Reconstruction of a convolution kernel in an integrodifferential problem with a fractional time derivative

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Davide Guidetti*

Dipartimento di Matematica,
Università di Bologna
Piazza di Porta S. Donato 5,
40126 Bologna, Italy.
E-mail: davide.guidetti@unibo.it

1 Introduction

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Problem:

$$\begin{cases} D_t^\alpha u(t) = Au(t) + \int_0^t k(t-s)Bu(s)ds + F(t), \quad t \in [0,T], \\ u(0) = u_0, \\ \Phi(u(t)) = g(t), \quad t \in [0,T]. \end{cases} \tag{1.1}$$

eq1.1

Unknowns: u, k.

Basic assumptions:

(A1) X is a complex Banach space with norm $\|\cdot\|$, $\alpha \in (0,2)$, $D_t^\alpha u$ is the Caputo derivative of u with respect to t.

(A2) $A : D(A) \to X$ is a linear operator; there exist, M, $R \in \mathbb{R}^+$, such that $\{\lambda \in \mathbb{C} : |\lambda| \geq R, |Arg(\lambda)| \leq \frac{\alpha\pi}{2}\} \subseteq \rho(A)$, and, for λ in this set,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\lambda|^{-1},$$

$B \in \mathcal{L}(D(A), X)$.

(A3) $\Phi \in X'$.

Notation: if $\theta \in (0,2)$,

$$D_\phi(A) = \begin{cases} (X, D(A))_{\phi,\infty} & \text{if} \quad \phi \in (0,1), \\ D(A) & \text{if} \quad \phi = 1, \\ \{x \in D(A) : Ax \in (X, D(A))_{\phi-1,\infty} & \text{if} \quad \phi \in (1,2). \end{cases}$$

The following characterization of $D_\theta(A)$ $(0 < \theta < 1)$ holds (see ....)

*The author is member of GNAMPA of Istituto Nazionale di Alta Matematica

Theorem 1.1. *Suppose that S satisfies the condition (A2). Let $\theta \in (0,1)$. Then*

$$D_\theta(A) = \{x \in X : \sup_{\xi \geq R} \xi^\theta \|A(\xi - A)^{-1}x\| < \infty\}.$$

An equivalent norm in $D_\theta(A)$ is

$$\|x\|_\theta := \sup\{\|x\| + \xi^\theta \|A(\xi - A)^{-1}x\| : \xi \geq R\} = \sup\{\|x\| + \xi^\theta \|A(\xi - A)^{-1}x\| : \xi \geq R, \xi \in \mathbb{Q}\}.$$

Lemma 1.2. *Let (Ω, μ) a measure space and let $f : \Omega \to X$ be measurable. Then*
(I) the function $t \to \|f(t)\|_\theta$ is measurable ($\|f(t)\|_\theta = \infty$ if $f(t) \notin D_\theta(A)$);
(II) if $\int_\Omega \|f(t)\|_\theta d\mu < \infty$, $\int_\Omega f(t) d\mu \in D_\theta(A)$ and

$$\| \int_\Omega f(t) d\mu \|_\theta \leq \int_\Omega \|f(t)\|_\theta d\mu$$

Proof. (I) It follows from $\|f(t)\|_\theta = \sup_{\xi \geq R, \xi \in \mathbb{Q}} g_\xi(t)$, with

$$g_\xi(t) = \|f(t)\| + \xi^\theta \|A(\xi - A)^{-1} f(t)\|.$$

(II) If $\xi \geq R, \xi \in \mathbb{Q}$,

$$\| \int_\Omega f(t) d\mu\| + \xi^\theta \|A(\xi - A)^{-1} \int_\Omega f(t) d\mu\| \leq \int_\Omega g_\xi(t) d\mu \leq \int_\Omega \|f(t)\|_\theta d\mu.$$

Taking the supremum in ξ, we obtain the assertion.

□

We shall employ the following

th1.1 **Theorem 1.3.** *Let $\alpha \in (0,2)$. Consider system*

$$\begin{cases} D_t^\alpha v(t) = Av(t) + f(t), \quad t \in [0,T], \\ v^{(k)}(0) = v_k, \quad k < \alpha, \end{cases} \tag{1.2}$$ eq1.2

supposing that (A1)-(A2) hold; then:
*(I) (**??**) has, at most, one solution, for every $f \in C([0,T];X)$, $u_0 \in D(A)$, $u_1 \in X$ in case $\alpha > 1$ (solution means $D_t^\alpha v \in C([0,T];X)$, $v \in C([0,T];D(A))$).*
*(II) Let $\theta \in (0,1)$, $\alpha\theta \neq 1$. Then necessary and sufficient conditions implyng that (**??**) has a strict solution v such that $D_t^\alpha v$ and Av are bounded with values in $D_\theta(A)$) are :*

$$u_k \in D_{1+\theta-\frac{k}{\alpha}}(A) (k < \alpha), \quad f \in C([0,T];X) \cap B([0,T];D_\theta(A)).$$

