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Learning from Euler. From Mathematical Practice to Mathematical Explanation

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Résumé : Dans son « Découverte d'un nouveau principe de mécanique » (1750) Euler a donné, pour la première fois, une preuve du théorème qu'on appelle aujourd'hui le Théorème d'Euler. Dans cet article je vais me concentrer sur la preuve originale d'Euler, et je vais montrer comment la pratique mathématique d'Euler peut éclairer le débat philosophique sur la notion de preuves explicatives en mathématiques. En particulier, je montrerai comment l'un des modèles d'explication mathématique les plus connus, celui proposé par Mark Steiner dans son article « Mathematical explanation » (1978), n'est pas en mesure de rendre compte du caractère explicatif de la preuve donnée par Euler. Cela contredit l'intuition originale du mathématicien Euler, qui attribuait à sa preuve un caractère explicatif spécifique.

Abstract: In his "Découverte d'un nouveau principe de mécanique" (1750) Euler offered, for the first time, a proof of the so-called Euler's Theorem. In this paper I will focus on Euler's original proof and I will show how a look at Euler's practice as a mathematician can inform the philosophical debate about the notion of explanatory proofs in mathematics. In particular, I will show how one of the major accounts of mathematical explanation, the one proposed by Mark Steiner in his paper "Mathematical explanation" (1978), is not able to account for the explanatory character of Euler's proof. This contradicts the original intuitions of the mathematician Euler, who attributed to his proof a particular explanatory character.

1 Introduction

Does philosophy of mathematics have something to do with the way mathematicians do their job? Can philosophy of mathematics learn from the observation and the analysis of the way mathematics is done? As it is well known,

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a positive answer to these questions came from the philosophy of mathematics itself, with the emergence during the sixties of a strong opposition to the classical foundational programs (logicism, Hilbert program and intuitionism). This reaction against the "dogmas of foundationalism" [Tymoczko 1998, 95], which started with Imre Lakatos and which has been pursued by what Aspray and Kitcher defined as "a maverick group of philosophers" [Aspray & Kitcher 1988], gave a central importance to the history of mathematics and assumed mathematical practice as a driving force in the philosophical research.

The research questions posed by the "maverick" philosophers were more oriented on the heuristics of mathematics, the way in which mathematics grows, the notion of explanation in mathematics, the distinction between formal and informal proofs and reasonings in mathematics [Aspray & Kitcher 1988, 17]. Furthermore, questions concerning the dynamics of mathematical discovery and the historical development of mathematics were also addressed [Kitcher 1984].

It is without a doubt that such research questions are still urgent and central to the agenda of contemporary philosophers of mathematics ([Mancosu, Jørgensen & Pedersen 2005]; [Mancosu 2008]). And it can be noted that similar claims about the importance of scientific practice in the philosophical investigation come from the general philosophy of science. [1] But how can the questions of the traditional philosophy of science be dissolved or moved by the examination of scientific practice? And, for what specifically concerns the topic touched by the present issue of this journal, how can an examination of the practice of mathematicians help in informing topics regarding the philosophy of mathematics? In this paper I will propose a possible answer to the latter question. By focusing on the notion of mathematical explanation, I will show that an analysis of Euler's scientific practice does provide not only an evaluation of a philosophical model (Steiner's model of mathematical explanation in mathematics), but it also suggests new directions of investigations that might come out as fruitful. In this sense, the aim of the paper will be twofold: it will provide a testing of one particular account of mathematical explanation on a specific case-study; it will show that an investigation which takes scientific practice as a starting point for philosophical analysis can have strong repercussions on debates which are central to the contemporary philosophy of mathematics and to philosophy of science in general.

It should be pointed out that this practice-driven methodology in the study of the notion of explanation, i. e. a methodology according to which a theory of explanation must be tested and refined starting from the analysis of scientific

1. For instance, in their paper on scientific understanding, Henk De Regt and Dennis Dieks write: "Nowadays few philosophers of science will contest that they should take account of scientific practice, both past and present. Any general characteristic of actual scientific activity is in principle relevant to the philosophical analysis of science" [De Regt & Dieks 2005, 139].

practice itself, has been adopted by other authors as well ([Hafner & Mancosu 2008]; [Mancosu 2008]). The same methodology is welcomed by some members of the so-called "Stanford School of History and Philosophy of Science", notably Nancy Cartwright and Margaret Morrison [Hoefer 2008]. These authors agree that it is possible to have a better comprehension of mathematical explanation focusing on particular case-studies and taking the test-case itself as a starting point for philosophical considerations.

In this study I will focus on mathematical explanation within mathematics, leaving aside the different topic of mathematical explanation in natural and social sciences [Mancosu 2008, 134]. But what do we refer to with the expression 'mathematical *explanation* of a mathematical fact'? To answer this question, it is important to note that mathematical activity cannot be reduced to a purely justificatory exercise. A great part of this activity is driven by factors other than justificatory aims such as establishing the truth of a mathematical fact. In many cases, when confronted with a theorem or a problem, the mathematician prefers a particular proof-strategy or procedure because it provides more than a mere justification of the mathematical truth ([Kitcher 1984]; [Sandborg 1998]; [Hafner & Mancosu 2005]; [Tappenden 2005]). In other words, that particular proof-strategy or procedure goes beyond the simple reason that 'it makes the mathematical truth evident'. It also provides an 'explanation' of *why* that mathematical result is evident (the reason *why*).

