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Analytical solution for channel flow of a Giesekus fluid with non-zero solvent viscosity

Irene Daprà [a,*], Giambattista Scarpi [b]

[a] DICAM University of Bologna, Italy
[b] University of Bologna, Italy

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ABSTRACT

A semi analytical solution is obtained here for the fully developed plane Poiseuille flow of a Giesekus fluid with a Newtonian solvent. The fluid behaviour is described using the Deborah number, the mobility factor and an appropriate ratio of fluid viscosity to total viscosity as parameters. The given solution shows that the velocity increases significantly with rising the polymer concentration, confirming that dilution of the solution produces the same effect as an increase in resistance. The analysis demonstrates that there are limiting values of Deborah number related to the mobility parameter.

1. Introduction

Viscoelastic fluids have important industrial applications such as filament stretching, plastic extrusion, injection moulding, oil well drilling, container filling, slurry suspension. In addition, they are sometimes used in heat transfer equipment, in pharmacology, in foodstuffs (Caglar Duvarci et al. [1]), in inkjet printers and many other cases. These fluids exhibit non-linear behaviour and therefore an exact solution is almost impossible even for simple geometries. There are a large number of possible non-linear rheological models to adequately describe the behaviour of complex polymers, one of these is the Giesekus model that uses three physical parameters: viscosity, mobility factor, and relaxation time to characterise the fluid (Giesekus [2,3]). The experimental determination of the mobility factor is well documented in Debbaut and Burhin [4] and in Calin et al. [5] where a series of experiments are performed to characterize a high-density polyethylene fluid, with direct reference to a Giesekus model. In [6], Rehage and Fuchs perform a series of rheological experiments using steady-state shear flow and large amplitude oscillating shear regime to examine the correspondence of the theoretical behaviour of the Giesekus fluid model with the measured data and they find excellent agreement in steady state flow, while when a large oscillating amplitude is applied the fluid shows instability.

The rheological law of non-Newtonian fluids is non-linear, and thus the practical problems involving them are difficult to solve analytically due to their complexity. For this reason, numerical methods for the simulation of non-Newtonian fluid flows have been an important branch of research. Restricting ourselves to analytical solutions, amongst the first to deal with plane Couette and Poiseuille flows of Giesekus fluid, Yoo and Choi [7] provide a careful analysis on the existence of the solution by distinguishing two ranges of variation of the mobility parameter and giving for each the limit values of the Deborah number beyond which no solution can be found. Schleiniger and Weinacht [8] analyse the solutions for the Poiseuille flow with and without the viscosity contribution of the solvent and determine, although in implicit form, the one with physical significance for plane and axi-symmetric flow.

The available explicit solutions are usually restricted to linear approximation, thus limiting their validity to a small range of the involved parameters' values as in Raisi et al. [9]. Daprà and Scarpi [10] provide a semi-analytical solution for a plane Poiseuille flow: the presented solution applies to the entire range of values of the physical and geometric parameters involved. Ferrás et al. [11] propose a semi-analytical solution for a channel flow with wall slip.

The addition of a solvent amplifies the effects of the non-linear terms in the constitutive equation increasing the already considerable difficulties in finding an analytical solution even in cases of simple geometries. As an example, with different models, Cruz et al. [12] provide an analytical solution for fluids whose polymer rheological behaviour is described by the PTT and FENE-P models and the solvent contribution is due to a Newtonian fluid. They develop in detail the case of the flow in a circular cross section pipe and extend the solution to the channel flow.

In their work [13] Araujo et al. show a semi-analytical method to obtain the streamwise velocity component and the components of the

* Corresponding author.
E-mail address: irene.dapra@unibo.it (I. Daprà).

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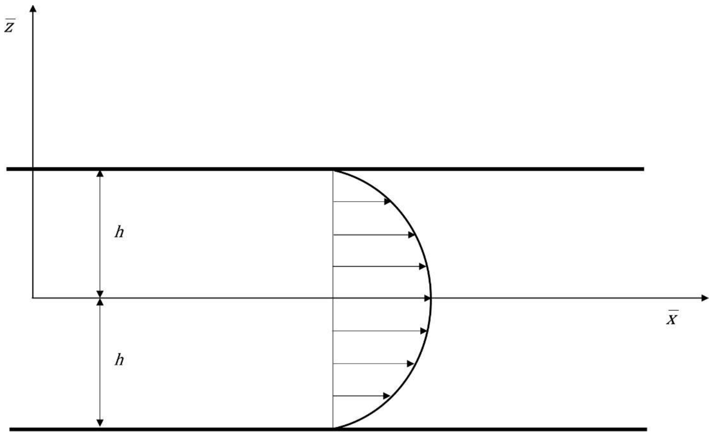

Fig. 1. Scheme of the channel.

extra-stress tensor using the LPTT viscoelastic fluid model for channel and pipe flow without simplifications and considering the solvent contribution in the homogeneous mixture.

Considering a Giesekus fluid and using a different approach from the classical one, in which the independent variable is the distance from the centre of the channel, da Silva Furlan et al. [14] solve the Poiseuille flow analytically. They rewrite the system of equations by choosing a component of the stress tensor as the independent variable and compare the results obtained first with those of Schleiniger and Weinacht and then with the solutions obtained by a numerical simulation of the governing equations, the Navier–Stokes and the constitutive equations, using high-order methods to obtain the solution. With this approach, the stress tensor components are obtained analytically, whereas the velocity profile is obtained with a higher-order numerical integration method.

The aim of the present work is to elaborate an analytical solution for a plane flow of a Giesekus fluid, considering both the full non-linearity of the constitutive equation and the contribution of a Newtonian solvent. The problem is defined by specifying the geometry and the governing equations in dimensionless form. The velocity distribution can be calculated analytically as a function of the mobility factor, of the Deborah number and of the ratio between the viscosity of the solvent and the total viscosity. The existence and validity range of the solution are also analysed. Finally, the obtained results are presented and discussed in detail.

2. Problem setting and governing equations

A plane layer of constant thickness $2h$ filled with an incompressible Giesekus fluid is considered; the motion is steady and laminar, under the action of a constant pressure gradient (Fig. 1).