(III) If $T_0 \in \mathbb{R}^+$, there exists $C(T_0) \in \mathbb{R}^+$ such that, if $0 < T \leq T_0$,

$$\|D_t^\alpha v\|_{B([0,T];D_\theta(A))} + \|v\|_{B([0,T];D_{1+\theta}(A))} \leq C(T_0)(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0,T];D_\theta(A))}).$$

Proof. Concerning (I)-(II), see ... . We show (III). We set $F : [0,T_0] \to D_\theta(A))$, $F(t) = f(t)$ if $0 \leq t \leq T$, $F(t) = f(t_0)$ if $T \leq t \leq T_0$. Let V be the solution of

$$\begin{cases} D_t^\alpha V(t) = AV(t) + F(t), \quad t \in [0,T_0], \\ V^{(k)}(0) = v_k, \quad k < \alpha. \end{cases}$$

Then $v = V_{|[0,T]}$, so that

$$\|D_t^\alpha v\|_{B([0,T];D_\theta(A))} + \|v\|_{B([0,T];D_\theta(A))}$$

$$\leq \|D_t^\alpha V\|_{B([0,T];D_\theta(A))} + \|V\|_{B([0,T];D_\theta(A))}$$

$$\leq C(T_0)(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|F\|_{B([0,T_0];D_\theta(A))})$$

$$= C(T_0)(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0,T];D_\theta(A))}).$$

Moreover, by ...

$$\|v\|_{C^\alpha([0,T];D_\theta(A))} \leq C(\alpha)\|D^\alpha v\|_{B([0,T];D_\theta(A))},$$

and $D(A) \in J_{1-\theta}(D_\theta(A), D_{1+\theta}(A))$, so that, if $0 \leq s < t \leq T$,

$$\|v(t) - v(s)\|_{D(A)} \leq C\|v(t) - v(s)\|_{D_\theta(A)}^\theta \|v(t) - v(s)\|_{D_{1+\theta}(A)}^{1-\theta}$$

$$\leq C_1(T_0)(t-s)^{\alpha\theta}(\sum_{k<\alpha} \|v_k\|_{D_{1+\theta-\frac{k}{\alpha}}(A)} + \|f\|_{B([0,T];D_\theta(A))}).$$

□

v can be represented in the form

$$v(t) = \sum_{k<\alpha} S_k(t)v_k + \int_0^t T(t-s)f(s)ds, \tag{1.3}$$ eq1.3

with

$$S_k(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1-k}(\lambda^\alpha - A)^{-1}d\lambda,$$

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda,$$

and Γ describing the boundary of

$$\{\lambda \in \mathbb{C} : |\lambda| \geq R^{\frac{1}{\alpha}}, |Arg(\lambda)| \leq \frac{\pi}{2} + \epsilon\},$$

with ϵ positive suitably small, oriented from $\infty e^{-i(\frac{\pi}{2}+\epsilon)}$ to $\infty e^{i(\frac{\pi}{2}+\epsilon)}$

le1.2 **Lemma 1.4.** *Suppose that (A1)-(A2) hold. Let $f_0 \in D_{\theta'}(A)$, with $\theta < \theta'$ and let*

$$z(t) = \int_0^t T(t-s)f(s)ds.$$

Then $Av \in C^1((0,T];X)$ and $\|(Av)'(t)\|_{D_\theta(A)} \leq Ct^{\alpha(\theta'-\theta)-1}$.

Proof. From (**??**), we have $z'(t) = T(t)f_0$ and, if $\in (0,T]$,

$$Az'(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\lambda^\alpha - A)^{-1}f_0 d\lambda.$$

We can assume $\theta' \in (\theta, 1)$. So, $D_{\theta'}(A) = (D_\theta(A), D_{1+\theta}(A))_{\theta'-\theta,\infty}$. This implies, for, $|\mu| \geq R$, $|Arg(\mu)| \leq \frac{\alpha\pi}{2}$,

$$\|A(\mu - A)^{-1}f_0\|_{D_\theta(A)} \leq C|\mu|^{\theta-\theta'}.$$

So

$$Az'(t) = \frac{1}{2\pi it}\int_{\Gamma'} e^\lambda A(\lambda^\alpha t^{-\alpha} - A)^{-1}d\lambda.$$

$$\|Az'(t)\|_{D_\theta(A)} \leq C_0 t^{-1}\int_{\Gamma'} e^{Re(\lambda)}|\lambda^\alpha t^{-\alpha}|^{\theta-\theta'}|d\lambda| \leq C_1 t^{\alpha(\theta'-\theta)-1}.$$