As Paolo Mancosu has shown in one of his studies on explanation in mathematics, even if there is a renewed interest on the notion of mathematical explanation, this interest does not represent a novelty in philosophy and the attention to the topic can be traced back to Aristotle [Mancosu 2000]. Although there seems to be enough evidence that mathematical explanations occur within mathematics, however, the notion has not been sufficiently studied and much more investigation is needed to adequately capture it. Mark Steiner's account of mathematical explanation, proposed by the author in his paper "Mathematical explanation" [Steiner 1978a], represents the first explicit contribution to the study of the notion in analytic philosophy. More precisely, with his model Steiner attempts to account for explanations in mathematics which come under the form of proof. [2]

The outline of the paper will be the following. First of all, I will illustrate Steiner's account of mathematical explanation in mathematics. Then, in Section 3, I will concentrate on Euler's mathematical practice and I will present his geometrical proof of the so-called 'Euler's theorem' for the existence of an instantaneous axis of rotation in rigid body kinematics. Finally, I will evaluate Steiner's account on this particular test-case and I will show how

2. Although "proofs are often vehicles for mathematical explanation" [Sandborg 1998, 616], not all explanations in mathematics come under this form. For cases of explanation in mathematics which do not come under the form of proof see [Kitcher 1984], [Sandborg 1998] and [Mancosu 2001].

the account does not consider Euler's geometrical proof as a genuine explanation, thus contradicting the intuitions of Euler himself. The final Section will contain my conclusions.

2 Steiner's account of explanation in mathematics

Mark Steiner has presented his account of mathematical explanation (coming under the form of proof) in his paper "Mathematical explanation" [Steiner 1978a]. The starting point in the building of his original approach is the following observation: "to explain the behaviour of an entity, one deduces the behavior from the essence or nature of the entity" [Steiner 1978a, 143]. The previous remark is aimed to face a well-known problem: mathematical truths are commonly regarded as necessary, then it is meaningless to speak of essential properties of a mathematical entity. Thus, in order to escape all the difficulties related to the definition of an essential property of a mathematical entity x, i. e. a property x enjoys in all possible worlds, Steiner introduces the relative notion of *characterizing property*:

> Instead of 'essence', I shall speak of 'characterizing property', by which I mean a property unique to a given entity or structure within a *family* or domain of such entities or structures. (I take the notion of a family or domain undefined in this paper; examples will follow shortly.) We thus have a relative notion, since a given entity can be part of a number of different domains or families. Even in a single domain, entities may be characterized multiply. [Steiner 1978a, 143]

For Steiner, an explanatory proof depends on such property, while a non-explanatory proof does not. In particular, "an explanatory proof makes reference to a characterizing property of an entity mentioned in the theorem, such that from the proof it is evident that the result depends on that property" [Steiner 1978a, 143]. The dependence characterizing property-result comes from the fact that if we try to manipulate the proof, by substituting in it a different object of the same domain, the theorem collapses. This introduces us to Steiner's second core-notion about explanation by proofs: *generalizability*—through the variation of a characterizing property. If we deforme the proof by varying a certain characterizing property of a related entity, what we obtain in response is a change of the theorem. To every deformation of the proof there corresponds a deformation in the theorem, i. e. to an array of proofs there corresponds an array of theorems. [3] The theorems obtained are

3. Although Steiner offers some examples, the notion of 'deformation' is left undefined in his discussion. He writes: "Deformation is similarly undefined—it implies

proved and explained by the deformations of the original proof. This is what Steiner takes for an explanatory proof to be generalizable [Steiner 1978a, 143].

To sum up, Steiner offers two criteria for considering a proof being explanatory:

C_1 dependence on a characterizing property of an entity mentioned in the theorem (dependence criterion)

C_2 possibility to deform the proof by "substituting the characterizing property of a related entity" and getting "a related theorem" (generalizability criterion).

Steiner's examples of explanatory proofs include two different proofs of the sum of the first n integers and the proof of the irrationality of $\sqrt{2}$ involving the fundamental theorem of arithmetic. In another paper, which is devoted to the distinct topic of mathematical explanations in science, Steiner considers as explanatory a proof of a particular theorem as given in linear algebra [Steiner 1978b]. Since the test case I am going to present in the next Section concerns the very same theorem (as discussed by Euler), let me consider here this example as illustrative of his account. The proof requires some familiarity with the language of linear algebra, and therefore it is first necessary to give some basic terminology.

An orthogonal matrix $\mathbf{A}$ is a real square matrix with the following property: $\mathbf{A^t} = \mathbf{A^{-1}}$ (the transpose matrix of $\mathbf{A}$ coincides with the inverse of $\mathbf{A}$).[4] The class of $n \times n$ orthogonal matrices is a group under matrix multiplication.[5] The group of real orthogonal $n \times n$ matrices is called the *orthogonal group*, and it is denoted by $O(n)$. The property $\mathbf{A^t} = \mathbf{A^{-1}}$ clearly holds for every real orthogonal matrix which belongs to $O(n)$. Since $\mathbf{AA^t} = \mathbf{I}$, we have: $|\mathbf{AA^t}| = |\mathbf{A}||\mathbf{A}^t| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2 = |\mathbf{I}| = 1$.[6] Thus $|\mathbf{A}| = \pm 1$ for every member of $O(n)$. The subgroup of matrices with determinant $+1$ is called the *special orthogonal group* and it is denoted by $SO(n)$. We can regard a matrix as a representation of a linear mapping α of the Euclidean space ($\alpha : V \longrightarrow V$), then n is the dimension of the space ($n = dim(V)$). The *characteristic polynomial* $P_\alpha(t)$ of the linear map $\alpha : V \longrightarrow V$ is the polynomial $|\mathbf{A} - t\mathbf{I}|$, where $\mathbf{A}$ is our matrix representation of α. The scalar λ is an *eigenvalue* of the linear mapping α if and only if $|\mathbf{A} - \lambda\mathbf{I}| = 0$ (this is a theorem), where the equation

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not just mechanical substitution, but reworking the proof, holding constant the proof idea" [Steiner 1978a, 147].

4. The inverse $\mathbf{A^{-1}}$ of $\mathbf{A}$ satisfies $\mathbf{A^{-1}A} = \mathbf{I} = \mathbf{AA^{-1}}$. The transpose matrix of $\mathbf{A}$, denoted by $\mathbf{A^t}$, is the matrix obtained by exchanging $\mathbf{A}$'s rows and columns: $(\mathbf{A^t})_{ij} = \mathbf{A}_{ji}$.