For a steady rectilinear flow in the direction of $\overline{x}$ axis, the continuity and momentum equations are

$$\frac{dv}{\partial \overline{x}} = 0 \tag{1}$$

$$-\nabla \overline{p} + \nabla \cdot T = 0 \tag{2}$$

where v is the vector velocity, $\overline{p}$ the pressure, T the total stress tensor and $-h \leq \overline{z} \leq h$.

The total stress tensor for a Giesekus fluid with non-zero solvent viscosity can be written as

$$T = T_p + T_s \tag{3}$$

the first part refers to the polymer contribution and satisfies the equation

$$T_p + \lambda \left[\frac{\partial T_p}{\partial \overline{t}} + v \cdot \nabla T_p - T_p \cdot \nabla v - (\nabla v)^T \cdot T_p \right] + \frac{\lambda \beta}{\mu} T_p \cdot T_p = \mu \left[\nabla v + (\nabla v)^T \right] \tag{4}$$

where μ is the zero-shear rate viscosity of the polymer, λ the stress relaxation time and β the dimensionless mobility parameter ($0 \leq \beta \leq 1$).

Main symbols adopted:	
De Deborah number	ε viscosity ratio
T_p polymer contribution to total stress tensor	η solvent viscosity
T_s solvent contribution to total stress tensor	μ polymeric viscosity
β mobility parameter	λ stress relaxation time

The solvent contribution to the total stress is

$$T_s = \eta \left[\nabla v + (\nabla v)^T \right] \tag{5}$$

where η is the viscosity of the solvent.

3. Semi analytical solution

In order to develop the foregoing analysis, we introduce the non-dimensional quantities: $x = \overline{x}/h$, $z = \overline{z}/h$, $u = v(\mu + \eta)/(Ph^2)$, $p = \overline{p}/(Ph)$, $t = \overline{t}Ph/(\mu + \eta)$, $\tau = T/(Ph)$, $De = \lambda Ph/\mu$, where $P = -\partial \overline{p}/\partial \overline{x}$. Eqs. (1) and (2) become respectively

$$\frac{du}{dx} = 0 \tag{6}$$

$$-\frac{\partial p}{\partial x} + \nabla \cdot (\tau_p + \tau_s) = -\frac{\partial p}{\partial x} + \nabla \cdot \tau_p + \varepsilon \nabla^2 u = 0 \tag{7}$$

where $\varepsilon = \eta/(\mu + \eta)$ is the viscosity ratio, $0 \leq \varepsilon \leq 1$: if $\varepsilon = 0$ the fluid consists only of the polymer; if $\varepsilon = 1$ only the solvent is present. The non-dimensional constitutive equation of the Giesekus fluid and of the solvent become

$$\tau_p + De(1-\varepsilon) \left[\frac{\partial \tau_p}{\partial t} + u \cdot \nabla \tau_p - \tau_p \cdot \nabla u - (\nabla u)^T \cdot \tau_p \right] + \beta De \tau_p \cdot \tau_p = (1-\varepsilon) \left[\nabla u + (\nabla u)^T \right] \tag{8}$$

$$\tau_s = \varepsilon \left[\nabla u + (\nabla u)^T \right] \tag{9}$$

For steady Poiseuille flow the continuity Eq. (6) is verified, and the momentum Eq. (7) has two scalar components. Except for $p = p(x, z)$ which depends linearly on x, all the other quantities depend only on z.

$$1 + \tau'_{pxz} = -\varepsilon u'' \tag{10}$$

$$\frac{\partial p}{\partial z} + \tau'_{pzz} = 0 \tag{11}$$

where a prime indicates d/dz. Eq. (11) allows to write $p(x, z)$ as

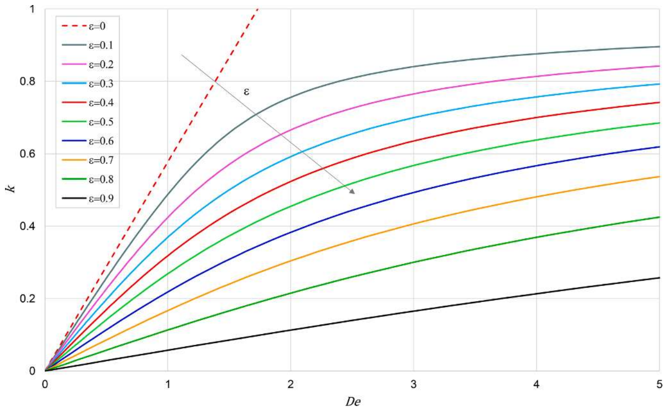

Fig. 2. The parameter k as a function of Deborah number for some values of ε for β=0.25.

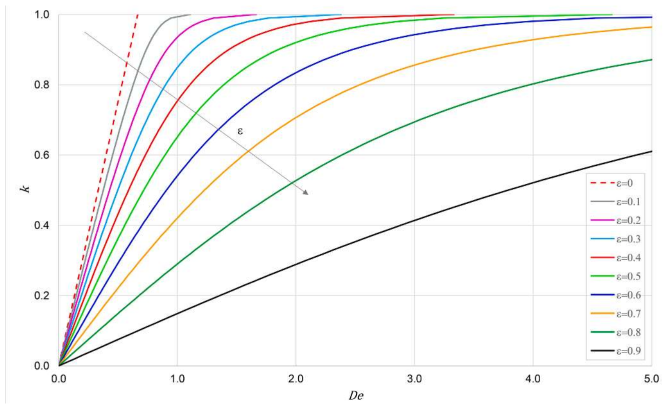

Fig. 3. The parameter k as a function of Deborah number for some values of ε for β=0.75.