□

le1.4 **Lemma 1.5.** *Suppose that (A1)-(A2), $\alpha \in (1,2)$, $\theta < \frac{1}{\alpha}$. Let $f_0 \in D_{\theta'}(A)$, with $\theta' > \theta + 1 - \frac{1}{\alpha}$ and let*

$$z(t) = S_1(t)f_0.$$

Then $Av \in C^1((0,T];X)$ and $\|(Av)'(t)\|_{D_\theta(A)} \leq Ct^{\alpha(\theta'-\theta-1)}$. Consequently, if $\theta' > \theta + 1 - \frac{1}{\alpha}$,

$$\int_0^T \|Az'(t)\|_\theta dt < \infty.$$

Proof. If $t > 0$, we have

$$Az'(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}A(\lambda^\alpha - A)^{-1}f_0 d\lambda.$$

$$= \frac{1}{2\pi i t^\alpha}\int_\Gamma e^{\lambda}\lambda^{\alpha-1}((\tfrac{\lambda}{t})^\alpha - A)^{-1}f_0 d\lambda$$

so that

$$\|Az'(t)\|_\theta \leq C_0 t^{-\alpha}\int_\Gamma e^{Re(\lambda)}|\lambda|^{\alpha-1-\alpha(\theta'-\theta)}t^{\alpha(\theta'-\theta)}\|x\|_{\theta'}|d\lambda| \leq C_1 t^{\alpha(\theta'-\theta-1)}.$$

□

pr1.3 **Proposition 1.6.** *We consider the problem*

$$\begin{cases} D_t^\alpha u(t) = Au(t) + F(t), & t \in [0,T], \\ u^{(k)}(0) = u_k, \end{cases}$$

with the following conditions:
(a) $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}v_{[\alpha]}$, with $G \in C^1([0,T];X)$, $G' \in B([0,T];D_\theta(A))$, $v_{[\alpha]} \in D_{1+\theta-\frac{[\alpha]}{\alpha}}(A)$;
(b) $u_0 \in D_{1+\theta}(A)$, $Au_0 + F(0) \in D_{\theta'}(A)$, for some $\theta' > \theta$.
Then $u(t) = U(t) + z(t)$, with:
(I) $U \in C^1([0,T];X)$, $v = U'$ solution of

$$\begin{cases} D^\alpha v(t) = Av(t) + G'(t), & t \in [0,T], \\ v(0) = v_0; \end{cases} \quad (1.4) \quad \text{eq1.4}$$

(II) z solution of

$$\begin{cases} D^\alpha z(t) = Az(t) + Au_0 + F(0), & t \in [0,T], \\ z(0) = 0. \end{cases} \quad (1.5) \quad \text{eq1.5A}$$

Proof. By Theorem **??**, (**??**) has a unique solution v, with $D^\alpha v, Av \in C([0,T];X) \cap B([0,T];D_\theta(A))$. We deduce

$$(1 * D^\alpha v)(t) = A(1 * v)(t) + G(t) - G(0), \quad t \in [0,T].$$

We set

$$J_\alpha g(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds.$$

Then $D^\alpha v = (J_\alpha)^{-1}(v - v_0)$. We deduce that

$$J_\alpha(1 * D^\alpha v) = 1 * J_\alpha(D^\alpha v) = 1 * (v - v_0) = 1 * v - tv_0$$

and

$$D^\alpha(1 * v) = D^\alpha(tv_0) + 1 * D^\alpha v = \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}v_0 + 1 * D^\alpha v,$$

$$D^\alpha(1 * v) = A(1 * v)(t) + G(t) - G(0) + \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}v_0, \quad t \in [0,T].$$

Setting

$$U(t) = (1 * v)(t) + u_0,$$

we deduce

$$D^{\alpha}U(t) = AU(t) + G(t) - F(0) - Au_0 + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} v_0 = F(t) - F(0) - Au_0, \quad t \in [0,T].$$

The conclusion follows.

□

co1.4 **Corollary 1.7.** *Suppose that (A1)-(A2) hold. Suppose, moreover, that*
(a) $k \in C([0,T])$,
(b) $u \in C^1((0,T];D(A))$, $\|Au'(t)\|_{D_\theta(A)} \le Ct^{\epsilon-1}$, *for some* $\epsilon \in \mathbb{R}^+$;
(c) u *is a strict solution to*

$$\begin{cases} D^{\alpha}u(t) = Au(t) + \int_0^t k(t-s)Au(s)ds + F(t), & t \in [0,T], \\ u(0) = u_0, \quad t \in [0,T], \end{cases}$$

with $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v_0$, $G \in C^1([0,T];X)$, $G' \in B([0,T];D_\theta(A))$, $v_0 \in D_{1+\theta}(A)$, $u_0 \in D_{1+\theta}(A)$, $Au_0 + F(0) \in D_{\theta'}(A)$, $\theta' > \theta$.
Then $u(t) = U(t) + z(t)$, *with*
(I) $U \in C^1([0,T];X)$, $v = U'$ *solution of*

$$\begin{cases} D^{\alpha}v(t) = Av(t) + G'(t) + k(t)Au_0 + \int_0^t k(t-s)Au'(s)ds, & t \in [0,T], \\ v(0) = v_0; \end{cases} \tag{1.6}$$ eq1.5