5. A group is a set G, together with a binary operation $*$ on G, which has the following properties: 1) for all g and h in G, $g * h \in G$; 2) for all f, g and h in G, $f * (g * h) = (f * g) * h$; 3) there is an unique e in G such that for all g in G, $g * e = g = e * g$; 4) if $g \in G$ there is some h in G such that $g * h = e = h * g$.

6. We used two well-known properties of determinants: the fact that, for any square matrix $\mathbf{A}$, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ and $|\mathbf{A^t}| = |\mathbf{A}|$.

$|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called *secular equation.* An *eigenvector* of the transformation is a non-null vector v that is transformed in a scalar multiple of itself: $\alpha(v) = \lambda v$. If λ is any eigenvalue of α, the set of all vectors v of V with $\alpha(v) = \lambda v$ is a nonzero subspace of V which is called the *eigenspace* of λ; it consists of the zero vector and all the eigenvectors belonging to λ. Naturally, if the eigenspace has dimension 1, it is a line.

All the notions sketched in the previous paragraph belong to the basic background knowledge of linear algebra and group theory. Consider now a particular case of orthogonal matrices (or transformations): those matrices for which the condition $|\mathbf{A}| = +1$ holds, i. e. the members of the special orthogonal group $SO(n)$. Here is the theorem considered by Steiner, together with the relative proof:

Theorem 1. *Every matrix* $\mathbf{A} \in SO(3)$, *with* $\mathbf{A} \neq \mathbf{I_3}$, *has an eigenvalue* $+1$.

Proof. The proof of the existence of such eigenvalue is very short and could be stated via a speedy argument which requires no direct calculations but only some background notions.[7] We assume $\mathbf{A} \neq \mathbf{I_3}$ because $\mathbf{A} = \mathbf{I_3}$ is the uninteresting case of the identity transformation.

Consider a real matrix $\mathbf{A}$ member of $SO(3)$ and recall that for this matrix the determinant is equal to $+1$. Furthermore we know that, in general, the secular equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ has three roots which correspond to three eigenvalues. We want to prove that one of these eigenvalues is our $\lambda = +1$.
Suppose now λ_1, λ_2 and λ_3 are eigenvalues of $\mathbf{A}$. They are the roots of a cubic polynomial with real coefficients (the entries of the matrix are real). Thus, according to the fundamental theorem of algebra (FTA) and the complex conjugate root theorem (CCRT), one of the eigenvalue (say, λ_1) is real.[8] By CCRT, if λ_2 is not real, then its complex conjugate λ_2^* is also an eigenvalue ($\lambda_3 = \lambda_2^*$). Since $|\mathbf{A}| = \lambda_1\lambda_2\lambda_3$, we have two possibilities for the eigenvalues:

7. This proof is that to which Steiner refers to and it is given in Goldstein's *Classical Mechanics* [Goldstein 1957, 123]. The very same proof is common in textbooks of linear algebra and group theory. For instance, see [Grove & Benson 1985].

8. The FTA states that any polynomial has at least a complex root. If we accept the factorization of a polynomial of degree n, it has exactly n complex roots. The CCRT says that if a polynomial in one variable with real coefficients, such as $|\mathbf{A} - \lambda\mathbf{I}|$, has a complex root λ, then the complex conjugate λ^* of λ is also a root of the polynomial. This means that, according to CCRT, if λ is a solution of the polynomial equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$, also λ^* will be a solution of the same equation. In general, the FTA says that $|\mathbf{A} - \lambda\mathbf{I}| = 0$ will always have complex solutions, but not necessarily real solutions. However, we are considering a real orthogonal 3×3 matrix, then the polynomial $|\mathbf{A} - \lambda\mathbf{I}|$ is a real polynomial (the entries of the matrix $\mathbf{A}$ are real, and so are the coefficients of the polynomial) of *odd* degree (degree 3). Thus, by FTA, the polynomial equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ has an odd number of solutions. Now, according to CCRT, complex solutions come in conjugate pairs, and consequently there is an even number of them in $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Therefore the polynomial equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ has at least one real solution (i. e. at least one real eigenvalue).

a) $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = \pm 1$
b) $\lambda_1 = 1,\ \lambda_2 = \lambda_3^* \notin \mathbf{R}$ (observe here the change in notation for λ_3^*)

In either case we have that $+1$ is an eigenvalue.[9] □

In his paper "Mathematics, explanation and scientific knowledge" [Steiner 1978b], Steiner considers the proof above as explanatory. How then are his criteria for explanatoriness C_1 and C_2 supposed to operate in this case? Recall, first of all, that for Steiner a proof is explanatory only if it makes evident that the conclusion depends on a property of some entity or structure mentioned in the theorem (criterion C_1). In other words, the proof must involve a characterizing property.

Steiner's attention in his paper is not explicitly directed to explanations in pure mathematics, but the focus is elsewhere.[10] This is, perhaps, what precludes him from offering a detailed description of how C_1 and C_2 are fulfilled in the proof above. However, in a personal communication he suggests that we have to "pick the characterizing property of $SO(3)$ as having an odd dimension". This means that the entity (or structure) mentioned in the theorem is $SO(3)$, which belongs to the family $SO(n)$, while its characterizing property is 'having an odd dimension'. But in what sense then the previous proof depends on the odd dimensionality of $SO(3)$? If we concentrate, as Steiner suggests, on the proof strategy above, it is evident that the existence of the eigenvalue $\lambda = +1$ *depends* on the fact that n is odd. We also agree with Steiner that there is no necessity for any eigenvalue to be +1 [Steiner 1978a, 18], because this does not hold for all the elements of the family $SO(n)$ (in two or four dimensions, for instance, the number of real eigenvalues is even and the theorem does not hold).

Second, recall that for Steiner generalizability comes when we substitute the characterizing property of a related object and what we get is a related deformed theorem (criterion C_2). Concerning C_2, then, Steiner's idea is that the generalizability of the proof above is obtained by replacing the dimension 3 by some odd number: "There is an explanatory proof of this [existence of an eigenvalue $+1$] that extends to any Euclidean space of odd dimension" (personal communication). It is easy to see how we can use this strategy to get new theorems. By replacing 3 by some odd number we obtain our related

9. What interests us here is the *existence* of such eigenvalue. Observe, however, that the proof given also shows that there is only one eigenvalue $\lambda = +1$. The *unicity* of $\lambda = +1$ can also be proved as a corollary of the more general theorem of Cartan-Dieudonné, by showing that the dimension of the space $\mathrm{Fix}(\alpha)$ of the fixed points of the transformation is 1 [Grove 2002, 49].

