time horizon case, the equilibrium average production capacity is pushed towards the infinite-time horizon steady state y_∞ (cf. Theorem 4.11) by the investment effort, which is almost kept constant for a large portion of the time interval. However, we observe that, since the optimal control $\hat{u}$ is decreasing and verifies $\hat{u}_T = 0$, the capital deterioration effect returns to dominate in the last part of the time interval also in the long horizon case.

We compare these graphs with those shown in Fig. 2, where we compute the same three functions, starting at an initial production capacity $x = y_\infty$, i.e., at the steady state of the infinite time horizon case.

In this case, as the time horizon gets larger the capital deterioration effect is delayed by a higher investment level in the first part of the time interval. Nonetheless, the condition $\hat{u}_T = 0$ implies that in the end the capital deterioration effect prevails. We observe a similar effect also in the case in which the initial production capacity x is higher than y_∞ , as shown in Fig. 3, where we set $x = 15$. Note that after a sufficiently large time, the average production capacity falls below the level y_∞ . We emphasize that in this case that the behaviour of the average production capacity and of the optimal control at equilibrium is different from the previous two cases, in which we observe that $\hat{q}$ is concave and $\hat{u}$ is decreasing. In general, the properties of these two functions depend on the parameters of the model, but it is not clear how they determine the shape of $\hat{q}$ and $\hat{u}$. This interesting question is, thus, left for future research.

Figures 1, 2, and 3 clearly suggest that the finite time horizon $\hat{q}$ and $\hat{u}$ converge to the corresponding functions in the infinite time horizon case. However, a different numerical method needs to be used in the latter setting. Indeed, a simple truncation at time T of the integro-differential Eq. (4.5), which is used to compute, in turn, $\hat{q}$ and $\hat{u}$, would imply that $z'_T = 0$ (as it is possible to see in both Figs. 1 and 2). Hence, even

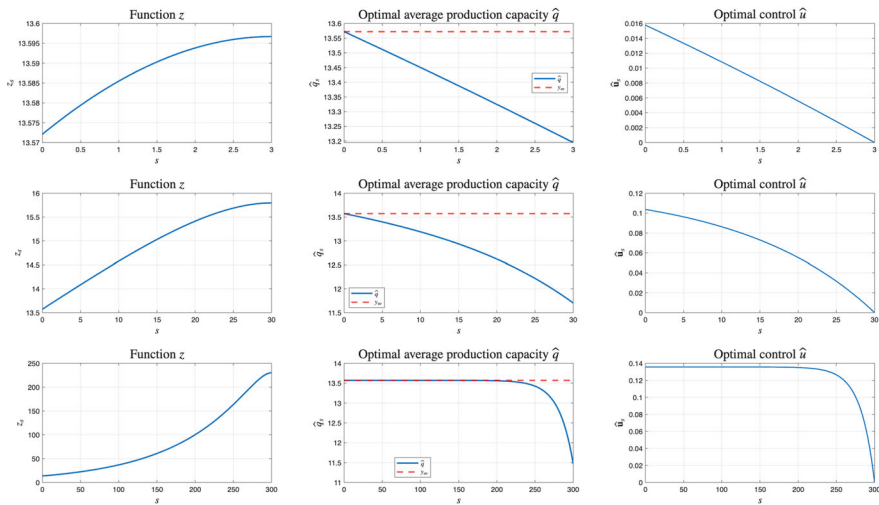


Fig. 2 The solution z to the integro-differential Eq. (4.5), the equilibrium average production capacity $\hat{q}$, and the equilibrium optimal investment strategy $\hat{u}$, in the short, medium, and long time horizon cases. Initial condition $x = y_\infty \approx 13.5721$

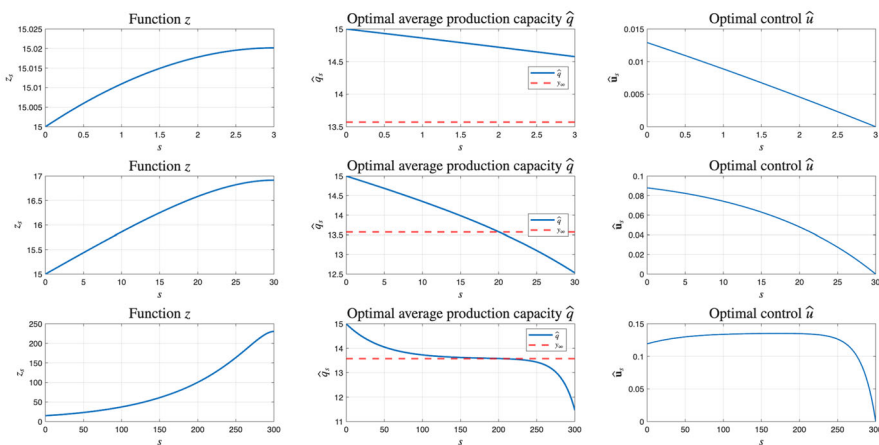


Fig. 3 The solution z to the integro-differential Eq. (4.5), the equilibrium average production capacity $\hat{q}$, and the equilibrium optimal investment strategy $\hat{u}$, in the short, medium, and long time horizon cases. Initial condition $x = 15$

for a large value of T , we would see a decreasing pattern of $\hat{q}$ in the long run, which is not consistent with the convergence to y_∞ proved in Theorem 4.11. We will discuss the details of such method in Sect. 6.2 below.