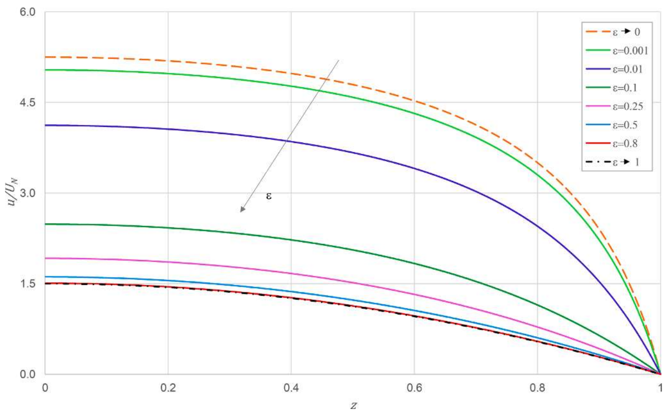

Fig. 4. Normalized velocity profile for some values of **ε** for **β** = 0.25 and = 1.5 .

$$p(x,z) = p_0 - x - \tau_{pzz}$$

here p_0 is the assigned pressure at a given point, e.g. at $(x,z) = (0,0)$.

The non-zero components of Eqs. (8) and (9) are

$$\tau_{pxx} - 2De\tau_{pxz}(1-\varepsilon)u' + \beta De\left(\tau_{pxx}^2 + \tau_{pxz}^2\right) = 0 \tag{12}$$

$$\tau_{pzz} + \beta De\left(\tau_{pxz}^2 + \tau_{pzz}^2\right) = 0 \tag{13}$$

$$\tau_{pxz} - De\tau_{pzz}(1-\varepsilon)u' + \beta De\tau_{pxz}\left(\tau_{pxx} + \tau_{pzz}\right) = (1-\varepsilon)u' \tag{14}$$

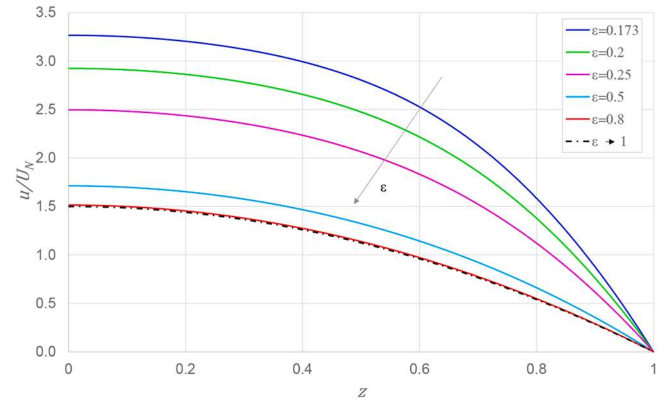

Fig. 5. Normalized velocity profile versus z for some values of **ε** for **β** = 0.75 and ***De*** = 1.5.

$$\tau_{sxz} = \varepsilon \frac{\partial u}{\partial z} \tag{15}$$

Eqs. (12)-(15) contain the mobility parameter β and the Deborah number De which encloses both physical parameters of the polymer (viscosity and relaxation time) and the pressure gradient, which gives rise to the motion.

Because of the symmetry of the flow field, only the region defined by $-1 \le z \le 0$ is analysed; the appropriate boundary conditions are

$$u(-1) = 0 \tag{16}$$

$$\left.\frac{du}{dz}\right|_{z=0} = 0 \tag{17}$$

Solving Eq. (17) with respect to τ_{pzz} gives

$$\tau_{pzz} = \frac{-1 \pm \sqrt{\varphi(\tau_{pxz})}}{2\beta De} \tag{18}$$

where

$$\varphi(\tau_{pxz}) = 1 - 4\beta^2 De^2 \tau_{pxz}^2 \tag{19}$$

obviously $\varphi(\tau_{pxz})$ must be non-negative. According to [8], to have a stable behaviour the positive sign should be taken in Eq. (18).

The other polymeric normal stress τ_{pxx} can be obtained from Eq. (14)

$$\tau_{pxx} = \frac{(1-\varepsilon)u'(1 + De\tau_{pzz})}{\beta De\tau_{pxz}} - \frac{1 + \beta De\tau_{pzz}}{\beta De} \tag{20}$$

therefore, both normal stresses can be expressed as a function of tangential stress.

As $\beta \to 0$ and $\varepsilon = 0$, Giesekus model reduces to the upper convected Maxwell model. As $\beta \to 0$ or $De \to 0$, $\tau_{pzz} \to 0$ and $\tau_{pxz} \to (1-\varepsilon)u'$. Even the normal stress $\tau_{pxx} \to 0$ as Deborah number tends to zero, while $\tau_{pxx} \to -De\tau_{pxz}^2$ as $\beta \to 0$.

The integration of Eq. (10) gives

$$\tau_{pxz} = -(z + \varepsilon u') \tag{21}$$

Following classical developments [8], from the previous relations an algebraic equation of sixth degree with respect to u' is obtained; if the solvent is present ($\varepsilon \neq 0$), this equation is not solvable in a closed form. However, if z is expressed as a function of τ_{pxz}, $z = -(\tau_{pxz} + \varepsilon u')$, a second-degree equation with respect to u' is derived. Therefore, based on Eqs. (12)-(14) and some algebraic steps given in detail in Appendix A, the equation for u' is as follows

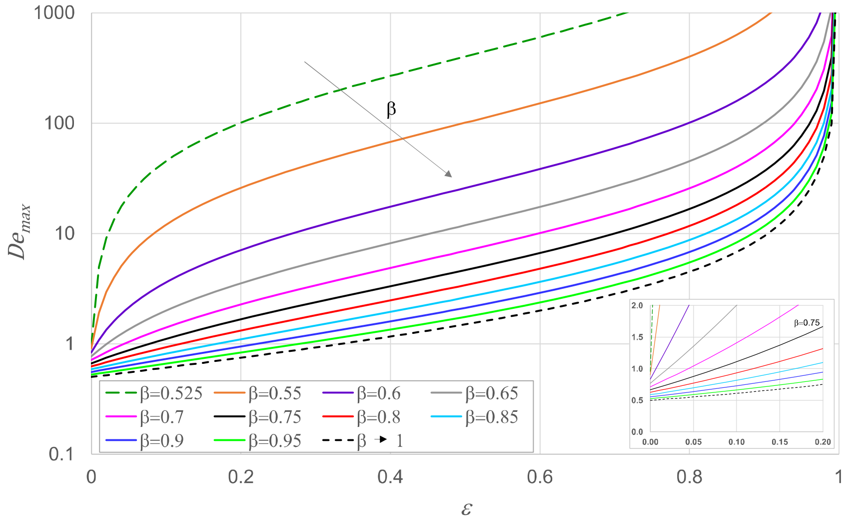

Fig. 6. The maximum value of De_{max} as a function of ε for some values of β.