10. In this paper Steiner is mainly concerned with the existence of mathematical explanations in science, and with the existential conclusions which follow once the existence of such explanations is accepted.

theorem, i. e. a theorem which states the existence of the eigenvalue +1 for every real matrix $\mathbf{A} \in SO(n)$ with n odd.[11]

After this short presentation of Steiner's theory, in the next Section I will concentrate on Euler and his geometrical proof of a particular result in kinematics of rigid body motion. The theorem in which this result appears, called Euler's theorem after Euler, states that the general displacement of a rigid body with a point fixed is a rotation about some axis. In other words, the theorem says that for every rotation of a rigid body with a fixed point there exists an instantaneous axis around which the rotation is made.[12] From an analysis of Euler's practice as a mathematician it will emerge that Euler looked at his geometrical proof as a mathematical 'explanation' of a mathematical truth, and not as a simple justificatory procedure.

3 Euler's proof for the existence of an instantaneous axis of rotation

Euler's contributions to mechanics are numerous and of primary importance. Among those, the use for the first time of a perpendicular Cartesian coordinate system to describe the motion of a rigid body in his *Recherches sur les corps célestes en général* [Euler 1747] and the introduction of the so-called 'Euler Angles' in chapter IV of his *Introductio in Analysis Infinitorum* (1748), while investigating surfaces in space. Euler was also the first to prove the existence of an instantaneous axis of rotation in the kinematics of rigid body motion. He obtained this result, for the first time, in his E177 *Découverte d'un nouveau principe de mécanique* [Euler 1750].[13]

In E177 Euler utilizes previous results in order to study the general motion of a rigid body with a fixed point and deduces the changes in the position and the velocity distribution from the given forces acting on the body. His enterprise in the dynamics of rigid body motion in space was stimulated by the problem of the rotation of the Earth around its axis (as to explain the precession of equinoxes). Newton had given a first explanation of the phenomenon in book III of the *Principia*, while D'Alembert tried to improve Newton's study

11. For instance, this theorem appears in Hermann Weyl's *The Classical Groups* [Weyl 1973, 58].

12. Note that the theorem seen above, used by Steiner to illustrate his account, is just a modern algebraic formulation of Euler's theorem, once the appropriate physical interpretation is established. This is, for instance, how the theorem appears in Goldstein's textbook: "The real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has the eigenvalue +1" [Goldstein 1957, 119].

13. In 1910 and 1913, the Swedish mathematician Gustaf Eneström completed the first comprehensive survey of Euler's works. Each work is classified by using the letter E and a number (the works are referred to as "Eneström number").

in his 1749 *Recherches sur la précession des équinoxes et sur la nutation de l'axe de la Terre* [Wilson 1987]. Furthermore, Euler's studies in hydraulics and in the motion of ships during the 1740's permitted him a new and different approach to mechanics.[14] The separation of the progressive motion of the center of gravity from the rotatory motion obtained in his *Scientia Navalis* (written between 1737 and 1740, and published in 1749) is applied here in a kinematical version and it permits Euler to reach the following conclusion: the rotatory motion of the body is independent of its progressive motion. As Euler affirms in the Introduction:

> Quelque composé que soit le mouvement d'un corps solide, on le peut toujours décomposer en un mouvement progressif et en un mouvement de rotation. [...] on peut entreprendre la recherche du mouvement de rotation, tout comme si le corps n'avoit aucun mouvement progressif. [Euler 1750, 82]

The separation of motions gives Euler the opportunity to concentrate only on rotatory motion and state the existence of an instantaneous axis of rotation:

> Supposant donc le centre de gravité d'un corps solide quelconque en repos, ce corps sera néanmoins susceptible d'une infinité de mouvemens différens. Or je démontrerai dans la suite, que, quel que soit le mouvement d'un tel corps, ce sera pour chaque instant non seulement le centre de gravité qui demeure en repos, mais il y aura aussi toujours une infinité de points situés dans une ligne droite, qui passe par le centre de gravité, dont tous se trouveront également sans mouvement. [Euler 1750, 83]

In the Section *Détermination du mouvement en général, dont un corps solide est susceptible, pendant que son centre de gravité demeure en repos*, in order to study the general motion of the body, Euler introduces a Cartesian system fixed in absolute space and assumes that a point Z of the body with coordinates x, y, z has velocities P, Q, R in the direction of the axes. The components of the velocity P, Q, R are functions of x, y, z. Euler's final purpose is to find these functions. He considers another point z "infiniment proche du précédent Z", of coordinates $x + dx$, $y + dy$, $z + dz$ and velocities $P + dP$, $Q + dQ$, $R + dR$. After an analytic treatment, Euler is able to conclude that there are points, which have coordinates $(Cu, -Bu, Au)$, that do not move during time dt.[15] These points are on a straight line through the origin, which is called the "instantaneous axis of rotation".[16]

14. See [Maltese 2000] on this phase of Euler's work in mechanics, concerning the relativity of motion.
15. The letters A, B, C stand for indeterminate coefficients.
16. Euler does not use the word *instantaneous*. He refers to it simply as *axe de rotation*.

