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Regularization by noise for Gevrey well-posedeness of a weakly hyperbolic operator

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Abstract

We present an example of a linear partial differential equation whose Cauchy problem becomes well-posed when perturbed by noise. Specifically, we make clear how a suitable multiplicative Stratonovich perturbation of Brownian type renders a weakly hyperbolic operator with double involutive characteristics well-posed in the C^∞-category, while its deterministic counterpart is only well-posed in the Gevrey s classes with $1 \le s < 2$.

Keywords Regularization by noise · Weakly hyperbolic operator · Itô-Stratonovich integration

Mathematics Subject Classification 60H15 · 35L10 · 60H50

1 Introduction and statement of the main result

It is well-known that the Cauchy problem

$$\begin{cases} \mathrm{P}(t, x, \partial_t, \partial_x)u(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n; \\ \partial_t^j u(0, x) = u_j(x), & x \in \mathbb{R}^n,\ j = 0, \dots, m-1, \end{cases} \tag{1.1}$$

where P is a linear hyperbolic differential operator of order $m \in \mathbb{N}$ with analytic coefficients, may fail to be well-posed in the C^∞-category if P is weakly hyperbolic, that is if some of its characteristic roots coincide. When this happens, and $r \in \mathbb{N}$ denotes the largest multiplicity of these roots, then a classical result in [9] shows that for arbitrary lower order terms of the operator P, one can in general only expect well-posedness in the Gevrey classes $\gamma^{(s)}$, with $1 \le s < \frac{r}{r-1}$. Moreover, the threshold $\frac{r}{r-1}$

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is usually optimal. We recall that $f \in \gamma^{(s)}(\mathbb{R}^n)$, the Gevrey class of order $s \geq 1$, if for any compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $h > 0$ such that

$$\left|\partial_x^\alpha f(x)\right| \leq C h^{-|\alpha|}(\alpha!)^s \quad \text{for all } x \in K \text{ and } \alpha \in \mathbb{N}^n.$$

We also denote $\gamma_0^{(s)}(\mathbb{R}^n) = \gamma^{(s)}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$.

In this paper we study an explicit example of a weakly hyperbolic operator with a manifold of involutive double characteristics $\Sigma_2 = \{t \in [0, T], x \in \mathbb{R} : (t, x, \tau = 0; \xi \neq 0)\}$ and $m = r = 2$, i.e. $T_\rho \Sigma_2^\sigma \subset T_\rho \Sigma_2$ the symplectic dual of the tangent space at $\rho \in \Sigma_2$ is contained in $T_\rho \Sigma_2$. The operator reads as

$$\begin{cases} (\partial_t^2 + i\partial_x)u(t,x) = 0, & t \in [0, T], x \in \mathbb{R}; \\ u(0,x) = \varphi_0(x), \partial_t u(0,x) = \varphi_1(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Albeit (1.2) is admittedly very simple, its principal part is nonetheless one of the only three possible cases in which a hyperbolic quadratic form can be invariantly represented in suitable symplectic coordinates. Thus, (1.2) is actually the microlocal model for *any* hyperbolic operator with double involutive characteristics. Many physical phenomena are driven by equations of the type (1.2): conical refraction of light rays through prisms, see e.g [22] and [26], dynamics of wet-dry regimes in linearized Saint-Venant shallow waters systems see e.g. [25] and [21], to name just a few.
To be more specific, localizing the principal symbol at a double point, we end up with a symplectic quadratic form of one of these three invariant cases, as proved in [14] and [15]:

$$\begin{aligned} &\bullet\ p(t, \tau, x; \xi) = 2\lambda t\tau,\ \lambda > 0; \\ &\bullet\ p(t, \tau, x; \xi) = -\tau^2; \\ &\bullet\ p(t, \tau, x; \xi) = -\tau^2 + 2\tau\xi_1 + x_1^2. \end{aligned} \tag{1.3}$$

The first case in (1.3) corresponds to the so called effectively hyperbolic case, while the third one is the only invariant case where its Hamiltonian matrix has a Jordan block of size 4. Stochastic generalizations of the first item in (1.3) were recently studied in [5], while some results for associated stochastic partial differential equations in this last case where also obtained in [4]. For the analysis of other weakly hyperbolic partial differential equations perturbed by noise we refer the reader to [1] and [2].
It is immediate to check that the lower order term in (1.2) does not satisfy the necessary Ivrii-Petkov-Hörmander condition, which for this model states that the sub-principal symbol of P must vanish on Σ_2, see [13] and [16]. This implies that the Gevrey threshold $s = 2$ is the natural barrier for equation (1.2).

Aim of the present manuscript is to understand the effect of noise on this particular setup. Is it possible to gain in-mean well-posedness for a suitable stochastic perturbation of (1.2)? The answer presented and proved here is affirmative, positioning this outcome among the already vast array of results capped under the general title of regularization by noise.

Dating back to results of Khasminskii [17], where examples of unstable systems of ordinary differential equations become asymptotically stable when their parameters are perturbed by white noise, passing through the works of Zvonkin [28] and Veretennikov [27], that prove existence of a unique strong solution for stochastic differential equation with merely bounded and measurable drift, and reaching for instance to the theory of transport-type stochastic perturbations and the foundational work of Flandoli–Gubinelli–Priola [11], where we can see that, in those setting, noise restores uniqueness for transport equations below deterministic thresholds, we do know that there are many instances of noise–regularized behaviors. See also the survey [12] and lecture notes [10] for a broader overview of the phenomenon.

The general philosophy of regularization by noise is that stochastic perturbations may introduce effective coercivity or averaging mechanisms that are absent in the deterministic problem. In many examples, this becomes visible after converting the equation to Itô form and exploiting quadratic variation [11]. Something along these lines is happening here as well.

On a slightly different angle but in a similar vein too, we may also cite some recent results [20] on the unique continuation property for stochastic wave equations, which seem to present a similar scheme: stochastic Carleman estimates restore unicity properties near characteristic surfaces, which are known to usually fail in the deterministic case.

