HIGHER-ORDER FRACTIONAL LAPLACIANS: AN OVERVIEW

LAPLACIANI FRAZIONARI D'ORDINE SUPERIORE: UNA PANORAMICA

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Abstract. We summarize some of the most recent results regarding the theory of higher-order fractional Laplacians, *i.e.*, the operators obtained by considering (non-integer) powers greater than 1 of the Laplace operator. These can also be viewed as the nonlocal counterparts of polylaplacians. In this context, nonlocality meets polyharmonicity and together they pose new challenges, producing at the same time surprising and complex structures.

As our aim is to give a fairly general idea of the state of the art, we try to keep the presentation concise and reader friendly, by carefully avoiding technical complications and by pointing out the relevant references. Hopefully this contribution will serve as a useful introduction to this fascinating topic.

Sunto. Riepiloghiamo alcuni dei più recenti risultati che riguardano la teoria dei Laplaciani frazionari di ordine superiore, cioè gli operatori ottenuti considerando potenze (non intere) maggiori di 1 dell'operatore di Laplace. Questi possono anche essere interpretati come le controparti non locali dei polilaplaciani. In questo contesto la nonlocalità incontra la poliarmonicità e insieme pongono nuove sfide, producendo allo stesso tempo strutture sorprendenti e complesse.

Poichè il nostro scopo è quello di dare un'idea piuttosto generale dello stato dell'arte, cerchiamo di mantenere la presentazione concisa e di facile lettura, evitando accuratamente complicazioni tecniche e indicando i riferimenti rilevanti. Speriamo che questo contributo possa essere un'utile introduzione a questo affascinante argomento.

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1. Introduction

The higher-order fractional Laplacian $(-\Delta)^s,\ s>1$, is a nonlocal operator of (in most cases, non-integer) order $2s$, obtained by abstractly computing a positive power of the Laplace operator. The term *higher-order* is intended to stress the fact that the power s is greater than 1, opposed to the family of operators obtained by considering

$$(1)\qquad \begin{gathered}(-\Delta)^s u(x) := c_{n,s}\lim_{\varepsilon\downarrow 0}\int_{\{y:\in\mathbb{R}^n:|y|>\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,\\ c_{n,s}=\frac{2^{2s}\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|},\quad s\in(0,1).\end{gathered}$$

For a presentation of the operator defined in (1) we refer to the introductory papers by Di Nezza, Palatucci, and Valdinoci [18], Bucur and Valdinoci [13], Garofalo [29], and the author and Valdinoci [8]. Mind that the limitation $s<1$ in (1) is due to the singularity of the kernel $|x-y|^{-n-2s}$ and that, therefore, the same formula does not carry over to $s>1$.

Higher-order fractional Laplacians appear for example in Geometry (see the works of Graham and Zworski [31], Chang and González [16], and Yang [55]) in connection to the fractional Paneitz operator on the hyperbolic space, in the theory of the Navier-Stokes equation (see Katz and Pavlović [38] and Tao [54]) as a hyper-dissipative term, and in generalizations of the Lane-Emden equations (see Fazly and Wei [25]). For a very general introduction to hyper-singular integral operators we refer to the books by Samko [48] and Samko, Kilbas, and Marichev [49]. We would like to mention also the survey by Saldaña [47], which develops more into detail some of the results contained here as well, but concentrates on $s\in(1,2)$.

Beside the motivations and applications listed above, the interest in this class of operators is generated by the rich and complex structure they possess and that we aim at describing here. Indeed, they show counterintuitive and surprising oscillatory behaviours which originate from both their nonlocality and their high order, or even from a mixture of the two. Moreover, as they interpolate the polyharmonic operators (*i.e.*, $(-\Delta)^s$ with $s\in\mathbb{N}$), they could potentially shed more light on some open questions about classical polylaplacians. The theory of polyharmonic boundary value problems is covered in the monograph by Gazzola, Grunau, and Sweers [30].

The purpose of this note is to go over some recent results concerning boundary value problems driven by higher-order fractional Laplacians. In doing so, we will try to keep unnecessary technicalities to a minimum, at the expense of presenting the results not in their most general form. Also, we will skip proofs except in a few instructive cases; nevertheless, we will often try to sketch the main ideas of the arguments.

1.1. **Notations.** In the following, we will make use of the notations listed below without further notice.

We will denote by $\lfloor\cdot\rfloor$ the integer part of a real number, *i.e.*,

$$\lfloor s\rfloor := \max\{d \in \mathbb{Z} : d < s\} \qquad \text{for } s \in \mathbb{R} :$$

remark that, with this definition,

$$\lfloor s\rfloor = s - 1 \qquad \text{for } s \in \mathbb{Z}. \tag{2}$$

If $\mathbb{N}$ denotes the set of positive integer numbers, then $2\mathbb{N}$ and $2\mathbb{N}-1$ denote respectively the sets of even and odd positive integers, whilst $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Symbols $\vee$ and $\wedge$ will denote respectively the max and min operations between real numbers, namely

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\} \qquad \text{for } a, b \in \mathbb{R}.$$

The Euler Gamma function will be denoted by Γ, as customary. Open balls in $\mathbb{R}^n$ will be expressed as $B_r(x)$, where $x \in \mathbb{R}^n$ is the center and $r > 0$ the radius; for balls centered at the origin we drop the indication of the center and we simply write B_r. Finally, ω_n will be the n-dimensional Lebesgue measure of the unit ball, namely

$$\omega_n := |B_1| = \frac{2\pi^{n/2}}{n\,\Gamma(n/2)}.$$

For a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we will write the positive and negative part of u respectively as

$$u_+ := u \vee 0 \quad \text{and} \quad u_- := u \wedge 0.$$

2. Definition of the operator

We start of course from the very definition of $(-\Delta)^s$, $s>1$. This can be given in several equivalent ways, and each one has its own motivation, importance, and also limitations. The definitions that we list below are the higher-order analogues of some of the ones for the fractional Laplacian, see Kwaśnicki [40].

2.1. As a hypersingular integral.

2.1. As a hypersingular integral. As we have already mentioned above, definition (1) does not make sense for $s>1$. In order to balance the high singularity of the kernel, one option consists in increasing the order of the finite difference which is averaged by the kernel. This reads

$$(-\Delta)^s u(x) := c_{n,m,s} \int_{\mathbb{R}^n} \frac{\displaystyle\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} u(x+jy)}{|y|^{n+2s}}\, dy \qquad \text{for } m\in\mathbb{N} \text{ and } s\in(0,m), \tag{3}$$

where

$$c_{n,m,s} := \begin{cases} \dfrac{2^{2s}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(-s)} \left(\displaystyle\sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} j^{2s} \right)^{-1} & \text{for } s\in(0,m)\setminus\mathbb{N}, \\[2ex] \dfrac{2^{2s}\Gamma(n/2+s)\, s!}{2\pi^{n/2}} \left(\displaystyle\sum_{j=2}^{m} (-1)^j \binom{2m}{m-j} j^{2s}\ln j \right)^{-1} & \text{for } s\in(0,m)\cap\mathbb{N}. \end{cases} \tag{4}$$

The definition is actually independent of the choice of the parameter $m\in\mathbb{N}$ as it will follow from the following paragraph. The integral in (3) is finite whenever, for example,

$$u\in C^{2m}(\mathbb{R}^n)\cap\mathcal{L}^1_s, \qquad \mathcal{L}^1_s := \left\{ v\in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}\, dx < \infty \right\}.$$

If one chooses $m=1$, it can be checked that (3) agrees with (1).