(II) z *solution of*

$$\begin{cases} D^{\alpha}z(t) = Az(t) + Au_0 + F(0), & t \in [0,T], \\ z(0) = 0. \end{cases}$$

Proof. From the assumptions,

$$(k * Au)(t) = k(t)Au_0 + \int_0^t k(t-s)Au'(s)ds$$

belonging to $C([0,T];X) \cap B([0,T];D_\theta(A))$. So the conclusion follows from Proposition **??**.

□

re1.5 **Remark 1.8.** On account of Lemma **??**, (**??**) can be written also in the form

$$\begin{cases} D^{\alpha}v(t) = Av(t) + G'(t) + k(t)Au_0 + \int_0^t k(t-s)Av(s)ds + \int_0^t k(t-s)Az'(s)ds, & t \in [0,T], \\ v(0) = v_0. \end{cases} \tag{1.7}$$ eq1.6

We set

$$S(v,k)(t) := (k * A(v + z'))(t). \tag{1.8}$$ eq1.7

le1.6 **Lemma 1.9.** *Suppose that the assumptions of Corollary* **??** *are satisfied. Let* $\Phi \in X'$. *We set*

$$h(t) = g(t) - \Phi(z(t)).$$

We suppose $\Phi(Au_0) \ne 0$ *and set*

$$\chi := \Phi(Au_0)^{-1}.$$

Then $h \in C^1([0,1])$, $D^{\alpha}h'$ *is defined and*

$$k(t) = K_0(t) - \chi\Phi(Av(t)) - R(v,k)(t), \quad t \in [0,T], \tag{1.9}$$ eq1.8

with

$$K_0(t) = \chi[D^\alpha h'(t) - \Phi(G'(t))], \tag{1.10}$$

$$R(v,k) = -\chi\{k * \Phi[A(v+z')]\}(t) = -\chi\Phi[S(v,k)(t)]. \tag{1.11}$$ eq1.10

On the other hand,suppose that

$$\Phi(u_0) = h(0), \quad \Phi(v_0) = h'(0). \tag{1.12}$$ eq1.11

Let (v,k) be a strict solution to (??)-(??), with $v \in C([0,T];D(A)) \cap B([0,T];D_{1+\theta}(A))$ and $k \in C([0,T])$. We set

$$u := u_0 + 1 * v + z.$$

Then, $u \in C^1([0,T];D(A))$, $\|Au'(t)\|_{D_\theta(A)} \le Ct^{\alpha(\theta'-\theta)-1}$ $\forall t \in (0,T]$ and (u,k) is a solution to (??).

Proof. Applying Φ to the first equation in (??), we easily deduce (??).

On the other hand, let (v,k) be a strict solution to (??)-(??), with $v \in C([0,T];D(A)) \cap B([0,T];D_{1+\theta}(A))$ and $k \in C([0,T])$. Then

$$k(\cdot)Au_0 + k * A(v+z') = k(\cdot)Au_0 + k * Au'.$$

So, by Corollary ??, the two first conditions in (??) are satisfied.

It remains to show that $\Phi(u) = g$. Applying Φ to the first equation in (??) and comparing with (??), we deduce

$$D^\alpha(\Phi v)(t) = \Phi(D^\alpha v(t)) = D^\alpha h'(t), \quad t \in [0,T].$$

From (??), we deduce $\Phi v = h'$ and $\Phi U = h$. We deduce that

$$\Phi(u) = \Phi(U) + \Phi(z) = g.$$

□

In conclusion, we are reduced to study the system (??)-(??), which we write in the equivalent form

$$\begin{cases} D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(v,k)(t), \quad t \in [0,T], \\ v(0) = v_0, \\ k(t) = K_0(t) + \Psi(Av(t)) - R(v,k)(t), \quad t \in [0,T], \end{cases} \tag{1.13}$$ eq1.12

with

$$G_1(t) = G'(t) + K_0(t)Au_0,$$

$$\Psi = -\chi\Phi,$$

$$S_1(v,k)(t) = R(v,k)(t)Au_0 + S(v,k)(t), \tag{1.14}$$ eq1.13A

le1.7 **Lemma 1.10.** *Suppose that (A1)-(A2) hold. We consider the problem*

$$\begin{cases} D^\alpha v(t) = Av(t) + \Psi(Av(t))f_0 + f(t), \quad t \in [0,T], \\ v(0) = v_0, \end{cases} \tag{1.15}$$ eq1.13