We may now state our main result whose proof is postponed to Sect. 3 below. We first rewrite (1.2) as

$$\begin{cases} \partial_t u(t,x) = v(t,x), & t \in (0,T], x \in \mathbb{R}; \\ \partial_t v(t,x) = -i\partial_x u(t,x), & t \in (0,T], x \in \mathbb{R}; \\ u(0,x) = \varphi_0(x), v(0,x) = \varphi_1(x), & x \in \mathbb{R}; \end{cases} \tag{1.4}$$

then, we consider the stochastic partial differential equation

$$\begin{cases} dU(t,x) = V(t,x)dt + \sigma\partial_x U(t,x) \circ dB(t), & t \in (0,T], x \in \mathbb{R} \\ dV(t,x) = -i\partial_x U(t,x)dt & t \in (0,T], x \in \mathbb{R} \\ U(0,x) = \varphi_0(x), V(0,x) = \varphi_1(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where σ is a positive real number and $\{B(t)\}_{t\geq 0}$ is a standard one dimensional Brownian motion. As usual by $\circ$ we denote Stratonovich-type integration, see for instance [18]. Equation (1.5) represents a stochastic perturbation of (1.4).
The perturbation in (1.5) is a Stratonovich transport noise in the same spatial direction as the lower-order term responsible for the deterministic Gevrey obstruction. After Fourier transform, it becomes the multiplicative noise $i\sigma\xi\widehat{U} \circ dB_t$. Passing to Itô form produces the correction $-\frac{\sigma^2}{2}\xi^2\widehat{U}\,dt$, and this correction yields the damping term $-\frac{\sigma^2}{2}\xi^2 m_2$ in the equation for the mixed moment. This term is the source of the uniform weighted energy estimate. The fact that the noise acts only on the U-component is part of the structure of the model: it gives a minimal stochastic transport perturbation and leads to a closed three-dimensional system for the second moments.

Other placements of the noise lead to different moment systems and are not treated here.

The next theorem states the better well-posedness of (1.5) over (1.4); in the sequel we set $H_x^\infty := \bigcap_{k\geq 0} H_x^k$ and H_x^k denoting the usual Sobolev space in the x variable.

Theorem 1.1 *Let $T > 0$, let $\sigma > 0$, and let $\{B(t)\}_{t\geq 0}$ be a standard one-dimensional Brownian motion. For every deterministic initial datum*

$$(\phi_0, \phi_1) \in H_x^\infty \times H_x^\infty,$$

the Cauchy problem (1.5) admits a unique adapted solution process (U, V) such that

$$U \in \bigcap_{k\geq 0} C([0,T]; L^2(\Omega; H_x^k)), \qquad V \in \bigcap_{k\geq 0} C([0,T]; L^2(\Omega; H_x^{k-\frac{1}{2}})).$$

Moreover, the solution depends continuously on the initial data in these norms.

Remark 1.2 This result can be compared with the known result, a proof of which is sketched in the next section, which states that for $s > 2$ the Cauchy problem for (1.4) is not locally solvable in $\gamma^{(s)}$; in particular, the Cauchy problem for (1.4) is not C^∞-solvable.

The proof of Theorem 1.1 proceeds entirely at the level of Fourier space and second moments. After Fourier transform in x, equation (1.5) becomes a two-dimensional linear Stratonovich system for $(\widehat{U}, \widehat{V})$. Passing to Itô form produces a correction proportional to $-\sigma^2\xi^2\widehat{U}$. We then consider the second moments

$$m_1 = \mathbb{E}|\widehat{U}|^2, \qquad m_2 = \mathbb{E}\Re(\overline{\widehat{U}}\widehat{V}), \qquad m_3 = \mathbb{E}|\widehat{V}|^2.$$

These moments satisfy a closed linear ODE system. The spectral analysis of this system shows that the weighted energy

$$F(t;\xi) = m_1(t;\xi) + \langle\xi\rangle^{-1} m_3(t;\xi)$$

satisfies a uniform estimate in ξ. Multiplying this bound by $\langle\xi\rangle^{2s}$ and integrating in ξ, we obtain Sobolev estimates in $H_x^s \times H_x^{s-\frac{1}{2}}$. Finally, the continuity in time follows from the continuity of each Fourier SDE and dominated convergence.

Remark 1.3 The argument developed here is model-specific: it relies on the exact closure of the second-moment system after Fourier transform. This feature allows us to reduce the stochastic PDE to a finite-dimensional deterministic ODE system for the moments. The same strategy may be applicable to other weakly hyperbolic models with involutive characteristics when a comparable microlocal or spectral reduction is available and the stochastic perturbation produces a closed moment system with uniformly bounded growth exponent. However, we do not claim that the present argument directly yields a general theorem for arbitrary weakly hyperbolic operators with variable coefficients. In such cases one would likely need a genuinely microlocal stochastic energy method.

Remark 1.4 The regularization phenomenon considered here should be distinguished from the stochastic restoration of unique continuation properties. Unique continuation concerns whether a solution is determined by partial vanishing information, for instance across a hypersurface. By contrast, the present paper studies the Cauchy problem and proves existence, uniqueness, and continuous dependence on the initial data in a mean-square Sobolev scale. Nevertheless, the two phenomena are related in spirit: in both cases the stochastic perturbation produces, through Itô correction or quadratic variation, a positive term that is absent in the deterministic problem. In stochastic unique continuation this enters through Carleman estimates, while here it enters through the spectral properties of the closed second-moment system.

Remark 1.5 The constants in the stochastic energy estimate are not stable as $\sigma \to 0$. Indeed, the spectral estimate for the moment system degenerates like a negative power of σ. This is consistent with the deterministic limit: for $\sigma = 0$, the equation reduces to the original weakly hyperbolic Cauchy problem, which is not C^∞-well-posed. Hence one cannot expect estimates uniform in σ down to zero in the Sobolev C^∞ scale.

The paper is organized as follows: in Sect. 2 we review the known facts that fix the Gevrey index $s = 2$ as the natural threshold for the deterministic model (1.4). In Sect. 3 we prove the main result, which is direct consequence of Theorem 3.7 below and of the stochastic energy estimates of Lemma 3.8.

