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Global boundedness of weak solutions to a class of nonuniformly elliptic equations

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Abstract

We consider second order elliptic equations in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x,u,Du) = b(x,u,Du), \qquad x \in \Omega,$$

where Ω is a bounded open set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. Our aim is to give conditions on the vector field $a(x,u,Du) = \left(a^i(x,u,Du)\right)_{i=1,\dots,n}$ and on the right hand side $b(x,u,Du)$ in order to obtain the *global boundedness* in $\overline{\Omega}$ of weak solutions u to the Dirichlet problem associated to the previous differential equation, when a boundary condition $u = u_0 \in L^\infty(\Omega)$ has been fixed on $\partial\Omega$. We do not assume *structure conditions* on the vector field $a(x,u,Du)$, nor *sign assumptions* on $b(x,u,Du)$; we only consider *ellipticity* and *growth conditions* on a and b.

A main novelty with respect to the literature about this subject is that we assume *general* $p,q-$*growth conditions* for the principal part of the differential equation; however *we do not need an upper bound for the ratio* $\frac{q}{p}$, but nothing more than $1 \le p \le q$.

Key words: Elliptic equations, Regularity of solutions, Global boundedness, p,q-growth conditions, General growth conditions.

Mathematics Subject Classification (2020): Primary: 35D30, 35J15, 35J60; Secondary: 49N60.

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1 Introduction

With reference to the nonlinear elliptic differential equation $-\Delta u = \lambda u + u^{\frac{n+2}{n-2}}$ in an open starshaped set $\Omega \subset \mathbb{R}^n$, Pohožaev [88] in 1965 proved that the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u = 0 \quad \text{on } \Omega \end{cases} \tag{1.1}$$

does not admit positive solutions in Ω when $\lambda = 0$. In 1983 Brezis and Nirenberg [11] found an interval for λ of positive real numbers, i.e. $\lambda \in (0, \lambda_1)$, with a different situation: when $n \geq 4$ the Dirichlet problem (1.1) has a positive solution if $\lambda \in (0, \lambda_1)$, while it does not admit positive solutions when $\lambda \in \mathbb{R}$, $\lambda \notin (0, \lambda_1)$. The same Brezis and Nirenberg noted in [11] that $\frac{n+2}{n-2}$ is a *critical exponent*. Precisely, if as usual we denote by p^* the *Sobolev exponent* of p, i.e. $p^* = \frac{np}{n-p}$ when $p < n$, then if $p = 2$:

$$p^* - 1 = \frac{np}{n-p} - 1 \underset{\text{if } p=2}{=} \frac{n+2}{n-2}\,.$$

If we change the Laplace operator Δ in the equation $(1.1)_1$ into the more general $p-$Laplace operator, then the previous Dirichlet problem (1.1) takes the form

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (|Du|^{p-2} u_{x_i}) = \lambda u + u^{p^*-1} & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Therefore, on looking at the above situation, for the right hand side an appropriate *critical* growth condition with respect to u happens with the power u^{p^*-1} when u is nonnegative, or more generally with $|u|^{p^*-1}$.

In this paper we consider a general elliptic second order Dirichlet problem of the form

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, u, Du) = b(x, u, Du)\,, & x \in \Omega\,, \\ u = u_0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{1.3}$$

with Ω bounded open set in $\mathbb{R}^n$, $n \geq 2$, and $u, u_0 : \Omega \to \mathbb{R}$. Our aim is to give conditions on the vector field $a(x, u, \xi) = \left(a^i(x, u, \xi)\right)_{i=1,\ldots,n}$ and on the right hand side $b(x, u, \xi)$ which guarantee the *global boundedness* in $\overline{\Omega}$ of weak solutions u to the Dirichlet problem (1.3). We do not assume *structure conditions* on the vector field $a(x, u, \xi)$, nor *sign assumptions* on $b(x, u, \xi)$; we only consider *coercivity* and *growth conditions* on a and b.

We already discussed about the *critical* growth condition with respect to u for the right hand side $b(x, u, \xi)$. In the model Dirichlet problem (1.2) the *critical power* is $|u|^{p^*-1}$; in the general context (1.3) we assume the following growth condition on the right hand side, $b(x, u, \xi)$

$$|b(x, u, \xi)| \leq M \left(|\xi|^{r-1} + |u|^{t-1}\right) + b_0(x)\,, \tag{1.4}$$

for a function $b_0 \in L^{s_0}(\Omega)$, $s_0 > \frac{n}{p}$, and for some $r, t \in \mathbb{R}$ with $1 \le r < p + \frac{p}{n}$ and $1 \le t < p^*$ (there is not upper bound on t if $p \ge n$).

One of the main novelties of this manuscript is that we assume general $p, q-$conditions on the vector field $a(x, u, \xi)$. Precisely when $1 \le p \le q$ (for simplicity of exposition we consider here $p \le n$, but in the next sections we study the unconstrained case $1 \le p \le q$) we assume the following *coercivity* and *growth conditions* for $a(x,u,\xi) = \left(a^i(x,u,\xi)\right)_{i=1,\dots,n}$

$$\begin{cases} (a(x,u,\xi), \xi) \ge c_1 |\xi|^p - c_2 |u|^\theta - b_1(x) \\ |a(x,u,\xi)| \le c_3 |\xi|^{q-1} + c_4 |u|^{q^*/q'} + b_2(x) \end{cases} \tag{1.5}$$

where as usual $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$, $\frac{1}{q'} := \frac{q-1}{q}$. Similarly as for p^*, q^* denotes the Sobolev exponent $q^* := \frac{nq}{n-q}$ if $q < n$ (q^* equal to any fixed real number greater than q if $q \ge n$). Here $0 \le \theta < p^*$ (there is not upper bound on θ and q^*/q' if $p \ge n$), $b_i \in L^{s_i}(\Omega)$, with $i \in \{1,2\}$ and $s_1 > \frac{n}{p}$, $s_2 \ge \frac{q}{q-1}$.