> [...] tous les points du corps, qui sont contenus dans ces formules $x = Cu$, $y = -Bu$, $z = Au$ demeureront en repos pendant le temps dt. Or tous ces points se trouvent dans une ligne droite, qui passe par le centre de gravité O; donc cette ligne droite demeurant immobile sera l'axe de rotation, autour duquel le corps tourne dans le présent instant. [Euler 1750, 95]

After the analytic argument, Euler adds a purely geometric (i. e. non analytic) proof of the existence of the instantaneous axis of rotation, discussing the infinitesimal motion of a spherical surface with a fixed point. He embarks on the new proof after the following remark: "Sans entrer dans le détail du calcul, que je viens de déveloper, on peut aussi prouver la même vérité par la seule Géométrie" [Euler 1750, 96]. Keep in mind the theorem: When a rigid body is moved around its center it is always possible to find a line, passing through the center, whose position is the same as before the motion. His geometrical proof runs as follows [17]:

Euler's geometrical proof. In time dt, the rigid body with a fixed point (the center of gravity) will have moved from one position to another. Now, consider in the moving body a spherical layer ("couche sphérique") whose center coincides with the center of gravity (in Figure 1 I have indicated the center of gravity with a cross '×', but this point is not showed in Euler's original diagram). As Euler observes, to know the motion of the spherical surface defined by this spherical layer amounts to determine the motion of the entire body.

If we consider an arc AB of a grand circle on the spherical surface, after time dt it will have moved to the new position ab. Recall the body is rigid, then $AB = ab$. Euler remarks that, from the study of this configuration, it will be possible to find the positions of *all* the points of the spherical surface after time dt [Euler 1750, 96].

Let's look at the diagram (Figure 1). If we prolong the two arcs BA and ba they will meet in a point, say C. If we take point a on the arc bcC such that $ac = AC$, after time dt point C (on the arc BA) will be transported in position c (on the arc ba), and thus the arc CAB in cab. For simplicity, call g_1 the grand circle BAC and g_2 the other circle bcC. Consider now a point M outside the grand circle g_1, and trace the arc MC. After time dt the point M will move to the new position, say m. If we want to know the position m, we have simply to trace the arc $cm = CM$ such that $\angle acm = \angle ACM$.

Now, if we prolong the two arcs CM and cm, they will intersect in a point O. Thus, after time dt, the arc CMO will be transported to the arc cmO. If $CMO = cmO$, the point O will not move in time dt and will be a fixed point. Nevertheless, to say that arcs CMO and cmO are equal amounts to say that the spherical triangle cCO is isosceles. Therefore the problem to see

17. For a similar reconstruction see [Koetsier 2007, 184–185].

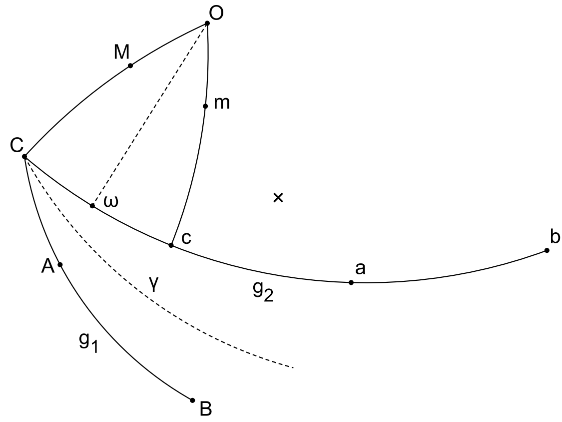

FIGURE 1: Reconstruction of Euler's original diagram as it appears in E177.

if there exists such a fixed point O corresponds to the problem of finding a possible configuration in which the spherical triangle cCO is isosceles. This configuration, as Euler points out, can *always* be found:

> Il est certain qu'on peut toujours constituer en sorte l'arc CMO, qu'ayant décrit son arc correspondant cmO, il soit $cmO = CMO$. Car pour que cela arrive on n'a qu'à constituer l'arc CMO en sorte, que l'angle cCO devienne égal à l'angle CcO; afin que le triangle sphérique COc devienne isoscèle et partant les côtés CO et cO égaux entr'eux. [Euler 1750, 96]

In order the spherical triangle to be isosceles it suffices that the angles $\angle cCO$ and $\angle CcO$ be equal. How do we choose M in the right way to obtain this configuration? Euler's reasoning is very simple. Consider the configuration in which the angles are equal: $\angle cCO = \angle CcO$. Observe that:

$$\angle cCO = \angle ACO - \angle ACc \quad \text{and} \quad \angle CcO = \angle 180 - \angle acO \tag{4}$$

But we also want that

$$\angle ACO = \angle acO \quad \text{and} \quad \angle cCO = \angle CcO \tag{5}$$

Thus, by simple substitutions and addition of the angles, we have:

$$\angle 2cCO = \angle 180 - \angle ACc \tag{6}$$

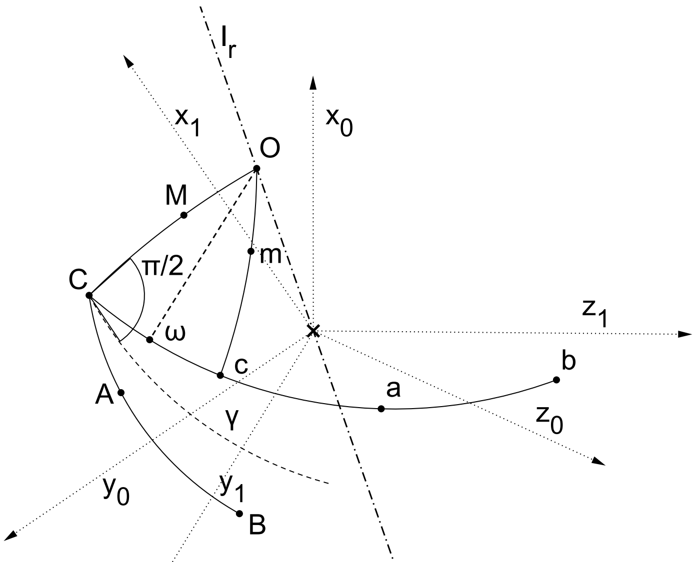

FIGURE 2: The instantaneous axis of rotation.

or

$$\angle cCO = \angle 90 - \angle \frac{ACc}{2} \tag{7}$$

The previous expression means that the angle formed by $\angle cCO$ and $\angle \frac{ACc}{2}$ is a right angle. Hence, if we take the angular bisector γ of the angle $\angle ACc$, we can conclude that, in order the spherical triangle OCc to be isosceles, point M must be chosen such that CM be perpendicular to γ. Then O is fixed in the motion and it belongs to a line (passing through the center of the surface) which does not move. □

Although Euler does not do this, it is immediate to see the instantaneous axis just by tracing a line passing through the center of gravity (which is at rest in the motion) and the point O. In Figure 2, I traced the instantaneous axis I_r and the system of reference fixed with the rigid body before and after the infinitesimal motion—respectively, (x_0, y_0, z_0) and (x_1, y_1, z_1).