2 A brief review of the deterministic case

The next two subsections present simple self-contained arguments showing how the threshold Gevrey $s = 2$ appears naturally for this model. Needless to say every result given here is well-known and belongs to a historically well-established theoretical framework. An exhaustive and complete collection of results can be found for instance in [24], source of some of the theorems adapted and used below.

2.1 Sufficiency

We present a very short and sketchy review of how to get Gevrey energy estimates for the model (1.2). We consider here, with Fourier derivatives $D_y := \frac{1}{i}\partial_y$, the slightly more general symbol

$$\mathrm{P}(x, D) = -D_t^2 + AD_x$$

with $A \in \mathbb{C}$. Taking the Fourier transform in x and setting

$$\widehat{u}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(x)dx, \quad \xi \in \mathbb{R},$$

we get

$$2i\Im\langle \mathrm{P}\widehat{u}, -D_t\widehat{u}\rangle = D_t|D_t\widehat{u}|^2 + 2i\Im\xi\langle A\widehat{u}, D_t\widehat{u}\rangle$$

and

$$2i\Im\langle D_t\widehat{u}, \widehat{u}\rangle = D_t|\widehat{u}|^2; \tag{2.1}$$

here $\Im z$ stands for the imaginary part of the complex number z. With the proviso, not affecting the gist of the argument, that ξ is positive and large and with $\lambda > 0$ to be suitably chosen, define the Gevrey norm

$$|||v|||_s^2(\xi) := \int_0^\infty e^{2\lambda t \xi^{1/s}} |v(t;\xi)|^2 dt, \quad \xi \in \mathbb{R},$$

and inner product

$$\langle f, g\rangle_s(\xi) = \int_0^\infty e^{2\lambda t \xi^{1/s}} f(t;\xi)\bar{g}(t;\xi) dt, \quad \xi \in \mathbb{R}.$$

Then, multiplying the two previous identities by $e^{2\lambda t \xi^{1/s}}$ and integrating by parts we get

$$2\Im\langle P\widehat{u}, -D_t\widehat{u}\rangle(\xi) \geq 2\lambda\xi^{1/s}|||D_t\widehat{u}|||_s^2(\xi) + 2\Im\xi\langle A\widehat{u}, D_t\widehat{u}\rangle(\xi). \tag{2.2}$$

Since we also have from (2.1) that

$$|||D_t\widehat{u}|||_s^2(\xi) \geq \lambda^2\xi^{2/s}|||\widehat{u}|||_s^2(\xi),$$

using Cauchy-Schwarz inequality and

$$2 - \frac{1}{s} < \frac{3}{s}$$

we see that we can estimate the last summand in (2.2) with a choice of λ and therefore, after an inverse Fourier transform, we have reached a Gevrey s with $s < 2$ energy estimate. Standard arguments allow us then to conclude. For more details on this derivation we refer the interested reader to the papers [3, 6–8].

Remark 2.1 For the sake of completeness we can also remark that the Cauchy problem for (1.2) fails when $s = 2$, albeit in a slightly weaker sense. Since the main focus here is however on the lack of solvability for $s > 2$, we point the interested reader to some classical literature for these cases: [24].

2.2 Sharpness

Now we want to verify that if $s > 2$ the Cauchy problem for (1.2) is ill-posed. This is done in Theorem 2.3 below. We follow the arguments of [24] Section 6.3, starting with the following important definition.

Definition 2.2 We say that the Cauchy problem for the second order differential operator P is locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (\varphi_0, \varphi_1) \in \left(\gamma^{(s)}(\mathbb{R}^n)\right)^2$, there exists a neighborhood U_Φ of the origin such that there exists $u \in C^\infty(U_\Phi)$ satisfying

$$\begin{cases} \mathrm{P}u = 0, & \text{in } U_\Phi; \\ D_0^j u\left(0, x'\right) = u_j\left(x'\right), \, j = 0, 1, & \left(0, x'\right) \in U_\Phi \cap \{x_0 = 0\}. \end{cases}$$

Then our main result in this subsection reads as follows.

Theorem 2.3 *If $s > 2$ then the Cauchy problem (1.2) is not locally solvable in $\gamma^{(s)}$. In particular the Cauchy problem (1.2) is not C^∞ solvable.*

To prove Theorem (2.3) we recall the following result (see e.g. [24], [19] or [23] Proposition 4.1).

Proposition 2.4 *Let K be a compact set of $\mathbb{R}^n$ and $h > 0$ be fixed. Assume that the Cauchy problem for P is locally solvable in $\gamma^{(s)}$ at the origin. Then there is $\delta > 0$ such that for any $\left(\varphi_0\left(x'\right), \varphi_1\left(x'\right)\right) \in \left(\gamma^{(s),h}(K)\right)^2$ there exists a unique $u(x) \in C^2(D_\delta)$ verifying $Pu = 0$ in D_δ and $D_0^j u\left(0, x'\right) = u_j\left(x'\right)$ on $D_\delta \cap \{x_0 = 0\}$.*

We also need the following proposition.

Proposition 2.5 *Assume that $\theta \in C_0^\infty(\mathbb{R})$ is an even function such that $\theta \notin \gamma_0^{(2)}(\mathbb{R})$. Let Ω be a neighborhood of the origin of $\mathbb{R}$ such that* $\operatorname{supp}\theta \subset \Omega \cap \{x_0 = 0\}$. *Then the Cauchy problem*

$$\begin{cases} (\partial_t^2 + i\partial_x)u(t,x) = 0, & t \in (0, T], x \in \mathbb{R}; \\ u(0,x) = 0, \, V(0,x) = \theta(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

has no $C^2(\Omega)$ solution.