Finally, we would like to underline how (3) is valid *also* for $s\in\mathbb{N}$, therefore providing with a nonlocal representation for local operators. We exemplify this by showing

$$-\Delta u(x) = c_{n,2,1} \int_{\mathbb{R}^n} \frac{u(x+2y)-4u(x+y)+6u(x)-4u(x-y)+u(x-2y)}{|y|^{n+2s}}\, dy. \tag{5}$$

Due to translation invariance, we can reduce ourselves to proving (5) at $x = 0$. Consider now $u \in C^3(\overline{B}_1)$ with

$$u(0) = |\nabla u(0)| = 0 \tag{6}$$

for simplicity (and without loss of generality). Assume first that

$$u(x) = 0 \qquad \text{for } |x| < r, \tag{7}$$

for some $r > 0$. Under these assumptions on u, the right-hand side of (5) vanishes at 0 regardless the size of r. Indeed,

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{u(2y) - 4u(y) - 4u(-y) + u(-2y)}{|y|^{n+2}}\, dy = \\ &\qquad = 2\int_{\mathbb{R}^n \setminus B_{r/2}} \frac{u(2y)}{|y|^{n+2}}\, dy - 8\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)}{|y|^{n+2}}\, dy \\ &\qquad = 2\int_{\mathbb{R}^n \setminus B_r} \frac{2^{n+2}\, u(Y)}{2^n\, |Y|^{n+2}}\, dY - 8\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)}{|y|^{n+2}}\, dy = 0. \end{aligned} \tag{8}$$

This proves (5) under assumption (7), that we are now going to remove. For $r \in (0,1)$, let $\mathbf{1}_r$ be the characteristic function of B_r, *i.e.*, $\mathbf{1}_r(x) = 1$ if $|x| < r$ and $\mathbf{1}_r(x) = 0$ otherwise. Then, the right-hand side of (5) becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{u(2y) - 4u(y) - 4u(-y) + u(-2y)}{|y|^{n+2}}\, dy = 2\int_{\mathbb{R}^n} \frac{u(2y) - 4u(y)}{|y|^{n+2}}\, dy \\ &= 2\int_{\mathbb{R}^n} \frac{\mathbf{1}_r(2y)u(2y) - 4\,\mathbf{1}_r(y)u(y)}{|y|^{n+2}}\, dy + 2\int_{\mathbb{R}^n} \frac{(1-\mathbf{1}_r(2y))u(2y) - 4(1-\mathbf{1}_r(y))u(y)}{|y|^{n+2}}\, dy. \end{aligned}$$

The second addend above vanishes for any $r \in (0,1)$ thanks to (8), since $(1 - \mathbf{1}_r)u$ fulfills (7). For the first one, we have

$$\int_{\mathbb{R}^n} \frac{\mathbf{1}_r(2y)u(2y) - 4\,\mathbf{1}_r(y)u(y)}{|y|^{n+2}}\, dy = \int_{B_{r/2}} \frac{u(2y) - 4u(y)}{|y|^{n+2}}\, dy - 4\int_{B_r \setminus B_{r/2}} \frac{u(y)}{|y|^{n+2}}\, dy. \tag{9}$$

Now, we recall (6) and we notice that

$$|u(2y) - 4u(y)| \le \|u\|_{C^3(\overline{B}_1)}|y| \qquad \text{for } y \in B_1,$$

which in turn implies that

$$\left|\int_{B_{r/2}} \frac{u(2y) - 4u(y)}{|y|^{n+2}}\, dy\right| \;\le\; C\|u\|_{C^3(\overline{B}_1)}r. \tag{10}$$

On the other hand, a Taylor expansion of u and (6) yield

$$
\begin{aligned}
\int_{B_r\setminus B_{r/2}} \frac{u(y)}{|y|^{n+2}}\,dy &= \int_{r/2}^r \frac{1}{\rho^3}\int_{\partial B_1} u(\rho\theta)\,d\theta\,d\rho \\
&= \int_{r/2}^r \frac{1}{2\rho}\int_{\partial B_1}\big(D^2u(0)\theta\cdot\theta+\eta(\rho\theta)\big)\,d\theta\,d\rho \qquad (11)\\
&= \Delta u(0)\int_{r/2}^r \frac{1}{2\rho}\int_{\partial B_1}\theta_1^2\,d\theta\,d\rho + \int_{r/2}^r \frac{1}{2\rho}\int_{\partial B_1}\eta(\rho\theta)\,d\theta\,d\rho
\end{aligned}
$$

for some $\eta : B_1 \to \mathbb{R}$ such that $|\eta(x)| \le C|x|$, in view of identity

$$\int_{\partial B_1} D^2u(0)\theta\cdot\theta\,d\theta = \Delta u(0)\int_{\partial B_1}\theta_1^2\,d\theta.$$

From this, (9), and (10) we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{u(2y)+u(-2y)-4u(y)-4u(-y)}{|y|^{n+2}}\,dy = \\
&\qquad = -\frac{1}{2}\ln 2\int_{\partial B_1}\theta_1^2\,d\theta\,\lim_{r\downarrow 0}\int_{B_r\setminus B_{r/2}}\frac{u(y)}{|y|^{n+2}}\,dy = -\Delta u(0)\,\frac{\ln 2}{2}\int_{\partial B_1}\theta_1^2\,d\theta
\end{aligned}
$$

which justifies (5), up to checking the value of the involved constants. It is interesting to remark that the main contribution to prove (5) comes in this case from the intermediate ring in (11).

Intuitively, (3) implies that any power $s > 0$ of the Laplacian operator can be seen as another power $\theta \in (0,1)$ of a polylaplacian $(-\Delta)^m$ in such a way that $s = \theta m$. Indeed, with this notation, one can rewrite (3) as

$$(-\Delta)^s u(x) = c_{n,m,s}\int_{\mathbb{R}^n} \frac{\displaystyle\sum_{j=-m}^{m}(-1)^j\binom{2m}{m-j}u(x+jy)}{|y|^{2\theta m}}\,\frac{dy}{|y|^n}$$

which is a nonlocal average in $\mathbb{R}^n$ of the $2m$-th order difference quotient with an altered exponent.

2.2. **As a pseudo-differential operator.** In order to characterize (3) as a pseudo-differential operator, it is enough to compute its Fourier symbol from (3). It holds

$$\mathcal{F}\big[(-\Delta)^s u\big](\xi) = |\xi|^{2s}\,\mathcal{F}u(\xi). \tag{12}$$

Recall that the Fourier transform is

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\, u(x)\, dx.$$

Then, at least formally,

$$\begin{aligned}\mathcal{F}\big[(-\Delta)^s u\big](\xi) = c_{n,m,s} \int_{\mathbb{R}^n} \frac{\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} \mathcal{F}\big[u(\cdot + jy)\big](\xi)}{|y|^{n+2s}}\, dy \; = \\ = c_{n,m,s}\, \mathcal{F}u(\xi) \int_{\mathbb{R}^n} \frac{\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} e^{ij\xi\cdot y}}{|y|^{n+2s}}\, dy.\end{aligned}$$