Why did Euler add the previous geometrical proof if he had already obtained the result via an analytic procedure? Bear in mind the mathematical result: there exists a line which does not move in the infinitesimal transformation. As Euler affirms, we can prove that this is true without calculations and via geometry alone [Euler 1750, 96]. It seems then that Euler adds the geometrical proof to convince the reader of the validity of the previous analytic result. More precisely, as I am going to show in the next paragraph, Euler is convinced that the geometrical procedure provides us with a kind of comprehension of the result (it is not mere justification). The geometrical proof has an epistemic value that the analytic proof has not.

Even though Euler strongly contributed to the transition towards an analytical mechanics, by pushing forward the process of mathematization started with Newton, geometrical arguments still played a crucial role in his scientific practice. As an example, in his *De novo genere oscillationum* (presented in 1739), in the study of what we call today 'forced simple harmonic motion', Euler interpreted geometrically every quantity involved in the problem. [18] This, however, should not come as a surprise. As a consequence of the 17th century, geometrical concepts were still rooted in mechanics and the way in which the new mechanics developed did not entirely depend on the new mathematical techniques for conceptual support [Mahoney 1984]. In the Preface to his *Mechanica* (1736), Euler points out that in reading works such as Newton's *Principia* and Hermann's *Phoronomia* he had difficulties in solving problems slightly differents from the geometrical cases presented but he could "understand" the solutions to the particular geometrical problems "well enough" [Euler 1736, Preface]. [19] Here Euler attributed to geometrical proofs some specific epistemic virtue ("understanding"). [20] Observe that, while Euler stresses that the works composed "without analysis" do not provide a sufficient knowledge to solve problems slightly differents from a given (geometrical) example, he admits that geometrical methods "persuade of the truth of the things that are demonstrated" and lead to the comprehension of the solution [Euler 1736, Preface]. This is exactly the case of E177: he adds the geometrical proof because it *persuades* of the existence of point O (and then of the instantaneous axis) and, furthermore, leads to the *comprehension* of the solution. The analytic procedure lacks in this epistemic value. This is why Euler's geometrical proof marks a decisive point and it is considered by Euler of primary importance. To prove a geometrical theorem, Euler adds to the analytic proof a purely geometric proof. In considering a geometrical proof for the geometrical theorem, Euler restricts the conceptual resources used to prove the theorem to those which determine the content of the theorem, i. e.

18. Referring to this paper, Jerome Ravetz has observed: "symptomatic of his thinking at this stage is the description of the sine as a line, not a function" [Ravetz 1961, 13]

19. The original passage is in Latin. I adopt here the translation which appears in [Guicciardini 2004].

20. Euler uses the Latin verb 'percipio', which means 'to perceive', 'to understand'. The verbal form is translated as 'understanding' by [Guicciardini 2004, 245].

geometrical concepts. To put it roughly, the same geometrical concepts are used to enunciate and prove the theorem. Therefore Euler is concerned with 'purity of methods'.[21] And, in line with what Euler affirms in the Preface to his *Mechanica*, this purity increases the epistemic quality of the proof with respect to the analytic proof. The geometrical procedure provides us with a kind of comprehension of the result (it is not pure justification), thus increasing the epistemic quality of the proof with respect to the analytic proof. The qualitative type of knowledge that the geometrical proof provides is the knowledge of the basic reasons *why* the result is true. The purely geometric proof is explanatory because, by making possible to reach the result through the same resources which determine the content of the theorem, it provides us with the knowledge of why the instantaneous axis exists: the axis exists because we can easily construct it geometrically, by finding point O and tracing the line in the diagram. Geometrical reasoning defines a natural, 'organic' conceptual path which leads us from the content of the theorem to the result. This is how the purely geometric proof sheds light on why the result is true.[22] The analytic proof, although convincing of the correctness of the mathematical claim, does not offer this possibility.

Let me mention how the previous case regarding the geometrical proof in E177 does not represent an isolated example within Euler's mathematical activity. In 1775 Euler published E478, *Formulae generales pro translatione quacunque corporum rigidorum*. Here he reconsiders the more general problem of the description of the change of position of a rigid body. The difference with E177 is that here Euler is dealing with an arbitrary (finite) motion. In Section 20 of E478 Euler examines the following problem: does it exist a straight line which has its initial and final directions parallel (before and

21. On purity of methods see [Detlefsen 2008] and [Detlefsen & Arana 2011]. Detlefsen and Arana offer the following characterization of purity: "purity in mathematics has generally been taken to signify a preferred relationship between the resources used to prove a theorem or solve a problem and the resources used or needed to understand or comprehend that theorem or problem. In this sense, a pure proof or solution is one which uses only such means as are in some sense intrinsic to (a proper understanding of) a theorem proved or a problem solved" [Detlefsen & Arana 2011, 1].