Assume that $\Psi \in X'$, $f_0 \in D_\theta(A)$, $f \in C([0,T];X) \cap B([0,T];D_\theta(A))$, $v_0 \in D_{1+\theta}(A)$. Then (??) has a unique solution v in $C([0,T];D(A)) \cap B([0,T];D_{1+\theta}(A))$. Moreover, If $T_0 \in \mathbb{R}^+$, there exists $C(T_0) \in \mathbb{R}^+$ such that, if $0 < T \le T_0$,

$$\|v\|_{C^\alpha([0,T];D_\theta(A))} + \|v\|_{C^{\alpha\theta}([0,T];D(A))} + \|v\|_{B([0,T];D_{1+\theta}(A))} \le C(T_0)(\|v_0\|_{D_{1+\theta}(A)} + \|f\|_{B([0,T];D_\theta(A))}).$$

Proof. We set, for $0 < \tau \leq T$,

$$X_\tau := \{V \in C([0,\tau]; D(A)) : V(0) = v_0\},$$

which is a complete metric space with the distance

$$d(V_1, V_2) := \|V_1 - V_2\|_{C([0,T];D(A))}. \tag{1.16}$$ eq1.14A

If $V \in X(\tau)$, we consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + \Psi(AV(t))f_0 + f(t), & t \in [0,\tau], \\ v(0) = v_0, \end{cases} \tag{1.17}$$ eq1.14

which, by Theorem **??**, has a unique solution $v = v(V)$, belonging to $B([0,T]; D_{1+\theta}(A))$, with $D^\alpha v \in B([0,T]; D_\theta(A))$. Clearly, the solutions in $[0,\tau]$ are the fixed points of the mapping $V \to v(V)$. If $V_1, V_2 \in X_\tau$, we have, setting $v_j := v(V_j)$,

$$d(v_1, v_2) \leq C(T_0)\tau^{\alpha\theta}\|\Psi(A(V_1 - V_2))\|_{C([0,\tau])} \leq C_1(T_0)\tau^{\alpha\theta} d(V_1, V_2).$$

So, if τ is sufficiently small, (**??**) has a unique solution in $[0,\tau]$.

In order to extend it, we show that a solution with the desired regularity $\tilde{v}$ is given in $[0,\sigma]$, with $\sigma \in (0,T)$, it can be extended in a unique way to a solution, again with the prescribed regularity, in $[0,(\sigma+\delta) \wedge T]$. So we set now, for $\delta \in (0, T-\sigma]$,

$$Y_\delta :=:= \{V \in C([0,\sigma+\delta]; D(A)) : V_{|[0,\sigma]} = \tilde{v}\},$$

again equipped with the distance (**??**) (replacing T with $\sigma+\delta$). If $V \in Y_\delta$, we consider again the problem (**??**) in the interval $[0,\sigma+\delta]$. Again, by Theorem **??** we have a unique solution $v = v(V)$; by the uniqueness guaranteed by this theorem in $[0,\sigma]$, we deduce $v_{|[0,\sigma]} = \tilde{v}$, so that $v \in Y_\delta$. If $v_j = v(V_j)$, with $V_j \in Y_\delta$, $j = 1,2$, we deduce from Theorem **??** (III)

$$d(v_1, v_2) = \|v_1 - v_2\|_{C([0,\sigma+\delta];D(A))} \leq \delta^{\alpha\theta}\|v_1 - v_2\|_{C^{\alpha\theta}([0,\sigma+\delta];D(A))} \leq C(T_0)\delta^{\alpha\theta} d(V_1, V_2).$$

Choosing δ so small that $C(T_0)\delta^{\alpha\theta} < 1$ (independently of σ), we can extend in a unique way the solution to $[0,\sigma+\delta]$.

The remaining part of the proof is analogous to that of Theorem **??**.

□

Now we study problem (**??**). We indicate with V_0 the solution of the problem

$$\begin{cases} D^\alpha V_0(t) = AV_0(t) + G_1(t) + \Psi(AV_0(t)), & t \in [0,T], \\ v(0) = v_0 \end{cases} \tag{1.18}$$ eq1.16

and set

$$K_1(t) = K_0(t) + \Psi(AV_0(t)), \quad t \in [0,T],$$

Of course, the existence and uniqueness of a solution V_0 in $B([0,T]; D_\theta(A))$ is guaranteed by Lemma **??**. We begin with the existence and, to some extent, uniqueness of a solution in a small interval:

Lemma 1.11. *Let $\delta \in \mathbb{R}^+$. Then there exists $\tau(\delta) \in (0,T]$, such that, if $0 < \tau \leq \tau(\delta)$ (**??**) has a unique solution (v,k) with $D^\alpha v$, Av in $B([0,\delta]; D_\theta(A))$, $k \in C([0,\delta])$ and*

$$\max\{\|v - V_0\|_{B([0,\tau];D_\theta(A))}, \|k - K_0\|_{C([0,\tau])} \leq \delta.$$

Proof. We set, for $\tau \in (0,T]$,

$X_{\delta,\tau} := \{(V,H) \in (C([0,\tau];D(A)) \cap B([0,\tau];D_{1+\theta}(A))) \times C([0,\tau]) : \max\{\|v - V_0\|_{B([0,\tau];D_\theta(A))}, \|H - K_1\|_{C([0,\tau])} \leq \delta\}$,

which is a complete metric space with the distance

$$d((V_1,H_1),(V_2,H_2)) = \max\{\|V_1 - V_2\|_{B([0,\tau];D_{1+\theta}(A))}, \|K_1 - K_2\|_{C([0,\tau])}\}.$$

Given (V,H) in $X_{\delta,\tau}$, we consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(V,H)(t), \quad t \in [0,T], \\ v(0) = v_0, \\ k(t) = K_0(t) + \Psi(Av(t)) - R(V,H)(t), \quad t \in [0,T], \end{cases} \tag{1.19}$$ eq1.17

By Lemma **??**, (**??**) has a unique solution (v,k) with the prescribed regularity. Clearly, as usual, solving (**??**) is equivalent to find a fixed point of $(V,H) \to (v,k)$.

From (**??**), we get

$$\begin{cases} D^\alpha (v - V_0)(t) = A(v - V_0)(t) + \Psi(A(v - V_0)(t))Au_0 + S_1(V,H)(t), \quad t \in [0,T], \\ (v - V_0)(0) = 0, \\ k(t) - K_1(t) = \Psi(A(v - V_0)(t)) - R(V,H)(t), \quad t \in [0,T], \end{cases}$$

so that

$$\|v - V_0\|_{B([0,\tau];D_{1+\theta}(A))} \leq C(T)\|S_1(V,H)\|_{B([0,\tau];D_\theta(A))}$$

We estimate $\|\|S_1(V,H)\|_{B([0,\tau];D_\theta(A))}$. By (**??**), (**??**), (**??**) and Lemma **??**we have

$$\|S_1(V,H)\|_{B([0,\tau];D_\theta(A))} \leq C_0\|S(V,H)\|_{B([0,\tau];D_\theta(A))} \leq C_1\|H\|_{C([0,\tau]}(\tau\|V\|_{B([0,\tau];D_{1-\theta}(A)} + \tau^{\alpha(\theta'-\theta)})$$
$$\leq C_1(\|K_1\|_{C([0,T)} + \delta)[\tau(\|V_0\|_{B([0,\tau];D_{1-\theta}(A)} + \delta) + \tau^{\alpha(\theta'-\theta)}] := \omega_0(\delta,\tau).$$

So

$$\|v - V_0\|_{B([0,\tau];D_{1+\theta}(A))} \leq C(T)\omega_0(\delta,\tau).$$

We have also

$$\|k - K_1\|_{C([0,\tau])} \leq C_1\|v - V_0\|_{B([0,\tau];D_{1+\theta}(A))} + \|R(V,H)\|_{C([0,\tau])} \leq C_2\omega_0(\delta,\tau).$$

As $\lim_{\tau\to 0}\omega_0(\delta,\tau) = 0$, if $\tau \leq \tau_0(\delta)$ and $(V,H) \in X_{\delta,\tau}$, $(v,k) \in X_{\delta,\tau}$.

Let now $(V_1,H_1),(V_2,H_2)$ belong to $X_{\delta,\tau}$. We indicate with (v_j,k_j) $(j = 1,2)$ the corresponding solutions of (**??**). It follows

$$\begin{cases} D^\alpha (v_1 - v_2)(t) = A(v_1 - v_2)(t) + \Psi(A(v_1 - v_2)(t))Au_0 + S_1(V_1,H_1)(t) - S_1(V_2,H_2)(t), \quad t \in [0,\tau], \\ (v_1 - v_2)(0) = 0, \\ k_1(t) - k_2(t) = \Psi(A(v_1 - v_2)(t)) - (R(V_1,H_1)(t) - R(V_2,H_2)(t)), \quad t \in [0,\tau]. \end{cases}$$