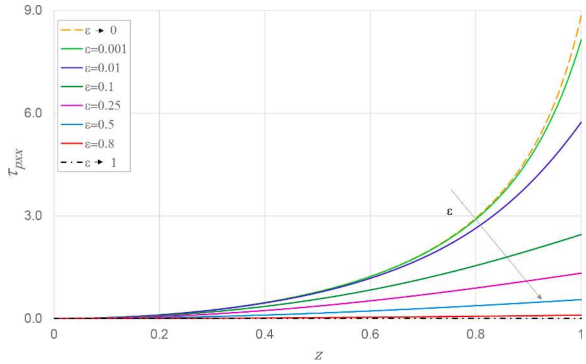

Fig. 7. Normal stress τ_{pxx} versus distance from channel axis for some values of ε for $\beta = 0.25$ and $De = 1.5$.

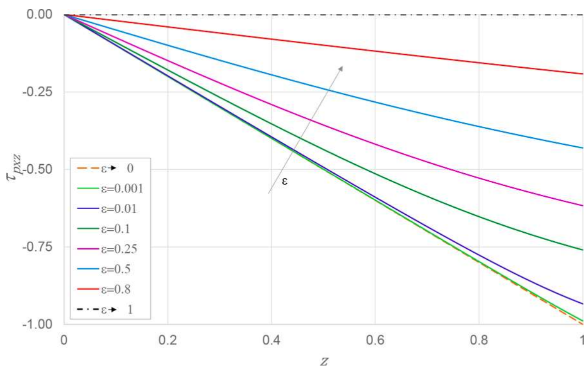

Fig. 8. Tangential stress τ_{pxz} versus z for some values of ε for $\beta = 0.25$ and $De = 1.5$.

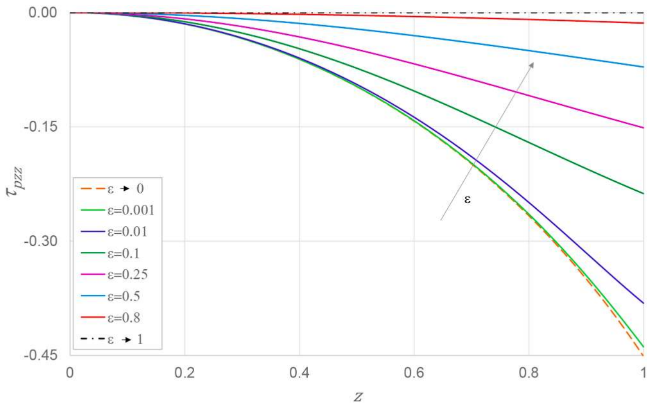

Fig. 9. Normal stress τ_{pzz} versus z for some values of ε for $\beta = 0.25$ and $De = 1.5$.

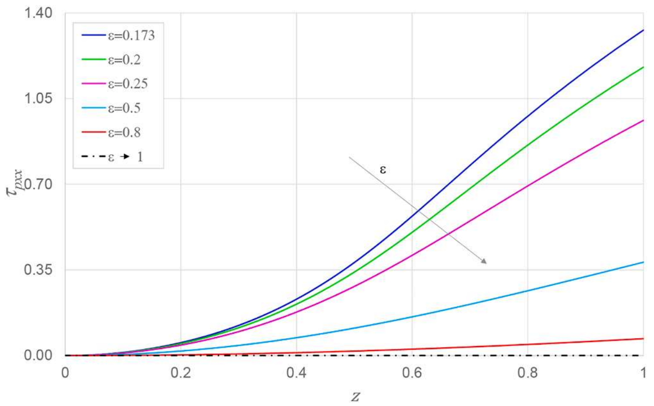

Fig. 10. Normal stress τ_{pxx} versus z for some values of ε for $\beta = 0.75$ and $De = 1.5$.

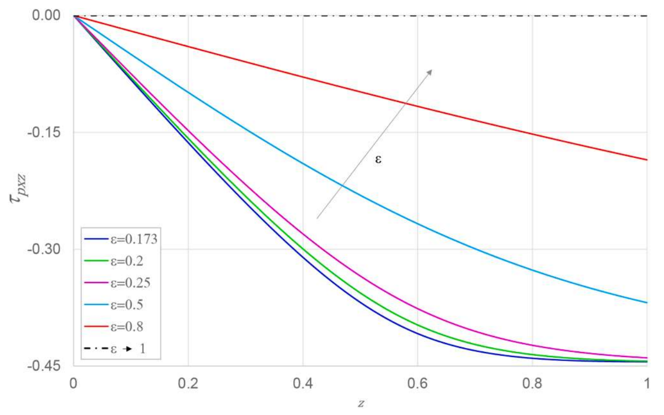

Fig. 11. Tangential stress $\boldsymbol{\tau}_{pxz}$ versus z for some values of **ε** for $\boldsymbol{\beta} = 0.75$ and $\boldsymbol{De} = 1.5$.