22. Although the connection purity of methods-explanation has not been sufficiently investigated yet, purity and explanation are not absolutely separate matters and the two notions seem to be closely connected [Mancosu 2011]. In particular, the idea that purity of methods can increase the epistemic quality of the proof, and more particularly that they can have explanatory value, has been expressed by some authors (cf. [Detlefsen & Arana 2011, 7–8]). Even though in this paper I am pointing to such connection (purity of methods-explanatory value) for the case of Euler's E177, it should be noted that the fact that a proof be 'pure' is not regarded as a necessary condition for that proof to be explanatory. Other features of a proof, for instance the virtue a proof has to be visualizable or its being part of a particular theory or framework, have been considered as plausible sources of explanatory power as well [Mancosu 2011], and this despite the 'pure' or 'impure' character of the proof.

after the translation)? In order to search this line he tries to solve a system of equations, but the problem is analytically too complex and he is not able to find a solution.[23] Thus, once more, he gives a geometrical proof, similar to that presented above.[24] Euler is convinced of the validity of his analytic (unsolved) procedure, and thus of the existence of the line, because he has a geometrical proof which confirms this "vérité". Again, the geometrical proof *persuades* of the *truth* of the things that are demonstrated. More precisely, Euler observes how sometimes rules of analytical formulas "hide" the truth of excellent properties such as the existence of the line under investigation:

> As it thus has been seen by the most solid reasoning, that in every translated situation there is always given such a straight line iz, whose direction does not differ from that which the same line IZ held in its initial situation [...] On this account this excellent property, the truth of which is so easily shown geometrically, will be most hidden by rules of analytical formulas; and owing to this we can anticipate the rules themselves to be the most important advancement for the whole science of mechanics. [Euler 1775, 96]

Interestingly, at the end of his paper Euler refers to the difficulty concerning the analytical problem and writes:

> But truly nobody who is easily stunned will undertake this [analytic] work; on this account this extraordinary property of all rigid bodies is judged as much more difficult, and can provide the best opportunity for the geometers to exercise their powers by thoroughly explaining this property. [Euler 1775, 98][25]

To sum up, Euler informs the reader why the particular geometrical proof was taken, beyond the simple reason that it makes the proof work. This is what happens in the geometrical proof presented above, where the proof is given by Euler to "easily" show the existence of the instantaneous axis. By making obvious how the fixed point O can be found, this particular proof convinces us of the result. Nevertheless, it provides not only a simple justification of the result, but it also provides an explanation of why the instantaneous axis does exist (the geometer explains this property through the conceptual instruments of geometry). And the previous quotations show how Euler is convinced that the (pure) geometrical proof has such particular (explanatory) virtue.

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23. For a presentation of E478, see [Koetsier 2007].

24. As Koetsier has pointed out, it is possible that Euler did not realize that his geometrical proof for the instantaneous motion, in E177, also holds for the case of finite motion [Koetsier 2007, 185].

25. John Sten's translation renders the Latin verb 'enucleare' as 'to explain'. This is the rendering as it is given in a standard Latin-English dictionary, as for instance in Charlton T. Lewis and Charles Short's *A Latin Dictionary* (Oxford: Clarendon Press, 1879).

4 Testing Steiner's account on Euler's proof

Steiner's model for explanation in mathematics has been discussed by various authors ([Resnik & Kushner 1987]; [Weber & Verhoeven 2002]; [Hafner & Mancosu 2005]). However, a fine-grained analysis of it has been carried out only by Resnik and Kushner [Resnik & Kushner 1987], and Hafner and Mancosu [Hafner & Mancosu 2005]. Both the criticisms proposed by these authors point to a major defect of Steiner's account: without any constraint on what a 'family' (or 'domain') of mathematical entities is, the choice of 'characterizing property' in a proof is subject to arbitrariness. Furthermore, and more interesting for what follows here, Hafner and Mancosu choose as test-case a proof recognized as explanatory in mathematical practice, coming from the work of Alfred Pringsheim in the theory of infinite series, and show how Steiner's criteria of explanatoriness fails in considering this proof as explanatory. Hafner and Mancosu's moral is that Steiner's model does not reflect the practice of mathematicians, and for this reason must be rejected or improved to account for the intuitions of the practicing mathematicians.

Let's now consider Euler's geometrical proof given in the previous Section. If Euler's geometrical proof is explanatory, as Euler's mathematical practice seems to suggest us, Steiner's model of explanation should reflect this intuition in recognizing it as an explanatory proof. In other words, criteria C_1 and C_2 should be applied successfully.

Recall Steiner's example discussed in Section 2.[26] First of all, we must identify the characterizing property. Steiner does not provide any precise indication on how to perform such identification, but he suggests that this property should characterize something referred to in the theorem such that from the proof it is evident that the conclusion depends on this property. What does Euler's theorem refer to? If we come back to the theorem, we can observe that it is about a spherical surface which remains rigid in the transformation (the distance between any two given points of the surface remains constant in the transformation), a point which is the center of such surface and which does not move during the transformation, and a line whose position is the same as before the transformation. Now, the object line cannot be taken into account as an object possessing a characterizing property, and this because it represents the result we want to prove. In fact, recall that according to Steiner the result must depend on the characterizing property. Therefore it would be a nonsense to say that the result depends on a characterizing property of the result itself. It can be thought that the characterizing property is the property the surface has to have a fixed point, or that the characterizing property is the property the point center has to remain in the same position. The proof and the result

26. In that case, the characterizing property of $SO(3)$ was 'having an odd dimension', and the generalizability of the theorem came from deforming the proof by replacing the dimension 3 by some odd number.

clearly depend on both these conditions. However, even if we assume that these properties characterize uniquely the two objects spherical surface and point—within the two families 'spherical surfaces subject to a transformation' and 'points subject to a transformation'—, it is difficult to see how criterion C_2 might apply. How do we deform the proof by substituting a characterizing property of a related entity? If we vary one of the two properties, the proof (and thus the theorem) will collapse. And this without telling us anything about new theorems.