Proof We define

$$U_\rho(t,x) = e^{-i\rho^2 x + iz\rho(T-t)}$$

and we choose $z = -i$ so that $U_\rho(t,x)$ is a particular solution of the homogeneous equation. Suppose that (2.3) has a $C^2(\Omega)$ solution. Applying Holmgren's Theorem (see Proposition 6.3 in [24]) we conclude that we can assume $u(x) = 0$ if $|t| \leq T$ and $|x| \geq r$ with some small $T > 0$ and $r > 0$.
We note that, with $(u, v) = \int_{\mathbb{R}} u(x)\bar{v}(x)dx$,

$$\begin{aligned} \int_0^T \left(\mathrm{P}U_\rho, u\right) dt = & \int_0^T \left(U_\rho, \mathrm{P}u\right) dt + i\left(D_t U_\rho(T), u(T)\right) \\ & + i\left(U_\rho(T), D_t u(T)\right) - i\left(U_\rho(0), D_t u(0)\right) \end{aligned}$$

because $u(0) = 0$. From this we have

$$\left(D_t U_\rho(T), u(T)\right) + \left(U_\rho(T), D_t u(T)\right) = \left(U_\rho(0), D_t u(0)\right). \tag{2.4}$$

We see that the left-hand side on (2.4) is $O(\rho)$. On the other hand the right-hand side is

$$e^{\rho T}\hat{\theta}(-\rho^2)$$

where $\hat{\theta}$ is the Fourier transform of θ. This comparison yields that, since θ is even

$$|\hat{\theta}(\rho)| \le C\rho^{1/2}e^{-c\rho^{1/2}} \le C'e^{-c'\rho^{1/2}}$$

with some $c' > 0$ and ρ large. We deduce that $\theta \in \gamma_0^{(2)}(\mathbb{R})$ which is a contradiction. □

Proof of Theorem 2.3 Suppose that the Cauchy problem (1.2) is locally solvable in $\gamma^{(s)}$ at the origin with some $s > 2$. Choose s' so that $s > s' > 2$, and choose a compact neighborhood K of the origin of $\mathbb{R}$ and a positive $h > 0$. Then from Proposition 2.4 there exists D_δ such that the Cauchy problem for P has a $C^2(D_\delta)$ solution for any Cauchy data in $\left(\gamma^{(s),h}(L)\right)^2$. We now choose $\theta \in \gamma_0^{(s')}(\mathbb{R})$ so that $\operatorname{supp}\theta \subset K \cap (D_\delta \cap \{x_0 = 0\})$ which satisfy the conditions in Proposition 2.5 below. Since it is clear that $\theta \in \gamma_0^{(s),h}(K)$ because $s > s'$ one can apply Proposition 2.4 to conclude that there is a $C^2(D_\delta)$ solution to (6.16), while this contradicts proposition 2.5. □

3 The stochastic energy estimate

In this section we prove the main theorem of our work, i.e. Theorem (1.1). We start taking the Fourier transform with respect to x in (1.5) to get

$$\begin{cases} d\widehat{U}(t;\xi) = \widehat{V}(t;\xi)dt + i\sigma\xi\widehat{U}(t;\xi)\circ dB(t), & t \in (0,T], \xi \in \mathbb{R}; \\ d\widehat{V}(t;\xi) = \xi\widehat{U}(t;\xi)dt, & t \in (0,T], \xi \in \mathbb{R}; \\ \widehat{U}(0;\xi) = \widehat{\varphi}_0(\xi), \widehat{V}(0;\xi) = \widehat{\varphi}_1(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{3.1}$$

which in Itô's form reads

$$\begin{cases} d\widehat{U}(t;\xi) = \left[\widehat{V}(t;\xi) - \frac{\sigma^2}{2}\xi^2\widehat{U}(t;\xi)\right]dt + i\sigma\xi\widehat{U}(t;\xi)dB(t), & t \in (0,T], \xi \in \mathbb{R}; \\ d\widehat{V}(t;\xi) = \xi\widehat{U}(t;\xi)dt, & t \in (0,T], \xi \in \mathbb{R}; \\ \widehat{U}(0;\xi) = \widehat{\varphi}_0(\xi), \widehat{V}(0;\xi) = \widehat{\varphi}_1(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{3.2}$$

For $t \in [0, T]$ and $\xi \in \mathbb{R}$ we define

$$m_1(t;\xi) := \mathbb{E}|\widehat{U}(t;\xi)|^2, \qquad m_2(t;\xi) := \mathbb{E}\,\Re\left(\overline{\widehat{U}(t;\xi)}\widehat{V}(t;\xi)\right),$$
$$m_3(t;\xi) := \mathbb{E}|\widehat{V}(t;\xi)|^2. \tag{3.3}$$

The idea is to do an energy estimate, of the type of Section (2.1), at the expectation level. It is here that the role of the perturbation σ can be seen to play its role.

Lemma 3.1 (Second moments system) *For any $\xi \in \mathbb{R}$ the function*

$$t \mapsto m(t;\xi) = (m_1(t;\xi), m_2(t;\xi), m_3(t;\xi))^\top$$

defined in (3.3) solves the linear system of ordinary differential equations

$$\begin{cases} \dot{m}_1(t;\xi) = 2m_2(t;\xi), \\ \dot{m}_2(t;\xi) = \xi m_1(t;\xi) - \frac{\sigma^2}{2}\xi^2 m_2(t;\xi) + m_3(t;\xi), \\ \dot{m}_3(t;\xi) = 2\xi m_2(t;\xi); \end{cases} \tag{3.4}$$

here, $\dot{m}_i(t;\xi) := \frac{d}{dt}m_i(t;\xi)$ for $i = 1, 2, 3$.