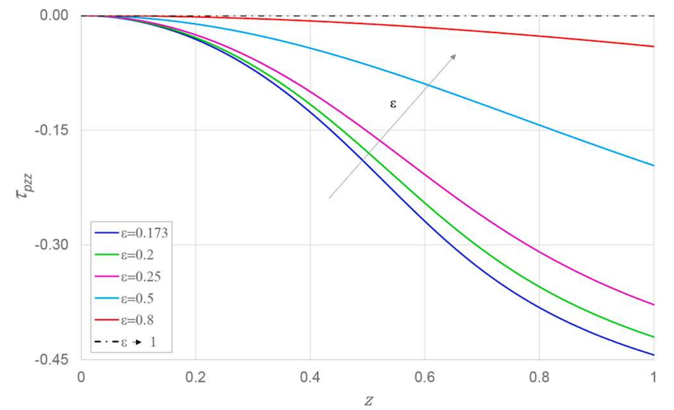

Fig. 12. Normal stress $\boldsymbol{\tau}_{pzz}$ versus z for some values of **ε** for $\boldsymbol{\beta} = 0.75$ and $\boldsymbol{De} = 1.5$.

$$u'^2(1-\varepsilon)^2\left(\tau_{pxz}^2 De^2\beta - 1 + \beta\right)^2 - u'\tau_{pxz}(1-\varepsilon)\left[\tau_{pxz}^2 De^2\beta\left(8\beta^2 - 8\beta + 1\right) - \beta + 1\right] + \tau_{pxz}^4 De^2\beta^2(2\beta - 1)^2 + \tau_{pxz}^2\beta(1-\beta) = 0 \tag{22}$$

which can be easily solved:

$$u' = \frac{\tau_{pxz}}{2(1-\varepsilon)\left(\tau_{pxz}^2 De^2\beta + \beta - 1\right)^2}\left[\begin{array}{l}\tau_{pxz}^2 De^2\beta\left(8\beta^2 - 8\beta + 1\right) + 1 - \beta \pm |2\beta - 1| \\ \left(\tau_{pxz}^2 De^2\beta - \beta + 1\right)\sqrt{1 - 4\tau_{pxz}^2 De^2\beta^2}\end{array}\right] \tag{23}$$

In order to decide which of the two solutions is the correct one, one has to analyse the behaviour of Eq. (23) varying β. If $\beta \to 0$ then $|2\beta - 1| = (1 - 2\beta)$, it results that $u' = 0$ everywhere if the minus sign is chosen, while if the plus sign is taken $\lim_{\beta\to 0} u' = \frac{\tau_{pxz}}{1-\varepsilon}$ which thus turns out to be an identity. The limit of Eq. (23) for $\beta \to 1$, $|2\beta - 1| = (2\beta - 1)$, yields $\lim_{\beta\to 1} u' = \frac{1 \pm \sqrt{1 - 4\tau_{pxz}^2 De^2}}{2(1-\varepsilon)\tau_{pxz} De^2}$; on the layer axis where $\tau_{pxz} = 0$, u' must also be zero and only the solution with the minus sign implies $u' = 0$ whereas the limit corresponding to the plus sign tends to infinity. For $\beta = 0.5$ the two expressions of u' overlap, so it follows that the plus sign is associated with $0 \le \beta \le 0.5$ and the minus sign with $0.5 \le \beta \le 1$. As stated above, the limiting value of u' tends to $\frac{\tau_{pxz}}{(1-\varepsilon)}$ as $De \to 0$.

$$De = \frac{k\delta}{2(1-\varepsilon)\left[k^2\beta\delta^2 - 1 + \beta\right]^2}\left\{\begin{array}{l}2k^4\beta^2\delta^4(1-\varepsilon) + k^2\beta\delta^2\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] \\ -\left[\beta(4-3\varepsilon) + \varepsilon - 2 - 2\beta^2(1-\varepsilon)\right] \pm \varepsilon|2\beta - 1|\left(k^2\beta\delta^2 - \beta + 1\right)\sqrt{1 - 4k^2\beta^2\delta^2}\end{array}\right\} \tag{27}$$

The shear stress of the polymer at the wall ($z = -1$), $\tau_{pxz}(-1)$ can be expressed as a fraction of τ_{pxzmax}, where from Eq. (19) $\tau_{pxzmax} = \frac{1}{2\beta De}$. However, according to Yoo and Choi [7], and recalling the thermodynamic considerations related to the entropy developed by Giesekus [15], the condition that should apply if $0 \le \beta \le 0.5$ is more restrictive: $\tau_{pxzmax} = \frac{1}{De}\sqrt{\frac{1-\beta}{\beta}}$, i.e. $\tau_{pxzmax} = \frac{\delta}{De}$ being $\delta = \sqrt{\frac{1-\beta}{\beta}}$ if $0 \le \beta \le 0.5$ and $\delta = \frac{1}{2\beta}$ if $0.5 \le \beta \le 1$.

The shear stress at the wall can be expressed as

$$\tau_{pxz}(-1) = \frac{k\delta}{De} \tag{24}$$

where $0 \le k \le 1$ represents the ratio between the wall shear-stress of the fluid and its limit-value τ_{pxzmax} when only the Giesekus fluid is present and then $\varepsilon = 0$.

If $De \to 0$ then $k \to 0$ and $\tau_{pxz}(-1) \to (1-\varepsilon)$. The same result is obtained if $\beta \to 0$ then $\tau_{pxz}(-1) \to (1-\varepsilon)$ and in both cases $u'(-1) \to 1$.

Substituting Eq. (24) in Eq. (23) gives

$$u'(-1) = \frac{k\delta}{2De(1-\varepsilon)\left(k^2\delta^2\beta + \beta - 1\right)^2}\left[\begin{array}{l}k^2\delta^2\beta\left(8\beta^2 - 8\beta + 1\right) + 1 - \beta \\ \pm |2\beta - 1|\left(k^2\delta^2\beta - \beta + 1\right)\sqrt{1 - 4k^2\delta^2\beta^2}\end{array}\right] \tag{25}$$

Eq. (21) can be solved with respect to u': recalling Eq. (24) it gives

$$u'(-1) = (1 - k\delta / De)/\varepsilon \tag{26}$$

substituting Eq. (26) in Eq. (25) a relation between the Deborah number De and k is obtained:

Again, the plus sign is associated with $0 \le \beta \le 0.5$ and the minus sign with $0.5 \le \beta \le 1$.

If $\beta = 0.5$ then Eq. (27) simplifies in

$$De = \frac{k\left[1 - (1-\varepsilon)k^2\right]}{(1-k^2)(1-\varepsilon)}$$

The parameter k, representing the ratio between the wall shear-stress of the fluid and its limit-value τ_{pxzmax}, is linked to Deborah number by Eq. (27). It can be seen, for example from Figs. 2 and 3, that De is an increasing function of k. If De grows, the pressure gradient grows, and hence the velocity. Thus k is qualitatively related to the value of fluid velocity. Furthermore, at the same De, i.e., pressure, viscosity and relaxation time, it can be verified that k depends almost linearly on the fluid mobility and increases as it increases. On the other hand, if De is constant, it can be observed that k decreases as ε increases.