A second main novelty is that we have here *general* $p, q-$*growth conditions* for the principal part of the differential equation in $(1.3)_1$; however *we do not need an upper bound for the ratio* $\frac{q}{p}$, but nothing more than $1 \le p \le q$. This aspect needs to be explained, if compared with the literature about regularity for general elliptic equations and systems under $p, q-$*growth conditions*. In fact for interior regularity an upper bound for $\frac{q}{p}$, of the type $\frac{q}{p} < 1 + O\left(\frac{1}{n}\right)$, or $\frac{q}{p} \le 1 + O\left(\frac{1}{n}\right)$, is always necessary. This is due to the Giaquinta-Marcellini counterexamples to regularity [56],[73],[75], who proved that if a condition of the type $\frac{q}{p} \le 1 + O\left(\frac{1}{n}\right)$ is not satisfied then the equation in $(1.3)_1$, even in the simpler form $\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(Du) = 0$, may have *unbounded* weak solutions in Ω. More precisely (see [75, Theorem 6.1]) an *unbounded* weak solution to the equation $\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(Du) = 0$ may exist in Ω if $n > 2$, $1 < p < n-1$ and $\frac{q}{p} > \frac{n-1}{n-1-p}$. The last condition is equivalent to $q > [p^*]_{n-1}$, when in this case $[p^*]_{n-1} := \frac{(n-1)p}{n-1-p}$ is the Sobolev exponent computed in dimension $n-1$. Related to this we mention the recent result by Peter Bella and Mathias Schäffner [2]: they apply the original 1991 method in [75] with a modification in the use of a smart version of the Sobolev inequality; i.e., a Sobolev inequality on spheres, surfaces of balls, instead of balls (see Lemma 3 in [2]); thus they gain in the dimensional parameter, $n-1$ instead of n. In this context see also [3] and the *local boundedness* theorem by Hirsch-Schäffner [65], with a sharp result when compared with the quoted example in [75, Theorem 6.1] of *existence of unbounded weak solutions.*

Does this contradict the *global boundedness result* stated below in Theorem 1.1? Stated *without upper bound on the ratio* $\frac{q}{p}$? The answer is that there is not contradiction, since the Giaquinta-Marcellini counterexample gives an unbounded weak solution in Ω, *unbounded up to the boundary* $\partial\Omega$. We also note that, for elliptic differential equations of the type $\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(Du) = 0$, to built a counterexample it is necessary to arrive with a singularity up to the boundary; in fact, if it would be possible to exhibit a weak solution which is singular only at a single point x_0 in Ω and smooth outside, then we could apply the *bounded slope condition method* in a ball $B(x_0)$ centered in x_0 and contained in Ω, arriving to a smooth (Lipschitz continuous) weak solution in $B(x_0)$, which contradicts the existence of an *isolated* singularity.

Our global boundedness result is the following.

Theorem 1.1 *We consider the Dirichlet problem (1.3) with a boundary datum* $u_0 \in L^{\infty}(\Omega) \cap W^{1,q}(\Omega)$*, under the coercivity and growth assumptions (1.4),(1.5). Then every weak solution* $u \in u_0 + W_0^{1,q}(\Omega)$ *to (1.3) is bounded in* $\overline{\Omega}$ *and there exist positive constants* α *and* c *such that*

$$\|u\|_{L^{\infty}(\Omega)} \le c\left(1 + \|u_0\|_{L^{\infty}(\Omega)}\right) \cdot \left(1 + \|u\|_{L^{p^*}(\Omega)}\right)^{\alpha} . \tag{1.6}$$

We emphasize here a particular case (and direct consequence) of Theorem 1.1 under conditions often considered in the literature.

Corollary 1.2 *We consider the Dirichlet problem with* $u_0 \in L^{\infty}(\Omega) \cap W^{1,q}(\Omega)$

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x, u, Du), & x \in \Omega, \\ u = u_0 \quad on \ \partial\Omega, \end{cases} \tag{1.7}$$

under the following coercivity and growth assumptions

$$\begin{cases} (a(x,\xi), \xi) \ge c_1 |\xi|^p - b_1(x) \\ |a(x,\xi)| \le c_3 |\xi|^{q-1} + b_2(x) \\ |b(x,u,\xi)| \le M\left(|\xi|^{r-1} + |u|^{t-1}\right) + b_0(x) \end{cases} \tag{1.8}$$

were $1 \le p \le q$, $1 \le r < p + \frac{p}{n}$, $(1 \le r < p+1$ *if* $p \ge n)$, $1 \le t < p^*$ *(without upper bound on* t *if* $p \ge n$*) and* $b_i \in L^{s_i}(\Omega)$, $i \in \{0,1,2\}$, *with* $s_0, s_1 > \frac{n}{p}$, $s_2 \ge \frac{q}{q-1}$. *Then every weak solution* $u \in u_0 + W_0^{1,q}(\Omega)$ *to (1.7) is bounded in* $\overline{\Omega}$ *and the* $L^{\infty}-$ *estimate (1.6) holds.*

A few words about the well-known classical results on regularity for weak solutions to elliptic equations in divergence form. A main tool is the fundamental Hölder continuity result by De Giorgi, which has been extensively considered also in the book by Ladyzhenskaya-Ural'tseva [68, Chapter 4]; the celebrated Moser's *iteration scheme* [84] is closely related. We also refer to the article by Evans [51], with explicit dependence of the vector field $\left(a^i\right)_{i=1,\ldots,n}$ only on the gradient variable and with right hand side $b = 0$; the relevant paper by DiBenedetto [40] on the $C^{1,\alpha}-$regularity for weak solutions of a class of degenerate elliptic equations; the famous $C^{1,\alpha}-$regularity result by Tolksdorf [95]; the article by Manfredi [71] on the $p-$Laplacian type integrals of the Calculus of Variations. Later, see also the articles by Lieberman [69] and Marcellini [75], the book by Giusti [60], the articles by Mingione [82], Pucci-Serrin [89] and Pucci-Servadei [90] (see in particular Theorem 2.2 about the local boundedness), the results by Duzaar-Mingione [44] and Cianchi-Maz'ya [18].

We emphasize that a starting condition in many of the above-quoted classical regularity results is the a-priori assumption either of the local or the global boundedness of the weak solutions. This is a motivation to consider here the boundedness stated in Theorem 1.1 above. A further motivation comes from the studies about *multiplicity* of solutions, which often need regularity of weak solutions up to the boundary. The researches in these fields nowadays is so wide that it is not possible to give a full landscape of all the papers related to multiplicity and to existence of nontrivial solution; we limit to the context of *general growth conditions* and we quote at least Mihăilescu-Pucci-Rădulescu [81], Zhang-Radulescu [100], Papageorgiou-Rădulescu-Zhang [86], Fang-Rădulescu-Chao Zhang-Xia Zhang [52], and the references therein.