Proof Applying Itô's formula to $|\widehat{U}(t;\xi)|^2 = \widehat{U}(t;\xi)\overline{\widehat{U}(t;\xi)}$ gives

$$d|\widehat{U}(t;\xi)|^2 = \overline{\widehat{U}(t;\xi)}d\widehat{U}(t;\xi) + \widehat{U}(t;\xi)d\overline{\widehat{U}(t;\xi)} + d\widehat{U}(t;\xi)d\overline{\widehat{U}(t;\xi)}.$$

Now, if we substitute $d\widehat{U}(t;\xi)$ and $d\overline{\widehat{U}(t;\xi)}$ with the corresponding expressions from (3.2), we find

$$d|\widehat{U}(t;\xi)|^2 = 2\Re(\overline{\widehat{U}(t;\xi)}\widehat{V}(t;\xi))dt.$$

Taking expectations yields $\dot{m}_1(t;\xi) = 2m_2(t;\xi)$, i.e. the first equation in (3.4). Moreover, using Itô's product rule we get

$$d(\overline{\widehat{U}(t;\xi)}\widehat{V}(t;\xi)) = \widehat{V}(t;\xi)d\overline{\widehat{U}(t;\xi)} + \overline{\widehat{U}(t;\xi)}d\widehat{V}(t;\xi) + d\overline{\widehat{U}(t;\xi)}d\widehat{V}(t;\xi).$$

Since $d\widehat{V}(t;\xi)$ has no Brownian part, the last term vanishes and, upon taking real parts and expectations, we obtain

$$\begin{aligned} \dot{m}_2(t;\xi) &= \mathbb{E}|\widehat{V}(t;\xi)|^2 + \xi\mathbb{E}|\widehat{U}(t;\xi)|^2 - \frac{\sigma^2}{2}\xi^2\mathbb{E}\Re(\overline{\widehat{U}(t;\xi)}\widehat{V}(t;\xi)) \\ &= \xi m_1(t;\xi) - \frac{\sigma^2}{2}\xi^2 m_2(t;\xi) + m_3(t;\xi). \end{aligned}$$

This corresponds to the second line in (3.4).
Lastly, applying Itô's formula to $|\widehat{V}(t;\xi)|^2$ and recalling that $dV(t;\xi)$ has no stochastic part, we see that

$$d|\widehat{V}(t;\xi)|^2 = 2\xi\,\Re(\overline{\widehat{U}(t;\xi)}\widehat{V}(t;\xi))dt.$$

Taking expectations yields $\dot{m_3}(t;\xi) = 2\xi m_2(t;\xi)$ thus completing the proof. □

Eigenvector decomposition
The eigenvalues of the matrix

$$A(\xi) := \begin{bmatrix} 0 & 2 & 0 \\ \xi & -\frac{\sigma^2}{2}\xi^2 & 1 \\ 0 & 2\xi & 0 \end{bmatrix} \tag{3.5}$$

associated with (3.4) are given by

$$\lambda_0 = 0, \qquad \lambda_\pm(\xi) = -\frac{\sigma^2\xi^2}{4} \pm \frac{1}{4}\sqrt{\sigma^4\xi^4 + 64\xi}, \tag{3.6}$$

with the square root interpreted as the complex square root when the argument is negative and the branch with $\sqrt{x} > 0$ if $x > 0$ is chosen. A simple computation shows the very important result that the solutions of the system are uniformly bounded in $|\xi|$. More precisely, we have the following result.

Lemma 3.2 (Uniform bound) *For all $\xi \in \mathbb{R}$ and $\sigma \neq 0$,*

$$\Re\lambda_+(\xi) \le \frac{2}{\sigma^{2/3}} = \lambda_+\left(\frac{2}{\sigma^{4/3}}\right);$$

moreover,

$$\xi \le 0 \quad \textit{implies} \quad \Re\lambda_+(\xi) = -\frac{\sigma^2\xi^2}{4} \le 0.$$

Notice that an eigenbasis for (3.5) is

$$v_0(\xi) = \begin{pmatrix} 1 \\ 0 \\ -\xi \end{pmatrix}, \qquad v_\pm(\xi) = \begin{pmatrix} 1 \\ \frac{\lambda_\pm(\xi)}{2} \\ \xi \end{pmatrix}.$$

Moreover, from (3.3) and Cauchy-Schwarz inequality we get

$$|m_2(t;\xi)| \le \sqrt{m_1(t;\xi)m_3(t;\xi)} \le \frac{m_1(t;\xi) + m_3(t;\xi)}{2}, \quad t \in [0,T];\ \xi \in \mathbb{R}, \tag{3.7}$$

thus implying that the second component of the vector $m(t) = (m_1(t;\xi), m_2(t;\xi), m_3(t;\xi))^\top$ can be controlled by the other two.
We now define the weighted energy

$$F(t;\xi) := m_1(t;\xi) + \langle\xi\rangle^{-1} m_3(t;\xi) \quad \text{with} \quad \langle\xi\rangle := \sqrt{1+\xi^2}. \tag{3.8}$$

To ease the notation we also write

$$\gamma := \frac{\sigma^2}{2}\xi^2 \quad \text{and} \quad \Delta := \sqrt{\gamma^2 + 16\xi}$$

so that

$$\lambda_\pm(\xi) = \frac{-\gamma \pm \Delta}{2}.$$

Writing the solution to (3.4) in eigenvector coordinates, i.e.

$$m(t;\xi) = q_0 v_0(\xi) + q_+(t;\xi) v_+(\xi) + q_-(t;\xi) v_-(\xi),$$

we have

$$q_0(t) = q_0(0), \qquad q_+(t;\xi) = e^{\lambda_+(\xi)t} q_+(0), \qquad q_-(t;\xi) = e^{\lambda_-(\xi)t} q_-(0). \tag{3.9}$$

Moreover,

$$\begin{aligned} m_1(t;\xi) &= q_0 + q_+(t;\xi) + q_-(t;\xi), \\ m_3(t;\xi) &= -\xi q_0 + \xi\big(q_+(t;\xi) + q_-(t;\xi)\big), \end{aligned}$$

and therefore using (3.8) we get

$$F(t;\xi) = \Big(1 - \frac{\xi}{\langle\xi\rangle}\Big) q_0 + \Big(1 + \frac{\xi}{\langle\xi\rangle}\Big)\big(q_+(t;\xi) + q_-(t;\xi)\big).$$

On the other hand, since $0 \le 1 \pm \frac{\xi}{\langle\xi\rangle} \le 2$, we obtain the basic estimate

$$F(t;\xi) \le 2|q_0| + 2|q_+(t;\xi)| + 2|q_-(t;\xi)|. \tag{3.10}$$

Notation In the sequel to ease the notation we will omit to write in the functions m, q_+, q_-, λ_+ and λ_- the explicit dependence on ξ.