We have

$$\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} \leq C_0(T)\|S_1(V_1,H_1)(t) - S_1(V_2,H_2)\|_{B([0,\tau];D_\theta(A))}$$

$$\leq C_1(T)\|S(V_1,H_1)(t) - S(V_2,H_2)\|_{B([0,\tau];D_\theta(A))}$$

$$\leq C_1(T)(\|(H_1 - H_2) * A(V_1 + z')\|_{B([0,\tau];D_\theta(A))} + \|H_2 * A(V_1 - V_2)\|_{B([0,\tau];D_\theta(A))})$$

$$\leq C_2(T)[\|H_1 - H_2\|_{C([0,\tau])}(\tau(\|V_0\|_{B([0,T];D_{1+\theta}(A))} + \delta) + \tau^{\alpha(\theta'-\theta)})$$

$$+\tau(\|K_1\|_{C([0,T])} + \delta)\|V_1 - V_2\|_{B([0,\tau];D_{1+\theta}(A)}]$$

$$\leq \omega_1(\delta,\tau)d((V_1,H_1),(V_2,H_2)),$$

with $\lim_{\tau\to 0}\omega_1(\delta,\tau) = 0$. It follows

$$\|k_1 - k_2\|_{C([0,\tau])}$$

$$\leq C_2(\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} + \|R(V_1,H_1) - R(V_2,H_2)\|_{C([0,\tau])})$$

$$\leq C_3(\|v_1 - v_2\|_{B([0,\tau];D_{1+\theta}(A))} + \|S(V_1,H_1)(t) - S(V_2,H_2)\|_{B([0,\tau];D_\theta(A))})$$

$$\leq C_3\omega_1(\delta,\tau)d((V_1,H_1),(V_2,H_2)).$$

So the conclusion follows from the contraction mapping theorem. □

We want to show that, in fact, (**??**) has a unique global solution. The key step is the following

Lemma 1.12. *Suppose that (A1)-(A3) hold. Consider problem (**??**), with $G_1 \in C(]0,T];X) \cap B([0,T];D_\theta(A))$, $u_0, v_0 \in D_{1+\theta}(A)$. Let $0 < \tau_0 \leq \tau_1 < \min\{2\tau_0, T\}$ and let (V,K) be a solution in $[0,\tau_1]$, with $V \in B([0,\tau_1];D_{1+\theta}(A))$, $K \in C([0,\tau_1])$. Then there exists δ positive, independent of τ_1, such that (**??**) has a unique solution (v,k) in $[0,(\tau_1+\delta) \wedge 2\tau_0 \wedge T]$ with $v \in B([0,(\tau_1+\delta) \wedge 2\tau_0 \wedge T];D_{1+\theta}(A))$, $k \in C([0,(\tau_1+\delta) \wedge 2\tau_0 \wedge T])$ and coinciding with (V,K) in $[0,\tau_1]$.*

Proof. Let $\delta \in \mathbb{R}^+$. We set

$$\tau(\delta) := (\tau_1 + \delta) \wedge (2\tau_0) \wedge T$$

and

$$X_\delta := \{(W,H) \in (C([0,\tau(\delta)];X) \cap B([0,\tau(\delta)];D_{1+\theta}(A))) \times C([0,\tau(\delta)])$$

$$: W_{|[0,\tau_1]} = V, H_{|[0,\tau_1]} = K\}.$$

For $(W,H) \in X_\delta$, we consider the problem

$$\begin{cases} D^\alpha v(t) = Av(t) + G_1(t) + \Psi(Av(t))Au_0 + S_1(W,H)(t), & t \in [0,(\tau_1+\delta) \wedge 2\tau_0]], \\ v(0) = v_0, \\ k(t) = K_0(t) + \Psi(Av(t)) - R(V,H)(t), & t \in [0,(\tau_1+\delta) \wedge 2\tau_0], \end{cases} \tag{1.20}$$

eq1.19

For any $(W,H) \in X_\delta$, (**??**) has a unique solution (v,k) with $v \in B([0,\tau(\delta)];D_{1+\theta}(A))$, $k \in C([0,\tau(\delta)])$. We observe that, by the uniqueness of the solution of (**??**), $v_{|[0,\tau_1]} = V$ and $k_{|[0,\tau_1]} = K$. We deduce that $(v,k) \in X_\delta$, which we equip with the usual distance

$$d((V_1,H_1),(V_2,H_2)) = \max\{\|V_1 - V_2\|_{B([0,\tau(\delta)];D_{1+\theta}(A)))}, \|H_1 - H_2\|_{C([0,\tau(\delta)])}\}.$$

Now we look for conditions ensuring that the mapping $(W,H) \to (v,k)$ is a contraction in X_δ. As usual, we get

$$d((v_1,k_1),(v_2,k_2)) \leq C(T)\|S(V_1,H_1) - S(V_2,H_2)\|_{B([0,\tau(\delta)];D_\theta(A)))}.$$