It is possible to find in the literature *global boundedness results* when $q = p$. Most of them are related to a Dirichlet problem for the $p-$Laplacian, a right hand side similar to (1.1),(1.2) and a zero boundary datum. For instance we refer to the global boundedness in the estimate (1.19) in Guedda-Veron [63] dated 1989, the Appendix of the article published in 1998 by Egnell [46], Talenti [94] in 1979 and Cianchi [17], with fine results based on rearrangements. Only few cases of global boundedness are known in the not uniformly elliptic case with $q > p$, starting by Kolodiĭ [67] in 1970 in the specific case of some *anisotropic* elliptic equations. Boccardo-Marcellini-Sbordone [6] studied a Dirichlet problem for the anisotropic differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u_{x_i}|^{p_i - 2} u_{x_i} \right) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \qquad x \in \Omega \,,$$

with $p = \min \{p_i : \; i = 1, 2, \ldots, n\}$ and $q = \max \{p_i : \; i = 1, 2, \ldots, n\}$. Later see also the case considered by Stroffolini [93] under some similar more general assumptions. More recent *global boundedness results* with general growth and $q > p$ have been given by Winkert-Zacher [99] for variable exponents, Perera-Squassina [87, Proposition 2.4] specific for double-phase equations, and by Ho-Winkert [66, Theorem 4.2] for double phase problems with variable exponents.

A special mention to the *classical maximum principle* for weak solutions to elliptic pde's, which for *divergence form operators* as in the case of equation $(1.3)_1$, has been treated for instance in the book by David Gilbarg and Neil S. Trudinger [59, Chapter 10], and extensively in the more recent monograph by Patrizia Pucci and James Serrin [89, Chapter 6]. Precisely we refer to Gilbarg-Trudinger [59, Theorem 10.9] and to the Pucci-Serrin's results [89, Theorems 6.1.1 and 6.1.2], where L^∞ a-priori estimates for weak solutions to solutions of the Dirichlet problem (1.3) have been obtained under *coercive properties* for $a\,(x, u, \xi)$ as in $(1.5)_1$ and bounds on the term $b\,(x, u, \xi)$ similar to (1.4). Their proofs are both based on the well known *Moser's iterative method* and involve *subsolutions* of $p-$Laplace type equations, or *minimal surface* problems. In the context of $p, q-$growth the authors of this manuscript recently used the

Moser's approach, for instance in [28],[75],[80], for *local* $L^{\infty}-$*gradient bound* of the weak solutions. The proof here of the global boundedness of weak solutions to the Dirichlet problem (1.3), on the contrary, is based on the *De Giorgi's iterative method*, starting from a *Caccioppoli's estimate*; full details in Section 3. Presumably, without dramatic changes, the proofs by Gilbarg-Trudinger [59, Theorem 10.9] in 1977 and by Pucci-Serrin [89, Theorems 6.1.1 and 6.1.2] in 2007 could be used also in this general context of $p,q-$growth; of course a proper definition of weak solution should be adapted to the new assumptions, when q is different from p.

In the mathematical literature *interior* boundedness can be find under the so-called natural growth conditions $q=p$, starting from the well known local boundedness (and then Hölder continuity too) by Giaquinta-Giusti [57] in 1982; we refer in particular to the *scalar case* treated in [57, Section 2]. Being more related to this manuscript, we mention some references about the *local boundedness* of solutions to elliptic equations and systems with *general* and $p,q-$*growth conditions*, with $q\geq p$. The local boundedness of solution to classes of anisotropic elliptic equations or systems have been investigated by the authors [24]–[28] and by DiBenedetto-Gianazza-Vespri [41]. Other results on the boundedness of solutions of PDEs or of minimizers of integral functionals can be found in Fusco-Sbordone [53],[54], Cupini-Leonetti-Mascolo [23], Carozza-Gao-Giova-Leonetti [13], Granucci-Randolfi [62], Biagi-Cupini-Mascolo [4].

Interior $L^{\infty}-$*gradient bound*, i.e., the *local Lipschitz continuity*, of weak solutions to nonlinear elliptic equations and systems under non standard growth conditions have been obtained since 1989 in [74]–[78]. Specific for the so-called *double phase operators*

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|Du|^{p-2}\,u_{x_i}\right)+a\left(x\right)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|Du|^{q-2}\,u_{x_i}\right),\qquad x\in\Omega\,, \tag{1.9}$$

with $a\left(x\right)\geq 0$ in Ω, are the articles by Colombo-Mingione [22], Baroni-Colombo-Mingione [1], Eleuteri-Marcellini-Mascolo [47]. See also the recent references for other Lipschitz regularity results: DiMarco-Marcellini [43], Bousquet-Brasco [10], DeFilippis-Mingione [35], Caselli-Eleuteri-Passarelli di Napoli [14], Gentile [55], Eleuteri-Marcellini-Mascolo-Perrotta [48]; see also [27],[28],[29],[30], [39],[79],[80]. The local boundedness of the solution u can be used to achieve further regularity properties, as the Hölder continuity of u or of its gradient Du; we limit here to cite Bildhauer-Fuchs [5], Liskevich-Skrypnik [70], Düzgun-Marcellini-Vespri [45], DiBenedetto-Gianazza-Vespri [41], Byun-Oh [12], Ciani-Skrypnik-Vespri [21] as examples of this approach. *Variable exponents* and double phase problems are considered by Byun-Oh [12], Ragusa-Tachikawa [91]. For *Orlicz-Sobolev spaces* see Diening-Harjulehto-Hasto-Ruzicka [42], Chlebicka [15], Chlebicka-DeFilippis [16], Hästö-Ok [64]. About *quasiconvex integrals* of the calculus of variations see in particular [7],[31],[72], and Schmidt [92], DeFilippis [32], DeFilippis-Stroffolini [38], Gmeineder-Kristensen [61] about *partial regularity* in the general *quasiconvex case*. Being a $p,q-$growth problem for the double phase (1.9), specific for the difficult case $p = 1$ (however when a is a positive constant), we mention the related C^1-regularity result due to Giga-Tsubouchi [58], obtained by mean of smart use of *methods of convex analysis* and the *maximum principle*, for *convex weak solutions* to equations with the sum of the $1-$Laplace operator and the $p-$Laplace operator; then generalized by Tsubouchi [96],[97],[98], with a different proof sometime related to that one by Bögelein-Duzaar-Giova-Passarelli di Napoli [8], for a class of degenerate parabolic systems.

For recent *boundary regularity* in the context considered in this manuscript we mention Cianchi-Maz'ya [18],[19], Bögelein-Duzaar-Marcellini-Scheven [9], DeFilippis-Piccinini [37]. We also refer to De Filippis-Mingione [36], Mingione-Rădulescu [83], who have outlined the recent trends and advances in the regularity theory for variational problems with non-standard growths and non-uniform ellipticity. Very recently we mention the interesting paper by Cianchi-Schäffner [20], specifically related to the local boundedness of minimizers under unbalanced Orlicz growth conditions, the sharp higher differentiability for minimizers of variational integrals by DeFilippis-Koch-Kristensen [33], the boundedness, Hölder continuity, Harnack inequality results for *local quasiminima* to elliptic double phase problems by Nastasi-Pacchiano Camacho [85], the Leray-Lions existence theorem extended to the context of general growth conditions in [29].