On the other hand, we can turn our attention to the proof itself and observe how it depends on a great number of geometrical objects (lines, points, angles, arcs, ...). Although each of those elements belongs to a family of similar entities and can be characterized in terms of some specific property (according to Euclidean geometry), they all contribute to the proof-strategy and are therefore necessary to the validity of the proof. Is it possible to identify a characterizing property among those objects? Perhaps, a natural move in the identification of the characterizing property would be to consider the particular importance attributed by Euler to some particular step in the proof. As we have seen above, Euler's key step in his proof-strategy is to find a geometrical configuration in which point O has the same position before and after the motion of the spherical surface. To find this configuration, as Euler points out, amounts to find a configuration in which the spherical triangle cCO is isosceles. It is obvious that the object triangle is the focus of Euler's proof. And it is evident that the proof, together with the result, depends on the property this object has to be isosceles (if spherical triangle cCO were not isosceles, point O would not be fixed in the transformation and the theorem would collapse). This suggests that we can choose the characterizing property of triangle cCO as that of being isosceles, i. e. as 'having the angles $\angle cCO$ and $\angle CcO$ equal' (or, equivalently, 'having the two sides CMO and cmO equal'). This condition, which uniquely characterizes the triangle (it picks out that particular triangle within the family of triangles), marks a crucial step in Euler's proof and then it deserves a particular attention. Unfortunately, although this choice would reflect Steiner's core intuition because the proof depends on this particular property of the spherical triangle cCO, neither C_1 nor his second criterion C_2 can be applied. Concerning C_1, the entity triangle cCO is not mentioned in the theorem, and therefore this criterion is not satisfied. Moreover, recall that, according to C_2, if we deform the proof by "substituting the characterizing property of a related entity" we get "a related theorem". Here the entity under investigation is a triangle, and its characterizing property is 'having the angles $\angle cCO$ and $\angle CcO$ equal'. If we consider a related entity (another triangle), it is not clear how a substitution in its characterizing property could be made. According to Euler's constructive proof, if we consider a spherical triangle cCO with two angles not equal, the theorem fails to hold and we do not obtain new theorems.

There is, however, a second object mentioned in the proof that should be regarded as good candidate for having a characterizing property. In order the spherical triangle OCc to be isosceles, point M must be chosen such that CM be perpendicular to γ. It might be thought, then, that the proof depends on the property the arc CM has to be perpendicular to the angular bisector γ. Or, which amounts to the same thing, that the proof depends on the property the angle formed by $\angle cCO$ and $\angle \frac{ACc}{2}$ has to be a right angle. Although they both represent reasonable choices for a characterizing property, and they uniquely characterize the theorem, even in this case the proof would not meet Steiner's criteria. The two entities (the arc CM and the angle formed by $\angle cCO$ and $\angle \frac{ACc}{2}$) are not mentioned in the theorem, contrary to what is demanded by C_1. Furthermore, it is not clear how the deformability criterion C_2 is supposed to operate. How do we pick out a related entity with a replaced characterizing property? Again, it seems that Steiner's criterion C_2 cannot be applied simply because it is not possible to deform the proof in his sense.

My choices above were motivated by the fact that the individuation of a configuration in which the spherical triangle OCc be isosceles represents a crucial step in Euler's proof strategy. If there is a characterizing property, it is reasonable to concentrate on that step. Furthermore, this well reflects Steiner's idea about the dependence characterizing property-result: it must be evident, from the proof, that the result depends on the characterizing property. There are, of course, others objects in the proof which can be picked out as entities possessing a specific characterizing property. For instance, the property angles $\angle cCO$ and $\angle CcO$ have according to the expression (4), or the property the grand circle AB has to remain unaffected in length when moved to the new position ab. However, even if we consider these possibilities, Steiner's criterion C_1 is not satisfied (the entities in question are not mentioned in the theorem) and it is hard to see how C_2 might be applied. Neither the theorem nor our proof seems to be 'deformable', in Steiner's sense, to yield genuinely new results. It is not possible to deform the proof by "holding constant the proof idea" [Steiner 1978a] and getting something like the new theorem that we found in Steiner's example of Section 2.

Let me summarize the results of this Section. For the proof to count as explanatory in Steiner's sense it must make plain that criterion C_1 and C_2 are satisfied. Although the choice of characterizing property in Euler's proof seems to be extremely subject to arbitrariness, as previous criticisms to the model have shown ([Resnik & Kushner 1987]; [Hafner & Mancosu 2005]), I have considered some possible candidates. My choice was motivated by the importance that a property of some geometrical objects has in Euler's proof strategy. However, neither criterion C_1 nor criterion C_2 apply for these characterizing properties, thus blocking Steiner's theory from recognizing the proof as explanatory. Surely other possibilities might be considered, but a general observation seems to show the inapplicability of the model for the particular test-case: Euler's constructive proof is about a particular result and it seems that no deformation of it will tell us anything about new theorems. The moral is that Steiner's account fails in considering this proof as explanatory and this contrasts with Euler's intuitions as a mathematician. [27] Obviously, the previous considerations shift the burden of the proof to Steiner. If his theory of explanation is right, Steiner must show that it is able to recognize Euler's geometrical proof as an explanatory proof.

5 Conclusions and perspectives

If my analysis is correct, Steiner's model cannot account for the explanatory character of the geometrical proof given by Euler and therefore it should be refined or even rejected in favour of a different approach. Perhaps, the lesson we learn from Euler is that new directions of investigations are needed to adequately capture the notion of mathematical explanation. We have seen that Euler appealed to geometry to show why a particular geometrical result is true. Moreover, I pointed to the crucial role that Euclidean geometry, and more precisely a pure (geometrical) method, played in his explanatory practice. This suggests that it might be more philosophically profitable to abandon Steiner's idea that an explanatory proof depends on a particular property of an entity mentioned in the theorem in favour of an approach which focuses on the preferences expressed by the mathematicians for some mathematical concepts or for the particular mathematical framework used to prove a theorem. On the other hand, it might be thought that the notion of explanatory proof cannot be captured *simpliciter*, as Steiner proposes, but that there is a variety of explanatory proof-practices in mathematics. In this sense, Euler's practice would represent just an example of such practices.

It would be interesting to pursue these issues here, but that is obviously something which represents a fertile terrain for another study. Rather, the modest aim of this paper was to show that Steiner's theory has difficulties in accounting for a case of genuine mathematical explanation. Furthermore, and more important for the ambitions of the present thematic issue, I hope to have provided an example of how mathematical practice can inform the philosophy of mathematics. About 250 years passed by, however we can still learn from Euler.

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27. Note that in considering Euler's geometrical proof as a mathematical explanation I did not endorse any particular account of mathematical explanation, such as Steiner's. The fact that this is a mathematical explanation comes from Euler's practice as a mathematician and physicist, as I showed in the previous Section.

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