Let $m(0) := (m_{10}, m_{20}, m_{30})^\top$. A direct computation yields

$$q_0(0) = \frac{1}{2} m_{10} - \frac{1}{2\xi} m_{30}, \tag{3.11}$$

$$q_+(0) = -\frac{\lambda_-}{2\Delta} m_{10} + \frac{2}{\Delta} m_{20} - \frac{\lambda_-}{2\xi\Delta} m_{30}, \tag{3.12}$$

$$q_-(0) = \frac{\lambda_+}{2\Delta} m_{10} - \frac{2}{\Delta} m_{20} + \frac{\lambda_+}{2\xi\Delta} m_{30}. \tag{3.13}$$

Note that, from (3.8) at $t = 0$,

$$m_{10} \le F(0;\xi), \qquad m_{30} \le \langle\xi\rangle\, F(0;\xi). \tag{3.14}$$

Also, by (3.7) at $t = 0$,

$$|m_{20}| \le \frac{m_{10} + m_{30}}{2}. \tag{3.15}$$

Lemma 3.3 (Large-$|\xi|$ coefficient bounds) *There exist $\xi_0 \ge 1$ and $C > 0$ (depending only on σ) such that for all $|\xi| \ge \xi_0$,*

$$\frac{1}{|\Delta|} \le \frac{C}{\xi^2}, \qquad \left|\frac{\lambda_-}{\Delta}\right| \le C, \qquad \left|\frac{\lambda_-}{\xi\Delta}\right| \le \frac{C}{|\xi|}, \qquad \left|\frac{\lambda_+}{\Delta}\right| \le \frac{C}{|\xi|^3}, \qquad \left|\frac{\lambda_+}{\xi\Delta}\right| \le \frac{C}{|\xi|^4}. \tag{3.16}$$

Proof Since $\gamma = \frac{\sigma^2}{2}\xi^2$ and $\Delta = \sqrt{\gamma^2 + 16\xi}$, we have $\Delta \sim \gamma \sim \xi^2$ as $|\xi| \to \infty$, hence $1/|\Delta| \lesssim 1/\xi^2$.
Moreover, $\lambda_- = \frac{-\gamma-\Delta}{2}$ satisfies $|\lambda_-| \sim \gamma \sim \xi^2$, so $|\lambda_-|/|\Delta| \lesssim 1$ and $|\lambda_-|/(|\xi||\Delta|) \lesssim 1/|\xi|$.
Finally, $\lambda_+ = \frac{-\gamma+\Delta}{2} = O(1/\xi)$ as $|\xi| \to \infty$, because $\Delta = \gamma\sqrt{1 + 16\xi/\gamma^2} = \gamma + O(1/\xi)$, hence $\lambda_+ = (-\gamma + \Delta)/2 = O(1/\xi)$. Therefore $|\lambda_+|/|\Delta| \lesssim (1/|\xi|)/\xi^2 = 1/|\xi|^3$ and similarly $|\lambda_+|/(|\xi||\Delta|) \lesssim 1/|\xi|^4$. □

Lemma 3.4 (Control by $F(0;\xi)$ of eigenvector coefficients) *There exist $\xi_0 \ge 1$ and $C > 0$ such that for all $|\xi| \ge \xi_0$,*

$$|q_0(0)| + |q_+(0)| + |q_-(0)| \le C\, F(0;\xi). \tag{3.17}$$

Proof *Step 1: bound $q_0(0)$.* By (3.11) and (3.14),

$$|q_0(0)| \le \tfrac{1}{2} m_{10} + \tfrac{1}{2|\xi|} m_{30} \le \tfrac{1}{2} F(0;\xi) + \tfrac{\langle\xi\rangle}{2|\xi|} \frac{m_{30}}{\langle\xi\rangle} \le C F(0;\xi),$$

since $\langle\xi\rangle/|\xi| \le \sqrt{2}$ for $|\xi| \ge 1$.

Step 2: bound $q_+(0)$. Using (3.12) and Lemma 3.3,

$$|q_+(0)| \le C m_{10} + \frac{C}{\xi^2}|m_{20}| + \frac{C}{|\xi|} m_{30}.$$

We bound each term by $F(0;\xi)$. First, $m_{10} \le F(0;\xi)$. Next, by (3.15) and (3.14),

$$\frac{1}{\xi^2}|m_{20}| \le \frac{1}{2\xi^2}(m_{10}+m_{30}) \le \frac{C}{\xi^2} F(0;\xi) + \frac{C}{\xi^2}\langle\xi\rangle F(0;\xi) \le \frac{C}{|\xi|} F(0;\xi) \le C F(0;\xi),$$

for $|\xi| \geq 1$. Finally,

$$\frac{1}{|\xi|} m_{30} = \frac{\langle \xi \rangle}{|\xi|} \frac{m_{30}}{\langle \xi \rangle} \leq C F(0; \xi).$$

Hence $|q_+(0)| \leq C F(0; \xi)$.