2 Assumptions and statement of the global boundedness result

With the aim to prove the global boundedness stated in Theorem 1.1, for the sake of clarity we list in this section, in a more detailed way, the assumptions and the result to be proved. We consider the Dirichlet problem

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x_i} a^i\left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right), \qquad x \in \Omega\,, \\ u(x) = u_0(x), \qquad x \in \partial\Omega\,, \end{cases} \tag{2.1}$$

with Ω bounded open set in $\mathbb{R}^n$, $n \geq 2$, and u_0 in $L^\infty(\Omega) \cap W^{1,q}(\Omega)$, $1 \leq p \leq q$. We say that a $u \in u_0 + W_0^{1,q}(\Omega)$ is a weak solution to (2.1) if

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x, u, Du)\varphi_{x_i} + b(x, u, Du)\varphi \right\} dx = 0\,, \quad \forall\, \varphi \in W_0^{1,q}(\Omega). \tag{2.2}$$

Since a-priori $u \in W^{1,q}(\Omega)$, then u is also globally bounded in Ω when $q > n$, as a well known consequence of the Sobolev-Morrey embedding theorem. Therefore, we could limit to consider the case $1 \leq p \leq q \leq n$. However we adopt the more general framework $p, q \in \mathbb{R}$, $1 \leq p \leq q$ with the aim to get a final global $L^\infty-$bound for u without upper restrictions on q.

Let us recall the assumptions on the vector field $a(x, u, \xi)$ and on the datum $b(x, u, \xi)$:

(i) there exist positive constants c_1, c_2 such that, for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$(a(x, u, \xi), \xi) \geq c_1 |\xi|^p - c_2 |u|^\theta - b_1(x) , \tag{2.3}$$

with $b_1 \geq 0$ and $b_1 \in L^{s_1}(\Omega)$;

(ii) for $r, t \geq 1$ and for a positive constant M

$$|b(x, u, \xi)| \leq M \left(|\xi|^{r-1} + |u|^{t-1} \right) + b_0(x) , \tag{2.4}$$

with $b_0 \geq 0$ and $b_0 \in L^{s_0}(\Omega)$.

Before stating the bounds on the other parameters, we remind that p^* denotes the usual Sobolev exponent $p^* := \frac{np}{n-p}$ if $p < n$ and p^* equal to any fixed real number greater than p if $p \geq n$. In particular, if $p \geq n$ we assume $1 \leq r < p+1$, $s_0, s_1 > 1$ and, without loss of generality, we choose

$$p^* > \max\left\{ \tfrac{p}{p-r+1};\ (r-1)\tfrac{p}{p-1};\ \theta;\ t;\ \tfrac{ps_0}{s_0-1};\ \tfrac{ps_1}{s_1-1} \right\}. \tag{2.5}$$

If $p < n$ the conditions on the exponents $\theta, t, r, s_0, s_1, s_2$ are

$$\text{on } \theta \geq 0 \text{ and } t \geq 1: \quad \theta, t < \tfrac{np}{n-p} =: p^* ; \tag{2.6}$$

$$\text{on } r \geq 1: \quad \tfrac{r}{p} < 1 + \tfrac{1}{n} ; \tag{2.7}$$

$$\text{on } s_0 \text{ and } s_1: \quad s_0, s_1 > \tfrac{n}{p}. \tag{2.8}$$

We note in particular that $s_0, s_1 > 1$.

Under the above general assumptions our boundedness result is the following.

Theorem 2.1 *Let $u \in u_0 + W_0^{1,q}(\Omega)$, for some $q \geq p \geq 1$, be a weak solution to the Dirichlet problem (2.1) with a bounded boundary datum $u_0 \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$. Assume that for every $\varphi \in W_0^{1,q}(\Omega)$,*

$$x \mapsto (a(x, u(x), Du(x)), D\varphi(x)) \text{ is in } L^1(\Omega). \tag{2.9}$$

If (2.3)-(2.8) hold, then $u \in L^\infty(\Omega)$ and there exists $\sigma > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq c \left(\|u_0\|_{L^\infty(\Omega)} + 1 \right) \cdot \left(1 + \|u\|_{L^{p^*}(\Omega)} \right)^{\frac{p^*-p}{\sigma}} , \tag{2.10}$$

where the constant c depends on the data, but it is independent of u.

The explicit expression for σ in (2.10) is

$$\sigma := p^* - \max\left\{ \tfrac{p}{p-r+1};\ \theta;\ t;\ \tfrac{p^*}{s_0} + 1;\ \tfrac{p^*}{s_1} \right\}, \tag{2.11}$$

and we note that $\sigma > 0$; this is due to the bound (2.5) if $p \geq n$, and to the bounds (2.6)–(2.8) if $p < n$; in particular we notice that $\frac{p}{p-r+1} < p^*$ if and only if $r < p + \frac{p}{n}$, i.e. (2.7) holds and, by (2.8) $s_i > \frac{n}{p} = \frac{p^*}{p^*-p} \geq \frac{p^*}{p^*-1}$ for every $i \in \{0, 1\}$.

We give the proof of Theorem 2.1 in Section 3.

Condition (2.9) ensures a well posed definition of u as a weak solution. It is important now to discuss the well posedness of the pairing

$$\int_{\Omega} (a(x,u,Du), D\varphi\,(x))\, dx\,,$$

in relationship with some explicit growth assumption. A sufficient condition to ensure that the pairing condition (2.9) holds, is the following growth assumption:

(iii) there exist positive constants c_3, c_4 such that, for a.e. $x \in \Omega$, every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|a\,(x,u,\xi)| \leq c_3\,|\xi|^{q-1} + c_4\,|u|^{q^*/q'} + b_2\,(x)\ , \tag{2.12}$$

with $b_2 \geq 0$ and $b_2 \in L^{s_2}(\Omega)$, $s_2 \geq q'$ with $\frac{1}{q'} := \frac{q-1}{q}$. Here $q^*/q' := \frac{nq}{n-q} \cdot \frac{q-1}{q}$ if $1 \leq q < n$, and $q^*/q' := \ell$, with ℓ nonnegative real number greater than $q-1$ if $q \geq n$.

Then, the following corollary of Theorem 2.1, which includes Theorem 1.1, follows.

Corollary 2.2 *Let u be as in Theorem 2.1 and assume (2.12), together with (2.3)-(2.8). Then $u \in L^{\infty}(\Omega)$ and (2.10) holds.*

Proof. We need only to prove that (2.12) is a sufficient condition for (2.9).