Using the identity (see [6, Lemma 2.1])

$$\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} e^{ij\xi\cdot y} = 2^m \big(1 - \cos(\xi\cdot y)\big)^m$$

we deduce

$$\begin{aligned}\mathcal{F}\big[(-\Delta)^s u\big](\xi) &= 2^m c_{n,m,s}\, \mathcal{F}u(\xi) \int_{\mathbb{R}^n} \frac{\big(1-\cos(\xi\cdot y)\big)^m}{|y|^{n+2s}}\, dy \\ &= 2^m c_{n,m,s}\, |\xi|^{2s} \mathcal{F}u(\xi) \int_{\mathbb{R}^n} \frac{\Big(1-\cos\big(\frac{\xi}{|\xi|}\cdot y\big)\Big)^m}{|y|^{n+2s}}\, dy \\ &= 2^m c_{n,m,s}\, |\xi|^{2s} \mathcal{F}u(\xi) \int_{\mathbb{R}^n} \frac{\big(1-\cos(y_1)\big)^m}{|y|^{n+2s}}\, dy.\end{aligned}$$

The computation is concluded by the identity

$$2^m c_{n,m,s} \int_{\mathbb{R}^n} \frac{\big(1-\cos(y_1)\big)^m}{|y|^{n+2s}}\, dy = 1 \qquad \text{for any } n, m \in \mathbb{N} \text{ and } s \in (0,m).$$

More details can be found in [6, Section 4] or [49, Chapter 5, Lemma 25.3 and Theorem 26.1].

In the spirit of the conclusion of the previous paragraph, the Fourier symbol of the operator can be seen as $|\xi|^{2s} = (|\xi|^{2m})^{\theta}$. With this interpretation, (12) implies the equivalence between considering a power θ of the polylaplacian of order $2m$ or a power τ of the polylaplacian of order $2p$, as long as $\theta m = \tau p$. Also, (12) directly implies (5).

2.3. **Recursive definition.** For $u \in C_c^\infty(\mathbb{R}^n)$ and $s > 1$, it is in principle possible to recursively define

$$(-\Delta)^s u = (-\Delta)^{s-1}(-\Delta)u = (-\Delta)(-\Delta)^{s-1}u.$$

Nevertheless, this representation is not able to capture the full range of functions on which (3) acts on. This is due to the integrability requirements which are naturally built in the operator and which change with the value of s. More generally, one could prove the following (see [6, Corollary 1]).

Proposition 2.1. *Let $\beta > s > 1$, $\Omega \subset \mathbb{R}^n$ be an open bounded Lipschitz domain, and $u \in C^{2\beta}(\Omega)$.*

i) If $u \in \mathcal{L}^1_{s-\lfloor s\rfloor}$, then $(-\Delta)^s u = (-\Delta)^{\lfloor s\rfloor}(-\Delta)^{s-\lfloor s\rfloor}u$ in Ω.
ii) If $\lfloor s\rfloor \in 2\mathbb{N}$ and $u \in \mathcal{H}^s_0(\Omega)$, then

$$(-\Delta)^s u = (-\Delta)^{\frac{\lfloor s\rfloor}{2}}(-\Delta)^{s-\lfloor s\rfloor}(-\Delta)^{\frac{\lfloor s\rfloor}{2}}u \qquad \text{in } \Omega.$$

iii) If $\lfloor s\rfloor \in 2\mathbb{N}-1$ and $u \in \mathcal{H}^s_0(\Omega)$, then

$$(-\Delta)^s u = (-\Delta)^{\frac{\lfloor s\rfloor-1}{2}}\operatorname{div}(-\Delta)^{s-\lfloor s\rfloor}\big(\nabla(-\Delta)^{\frac{\lfloor s\rfloor-1}{2}}u\big) \qquad \text{in } \Omega.$$

For the definition of the Sobolev space $\mathcal{H}^s_0$ see (14).

2.4. **As a Dirichlet-to-Neumann operator.** The first result in this direction is due to Chang and González [16] for $s \in (0, n/2)$: there the Caffarelli-Silvestre extension [14] is successfully generalized, via an iterative procedure, by considering the Euclidean space as the boundary of the hyperbolic one and the extension problem itself as a scattering operator. This has been later pushed further by Yang [55] to cover any $s \in (0,\infty)$. Cora and Musina [17] have analyzed the variational characterization of the extension for $s \in (0, n/2)$ in a suitable functional framework making use of homogeneous spaces.

As a model case, consider $s \in (1,2)$. Given $u : \mathbb{R}^n \to \mathbb{R}$, take the higher-order boundary value problem on the half-space $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$

$$\begin{cases} \Delta_b^2 U = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \qquad \Delta_b = \Delta + \dfrac{3-2s}{y}\,\partial_y, \\ U = u \quad \text{on } \partial\mathbb{R}^{n+1}_+, \\ \lim\limits_{y\downarrow 0} y^{3-2s}\partial_y U(x,y) = 0 \quad \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Then

$$\begin{aligned} (-\Delta)^s u(x) &= C_s \lim_{y\downarrow 0} \Big(y^{3-2s} \partial_y \Delta_b U(x,y) \Big) && \text{for } x \in \mathbb{R}^n, \\ C_s &= 2^{1-2(s-\lfloor s \rfloor)} \frac{\Gamma(\lfloor s \rfloor + 1)\,\Gamma(\lfloor s \rfloor + 1 - s)}{\Gamma(s)} && \text{for } s > 0. \end{aligned}$$

The value of the constant C_s has been computed in [17].

2.5. **Fractional Sobolev spaces and quadratic form.** It is possible to naturally associate a bilinear form to (3). This can be introduced in the following different yet equivalent ways

(13)
$$\begin{aligned} &\mathcal{E}_s(u,v) = \\ &= \int_{\mathbb{R}^n} |\xi|^{2s}\, \mathcal{F}u(\xi)\, \overline{\mathcal{F}v(\xi)}\, d\xi \\ &= \frac{c_{n,2m,s}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\displaystyle\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} u(x+jy) \sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} v(x+ky)}{|y|^{n+2s}}\, dy\, dx \\ &= \begin{cases} \mathcal{E}_{s-\lfloor s \rfloor}\Big((-\Delta)^{\frac{\lfloor s \rfloor}{2}} u, (-\Delta)^{\frac{\lfloor s \rfloor}{2}} v \Big) & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \mathcal{E}_{s-\lfloor s \rfloor}\Big(\nabla(-\Delta)^{\frac{\lfloor s \rfloor - 1}{2}} u, \nabla(-\Delta)^{\frac{\lfloor s \rfloor - 1}{2}} v \Big) & \text{if } \lfloor s \rfloor \in 2\mathbb{N} - 1. \end{cases} \end{aligned}$$

The fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \big\{ u \in L^2(\mathbb{R}^n) : \mathcal{E}_s(u,u) < \infty \big\}.$$

On a bounded smooth domain $\Omega \subset \mathbb{R}^n$ the Sobolev space associated with homogeneous boundary conditions is

$$\mathcal{H}^s_0(\Omega) = \big\{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}\big\}. \tag{14}$$

The issue of encoding the boundary conditions in the Sobolev space is a non-trivial one: for example, the space in (14) not always coincides with one obtained via the closure of $C^\infty_c(\Omega)$. These aspects can be checked, *e.g.*, in Grisvard [32] or in Chandler-Wilde, Hewett, and Moiola [15].