We conclude our discussion of the finite time horizon case, by looking at Figs. 4, 5, and 6, which provide sample trajectories of the equilibrium production capacity process $\hat{X} := X^{\xi, \hat{u}}$ (cf. Eq. (2.1)), in each of the three time horizons given above and for $\sigma = 0.001, 0.01, 0.1$. In both figures we consider $\xi = \hat{X}_0$ deterministic, thus

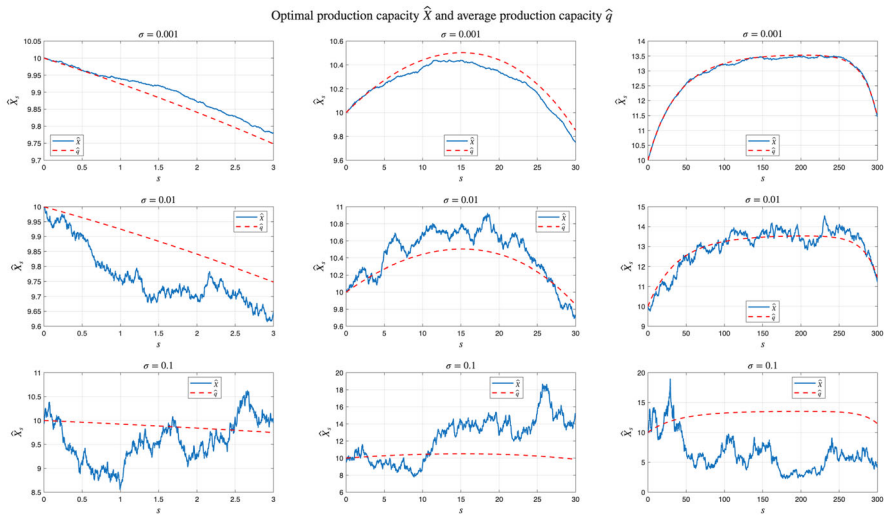


Fig. 4 Trajectories of $\hat{X}$ and the equilibrium average production capacity $\hat{q}$, in the short, medium, and long time horizon cases, and for $\sigma = 0.001, 0.01, 0.1$. Initial condition $x = 10$

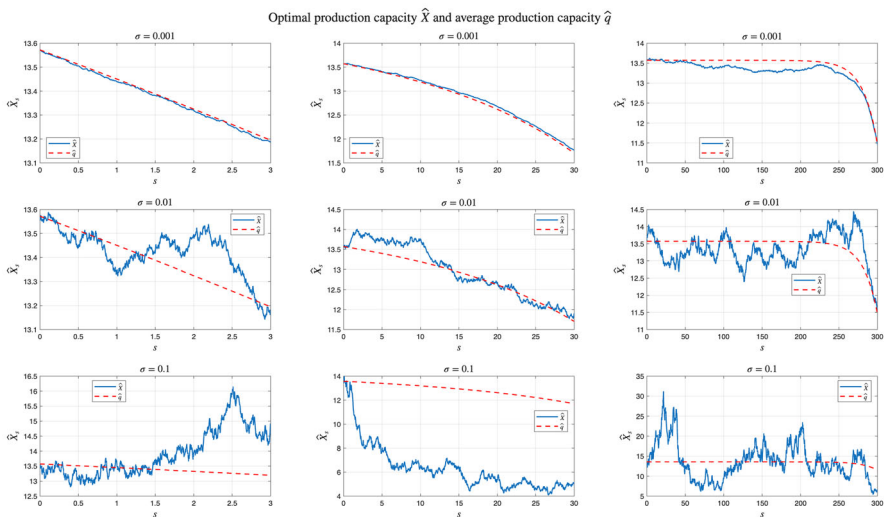


Fig. 5 Trajectories of $\hat{X}$ and the equilibrium average production capacity $\hat{q}$, in the short, medium, and long time horizon cases, and for $\sigma = 0.001, 0.01, 0.1$. Initial condition $x = y_\infty \approx 13.5721$

coinciding with the initial average production capacity x ; more precisely, in Fig. 4 we consider $\xi = x = 10$; in Fig. 5, we set $\xi = x = y_\infty$; in Fig. 6, we fix $\xi = x = 15$.