Since φ is a test function in $W_0^{1,q}(\Omega)$, we need to prove that $a(x,u,Du)$ is in $L^{q'}$, with as usual $\frac{1}{q} + \frac{1}{q'} = 1$. To do this we use the inequality (2.12). By the Sobolev embedding theorem $u \in L^{q^*}(\Omega)$, with $q^* := \frac{nq}{n-q}$ if $q < n$; therefore

$$\int_{\Omega} |u(x)|^{q^*/q'} |D\varphi(x)|\, dx \leq \|u\|_{L^{q^*}(\Omega)}^{q^*/q'} \|D\varphi\|_{L^q(\Omega;\mathbb{R}^m)} < \infty.$$

If instead $q \geq n$, then $u \in L^s(\Omega)$ for every $s > q$ and, as said, q^*/q' stands for a real number $\ell > q-1$. Then

$$\int_{\Omega} |u(x)|^{\ell} |D\varphi(x)|\, dx \leq \|u\|_{L^{\ell q'}(\Omega)}^{\ell} \|D\varphi\|_{L^q(\Omega;\mathbb{R}^m)} < \infty.$$

It remains to check that, if $b_2 \in L^{q'}(\Omega)$, then $b_2|D\varphi| \in L^1(\Omega)$. This trivially follows, because $b_2 \in L^{s_2}(\Omega)$, $s_2 \geq q'$.

The conclusion follows by Theorem 2.1. ■

Remark 2.3 (about the summability condition on b) *We discuss the correctness of*

$$\int_{\Omega} b(x,u,Du)\,\varphi\,(x)\ dx\,. \tag{2.13}$$

Since φ is a test function in $W_0^{1,q}(\Omega)$, by the imbedding of $W_0^{1,q}(\Omega)$ into $L^{q^*}(\Omega)$ we have $\varphi \in L^{q^*}(\Omega)$; here, as usual, $q^* := \frac{nq}{n-q}$ if $q < n$, otherwise is any number *greater than q. By (2.4) the integral (2.13) is correctly defined if $|Du|^{r-1}$, $|u|^{t-1}$, b_0 are in $L^{(q^*)'}(\Omega)$.*

Let us show that $b_0 \in L^{s_0}(\Omega) \subset L^{(q^*)'}(\Omega)$ or, equivalently, that $s_0 \geq (q^*)'$. Let us assume $1 \leq p < n$. Then, by (2.8), $s_0 > \frac{n}{p}$. Since $p \leq q$ then $p^* \leq q^*$ and

$$(q^*)' \leq (p^*)' = \tfrac{p^*}{p^*-1} = \tfrac{np}{np-n+p};$$

this last quantity is less than or equal to $\frac{n}{p}$ if and only if $\frac{p}{np-n+p} \leq \frac{1}{p}$, which is equivalent to $p^2 - (n+1)p + n \leq 0$; i.e. $1 \leq p \leq n$. Therefore $(q^*)' \leq \frac{n}{p} < s_0$.

If $p \geq n$ the inclusion $L^{s_0}(\Omega) \subset L^{(q^*)'}(\Omega)$ is trivially satisfied; indeed, by (2.5), $q^* \geq p^* > \frac{ps_0}{s_0-1} > \frac{s_0}{s_0-1}$, thus $s_0 > (q^*)'$.

As far as the $u-$dependence is concerned, we need $u \in L^{(t-1)\frac{q^*}{q^*-1}}(\Omega)$. By Sobolev embedding Theorem $u \in L^{q^*}(\Omega)$, therefore the needed summability for u is satisfied if $(t-1)\frac{q^*}{q^*-1} \leq q^*$, or equivalently, $t \leq q^*$. This last condition holds, because $t < p^*$, see (2.5) and (2.6).

We conclude by considering the product $|Du|^{r-1}\varphi$ that we want to be summable in Ω; this happens if $(r-1)\frac{q^*}{q^*-1} \leq q$. Let us consider three cases: $p \geq n$, the case $p \leq q < n$ and the case $p < n \leq q$. If $p \geq n$, then, by (2.5), $q^* \geq p^* > \frac{p}{p-r+1} \geq \frac{q}{q-r+1}$, that implies $(r-1)q^* < (q^*-1)q$ and, eventually, $(r-1)\frac{q^*}{q^*-1} \leq q$. If instead $p \leq q < n$ then we observe that $(r-1)\frac{q^*}{q^*-1} \leq q$ is equivalent to $\frac{r}{q} \leq 1 + \frac{1}{n}$ that holds true because $\frac{r}{q} \leq \frac{r}{p} < 1 + \frac{1}{n}$, see (2.7). If $p < n$ and $q \geq n$, then, $(r-1)(q^*)' < q$ is satisfied thanks to $q^* \geq p^*$ and (2.7). Indeed, (2.7) is equivalent to $p^* > \frac{p}{p-r+1}$, therefore

$$q^* \geq p^* > \frac{p}{p-r+1} > \frac{q}{q-r+1} = \left(\tfrac{q}{r-1}\right)'.$$

3 Proof of the global boundedness result

In this section we prove Theorem 2.1. We start by a Caccioppoli's inequality for the weak solutions of (1.3).

Proposition 3.1 *Let $u_0 \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$ and let $u \in u_0 + W_0^{1,q}(\Omega)$ be a weak solution to (1.3) under the assumptions* (2.3), (2.4) *and the related conditions on the exponents in Section 2. Consider, for every $k \in \mathbb{R}$, $k \geq k_0$, with $k_0 > \|u_0\|_\infty$, the super-level sets $A_k := \{x \in \Omega : u(x) > k\}$. Then there exists c depending only on the data, but neither on u nor on k, such that*

$$\begin{aligned}\int_{A_k} |Du|^p\,dx \leq\; & c\|b_0\|_{L^{s_0}(\Omega)}\|u-k\|_{L^{p^*}(A_k)}|A_k|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\ & + c\,\|u-k\|^{\theta}_{L^{p^*}(A_k)}|A_k|^{1-\frac{\theta}{p^*}} + c\,\|u-k\|^{t}_{L^{p^*}(A_k)}|A_k|^{1-\frac{t}{p^*}} \\ & + c\|u-k\|^{\frac{p}{p-r+1}}_{L^{p^*}(A_k)}|A_k|^{1-\frac{p}{p^*(p-r+1)}} \\ & + c\,(1+k^\tau)|A_k| + c\|b_1\|_{L^{s_1}(\Omega)}|A_k|^{1-\frac{1}{s_1}}, \end{aligned} \tag{3.1}$$

where

$$\tau := \max\{\theta, t\}\,. \tag{3.2}$$

Proof. For every $k \geq k_0$, with $k_0 > \|u_0\|_\infty$, we define the test function φ_k as follows

$$\varphi_k(x) := (u(x) - k)_+ \quad \text{for a.e. } x \in \Omega,$$

where $(u(x) - k)_+ = \max\{u(x) - k, 0\}$. Notice that $\varphi_k \in W_0^{1,q}(\Omega)$. The trace is 0, because $u(x) - k < u(x) - \|u_0\|_\infty$ on the boundary of Ω. Let us consider the super-level sets: $A_k := \{x \in \Omega : u(x) > k\}$. Using φ_k as a test function in (2.2) we get

$$\begin{aligned} I_1 :=& \int_{A_k} (a(x, u, Du), Du)\, dx \\ =& - \int_{A_k} b(x, u, Du)(u - k)\, dx =: I_2. \end{aligned} \tag{3.3}$$