3. Properties

3.1. Integration by parts. An integration by parts formula was computed by Grubb [34] for general pseudo-differential operators of order $2s$ with even symbol, therefore including $(-\Delta)^s$. A more specific formula for the fractional Laplacian can be found in [35]: for a bounded open set $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$ and $u, v : \mathbb{R}^n \to \mathbb{R}$ with $d^{1-s}u, d^{1-s}v \in C^\infty(\overline{\Omega})$ and $u, v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ it holds

$$\int_\Omega \big(v(-\Delta)^s u - u(-\Delta)^s v\big) = \frac{\Gamma(1+s)^2}{s} \int_{\partial\Omega} \left(\frac{u}{d^{s-1}}\,\partial_\nu\Big(\frac{v}{d^{s-1}}\Big) - \frac{v}{d^{s-1}}\,\partial_\nu\Big(\frac{u}{d^{s-1}}\Big)\right). \tag{15}$$

Here d stands for a smooth version of the distance to the boundary of $\partial\Omega$.

3.2. The Pohozaev identity. This result is due to Ros-Oton and Serra [46] and it is based on their preceding work [45] dealing with the case $s \in (0,1)$ and the regularity theory by Grubb [33]. Given a bounded open set $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$ and $u \in \mathcal{H}^s_0(\Omega)$ with $(-\Delta)^s u \in C^\infty(\overline{\Omega})$, it holds

$$2\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = (2s-n)\int_\Omega u(-\Delta)^s u - \Gamma(1+s)^2 \int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 x \cdot \nu \tag{16}$$

where ν stands for the outward unit normal to $\partial\Omega$.

Proof of (16). We give here a proof[1] of (16) as a consequence of (15).

[1] The presented proof for $s \in (0,1)$ was explained to the author by X. Ros-Oton in a private communication in 2014.

In the assumptions listed above, it holds $d^{-s}u \in C^\infty(\overline{\Omega})$ by the results in [33]. Take $v = x \cdot \nabla u$ in (15). Since

$$\partial_\nu\Big(\frac{u}{d^{s-1}}\Big)\Big|_{\partial\Omega} = -\frac{u}{d^s}\Big|_{\partial\Omega}$$

$$\frac{x\cdot\nabla u}{d^{s-1}}\Big|_{\partial\Omega} = \frac{d^s\, x\cdot\nabla(d^{-s}u)}{d^{s-1}}\Big|_{\partial\Omega} + \frac{d^{-s}u\, x\cdot\nabla d^s}{d^{s-1}}\Big|_{\partial\Omega} = (sd^{-s}u\, x\cdot\nabla d)\big|_{\partial\Omega} = -s\frac{u}{d^s}\Big|_{\partial\Omega} x\cdot\nu$$

and[2]

$$\begin{aligned}(-\Delta)^s(x\cdot\nabla u)(x) &= (-\Delta)^s\big(\partial_t\big|_{t=1} u(tx)\big) = \partial_t\big|_{t=1}(-\Delta)^s\big(u(t\cdot)\big)(x)\\ &= \partial_t\big|_{t=1}\big[t^{2s}(-\Delta)^s u(tx)\big] = \partial_t\big|_{t=1}(-\Delta)^s u(tx) + 2s(-\Delta)^s u(x)\\ &= x\cdot\nabla(-\Delta)^s u + 2s(-\Delta)^s u,\end{aligned}$$

one has

$$\begin{aligned}\int_\Omega (x\cdot\nabla u)(-\Delta)^s u &= \int_\Omega u(-\Delta)^s(x\cdot\nabla u) - \frac{\Gamma(1+s)^2}{s}\int_{\partial\Omega}\frac{x\cdot\nabla u}{d^{s-1}}\,\partial_\nu\Big(\frac{u}{d^{s-1}}\Big) =\\ &= \int_\Omega u\, x\cdot\nabla(-\Delta)^s u + 2s\int_\Omega u(-\Delta)^s u - \Gamma(1+s)^2\int_{\partial\Omega}\Big(\frac{u}{d^s}\Big)^2 x\cdot\nu.\end{aligned}$$

Using a classical integration by parts, one obtains

$$\int_\Omega u\, x\cdot\nabla(-\Delta)^s u = -n\int_\Omega u(-\Delta)^s u - \int_\Omega (x\cdot\nabla u)(-\Delta)^s u.$$

□

3.3. **Unique continuation.** In this direction there are two main contributions: one by Felli and Ferrero [26] on s-harmonic functions and one by García-Ferrero and Rüland [28] on Schrödinger type equations.

The statement of the main result in [26] goes as follows.

Theorem 3.1. *Let $s \in (1,2)$ with $2s < n$ and $\Omega \subset \mathbb{R}^n$ be an open domain. If*

$$u \in \mathcal{D}^{s,2}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathcal{E}_s}$$

[2]This identity can also be checked by means of the pseudo-differential definition. We are thankful to S. Jarohs for explaining this approach.

(the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the quadratic form $\mathcal{E}_s$) is a solution[3] *to*

$$(-\Delta)^s u = 0 \qquad \text{in } \Omega$$

and there exists $x_0 \in \Omega$ such that for any $k \in \mathbb{N}$ there exists $C_k > 0$ for which

$$|u(x)| \le C_k |x - x_0|^k \qquad \text{in a neighbourhood of } x_0,$$

then $u \equiv 0$ in $\mathbb{R}^n$.

On the other hand, [28] proves the following.

Theorem 3.2. *Let $s \in (0, \infty) \setminus \mathbb{N}$ and $u \in H^{2s}(\mathbb{R}^n)$ be a solution of*

$$(-\Delta)^s u + qu = 0 \qquad \text{in } \mathbb{R}^n$$

where

$$|q(x)| \le C|x|^{-2s} \qquad \text{for } x \in \mathbb{R}^n \text{ and some constant } C > 0.$$

If u vanishes of infinite order at 0, i.e.,

$$\lim_{r \downarrow 0} r^{-\alpha} \|u\|^2_{L^2(B_r)} = 0 \qquad \text{for any } \alpha > 0,$$

then $u \equiv 0$ in $\mathbb{R}^n$.

3.4. **Interaction of segregated functions.** Recalling the bilinear form in (13), if $u, v \in H^s(\mathbb{R}^n)$ with $uv \equiv 0$ in $\mathbb{R}^n$, then (see [5, Lemma 4.4])

$$\mathcal{E}_s(u, v) = \frac{2^{2s-1}\Gamma(n/2 + s)}{\pi^{n/2}\Gamma(-s)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x)\, v(y)}{|x - y|^{n+2s}}\, dx\, dy. \tag{17}$$

As a consequence,

$$\text{if} \quad u, v \ge 0,\ uv \equiv 0 \text{ in } \mathbb{R}^n, \qquad \text{then} \qquad \mathcal{E}_s(u, v) \begin{cases} \le 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \ge 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N} - 1, \end{cases} \tag{18}$$

[3] The precise notion of solution in this context can be looked up in [26].

where the inequalities are strict if both u and v are non-trivial, and

(19) if $u, v \geq 0,\ uv \equiv 0$ in $\mathbb{R}^n$,

$$\text{then}\ \ \mathcal{E}_s(u+v, u+v) \begin{cases} \leq \mathcal{E}_s(u-v, u-v) & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \geq \mathcal{E}_s(u-v, u-v) & \text{if } \lfloor s \rfloor \in 2\mathbb{N}-1. \end{cases}$$

Equation (17) can be computed by integrating by parts several times the last representation in (13) and moving the derivatives from u and v to the kernel.