4. Calculation of the velocity

By solving Eq. (21) with respect to u' it follows

$$u'(z) = -\frac{\tau_{pxz} + z}{\varepsilon} \tag{28}$$

For any z the shear stress τ_{pxz} can be written as $\tau_{pxz}(z) = m\tau_{pxz}(-1) = mk\tau_{max} = mk\delta/De$, with $-1 \le m \le 1$; when $m = \pm 1$ it results $z = \mp 1$; expressing $u'$and z as a function of m we have

$$u'(z) = \frac{du}{dz} = \frac{du}{dm}\frac{dm}{dz} = -\frac{1}{\varepsilon}\left(\frac{mk\delta}{De} + z(m)\right) \tag{29}$$

and thus

$$du = -\frac{1}{\varepsilon}\left(\frac{mk\delta}{De} + z(m)\right)\frac{dz}{dm}dm \tag{30}$$

and

$$u(m) = -\frac{1}{\varepsilon}\left[\frac{k\delta}{De}\int_1^m m\frac{dz}{dm}dm + \int_1^m z\frac{dz}{dm}dm\right] \tag{31}$$

which gives

$$u(m) = -\frac{1}{\varepsilon}\left[\frac{k\delta}{De}(mz(m) + 1 - M(m)) + \frac{z^2(m)}{2} - \frac{1}{2}\right] \tag{32}$$

being

$$M(m) = \int_1^m z(m)dm \tag{33}$$

The evaluation of the integral $M(m)$ is given in the Appendix B.

The function $z(m)$ can be obtained from Eq. (21) recalling Eq. (23) and putting $\tau_{pxz} = \frac{mk\delta}{De}$.

It follows

$$z(m) = \frac{mk\delta}{2(1-\varepsilon)De\left[m^2k^2\delta^2\beta + \beta - 1\right]^2}\left\{\begin{array}{l}\beta(4-3\varepsilon) - 2m^4k^4\delta^4\beta^2(1-\varepsilon) - 2\beta^2(1-\varepsilon) + \varepsilon - 2 - m^2k^2\delta^2\beta \\ \left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] \pm \varepsilon\left(\beta - m^2k^2\delta^2\beta - 1\right)|2\beta - 1|\sqrt{1 - 4m^2k^2\delta^2\beta^2}\end{array}\right\} \tag{34}$$

Case $0 \le \beta \le 0.5$

If $0 \le \beta \le 0.5$ it results $\delta = \sqrt{\frac{1}{\beta} - 1}$ and $|2\beta - 1| = 1 - 2\beta$; Eq. (27) becomes

$$De = \frac{k}{2(1-\varepsilon)\sqrt{\beta(1-\beta)}\left(1-k^2\right)^2}\left\{\begin{array}{l}2k^4(1-\beta)(1-\varepsilon) + k^2\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] - 2\beta(1-\varepsilon) - \varepsilon + 2 \\ +\varepsilon\left(1+k^2\right)(1-2\beta)\sqrt{1 - 4k^2\beta(1-\beta)}\end{array}\right\} \tag{35}$$

The function $z(m)$ is obtained from Eq. (34). It results

$$z(m) = \frac{mk}{2De\sqrt{\beta(1-\beta)}(1-\varepsilon)\left(1-m^2k^2\right)^2}\left\{\begin{array}{l}2\beta(1-\varepsilon) + \varepsilon - 2 - 2m^4k^4(1-\beta)(1-\varepsilon) - m^2k^2\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] \\ -\varepsilon\left(1+m^2k^2\right)(1-2\beta)\sqrt{1-4m^2k^2\beta(1-\beta)}\end{array}\right\} \tag{36}$$

Case $0.5 \le \beta \le 1$

If $0.5 \le \beta < 1$ then $\delta = \frac{1}{2\beta}$ and $|2\beta - 1| = 2\beta - 1$; Eq. (27) gives

$$De = \frac{k}{2\beta(1-\varepsilon)\left[k^2 - 4\beta(1-\beta)\right]^2}\left\{\begin{array}{l}k^4(1-\varepsilon) + 2k^2\beta\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] - 8\beta^2\left[2\beta^2(\varepsilon - 1) + \beta(4-3\varepsilon) + \varepsilon - 2\right] \\ -2\beta\varepsilon(2\beta - 1)\sqrt{\left(1-k^2\right)\left(k^2 - 4\beta^2 + 4\beta\right)}\end{array}\right\} \tag{37}$$

The following expression for $z(m)$ is obtained:

$$z(m) = \frac{mk}{2De\beta(1-\varepsilon)\left[m^2k^2 - 4\beta(1-\beta)\right]^2}\left\{\begin{array}{l}8\beta^2\left[\beta(4-3\varepsilon) - 2\beta^2(1-\varepsilon) + \varepsilon - 2\right] - m^4k^4(1-\varepsilon) - 2m^2k^2\beta\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right] \\ +2\beta\varepsilon\left[m^2k^2 + 4\beta(1-\beta)\right](2\beta - 1)\sqrt{1 - m^2k^2}\end{array}\right\} \tag{38}$$

If $\beta = 0.5$ then Eqs. (36) and (38) reduce to:

$$z(m) = \frac{mk\left[1 - m^2k^2(1-\varepsilon)\right]}{De(1-\varepsilon)\left(m^2k^2 - 1\right)}$$

5. Limiting values

The maximum value of De, De_{max} is obtained for $k = 1$.

If $0 < \beta \le 0.5$ and $\varepsilon \ne 0$ the limit becomes infinite as can be seen from Eq. (35); if $\varepsilon = 0$ (only polymer, no solvent) the limit is, as expected, $\sqrt{(1-\beta)/\beta}$; if $\varepsilon = 1$ only the Newtonian solvent is present and $De_{max} \to \infty$.