As it is clear from our results, the role of σ is just to measure the exposure of the production capacity level to the source of risk modeled by the Wiener process B appearing in SDE (2.1). Lower values of σ imply that the equilibrium production

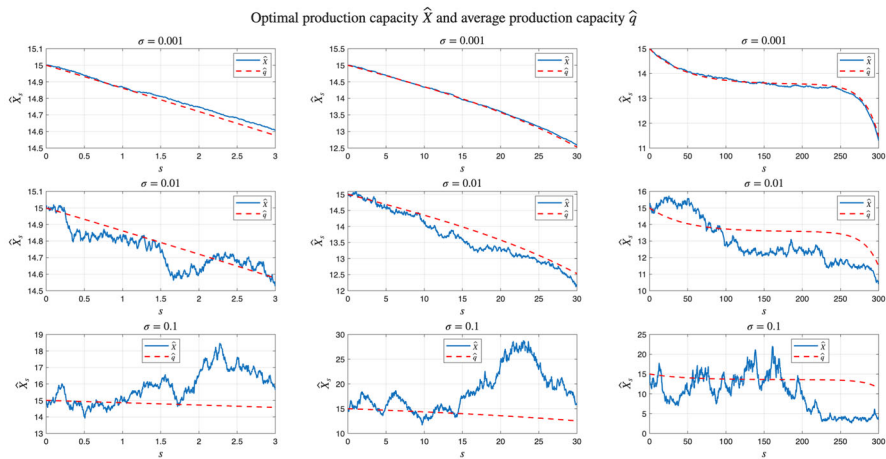


Fig. 6 Trajectories of $\hat{X}$ and the equilibrium average production capacity $\hat{q}$, in the short, medium, and long time horizon cases, and for $\sigma = 0.001, 0.01, 0.1$. Initial condition $x = 15$

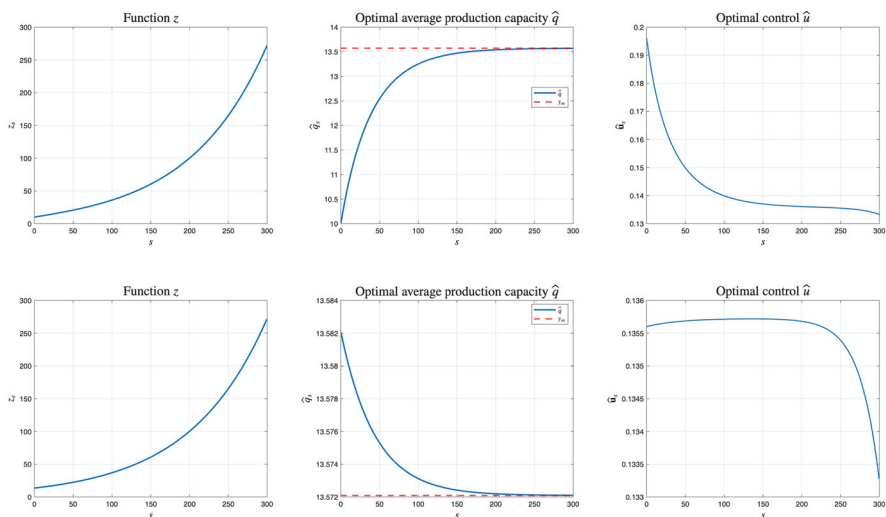


Fig. 7 The solution z to the integro-differential Eq. (4.5), the equilibrium average production capacity $\hat{q}$, and the equilibrium optimal investment strategy $\hat{u}$. Initial conditions: $x = 10$ in the first row; $x = y_\infty + 0.01 \approx 13.5821$ in the second row

capacity $\hat{X}$ follows more closely its average, while higher values of σ entail larger fluctuations, due to a higher sensitivity to risk.

6.2 The Infinite Time Horizon Case

To compute numerically the optimal average production capacity $\hat{q}$ and the optimal control $\hat{u}$ at equilibrium, we truncate the time domain $[0, +\infty)$ to the interval $[0, T]$,