Related inequalities have been found by Musina and Nazarov [43]. Consider $s \in (0, 3/2)$, then for any $u \in \mathcal{H}^s_0(\Omega)$ also $u_+, u_- \in \mathcal{H}^s_0(\Omega)$, see Bourdaud and Meyer [12]. It holds

$$\mathcal{E}_s(u_+, u_-) \begin{cases} \leq 0 & \text{if } s \in (0,1), \\ = 0 & \text{if } s = 1, \\ \geq 0 & \text{if } s \in (1, 3/2), \end{cases} \tag{20}$$

$$\text{and} \qquad \mathcal{E}_s(|u|, |u|) \begin{cases} \leq \mathcal{E}_s(u,u) & \text{if } s \in (0,1), \\ = \mathcal{E}_s(u,u) & \text{if } s = 1, \\ \geq \mathcal{E}_s(u,u) & \text{if } s \in (1, 3/2). \end{cases}$$

Again, the inequalities are strict if u_+ and u_- are non-trivial. The restriction $s < 3/2$ is not technical, but structural: indeed, for $s \geq 3/2$, in general $u_+, u_- \notin \mathcal{H}^s_0(\Omega)$ as showed by [12].

4. Representation formulas

4.1. **The fundamental solution.** For any $s > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, define

$$F_s(x) := \begin{cases} \kappa_{n,s} |x|^{2s-n} & \text{if } s - \dfrac{n}{2} \notin \mathbb{N}_0, \\ \kappa_{n,s} |x|^{2s-n} \ln |x| & \text{if } s - \dfrac{n}{2} \in \mathbb{N}_0, \end{cases} \tag{21}$$

where

$$\kappa_{n,s} := \begin{cases} \dfrac{\Gamma(\frac{n}{2} - s)}{2^{2s} \pi^{n/2} \Gamma(s)} & \text{if } s - \dfrac{n}{2} \notin \mathbb{N}_0, \\ \dfrac{2^{1-2s} (-1)^{s+1-n/2}}{(s - n/2)!\, \pi^{n/2} \Gamma(s)} & \text{if } s - \dfrac{n}{2} \in \mathbb{N}_0. \end{cases}$$

The function F_s is the fundamental solution of $(-\Delta)^s$ in the sense that

$$(-\Delta)^s F_s = \delta_0 \qquad \text{in } \mathbb{R}^n$$

where δ_0 denotes a Dirac delta centred at the origin. As a consequence, if for example $f \in L^1(\mathbb{R}^n)$ has compact support,

$$u(x) = (F_s * f)(x) = \int_{\mathbb{R}^n} F_s(x-y)\, f(y)\, dy \tag{22}$$
$$\text{is a distributional solution of} \quad (-\Delta)^s u = f \quad \text{in } \mathbb{R}^n.$$

Note that the above u is a *Riesz potential* [41] for $2s < n$. In this range, assertion (22) can be reversed: u is the only distributional solution satisfying

$$\lim_{|x|\uparrow\infty} u(x) = 0.$$

Mind that (22) has been long since known, see for example [49].

4.2. **Boggio's formula for the Green function.** In 1905, Boggio [11] explicitly computed the Green function of the ball for polyharmonic operators, by extending a formula previously obtained by Lauricella [42] for the particular case $s = 2$. For any $s \in \mathbb{N}$, it goes as follows

$$G_s(x,y) = k_{n,s}|x-y|^{2s-n} \int_0^{\rho(x,y)} \frac{v^{s-1}}{(v+1)^{n/2}}\, dv \qquad \text{for } x, y \in \mathbb{R}^n,\ x \neq y, \tag{23}$$

where

$$\rho(x,y) = \frac{(1-|x|^2)_+(1-|y|^2)_+}{|x-y|^2}, \qquad k_{n,s} = \frac{1}{n\omega_n}\frac{2^{1-2s}}{\Gamma(s)^2}. \tag{24}$$

Later, the very same formula was proved to hold for $s \in (0,1)$ by Blumenthal, Getoor, and Ray [10]. Independently and roughly at the same time, Dipierro and Grunau [19] and the author, Jarohs, and Saldaña [3] extended the validity of (23) also to all $s > 1$. Mind that the Green function is meant to choose the only[4] solution of

$$(-\Delta)^s u = f \quad \text{in } B_1, \qquad f \in L^2(B_1),$$

which lies in the space $\mathcal{H}^s_0(B_1)$ as defined in (14). In this sense G_s is uniquely determined.

[4] The uniqueness can be obtained by means of standard variational arguments.

Both the proofs in [19] and [3] rely on a splitting of the operator, although in different ways: formally, the proof in [19] goes like

$$\begin{aligned} (-\Delta)^s G_s(\cdot,y) &= (-\Delta)^{\lfloor s\rfloor}(-\Delta)^{s-\lfloor s\rfloor} G_s(\cdot,y) \\ \text{where}\quad (-\Delta)^{s-\lfloor s\rfloor} G_s(\cdot,y) &= G_{\lfloor s\rfloor}(\cdot,y) + \lfloor s\rfloor\text{-harmonic function} \end{aligned}$$

and is concluded by knowing that $G_{\lfloor s\rfloor}$ is the Green function for $(-\Delta)^{\lfloor s\rfloor}$ by the results in [11]; it must be mentioned that the explicit computations are simplified by the reduction to the case of the pole at the origin ($y=0$) and the application of Möbius transformations (inversions, translations, rotations, and compositions of them) to reconstruct the general case: these might be useful in many other contexts.

Instead, the proof of [3] is performed by induction on s via

$$\begin{aligned} (-\Delta)^s G_s(\cdot,y) &= (-\Delta)^{s-1}(-\Delta) G_s(\cdot,y) \\ \text{where}\quad (-\Delta) G_s(\cdot,y) &= G_{s-1}(\cdot,y) + (s-1)\text{-harmonic function} \end{aligned} \tag{25}$$

and exploits the inductive hypothesis to conclude. Here, the calculation of $(-\Delta)G_s(\cdot,y)$ is elementary, although long and tedious. Moreover, the proof also goes through for integer s and therefore could be interpreted as an alternative argument with respect to the original one in [11]. The $(s-1)$-harmonicity of the correction term in (25) is done by showing that it satisfies representation formulas for harmonic functions in terms of their boundary traces, which is what we discuss next.

4.3. **Boundary kernels.** In the polyharmonic theory, the number of boundary conditions has to increase with the order of the operators for a boundary value problem to be well posed. It holds[5]

$$\mathcal{H}^s_0(\Omega) = \left\{ u \in H^s(\Omega) : \frac{\partial^k u}{\partial \nu^k} = 0 \ \text{ on } \partial\Omega \ \text{ for } k = 0,\dots,s-1 \right\} \qquad \text{for } s \in \mathbb{N}. \tag{26}$$

This space corresponds to homogeneous *Dirichlet conditions* on the boundary: indeed, one way of imposing boundary conditions in the polyharmonic context is to prescribe the

[5] This can be seen by means of extension and density arguments.

values of

$$\frac{\partial^k u}{\partial \nu^k} \qquad \text{on } \partial\Omega, \text{ for } k = 0, \dots, s-1. \tag{27}$$

In the nonlocal fractional framework the above is replaced (on B_1) by the prescription of

$$\frac{\partial^k}{\partial \nu^k}\left(\frac{u(x)}{(1-|x|^2)^{s-\lfloor s\rfloor-1}}\right) \qquad \text{on } \partial B_1,\ k = 0, \dots, \lfloor s\rfloor. \tag{28}$$