Step 3: bound $q_-(0)$. Using (3.13) and Lemma 3.3,

$$|q_-(0)| \leq \frac{C}{|\xi|^3} m_{10} + \frac{C}{\xi^2} |m_{20}| + \frac{C}{|\xi|^4} m_{30} \leq C F(0; \xi),$$

using again (3.15) and (3.14) and $|\xi| \geq 1$. Summing the three bounds yields (3.17). □

Theorem 3.5 (Uniform weighted-energy bound for large $|\xi|$) *Fix* $T > 0$. *There exist constants* $C_1, C_2 > 0$ *and* $\xi_0 \geq 1$ *such that for all* $|\xi| \geq \xi_0$ *and all* $t \in [0, T]$,

$$F(t; \xi) \leq C_1 e^{C_2 t} F(0; \xi). \tag{3.18}$$

Proof From (3.10) and (3.9),

$$F(t; \xi) \leq 2|q_0(0)| + 2e^{\Re \lambda_+ t} |q_+(0)| + 2e^{\Re \lambda_- t} |q_-(0)|.$$

For $|\xi|$ large, $\Re\lambda_+ = O(1/|\xi|)$ and $\Re\lambda_- \sim -\gamma \leq 0$, so there exist ξ_0 and $C_2 > 0$ such that for all $|\xi| \geq \xi_0$,

$$e^{\Re\lambda_+ t} \leq e^{C_2 t}, \qquad e^{\Re\lambda_- t} \leq 1, \qquad t \in [0, T].$$

Therefore,

$$F(t; \xi) \leq 2e^{C_2 t} \big(|q_0(0)| + |q_+(0)| + |q_-(0)| \big).$$

Applying Lemma 3.4 gives (3.18). □

The large-$|\xi|$ estimate in Theorem 3.5 leaves open the compact region $|\xi| \leq \xi_0$. On this region we can bound the weighted energy by a uniform constant using compactness.

Lemma 3.6 (Bounded-ξ patch) *Fix* $T > 0$ *and* $\xi_0 \geq 1$. *There exists a constant* $M = M(T; \xi_0, \sigma) > 0$ *such that for all* $|\xi| \leq \xi_0$ *and all* $t \in [0, T]$,

$$F(t; \xi) \leq M\, F(0; \xi).$$

Consequently, for any $C_2 \geq 0$,

$$F(t; \xi) \leq M\, e^{C_2 t}\, F(0; \xi), \qquad t \in [0, T].$$

Proof Recalling the definition of $A(\xi)$ in (3.5), define the diagonal weight

$$W(\xi) := \mathrm{diag}\Big[1,\ 1,\ \langle \xi \rangle^{-1}\Big];$$

then for each ξ,

$$W(\xi)m(t) = W(\xi)e^{tA(\xi)}W(\xi)^{-1}\,W(\xi)m(0).$$

The map $(t;\xi) \mapsto B(t;\xi) := W(\xi)e^{tA(\xi)}W(\xi)^{-1}$ is continuous on the compact set $[0,T] \times [-\xi_0;\xi_0]$ (matrix exponential is continuous and $W(\xi)$ is smooth); hence

$$M := \sup_{t\in[0,T]}\sup_{|\xi|\le\xi_0} \|B(t;\xi)\|_{1\to1} < \infty.$$

Moreover, using that $m_1, m_3 \ge 0$, the definition (3.8) implies

$$F(t;\xi) = m_1(t) + \langle\xi\rangle^{-1}m_3(t) = \|W(\xi)m(t)\|_{\ell^1} - |m_2(t)| \le \|W(\xi)m(t)\|_{\ell^1},$$

and similarly $F(0;\xi) \ge 0$ and $\|W(\xi)m(0)\|_{\ell^1} = F(0;\xi) + |m_2(0)| \le 2F(0;\xi)$ by (3.7). Therefore,

$$F(t;\xi) \le \|W(\xi)m(t)\|_{\ell^1} \le \|B(t;\xi)\|_{1\to1}\,\|W(\xi)m(0)\|_{\ell^1} \le 2\,M\,F(0;\xi),$$

uniformly for $t \in [0,T]$ and $|\xi| \le \xi_0$. The last statement follows from $e^{C_2 t} \ge 1$. □

Combining Lemma 3.6 with Theorem 3.5 yields constants $C_1^*, C_2^* > 0$ such that for all $\xi \in \mathbb{R}$ and all $t \in [0,T]$,

$$F(t;\xi) \le C_1^* e^{C_2^* t} F(0;\xi).$$

Thus we have proved:

Theorem 3.7 *Fix $T > 0$. There exist constants $C_1, C_2 > 0$ such that for all $\xi \in \mathbb{R}$ and all $t \in [0,T]$,*

$$F(t;\xi) \le C_1 e^{C_2 t} F(0;\xi). \tag{3.19}$$

We now apply the standard Plancherel technique that upgrades the frequency estimate for $F(t;\xi)$ to Sobolev moment bounds in the x variable.

Lemma 3.8 (Sobolev moment bounds from the weighted energy) *Fix $s \in \mathbb{R}$ and $T > 0$. Assume that for all $\xi \in \mathbb{R}$ and all $t \in [0,T]$,*

$$F(t;\xi) \le C_1 e^{C_2 t} F(0;\xi), \tag{3.20}$$

where

$$F(t;\xi) = m_1(t;\xi) + \langle\xi\rangle^{-1}m_3(t;\xi),$$
$$m_1(t;\xi) = \mathbb{E}\,|\widehat{U}(t;\xi)|^2, \quad m_3(t;\xi) = \mathbb{E}\,|\widehat{V}(t;\xi)|^2.$$

Then,

$$\sup_{t\in[0,T]}\left[\mathbb{E}\|U(t)\|^2_{H^s_x}+\mathbb{E}\|V(t)\|^2_{H_x^{s-\frac{1}{2}}}\right]\le C_1e^{C_2T}\left[\|\varphi_0\|^2_{H^s_x}+\|\varphi_1\|^2_{H_x^{s-\frac{1}{2}}}\right]. \quad (3.21)$$

Proof Multiply the frequency estimate (3.20), which was proved in Theorem 3.7, by $\langle\xi\rangle^{2s}$ and integrate over $\xi\in\mathbb{R}$ to get

$$\int_{\mathbb{R}}\langle\xi\rangle^{2s}F(t;\xi)\,d\xi\le C_1e^{C_2t}\int_{\mathbb{R}}\langle\xi\rangle^{2s}F(0;\xi)\,d\xi.$$