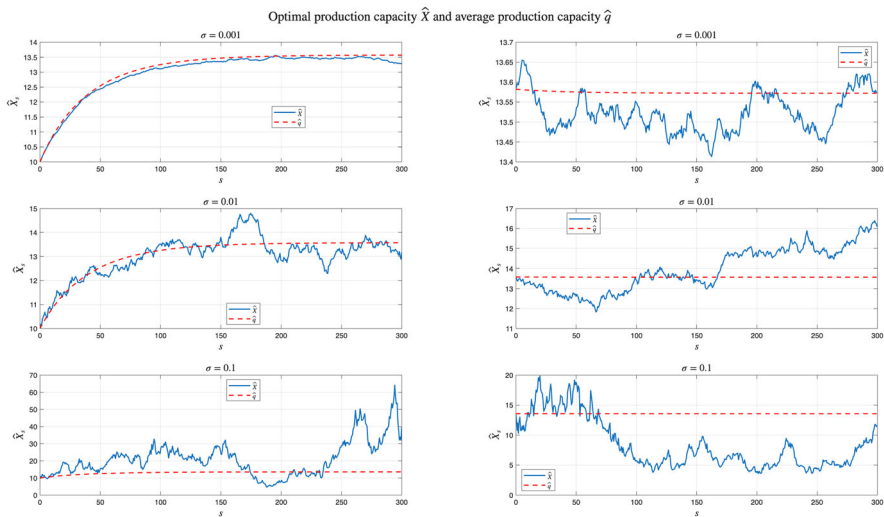


Fig. 8 Trajectories of $\hat{X}$ and the equilibrium average production capacity $\hat{q}$, in the infinite time horizon case, and for $\sigma = 0.001, 0.01, 0.1$. Initial conditions: $x = 10$ in the left column; $x = y_\infty + 0.01 \approx 13.5821$ in the right column

with $T = 300$. The specific choice of $T = 300$ is motivated by a balance between accuracy and stability of the *shooting method*, which is discussed below, and the need for a sufficiently long time horizon to show convergence of $\hat{q}$ to the steady state y_∞ given in Theorem 4.11. In addition, this value allows us to make a comparison with the longer finite time horizon case previously illustrated.

As discussed in Sect. 6.1, we do not rely on a simple truncation to T of the integro-differential Eq. (4.5), as this would artificially impose a decreasing behaviour of $\hat{q}$, as time approaches T . This would be inconsistent with the convergence of the average production capacity to y_∞ , as stated in Theorem 4.11.

We use, instead, the fact that any solution z to the integro-differential Eq. (4.5) also solves the second-order ODE (4.23), to which we can associate the initial value problem (IVP)

$$\begin{cases} z_s'' = (\rho + 2\delta)z_s' - e^{\delta(1+\beta)s}z_s^{-\beta}, & s \geq 0, \\ z_0 = x, \\ z_0' = \zeta > 0, \end{cases}$$

where x is the given initial average production capacity. From the proof of Theorem 4.11, we know that there exists a unique value of $\zeta > 0$ such that the IVP above has a unique solution in the class $\mathcal{C}_{x,\infty}$ defined in (4.20), which is also the unique solution to (4.5).

To approximate z numerically we employ the so-called *shooting method*, which, starting from an initial guess and using a sequence of successive approximations (via the bisection method), computes the value of ζ . This allows us to retain the correct properties of the solution z to IDE (4.4) and, in particular, the fact that it grows

exponentially at infinity at rate δ (cf. Fig. 7 below), a fact which is implied by the convergence of $\hat{q}$ to y_∞ .

To compute the optimal control $\hat{u}$, we observe that truncating the integral in (3.9) at T , would force $\hat{u}_T = 0$, which would yield a poor approximation as time approaches the truncation point. For this reason, we actually compute numerically the solution to IDE (4.4)—and, thus, the average production capacity $\hat{q}$ —on the larger time interval $[0, \tilde{T}]$, with $\tilde{T} = 400$, and then we use the intended time interval $[0, T] = [0, 300]$ to compute $\hat{u}$ and to draw the graphs.

In Fig. 7 we compare the three functions z , $\hat{q}$, and $\hat{u}$, choosing as initial conditions either $x = 10$, or $x = y_\infty + 0.01$, i.e., slightly above the steady state y_∞ .

As expected, we obtain convergence of the equilibrium average production capacity to the steady state y_∞ . The optimal control reaches an almost constant value in a large part of the time interval, before starting to decrease to zero, as implied by Eq. (3.9).

Also in the infinite time horizon case, we conclude our discussion with a simulation of sample trajectories of the equilibrium production capacity process $\hat{X} := X^{\xi, \hat{u}}$, whose dynamics are given in (2.1), for $\sigma = 0.001, 0.01, 0.1$. In Fig. 8 we show two cases, in which we consider $\xi = \hat{X}_0$ deterministic, thus coinciding with the initial average production capacity x ; more precisely, $\xi = x = 10$ and $\xi = x = y_\infty + 0.01$.

Similarly to the finite time horizon case, we observe that σ simply accounts for the exposure of the production capacity level to the risk factor modeled by the Brownian motion B in SDE (2.1). Note that in the case $x = y_\infty + 0.01$, $\sigma = 0.001$ (top plot on the right), the level of fluctuation of $\hat{X}$ with respect to the average $\hat{q}$ is only apparently high, as one can see from the scale of the vertical axis.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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