Note that, by (2), the above is formally equivalent (up to multiplicative constants) to (27). Recall also that this boundary term was already appearing in (15) with $k = \lfloor s\rfloor$. It is possible to explicitly construct s-harmonic functions[6] on B_1 using (28) and the boundary kernels

$$E_{k,s}(x,\theta) := \frac{1}{\omega_n}(1-|x|^2)^s \frac{(-1)^{\lfloor s\rfloor-k}}{(\lfloor s\rfloor-k)!}\frac{\partial^{\lfloor s\rfloor-k}}{\partial(|x|^2)^{\lfloor s\rfloor-k}}\bigg|_{y=\theta}\zeta_x(y) \tag{29}$$
$$\text{where} \quad \zeta_x(y) := \frac{|y|^{n-2}}{|x-y|^n},\ x, y \in B_1,\ \theta \in \partial B_1,\ k = 0, \dots, \lfloor s\rfloor,$$

which were already appearing in a slightly different form in Edenhofer [24] for the analysis in the classical theory. The representation formulas generated by (29) and (28) are contained in [4].

A particularly interesting case is

$$E_{\lfloor s\rfloor,s}(x,\theta) = \frac{1}{\omega_n}(1-|x|^2)^s\zeta_x(\theta) = \frac{1}{\omega_n}\frac{(1-|x|^2)^s}{|x-\theta|^n} \qquad \text{for } x \in B_1 \text{ and } \theta \in \partial B_1,$$

which coincides, up to multiplicative constants, with the *Martin kernel*

$$\lim_{B_1 \ni y \to \theta} \frac{G_s(x,y)}{G_s(0,y)} \qquad \text{for } x \in B_1 \text{ and } \theta \in \partial B_1.$$

This has been analyzed in [3].

The boundary kernels in (29) also yield a Hopf Lemma [4, Corollary 1.9] which, in turn, can be used alongside the Pohozev identity (16) to prove non-existence of positive solutions to some nonlinear problems, see for example Hernández-Santamaría and Saldaña [37, Proposition 1.2].

[6] Conversely, the functions satisfying the representation are s-harmonic.

4.4. **Nonlocal Poisson kernel.** The nonlocality of the operator $(-\Delta)^s$, which spreads its averaging action all over $\mathbb{R}^n$ (recall (3)), translates in boundary value problems into the prescription of the values of the solution *outside* the underlying domain. This is also a characteristic of the fractional Laplacian (1). This is usually done by setting (if the differential equation is given on the ball B_1)

$$u = h \qquad \text{in } \mathbb{R}^n \setminus \overline{B}_1 \tag{30}$$

for some given measurable $h : \mathbb{R}^n \setminus \overline{B}_1$. As it happens for the Dirichlet conditions (28), it is possible to provide an explicit kernel acting on $\mathbb{R}^n \setminus \overline{B}_1$ which reconstructs s-harmonic function in B_1. This one reads [4, equation (1.10)]

$$\begin{aligned} P_s(x,y) &= (-1)^{\lfloor s\rfloor} \frac{\gamma_{n,s}}{|x-y|^n} \left(\frac{1-|x|^2}{|y|^2-1}\right)^s \qquad && \text{for } x \in B_1 \text{ and } y \in \mathbb{R}^n \setminus \overline{B}_1, \\ \text{with} \quad \gamma_{n,s} &= \frac{2}{\omega_n\, \Gamma(s-\lfloor s\rfloor)\, \Gamma(1-s+\lfloor s\rfloor)} && \text{for } n \in \mathbb{N} \text{ and } s \in (0,\infty) \setminus \mathbb{N}. \end{aligned} \tag{31}$$

Note that P_s is negative (respectively, positive) whenever $\lfloor s\rfloor \in 2\mathbb{N}-1$ (respectively, $\lfloor s\rfloor \in 2\mathbb{N}$). Also, mind that $\gamma_{n,s}$ only depends on the value of $s-\lfloor s\rfloor$ and degenerates when $s \in \mathbb{N}$, trivializing therefore the kernel—this agrees with the fact that the operator localizes.

So, (30) accounts for a nonlocal Dirichlet condition. We should at this point also mention that a notion of higher-order Neumann condition has been proposed for $s \in (1,2)$ by Barrios, Montoro, Peral, and Soria [9], designed upon the one proposed by Dipierro, Ros-Oton, and Valdinoci [20] for (1).

We can resume the last paragraphs with the following (see [4, Theorem 1.4]).

Theorem 4.1 (Explicit solutions)**.** *Let $\alpha \in (0,1]$, $2s+\alpha \notin \mathbb{N}$, $f \in C^\alpha(\overline{B}_1)$, $h : \mathbb{R}^n \setminus \overline{B}_1 \to \mathbb{R}$ be measurable such that*

$$\int_{\mathbb{R}^n\setminus\overline{B}_1} \frac{|h(y)|}{(|y|-1)^s |y|^{n+s}}\, dy < \infty,$$

$g_k \in C^{\lfloor s\rfloor-k,0}(\partial B_1)$ for $k = 0,\dots,\lfloor s\rfloor$, and $u : \mathbb{R}^n \to \mathbb{R}$ be given by $u = h$ in $\mathbb{R}^n \setminus \overline{B}_1$ and

$$\begin{aligned} u(x) = &\int_{B_1} G_s(x,y) f(y)\, dy + \int_{\mathbb{R}^n\setminus\overline{B}_1} P_s(x,y)\, h(y)\, dy \\ &+ \sum_{k=0}^m \int_{\partial B_1} E_{k,s}(x,\theta)\, g_k(\theta)\, d\theta \end{aligned} \qquad \textit{for } x \in B_1. \tag{32}$$

Then, $u \in C^{2s+\alpha}(B_1)$, $(1-|x|^2)^{1+\lfloor s\rfloor - s}u \in C^{\lfloor s\rfloor}(\overline{B}_1)$, *and*

$$\begin{aligned}
&(-\Delta)^s u = f \quad \text{in } B_1, \qquad u = h \quad \text{in } \mathbb{R}^n \backslash \overline{B}_1, \\
&\frac{(-1)^k}{k!}\frac{\partial^k}{\partial(|x|^2)^k}\frac{u(x)}{(1-|x|^2)^{s-\lfloor s\rfloor-1}} = g_k \qquad \text{on } \partial B_1, \ \text{for } k = 0, 1, \ldots, \lfloor s\rfloor.
\end{aligned}$$

4.5. **Point inversions: From the ball to the half-space.** One of the useful tools of the potential theoretical approach that we have taken in this section is represented by point inversions and Kelvin transforms or, more generally, Möbius transformations.