Now, we separately consider and estimate I_i, $i = 1, 2$. By (2.4) there exists a positive constant M such that

$$I_2 \leq \int_{A_k} \left\{ M\,|Du|^{r-1}\,(u - k) + M|u|^{t-1}(u - k) + b_0(u - k) \right\}\, dx.$$

We estimate the right-hand side using the Young's inequality, with exponents $\frac{p}{p-r+1}$ and $\frac{p}{r-1}$; thus there exists $c > 0$, such that

$$\begin{aligned} I_2 \leq& \tfrac{c_1}{2} \int_{A_k} |Du|^p\, dx + c \int_{A_k} \left\{ (u - k)^{\frac{p}{p-r+1}} + |u|^{t-1}(u - k) \right\}\, dx \\ &+ c \int_{A_k} b_0(u - k)\, dx\,, \end{aligned} \tag{3.4}$$

where c_1 is the constant in (2.3). Collecting (3.3),(3.4) and using (2.3) we get

$$\begin{aligned} &\tfrac{c_1}{2} \int_{A_k} |Du|^p\, dx \\ \leq& c \int_{A_k} \left\{ |u|^\theta + (u - k)^{\frac{p}{p-r+1}} + |u|^{t-1}(u - k) + b_0(u - k) + b_1 \right\}\, dx. \end{aligned} \tag{3.5}$$

By the integrability assumptions on b_0,b_1,

$$\int_{A_k} b_0(u - k)\, dx \leq c\, \|b_0\|_{L^{s_0}(\Omega)} \|u - k\|_{L^{p^*}(A_k)} |A_k|^{1 - \frac{1}{s_0} - \frac{1}{p^*}}$$

and

$$\int_{A_k} b_1\, dx \leq \|b_1\|_{L^{s_1}(\Omega)} |A_k|^{1 - \frac{1}{s_1}}.$$

For a.e. $x \in A_k$, we have $|u|^{t-1}(u-k)_+ \leq c(u-k)^t + ck^t$ and $|u|^\theta \leq c(u-k)^\theta + ck^\theta$; therefore

$$\int_{A_k} \left\{ |u|^\theta + |u|^{t-1}(u - k) \right\}\, dx \leq c \int_{A_k} \{(u - k)^\theta + (u - k)^t\}\, dx + c(k^\theta + k^t)|A_k|.$$

Since the assumptions (2.5),(2.6) hold, we can use the Hölder inequalities with exponents $\frac{p^*}{\theta}$, $\frac{p^*}{t}$ and $p^* \frac{p-r+1}{p}$, so we get

$$\begin{aligned}
&\int_{A_k} \{(u-k)^\theta + (u-k)^t + (u-k)^{\frac{p}{p-r+1}}\}\, dx \\
&\quad \le c\, \|u-k\|^\theta_{L^{p^*}(A_k)} |A_k|^{1-\frac{\theta}{p^*}} + c\, \|u-k\|^t_{L^{p^*}(A_k)} |A_k|^{1-\frac{t}{p^*}} \\
&\qquad + c\|u-k\|^{\frac{p}{p-r+1}}_{L^{p^*}(A_k)} |A_k|^{1-\frac{p}{p^*(p-r+1)}}.
\end{aligned}$$

Collecting these inequalities we obtain

$$\begin{aligned}
\int_{A_k} |Du|^p\, dx \le{}& c\, \|b_0\|_{L^{s_0}(\Omega)} \|u-k\|_{L^{p^*}(A_k)} |A_k|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\
&+ c\, \|u-k\|^\theta_{L^{p^*}(A_k)} |A_k|^{1-\frac{\theta}{p^*}} + c\, \|u-k\|^t_{L^{p^*}(A_k)} |A_k|^{1-\frac{t}{p^*}} \\
&+ c\|u-k\|^{\frac{p}{p-r+1}}_{L^{p^*}(A_k)} |A_k|^{1-\frac{p}{p^*(p-r+1)}} \\
&+ c\,(1+k^\tau)|A_k| + c\|b_1\|_{L^{s_1}(\Omega)} |A_k|^{1-\frac{1}{s_1}}, \qquad (3.6)
\end{aligned}$$

with τ as in (3.2); i.e. $\tau := \max\{\theta, t\}$. By this estimate, the Caccioppoli's inequality (3.1) follows. ■

We proceed towards the proof of Theorem 2.1 by setting up the celebrated De Giorgi's iterative method. Given any real number $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$, we consider the (increasing) sequence

$$k_h := d(1 - \tfrac{1}{2^{h+1}}), \qquad h \in \mathbb{N} \cup \{0\} \qquad (3.7)$$

and we define a sequence $(J_h)_{h\ge 0}$ of non-negative numbers as follows:

$$J_h := \int_{A_{k_h}} (u - k_h)^{p^*}\, dx. \qquad (3.8)$$

Then the following result holds.

Proposition 3.2 *For every real number $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$,*

$$\begin{aligned}
J_{h+1} \le{}& c_* \left(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)}\right)^{\frac{p^*}{p}\max\left\{\frac{1}{s_0},\frac{1}{s_1}\right\}} \times \\
&\times \frac{1}{d^{\frac{p^*}{p}\sigma}} \left(2^{\frac{p^*}{p}} p^*\right)^h J_h^{\frac{p^*}{p}\left(1-\max\left\{\frac{1}{s_0},\frac{1}{s_1}\right\}\right)}, \qquad (3.9)
\end{aligned}$$

where σ is defined in (2.11); i.e., $\sigma := p^* - \max\left\{\frac{p}{p-r+1};\ \theta;\ t;\ \frac{p^*}{s_0}+1;\ \frac{p^*}{s_1}\right\}$, and c_* positive constant depending on the data, the L^{s_0} and the L^{s_1} norms of b_0, b_1, respectively, but it is independent of u and d.