Let $\tau_1 \le t \le \tau(\delta)$. Then

$$\|S(V_1, H_1)(t) - S(V_2, H_2)(t)\|_{D_\theta(A)}$$

$$\le \| \int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s) + z'(s))ds\|_{D_\theta(A)} + \| \int_0^t H_2(t-s)(A(V_1(s) - V_2(s))ds\|_{D_\theta(A)}.$$

We set $\tilde{v} := V_{|[0,\tau_0]}$, $\tilde{h} := H_{|[0,\tau_0]}$. Then we have, on account of $t - \tau_1 \le \tau_0$,

$$\int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s) + z'(s))ds = \int_0^{t-\tau_1} (H_1(t-s) - H_2(t-s))A(\tilde{v}(s) + z'(s))ds,$$

so that

$$\| \int_0^t (H_1(t-s) - H_2(t-s))A(V_1(s)+z'(s))ds\|_{D_\theta(A)} \le \|H_1 - H_2\|_{C([0,\tau(\delta)])}(\|\tilde{v}\|_{B([0,\tau_0];D_{1+\theta}(A))}\delta + C_0\delta^{\alpha(\theta'-\theta)}).$$

Analogously,

$$\| \int_0^t H_2(t-s)(A(V_1(s) - V_2(s))ds\|_{D_\theta(A)} = \| \int_{\tau_1}^t \tilde{h}(t-s)(A(V_1(s) - V_2(s))ds\|_{D_\theta(A)}$$

$$\le \delta \max(|\tilde{h}|)\|V_1 - V_2\|_{B([0,\tau(\delta)];D_{1+\theta}(A))}$$

We deduce that

$$\|v_1 - v_2\|_{B([0,\tau(\delta)];D_{1+\theta}(A))} \le \omega_0(\delta)d((V_1, H_1), (V_2, H_2)),$$

with $\lim_{\delta\to 0} \omega_0(\delta) = 0$. We observe that $\omega(\delta)$ does not depend on τ_1. We have also

$$\|k_1 - k_2\|_{C([0,\tau(\delta)]} \le \|\Psi(A(V_1 - V_2))\|_{C([0,\tau(\delta)]} + \|R(V_1, H_1) - R(V_2, H_2)\|_{C([0,\tau(\delta)]} \le \omega_1(\delta)d((V_1, H_1), (V_2, H_2)),$$

with $\lim_{\delta\to 0} \omega_1(\delta) = 0$, and the conclusion follows.

□

Now we are able to prove the main result of the paper:

th1.10 **Theorem 1.13.** *Suppose that (A1)-(A3). Consider problem* **??**, *with u, k unknown. Assume that the following further conditions are fulfilled:*

(a) $\alpha \in (0, 1]$;

(b) $F(t) = G(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}v_0$, with $G \in C^1([0, T]; X)$, $G' \in B([0, T]; D_\theta(A))$, $\theta \in (0, 1)$, $v_0 \in D_{1+\theta}(A)$;

(c) $u_0 \in D_{1+\theta}(A)$;

(d) $Au_0 + F(0) \in D_{\theta'}(A)$, with $\theta < \theta'$;

(e) $\Phi \in X'$;

*(f) if z if the solution of (**??**) and $h(t) = g(t) - \Phi(z(t))$, $D^{1+\alpha}h \in C([0, T])$, $h(0) = \Phi(u_0)$, $h'(0) = \Phi(v_0)$;*

(g) $\Phi(Au_0) \neq 0$.

*Then (**??**) has a unique solution (u, k) such that $u - z \in C^1([0, T]; D(A))$, $(u - z)' \in B([0, T]; D_\theta(A))$, $k \in C([0, T])$.*

Proof. If (u, k) is a solution with the required properties, $k * Au \in C^1([0, T]; X)$ and $(k * Au)' \in B([0, T]; D_\theta(A))$. So, by Corollary **??**, $u = U + z$, with $v = U'$ solution of (**??**), or, equivalently (**??**).

On the other hand, if v is a solution of (**??**), $u := U + z$, with $U := u_0 + 1 * v$, satisfies the two first equations in (**??**). From (**??**) we have also $\Phi(U) = h$ and $\Phi(D^\alpha v) = D^{1+\alpha}h$. Applying Φ to the first equation in (**??**), on account of (g), we deduce (**??**).

.....

□

[**?**] Problem of determination from final data (not convolution kernels).

Paper [**?**] Reconstruction of a kernel m such that $k = a + m$, applicable in case $\alpha \le 1$. Even in this case needed not so mich regularity, but also more compatibility conditions than here.

[**?**] Determination of order of derivation α and coefficient of the second order space derivative $\alpha \in (0, 1)$. Hilbert space setting. The operator A with conditions on the spectrum which are satisfied by a positive self-adjoint compcat operator. Assumptions on the Fourier coefficients on the data.

Determination of source term: [**?**],

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