If $0.5 \le \beta < 1$ from Eq. (37)

$$De_{max} = \frac{1}{2\beta} + \frac{\varepsilon}{(1-\varepsilon)(2\beta-1)^2} \tag{39}$$

if $\varepsilon = 0$ (only polymer, no solvent) Eq. (39) gives the expected value $De_{max} = \frac{1}{2\beta}$. If $\varepsilon \to 1$, $De_{max} \to \infty$. Solving Eq. (39) with respect to ε gives the minimum value of ε to obtain a given value of De_{max}:

$$\varepsilon_{min} = 1 - \frac{2\beta}{(2\beta-1)^2(2De_{max}\beta-1)+2\beta} \tag{40}$$

The shear stress τ_{pxz} and the normal stresses τ_{pzz} and τ_{pxx} can be now calculated using eqs. (21), (18) and (20).

To better explain how the proposed method can be used to calculate the fluid velocity, one can start by choosing the characteristics of the Giesekus fluid and the pressure gradient, i.e. De, β, and the amount of Newtonian solvent to set the viscosity ratio ε; using Eq. (35) or Eq. (37), based on the value of β, one can derive numerically the value of k. Fixed m, $-1 \le m \le 1$, the value of $z(m)$ is given by Eq. (36) or Eq. (38). Finally, the value of the fluid velocity can be obtained from Eq. (32) where the integral can be calculated using the equations in appendix B; $u'(z)$ can be obtained from Eq. (28), the tangential component of the polymer stress tensor τ_{pxz} and the normal components τ_{pxx} and τ_{pzz}, can be easily determined using Eqs. (21), (18) and (20).

6. Results and discussion

The main purpose of the following analysis is to illustrate the influence of the amount of a Newtonian solvent in a Giesekus fluid in a steady-state channel flow, using the results of the proposed solution.

In particular, interest focuses on the velocity and stress profiles when the solvent viscosity ratio ε rises from $\varepsilon = 0$, where only the Giesekus fluid is present, and $\varepsilon = 1$, where the fluid is reduced to a Newtonian one. The reference velocity represents the mean velocity of a Newtonian fluid in terms of pressure gradient, using as the viscosity the sum of the polymeric and the solvent viscosity $U_N = -\frac{\partial p}{\partial x}\frac{h^2}{3(\mu+\eta)}$. The first calculations are performed keeping Deborah number constant and using two different values of the mobility factor β: one lower than 0.5 and the other higher to illustrate the two possible situations named the upper and lower branch solution (Yoo and Choi [7]). Fig. 2 shows for $\beta = 0.25$ the behaviour of the parameter k as a function of Deborah number for some values of ε; k represent the ratio between the wall shear-stress of the polymer and its limit-value τ_{pxzmax}. It can be seen that for $\varepsilon = 0$, i.e. when only the Giesekus fluid is present, k grows linearly from 0 to 1 as De increases from 0 to $\sqrt{\frac{1-\beta}{\beta}}$. For $\varepsilon > 0$, k increases as De increases but the slope of the curve decreases increasing De. It is seen that k decreases as ε increases. Fig. 3 shows k as a function of De when $\beta = 0.75$; again if $\varepsilon = 0$, k grows linearly as Deborah number grows until $k = 1$ for $De = \frac{1}{2\beta}$. If $\varepsilon > 0$, k grows as De increases and reaches the value $k = 1$ when De_{max}. Fig. 4 shows the velocity profiles for increasing values of, with $De = 1.5$ and $\beta = 0.25$. The velocity has been normalized dividing it by the mean Newtonian velocity U_N. The purely polymeric fluid is represented by $\varepsilon \to 0$. As expected, when the amount of solvent increases, the velocity decreases considerably, on thickening of the fluid in laminar flow, an effect similar to the increase in drag. For $\varepsilon \to 1$, (dash-dot line), the velocity profile obtained differs from the parabolic profile representing Newtonian fluid by less than $1.5E-08$, confirming the correctness of the current solution. Fig. 5 illustrates the normalized velocity profiles for increasing values of ε as $De = 1.5$ and $\beta = 0.75$. If $\beta > 0.5$, there is a minimum value of ε to obtain the assigned value of Deborah number; for $De = 1.5$, $\varepsilon_{min} = 0.173$, which gives the maximum value of velocity at the axis of the layer. Comparison Fig. 5 with Fig. 4, shows that for the same ε, the velocity is greater for greater mobility factor. As $\varepsilon \to 1$, the velocity profile overlaps the Newtonian profile again. Fig. 6 shows, in a semi-logarithmic scale, the maximum value of De_{max} as a function of ε for some values of the mobility factor. If $0 < \beta \le 0.5$, $De_{max} \to \infty$, whereas for a given ε it decreases more and more as mobility increases. The small figure, in linear scale, allows to verify that for $De = 1.5$, $\varepsilon_{min} = 0.173$. Fig. 7, 8 and 9 show the axial normal component of the polymer stress tensor τ_{pxx}, the tangential component τ_{pxz} and the normal component τ_{pzz}, respectively, as the solvent viscosity ratio increases, keeping De and β constant. The stress components decrease, in absolute value, when ε increases, until for $\varepsilon \to 1$, the polymeric stresses vanish. Substituting the numerical values of the first derivative of velocity u' and those of total stress components into Eqs. (12), (13) and (14), the maximum errors are everywhere less than $2.0E-14$. The behaviour of the polymer stress components for $\beta = 0.75$ are shown in Fig. 10, 11, and 12. As expected, absolute values of stress decrease increasing ε; a comparation with Fig. 7, 8 and 9 shows that the stress values are lower as $\beta > 0.5$.

7. Conclusions

In the present work, the Poiseuille flow of a Giesekus fluid to which a Newtonian solvent is added has been analysed. The velocity profiles and the polymer stress components have been determined analytically as a function of the Giesekus fluid classical parameters and of the ratio ε of solvent viscosity to total viscosity. If the mobility factor β is greater of 0.5, it is emphasized that a solution exists for given β and De only if ε is greater than a suitable value.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jnnfm.2023.105152.