The first example is the inversion with respect to the unit sphere

$$\begin{aligned}
\kappa_{0,1} : \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{R}^n \setminus \{0\} \\
x &\longmapsto \frac{x}{|x|^2}
\end{aligned}$$

where, formally, the origin is sent to the point at infinity and *vice versa*. Changing the pole of the transformation and the size of the sphere gives rise to

$$\begin{aligned}
\kappa_{z,r} : \mathbb{R}^n \setminus \{z\} &\longrightarrow \mathbb{R}^n \setminus \{z\} \\
x &\longmapsto r\frac{x-z}{|x-z|^2} + z
\end{aligned}$$

which fixes $\partial B_r(z)$. A Möbius transformation is the composition of $\kappa_{0,1}$ with similarities (translations, rotations, dilations, and reflections). The Kelvin transform of a function $u \in C^\infty_c(\mathbb{R}^n \setminus \{z\})$ is

$$K_{s,z,r}u(x) = |x-z|^{2s-n}(u \circ \kappa_{z,r})(x)$$

and it satisfies (*cf.* [1, Proposition 2] and, for Möbius transformations, [19, Lemma 3]) for $x \in \mathbb{R}^n \setminus \{z\}$

$$(-\Delta)^s\big(K_{s,z,r}u\big)(x) = \frac{r^{2s}}{|x|^{4s}}K_{s,z,r}\big[(-\Delta)^s u\big](x) = \frac{r^{2s}}{|x-z|^{n+2s}}(-\Delta)^s u\big(\kappa_{z,r}(x)\big). \tag{33}$$

As we have already mentioned above, (33) has been exploited in [19] to reduce the study of the Boggio's formula to the case of pole to the origin: it is then possible to move the pole around B_1 by choosing the correct Möbius transformation. This relation can also be used to deduce the formulas for the Green and the Poisson kernel on the half-space and this was the purpose of [1]. For the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$ the kernels read

$$G_s^{\mathbb{R}^n_+}(x,y) = k_{n,s}|x-y|^{2s-n} \int_0^{\frac{4x_1y_1}{|x-y|^2}} \frac{v^{s-1}}{(v+1)^{n/2}}\,dv \qquad \text{for } x,y \in \mathbb{R}^n_+,\ x \neq y,$$

$$P_s^{\mathbb{R}^n_+}(x,y) = (-1)^{\lfloor s \rfloor} \frac{\gamma_{n,s}}{|x-y|^n} \left(\frac{x_1}{-y_1}\right)^s \qquad \text{for } x \in \mathbb{R}^n_+ \text{ and } y \in \mathbb{R}^n \setminus \overline{\mathbb{R}^n_+}.$$

5. Loss of maximum principles

We collect in this section the known counterexamples to the weak maximum principle

$$u \in \mathcal{H}^s_0(\Omega), \quad (-\Delta)^s u \geq 0 \quad \text{in } \Omega \qquad \Longrightarrow \qquad u \geq 0 \quad \text{in } \Omega. \tag{34}$$

Note first that the positivity of G_s in (23), or the one of F_s in (21) for $2s < n$, can be interpreted as a confirmation of (34) for $\Omega = B_1$ or respectively $\Omega = \mathbb{R}^n$. Also, for $s \in (0,1]$, (34) holds: if $u \in \mathcal{H}^s_0(\Omega)$ is such that $\mathcal{E}_s(u,v) \geq 0$ for every $v \in \mathcal{H}^s_0(\Omega)$ with $v \geq 0$, then

$$0 \leq \mathcal{E}_s(u,u_-) = \mathcal{E}_s(u_+,u_-) - \mathcal{E}_s(u_-,u_-) \leq \mathcal{E}_s(u_+,u_-) \leq 0$$

by (20) and therefore $u_- \equiv 0$. When $s > 1$, the above argument presents different issues: the first one is the stability of cut-offs, namely, in general, $u_- \notin \mathcal{H}^s_0(\Omega)$ (recall the discussion around (20)); the second one is the reversion of the sign of $\mathcal{E}_s(u_+,u_-)$, see again (20). This leaves the question of the validity of (34) open for $s > 1$.

5.1. **Recallings on the local theory.** Recall (26). When coupled with Dirichlet conditions, polylaplacians $(-\Delta)^s$, $s \in \mathbb{N} \setminus \{1\}$, present an oscillatory behaviour. This may be exemplified by the failure of (34). It was first remarked by Hadamard in 1908 [36] that, in dimension $n = 2$, there are annular domains for which the weak maximum principle fails for the bilaplacian $s = 2$.

It was then conjectured that this was *not* the case for convex domains: this is known in the literature as the *Boggio-Hadamard conjecture*. But this conjecture was proved to be false by Duffin [21] for some rectangular domains and, later on, several other counterexamples were built; these include for $s, n = 2$: an infinite strip (Duffin [22]), "most of" infinite wedges (Seif [50]), the punctured disk (Nakai and Sario [44]) which is the limiting case of an annulus, and eccentric ellipses (Garabedian [27]). As to this one last counterexample, it is worth mentioning that, on top of the simple geometry of the domain, it is possible to provide completely elementary counterexamples in terms of polynomials (Shapiro and Tegmark [51]): this highlights how the above mentioned oscillations are deeply written in the nature of the higher-order operators and they do not arise as singular phenomena.

For $s = 3$ we refer to Sweers [52] and for $s = 4$ to Sweers [53], for elementary explicit counterexamples. For $s \in \mathbb{N}$ even and any space dimension, the lack of maximum principles can be deduced by the analysis of the oscillations of the first eigenfunction carried out by Kozlov, Kondrat'ev, and Maz'ya [39].

5.2. **Disconnected domains.** The first considerations towards (34) directly follow from equations (18) and (19) for $\lfloor s \rfloor \in 2\mathbb{N} - 1$.

5.2.1. *A first counterexample.* Take for example $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ two disconnected non-empty open sets, define $\Omega = \Omega_1 \cup \Omega_2$, assume that (34) holds in Ω_1, and also

$$\inf \Big\{ |x - y| : x \in \Omega_1,\ y \in \Omega_2 \Big\} > 0. \tag{35}$$

Then take $M > 0$ to be determined, $u_1 \in \mathcal{H}^s_0(\Omega_1)$ solving $(-\Delta)^s u_1 = 1$ in Ω_1 (so that $u_1 \geq 0$ in Ω_1 by (34)), and $u_2 \in C^\infty_c(\Omega_2)$, $u_2 \geq 0$. Then, for any non-negative $v \in \mathcal{H}^s_0(\Omega)$, $v = v_1 + v_2$ with $v_j \in \mathcal{H}^s_0(\Omega_j)$ for $j = 1, 2$, we have

$$\mathcal{E}_s(Mu_1 - u_2, v) = M\mathcal{E}_s(u_1, v_1) - \mathcal{E}_s(u_2, v_1) + M\mathcal{E}_s(u_1, v_2) - \mathcal{E}_s(u_2, v_2)$$

$$= M \int_{\Omega_1} v_1(x)\, dx - \frac{2^{2s-1}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(-s)} \int_{\Omega_1} v_1(x) \int_{\Omega_2} \frac{u_2(y)}{|x-y|^{n+2s}}\, dy\, dx \tag{36}$$

$$+ M\frac{2^{2s-1}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(-s)} \int_{\Omega_2} v_2(x) \int_{\Omega_1} \frac{u_1(y)}{|x-y|^{n+2s}}\, dy\, dx - \int_{\Omega_2} v_2(x)\, (-\Delta)^s u_2(x)\, dx. \tag{37}$$

Up to choosing M sufficiently large independently of v_1, it is possible to make (36) positive: this because u_2 is fixed and Ω_1, Ω_2 are supposed (in (35)) to be at positive distance, so that the kernel in the second term of (36) is uniformly bounded. A similar argument can be applied to (37), where it is crucial to exploit the positivity of the product $\mathcal{E}_s(u_1, v_2)$.