By definition of F,

$$\int_{\mathbb{R}}\langle\xi\rangle^{2s}F(t;\xi)\,d\xi=\int_{\mathbb{R}}\langle\xi\rangle^{2s}\mathbb{E}|\widehat{U}(t;\xi)|^2d\xi+\int_{\mathbb{R}}\langle\xi\rangle^{2s-1}\mathbb{E}|\widehat{V}(t;\xi)|^2d\xi,$$

and using Fubini's theorem and Plancherel's identity in the x-variable gives

$$\int_{\mathbb{R}}\langle\xi\rangle^{2s}\mathbb{E}|\widehat{U}(t;\xi)|^2d\xi=\mathbb{E}\|U(t)\|^2_{H^s_x},\qquad\int_{\mathbb{R}}\langle\xi\rangle^{2s-1}\mathbb{E}|\widehat{V}(t;\xi)|^2\,d\xi=\mathbb{E}\|V(t)\|^2_{H_x^{s-\frac{1}{2}}},$$

since $\langle\xi\rangle^{2s-1}=\langle\xi\rangle^{2(s-\frac{1}{2})}$. The same identities hold at $t=0$ with $U(0)=\varphi_0$ and $V(0)=\varphi_1$. Taking the supremum over $t\in[0,T]$ yields the claim. $\square$

An immediate consequence of the Lemma 3.8 and Theorem 3.7 is:

Theorem 3.9 (Mean-square $H^s_x\times H_x^{s-\frac{1}{2}}$ well-posedness) *Assume that estimate* (3.21) *holds for a given* $s\in\mathbb{R}$. *Then the homogeneous Cauchy problem* (1.5) *is well-posed in* $H^s_x\times H_x^{s-\frac{1}{2}}$ *in the* mean-square sense, *namely:*

- *for every* $(\varphi_0,\varphi_1)\in H^s_x\times H_x^{s-\frac{1}{2}}$ *there exists a unique adapted solution* (U,V)*;*
- *the solution satisfies*

$$U\in L^\infty([0,T];L^2(\Omega;H^s_x)),\qquad V\in L^\infty([0,T];L^2(\Omega;H_x^{s-\frac{1}{2}}));$$

- *the solution map* $(\varphi_0,\varphi_1)\mapsto(U,V)$ *is continuous with respect to these norms.*

Corollary 3.10 (Mean-square C^∞ well-posedness) *Assume that estimate* (3.21) *holds for all* $s\ge 0$. *Then the Cauchy problem (1.5) is well-posed in* H^∞_x *in the mean-square sense:*

$$U\in\bigcap_{k\ge0}L^\infty([0,T];L^2(\Omega;H^k_x)),\qquad V\in\bigcap_{k\ge0}L^\infty([0,T];L^2(\Omega;H_x^{k-\frac{1}{2}})).$$

Due to the fact that (3.1) is a linear system we can also improve the regularity of the solution process from $L^\infty[0,T]$ to $C[0,T]$ utilizing the following proposition:

Proposition 3.11 (Time continuity in $L^2(\Omega; H^s_x)$ from mode-wise SDE continuity)*abellem:timeCont Fix $s \in \mathbb{R}$ and $T > 0$. Let (U, V) be solution to (1.5) whose x-Fourier transform $\widehat{U}(t;\xi)$ and $\widehat{V}(t;\xi)$ satisfy (3.1). Assume moreover the uniform Sobolev moment bounds*

$$\sup_{t\in[0,T]} \mathbb{E}\|U(t)\|^2_{H^s_x} < \infty, \qquad \sup_{t\in[0,T]} \mathbb{E}\|V(t)\|^2_{H_x^{s-\frac{1}{2}}} < \infty. \tag{3.22}$$

Then,

$$U \in C\big([0,T]; L^2(\Omega; H^s_x)\big), \qquad V \in C\big([0,T]; L^2(\Omega; H_x^{s-\frac{1}{2}})\big).$$

Proof We prove the statement for U; the proof for V is identical. Fix $t_0 \in [0,T]$ and let t tend to t_0. By Plancherel's identity in x,

$$\mathbb{E}\|U(t) - U(t_0)\|^2_{H^s_x} = \int_{\mathbb{R}} \langle\xi\rangle^{2s}\mathbb{E}\big|\widehat{U}(t;\xi) - \widehat{U}(t_0;\xi)\big|^2 d\xi. \tag{3.23}$$

For each fixed ξ, the assumed continuity of the mode process yields

$$\mathbb{E}\big|\widehat{U}(t;\xi) - \widehat{U}(t_0;\xi)\big|^2 \longrightarrow 0, \quad \text{as } t \to t_0.$$

Moreover,

$$\big|\widehat{U}(t;\xi) - \widehat{U}(t_0;\xi)\big|^2 \leq 2|\widehat{U}(t;\xi)|^2 + 2|\widehat{U}(t_0;\xi)|^2,$$

hence

$$\langle\xi\rangle^{2s}\mathbb{E}\big|\widehat{U}(t;\xi) - \widehat{U}(t_0;\xi)\big|^2 \leq 4 \sup_{\tau\in[t_0,T]} \langle\xi\rangle^{2s}\mathbb{E}|\widehat{U}(\tau;\xi)|^2.$$

The right-hand side is integrable in ξ because (3.22) is equivalent by Plancherel to

$$\sup_{\tau\in[t_0,T]} \int_{\mathbb{R}} \langle\xi\rangle^{2s}\mathbb{E}|\widehat{U}(\tau;\xi)|^2 d\xi < \infty.$$

Therefore dominated convergence applies in (3.23) and yields

$$\mathbb{E}\|U(t) - U(t_0)\|^2_{H^s_x} \longrightarrow 0 \quad \text{as } t \to t_0,$$

which is exactly $U \in C([0,T]; L^2(\Omega; H^s_x))$. ◻

Then Theorem 1.1 follows from the next corollary.

Corollary 3.12 (Continuity C^∞ well-posedness)*abelfinRes Assume that estimate* (3.21) *holds for all* $s \geq 0$*. Then the Cauchy problem* (1.5) *is well-posed in* H^∞_x *in the continuous sense:*

$$U \in \bigcap_{k\geq 0} C\big([0,T]; L^2(\Omega; H^k_x)\big), \qquad V \in \bigcap_{k\geq 0} C\big([0,T]; L^2(\Omega; H^{k-\frac{1}{2}}_x)\big).$$

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