Appendix A

Solving Eq. (14) with respect to τ_{pzz} gives

$$\tau_{pzz} = \frac{\beta De\tau_{pxx}\tau_{pxz} + \tau_{pxz} - v'}{De\left(v' - \beta\tau_{pxz}\right)} \tag{A1}$$

where

$$v' = (1-\varepsilon)u' \tag{A2}$$

Substituting in Eq. (13):

$$De^2\tau_{pxz}^2\beta^3\tau_{pxx}^2 + \beta De\tau_{pxz}\tau_{pxx}\left[\beta\tau_{pxz} + v'(1-2\beta)\right] + \beta De^2\tau_{pxz}^2\left(\beta\tau_{pxz} - v'\right)^2 + v'(1-\beta)\left(\tau_{pxz} - v'\right) = 0 \tag{A3}$$

Solving Eq. (12) with respect to τ_{pxx} gives

$$\tau_{pxx} = \frac{-1 \pm \sqrt{\left[4\beta\tau_{pxz}De^2\left(2v' - \beta\tau_{pxz}\right)\right]}}{2\beta De} \tag{A4}$$

Choosing the plus sign and substituting in Eq. (A3)

$$\tau_{pxz}(1-2\beta)\sqrt{\left[4\beta De^2\tau_{pxz}\left(2v' - \beta\tau_{pxz}\right)\right] + 1} + 2\beta De^2\tau_{pxz}^2 v' + \tau_{pxz} + 2(\beta-1)v' = 0 \tag{A5}$$

i.e.

$$\tau_{pxz}(1-2\beta)\sqrt{\left[4\beta De^2\tau_{pxz}\left(2v' - \beta\tau_{pxz}\right)\right] + 1} = -\left[2\beta De^2\tau_{pxz}^2 v' + \tau_{pxz} + 2(\beta-1)v'\right] \tag{A6}$$

Squaring both members gives

$$\tau_{pxz}^2(1-2\beta)^2\left\{\left[4\beta De^2\tau_{pxz}\left(2v' - \beta\tau_{pxz}\right)\right] + 1\right\} = \left[2\beta De^2\tau_{pxz}^2 v' + \tau_{pxz} + 2(\beta-1)v'\right]^2 \tag{A7}$$

which recalling Eq. (A2) finally gives Eq. (22).

Appendix B

Calculation of

$$M(m) = \int_1^m z(m)dm = M_1 \pm M_2 \tag{B1}$$

The plus sign if $0 \le \beta \le 0.5$, i.e. $\delta = \sqrt{[(1-\beta)/\beta]}$ and the minus sign if $0.5 \le \beta \le 1$, i.e. $\delta = 1/(2\beta)$.

$$M_1 = \int_1^m \left\{ \begin{array}{c} \dfrac{mk\delta}{2(1-\varepsilon)De\left[m^2k^2\delta^2\beta + \beta - 1\right]^2} \\ \left\{\beta(4-3\varepsilon) - 2m^4k^4\delta^4\beta^2(1-\varepsilon) - 2\beta^2(1-\varepsilon) + \varepsilon - 2 - m^2k^2\delta^2\beta\left[8\beta^2\varepsilon + 4\beta(1-3\varepsilon) + 5\varepsilon - 4\right]\right\} \end{array} \right\} dm$$
$$= \frac{1}{4Dek\beta\delta(\varepsilon-1)\left(k^2\delta^2\beta m^2 + \beta - 1\right)\left(k^2\delta^2\beta + \beta - 1\right)} \tag{B2}$$
$$\left\{ \begin{array}{l} \varepsilon\left(8\beta^2 - 8\beta + 1\right)\left(k^2m^2\delta^2\beta + \beta - 1\right)\left(k^2\beta\delta^2 + \beta - 1\right)\ln\left(\dfrac{k^2m^2\delta^2\beta + \beta - 1}{k^2\delta^2\beta + \beta - 1}\right) + 2k^2\beta\delta^2\left(1 - m^2\right) \\ \left[k^4m^2\beta^2\delta^4(\varepsilon - 1) + k^2\beta\delta^2\left(m^2+1\right)(1-\beta)(1-\varepsilon) - (1-\beta)\left[4\beta^2\varepsilon - \beta(3\varepsilon+1) + 1\right]\right] \end{array} \right\}$$

$$M_2 = \int_1^m \left\{ \frac{mk\delta}{2(1-\varepsilon)De\left[m^2k^2\delta^2\beta + \beta - 1\right]^2}\left[\varepsilon\left(\beta - m^2k^2\delta^2\beta - 1\right)|2\beta - 1|\sqrt{1 - 4m^2k^2\delta^2\beta^2}\right] dm \right\}$$
$$= \frac{\varepsilon|2\beta - 1|}{2Dek\beta\delta(\varepsilon-1)\left(k^2\delta^2\beta m^2 + \beta - 1\right)\left(k^2\delta^2\beta + \beta - 1\right)} \tag{B3}$$
$$\left\{\left[k^2m^2\delta^2\beta + 2(\beta-1)\right]\left(k^2\beta\delta^2 + \beta - 1\right)\sqrt{\left(1 - 4k^2m^2\beta^2\delta^2\right)} - \left[k^2\delta^2\beta + 2(\beta-1)\right]\left(k^2m^2\beta\delta^2 + \beta - 1\right)\sqrt{\left(1 - 4k^2\beta^2\delta^2\right)}\right\}$$
$$+ \frac{\varepsilon\left(8\beta^2 - 8\beta + 1\right)}{4Dek\beta\delta(\varepsilon - 1)}\ln\left\{\frac{\left[|2\beta-1| - \sqrt{\left(1 - 4k^2m^2\beta^2\delta^2\right)}\right]\left[|2\beta-1| + \sqrt{\left(1 - 4k^2\beta^2\delta^2\right)}\right]}{\left[|2\beta-1| + \sqrt{\left(1 - 4k^2m^2\beta^2\delta^2\right)}\right]\left[|2\beta-1| - \sqrt{\left(1 - 4k^2\beta^2\delta^2\right)}\right]}\right\}$$

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