We notice that, by assumptions (2.5),(2.8), $\frac{p^*}{p}\left(1-\max\left\{\frac{1}{s_0},\frac{1}{s_1}\right\}\right)>1$.

Proof of Proposition 3.2. Since $(k_h)_h$ is increasing, the sequence $(J_h)_h$ is decreasing: in fact

$$\begin{aligned}J_{h+1} &= \int_{A_{k_{h+1}}} (u-k_{h+1})^{p^*}\,dx \le \int_{A_{k_{h+1}}} (u-k_{h+1})^{p^*}\,dx \\ &\le \int_{A_{k_{h+1}}} (u-k_h)^{p^*}\,dx \le \int_{A_{k_h}} (u-k_h)^{p^*}\,dx = J_h\,. \end{aligned} \tag{3.10}$$

By taking into account the definitions of J_h and k_h we have

$$\begin{aligned}J_h &= \int_{A_{k_h}} (u-k_h)^{p^*}\,dx \ge \int_{A_{k_{h+1}}} (u-k_h)^{p^*}\,dx \\ &\ge (k_{h+1}-k_h)^{p^*}\left|A_{k_{h+1}}\right| = \left(\frac{d}{2^{h+2}}\right)^{p^*}\left|A_{k_{h+1}}\right|. \end{aligned} \tag{3.11}$$

Since $(u-k_{h+1})_+$ is in $W_0^{1,q}(\Omega)$ and in particular in $W_0^{1,p}(\Omega)$, then

$$\begin{aligned}J_{h+1}^{\frac{p}{p^*}} &= \left(\int_{A_{k_{h+1}}} (u-k_{h+1})^{p^*}\,dx\right)^{\frac{p}{p^*}} \\ &= \left(\int_\Omega ((u-k_{h+1})_+)^{p^*}\,dx\right)^{\frac{p}{p^*}} \\ &\le C_S^p \int_\Omega \left|D((u-k_{h+1})_+)\right|^p dx = C_S^p \int_{A_{k_{h+1}}} |Du|^p\,dx, \end{aligned} \tag{3.12}$$

where c_S is the Sobolev constant. To estimate the last integral in (3.12), we use the Caccioppoli estimate (3.1), so obtaining

$$\begin{aligned}&\int_{A_{k_{h+1}}} |Du|^p\,dx \le c\,\|b_0\|_{L^{s_0}(\Omega)} J_h^{\frac{1}{p^*}} |A_{k_{h+1}}|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\ &+ c\,J_h^{\frac{\theta}{p^*}} |A_{k_{h+1}}|^{1-\frac{\theta}{p^*}} + c\,J_h^{\frac{t}{p^*}} |A_{k_{h+1}}|^{1-\frac{t}{p^*}} + c\,J_h^{\frac{p}{p^*(p-r+1)}} |A_{k_{h+1}}|^{1-\frac{p}{p^*(p-r+1)}} \\ &+ c\,(1+k_{h+1}^\tau)|A_{k_{h+1}}| + c\,\|b_1\|_{L^{s_1}(\Omega)} |A_{k_{h+1}}|^{1-\frac{1}{s_1}}, \end{aligned} \tag{3.13}$$

with τ as in (3.2); i.e. $\tau := \max\{\theta, t\}$. By (3.11) we get

$$\left|A_{k_{h+1}}\right| \le \left(\frac{2^{h+2}}{d}\right)^{p^*} J_h = 4^{p^*}\left(\frac{2^h}{d}\right)^{p^*} J_h. \tag{3.14}$$

Collecting (3.13), (3.14), and using that $k_{h+1} \le d$, we get

$$\int_{A_{k_{h+1}}} |Du|^p\, dx \le\ c\, \|b_0\|_{L^{s_0}(\Omega)} \left(\frac{2^h}{d}\right)^{p^* - \frac{p^*}{s_0} - 1} J_h^{1-\frac{1}{s_0}}$$

$$+ c \left\{ \left(\frac{2^h}{d}\right)^{p^*-\theta} + \left(\frac{2^h}{d}\right)^{p^*-t} + \left(\frac{2^h}{d}\right)^{p^* - \frac{p}{p-r+1}} + (1+d^\tau) \left(\frac{2^h}{d}\right)^{p^*} \right\} J_h$$

$$+ c\|b_1\|_{L^{s_1}(\Omega)} \left(\frac{2^h}{d}\right)^{p^* - \frac{p^*}{s_1}} J_h^{1-\frac{1}{s_1}} \tag{3.15}$$

with a constant c depending on $n, p, q, \theta, t, r, s_0, s_1$ and the embedding Sobolev constant c_S, but depending neither on d, h nor on u. We now put together (3.12) and (3.15) and we obtain

$$J_{h+1}^{\frac{p}{p^*}} \le c\, \|b_0\|_{L^{s_0}(\Omega)} \left(\frac{2^h}{d}\right)^{p^* - \frac{p^*}{s_0} - 1} J_h^{1-\frac{1}{s_0}}$$

$$+ c \left\{ \left(\frac{2^h}{d}\right)^{p^*-\theta} + \left(\frac{2^h}{d}\right)^{p^*-t} + \left(\frac{2^h}{d}\right)^{p^* - \frac{p}{p-r+1}} + (1+d^\tau) \left(\frac{2^h}{d}\right)^{p^*} \right\} J_h$$

$$+ \|b_1\|_{L^{s_1}(\Omega)} \left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{1}{s_1}\right)} J_h^{1-\frac{1}{s_1}}.$$

Notice that $J_h \le \|u\|^{p^*}_{L^{p^*}(\Omega)}$ for every $h \in \mathbb{N}$, so that

$$\max\{J_h;\ J_h^{1-\frac{1}{s_0}};\ J_h^{1-\frac{1}{s_1}}\} \le (1 + \|u\|^{p^*}_{L^{p^*}(\Omega)})^{\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}} J_h^{1-\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}}.$$

Therefore

$$J_{h+1}^{\frac{p}{p^*}} \le c\, (1 + \|u\|^{p^*}_{L^{p^*}(\Omega)})^{\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}} \times$$

$$\times \left(1 + \|b_0\|_{L^{s_0}(\Omega)} + \|b_1\|_{L^{s_1}(\Omega)}\right) \times$$

$$\times \left\{ \left(\frac{2^h}{d}\right)^{p^* - \frac{p^*}{s_0} - 1} + \left(\frac{2^h}{d}\right)^{p^*-\theta} + \left(\frac{2^h}{d}\right)^{p^*-t} + \left(\frac{2^h}{d}\right)^{p^* - \frac{p}{p-r+1}} \right.$$

$$\left. + (1+d^\tau) \left(\frac{2^h}{d}\right)^{p^*} + \left(\frac{2^h}{d}\right)^{p^* - \frac{p^*}{s_1}} \right\} J_h^{1-\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}}. \tag{3.16}$$

Taking into account that $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$ and denoting σ as in (2.11), i.e.

$$\sigma := p^* - \max\left\{\frac{p}{p-r+1};\ \theta;\ t;\ \frac{p^*}{s_0} + 1;\ \frac{p^*}{s_1}\right\},$$

by inequality (3.16) we obtain

$$J_{h+1}^{\frac{p}{p^*}} \le c_0 \left(1 + \|b_0\|_{L^{s_0}(\Omega)} + \|b_1\|_{L^{s_1}(\Omega)}\right) \times$$

$$\times \frac{1}{d^\sigma} \left(2^{p^*}\right)^h (1 + \|u\|^{p^*}_{L^{p^*}(\Omega)})^{\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}} J_h^{1-\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}}.$$

Raising at the power $\frac{p^*}{p}$, we get (3.9) with

$$c_* := \max\left\{1, c_0^{\frac{p^*}{p}}\right\} \left(1 + \|b_0\|_{L^{s_0}(\Omega)} + \|b_1\|_{L^{s_1}(\Omega)}\right)^{\frac{p^*}{p}}. \tag{3.17}$$

■

We remind the following classical lemma of Real Analysis (see, e.g., [60, Lemma 7.1]).

Lemma 3.3 *Let $(z_h)_{h\geq 0}$ be a sequence of positive real numbers satisfying the following recursive relation*

$$z_{h+1} \leq L\,\zeta^h z_h^{1+\alpha} \qquad (h \in \mathbb{N} \cup \{0\}), \tag{3.18}$$

where $L, \alpha > 0$ and $\zeta > 1$. If $z_0 \leq L^{-\frac{1}{\alpha}} \zeta^{-\frac{1}{\alpha^2}}$, then $z_h \leq \zeta^{-\frac{h}{\alpha}} z_0$ for every $h \geq 0$. In particular, $z_h \to 0$ as $h \to \infty$.

With Proposition 3.2 and Lemma 3.3 at hand, we are ready to provide the proof of our global boundedness result.

Conclusion of the proof of Theorem 2.1. Let $u \in u_0 + W_0^{1,q}(\Omega)$ be a weak solution to (2.1), with $u_0 \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$, under the assumptions in Section 2. Let $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$ (to be chosen later on) and let $(J_h)_{h\geq 0}$ be the sequence defined in (3.8). Owing to Proposition 3.2, we have the estimate

$$J_{h+1} \leq \tilde{L}\,(2^{\frac{p^*}{p} p^*})^h\, J_h^{1+\alpha} \qquad (h \in \mathbb{N} \cup \{0\}), \tag{3.19}$$

where α is

$$\alpha := \tfrac{p^*}{p}\left(1 - \max\left\{\tfrac{1}{s_0}, \tfrac{1}{s_1}\right\}\right) - 1, \tag{3.20}$$

and the constant $\tilde{L}$ is given by

$$\tilde{L} := c_*\,(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)})^{\frac{p^*}{p}\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}}\, \frac{1}{d^{\frac{p^*}{p}\sigma}},$$

where σ is defined in (2.11) and c_*, independent of d, is defined as in (3.17). We notice that $\alpha > 0$ because s_0, s_1 satisfy (2.5),(2.8). If we define

$$\tilde{c} := \max\left\{c_*, (2\|u_0\|_{L^\infty(\Omega)} + 1)^{\frac{p^*\sigma}{p}}\right\},$$

then $\tilde{c} \geq c_*$ and we have

$$J_{h+1} \leq L\,(2^{\frac{p^*}{p} p^*})^h\, J_h^{1+\alpha} \qquad (h \in \mathbb{N} \cup \{0\}), \tag{3.21}$$

with the constant L given by

$$L := \tilde{c}\,(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)})^{\frac{p^*}{p}\max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}}\, \frac{1}{d^{\frac{p^*}{p}\sigma}} \geq \tilde{L}.$$

We claim that it is possible to choose $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$ in such a way that

$$J_0 := \int_{A_{\frac{d}{2}}} \left(u - \tfrac{d}{2}\right)^{p^*} \, dx \le L^{-1/\alpha} \, (2^{\frac{p^*}{p} p^*})^{-1/\alpha^2}. \tag{3.22}$$

In fact, by definition of J_0 and since $u \in L^{p^*}(\Omega)$, we have

$$J_0 \le \int_\Omega |u|^{p^*} \, dx < \infty;$$

thus condition (3.22) is clearly fulfilled if we choose

$$d := \tilde{c}^{\frac{p}{p^*\sigma}} \, 2^{\frac{p^*}{\alpha\sigma}} \, \left(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)}\right)^{\frac{p\alpha}{p^*\sigma} + \max\left\{\frac{1}{s_0}, \frac{1}{s_1}\right\}\frac{1}{\sigma}},$$

that is, taking into account (3.20),

$$d := \tilde{c}^{\frac{p}{p^*\sigma}} \, 2^{\frac{p^*}{\alpha\sigma}} \, \left(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)}\right)^{\frac{p^*-p}{p^*\sigma}}. \tag{3.23}$$

Notice that $d > 2\|u_0\|_{L^\infty(\Omega)} + 1$. With (3.22) at hand and d as in (3.23), we are entitled to apply Lemma 3.3. As a consequence, we obtain

$$\lim_{h\to\infty} J_h = \lim_{h\to\infty} \int_{A_{k_h}} (u - k_h)^{p^*} \, dx = \int_{A_d} (u - d)^{p^*} \, dx = 0\,. \tag{3.24}$$

Since, by definition, $u - d > 0$ on A_d, from (3.24) we then conclude that

$$\big|A_d\big| = 0, \qquad \text{whence} \quad u \le d \text{ for a.e. } x \in \Omega.$$

To prove that u is globally bounded from below, we can reason analogously, using the sub-level sets of u. So we obtain that there exists $\bar{c}$ such that $-u \le d'$ a.e. in Ω, with

$$d' := \bar{c}^{\frac{p}{p^*\sigma}} \, 2^{\frac{p^*}{\alpha\sigma}} \, \left(1 + \|u\|^{p^*}_{L^{p^*}(\Omega)}\right)^{\frac{p^*-p}{p^*\sigma}}.$$

We have so proved that $u \in L^\infty(\Omega)$ and (2.10) follows. ■

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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