Globally, the above construction produces a $u \in \mathcal{H}^s_0(\Omega)$ with $(-\Delta)^s u \geq 0$ in Ω in the weak sense, although the argument could also be slightly refined to have it in the pointwise sense. So, we must conclude that (34) fails in general at least for $\lfloor s\rfloor \in 2\mathbb{N}-1$, in particular for any $s \in (1,2)$.

On a more discursive note, one could also say that (34) fails in "most" disconnected sets, as the only real assumption we have taken is (35):

- The fact that we assume (34) on Ω_1 is all but necessary: if indeed (34) were already failing in Ω_1, then *a fortiori* it would fail in the whole Ω;.
- Also, mind that we do not need to assume that Ω_1 and Ω_2 are connected (although the notation used might suggest otherwise), so that Ω could be made up of several connected components.
- In conclusion, the presented construction only leaves out disconnected sets for which a partition as in (35) is not possible; nevertheless, we believe that this is only a matter of running more thorough estimates for (36) and (37).
- Let us only remark here that the disconnection of the domain is not a bizarre or exceptional feature, as the nonlocality of the operator is able to overrule it. Anyways, via a perturbative argument, it is possible to bridge the connected components and bring back a connected geometry, this has been performed in [4, Theorem 1.11].

5.2.2. *The sign of the Green function on a two-ball domain.* The discussion of the previous paragraph does not extend at all to the case $\lfloor s\rfloor \in 2\mathbb{N}$, for the sign of $\mathcal{E}_s(u_1,v_2)$ in (37) becomes negative. Actually, in this case, there are available examples of disconnected domains satisfying (34). Indeed, if $\Omega = B_1 \cup B_1(te_1)$ and

$$t > 2 + \left|\frac{2\sin(\pi s)}{\pi n}\right|^{1/n},$$

the Green function G_s^Ω of $(-\Delta)^s$ on Ω satisfies the following sign characterization ($s \notin \mathbb{N}$):

$$\begin{array}{llll}
G_s^\Omega(x,y) > 0 & \text{if} \quad x,y\in B_1, & & \text{or} \quad x,y\in B_1(te_1),\\
G_s^\Omega(x,y) > 0 & \text{if} \quad \lfloor s\rfloor\in 2\mathbb{N}, & \text{and } x\in B_1,\ y\in B_1(te_1), & \text{or} \quad x\in B_1(te_1),\ y\in B_1,\\
G_s^\Omega(x,y) < 0 & \text{if} \quad \lfloor s\rfloor\in 2\mathbb{N}-1, & \text{and } x\in B_1,\ y\in B_1(te_1), & \text{or} \quad x\in B_1(te_1),\ y\in B_1.
\end{array}$$

The details of the proof can be found in [4, Theorem 1.10]. Let us underline how

$$0 \leq \left|\frac{2\sin(\pi s)}{\pi n}\right|^{1/n} \leq 1.$$

5.2.3. *Partial recovery of maximum principles.* The analysis carried out in [4, Theorem 1.10] can be pushed to a somewhat more precise result. To fix ideas, consider $s \in (1,2)$ and $\Omega = B_1 \cup B_1(te_1)$: Paragraph 5.2.1 states how (34) fails for any $t > 2$; Paragraph 5.2.2 describes how the sign of the Green function behaves for $t > 3$; in a forthcoming work the author and Jarohs [2], for $t > 4$, give an example where (34) is recovered. It goes as follows: consider $\tau : \mathbb{R}^n \to \mathbb{R}^n$ the inversion of $\mathbb{R}^n$ transforming B_1 into $B_1(te_1)$ and *vice versa*, namely

$$\begin{aligned} \tau : \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} &\longrightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \\ x = (x_1, x') &\longmapsto (t - x_1, x'); \end{aligned}$$

let $g \in L^2(B_1),\ g \geq 0$, and $f = g - (-1)^{\lfloor s \rfloor} g \circ \tau$; then the weak solution $u \in \mathcal{H}^s_0(\Omega)$ of

$$(-\Delta)^s u = f \qquad \text{in } \Omega$$

is of the form $u = v - (-1)^{\lfloor s \rfloor} v \circ \tau$ for some $v \in \mathcal{H}^s_0(B_1)$ with $v \geq 0$. In particular, when $\lfloor s \rfloor \in 2\mathbb{N} - 1$, both $f \geq 0$ and $u \geq 0$, which amounts to be a particular case where (34) is valid. A key step in the proof consists in rephrasing the problem in terms of the Green function and proving

$$G^\Omega_s(x,y) - (-1)^{\lfloor s \rfloor} G^\Omega_s\big(x, \tau(y)\big) \geq 0 \qquad \text{for } x, y \in B_1.$$

One consequence is for example the fact that the torsion function of $\Omega = B_1 \cup B_1(te_1)$, *i.e.*, the solution of

$$u \in \mathcal{H}^s_0(\Omega), \qquad (-\Delta)^s u = 1 \quad \text{in } \Omega,$$

is positive for any $s > 0$.

5.3. **Ellipsoids.** So far we have not presented any counterexample to (34) for $\lfloor s \rfloor \in 2\mathbb{N}$. In this section we partially fill this gap, by providing a construction covering $s \in (2,3)$. The main reference is the one by the author, Jarohs, and Saldaña [7]. For simplicity we limit ourselves to the two-dimensional case $n = 2$, although this is not really necessary.

Consider $a > 1$, define the open ellipse with semi-axes 1 and $a^{-1/2}$, *i.e.*,

$$E_a = \big\{(x,y) \in \mathbb{R}^2 : x^2 + ay^2 < 1\big\},$$

and

$$u_\beta(x,y) = \begin{cases} \big(1 - x^2 - ay^2\big)^\beta & \text{for } (x,y) \in E_a,\ \beta > -1, \\ 0 & \text{for } (x,y) \in \mathbb{R}^2 \setminus E_a, \end{cases} \tag{38}$$

The restriction $\beta > -1$ is only aimed at guaranteeing $u_\beta \in L^1_{loc}(\mathbb{R}^2)$, so that it is possible to evaluate (3) on u_β.

The first tool we need is the torsion function of E_a, *i.e.*, the solution $u \in \mathcal{H}^s_0(E_a)$ of

$$(-\Delta)^s u = 1 \qquad \text{in } E_a.$$

This is given by a constant multiple of u_s as defined in (38). The precise computations are contained in [7, Corollary 3.3] and are based on the analogous ones for the ball and $s \in (0,1)$, see Dyda [23]. The same calculations can also be extended to compute $(-\Delta)^s u_\beta$ for any $\beta > -1$ and give, in particular, that for any $j \in \mathbb{N}$

$$(-\Delta)^s u_{s+j} \quad \text{is a polynomial of degree } 2j \text{ in } E_a.$$

Next, another tool are relations of the type[7]

$$\begin{aligned}
(-\Delta)^s\big(x\,u_\beta(x,y)\big) &= -\frac{1}{2(\beta+1)}\,\frac{\partial}{\partial x}(-\Delta)^s u_{\beta+1}(x,y) \\
(-\Delta)^s\big(x^2 u_\beta(x,y)\big) &= \frac{1}{2(\beta+1)}\,\frac{\partial}{\partial x}(-\Delta)^s u_{\beta+1}(x,y) + \\
&\qquad + \frac{1}{4(\beta+1)(\beta+2)}\,\frac{\partial^2}{\partial x^2}(-\Delta)^s u_{\beta+2}(x,y)
\end{aligned}$$

[7] We only name a few examples, more are needed in the complete argument.

which imply that, if $p(x,y)$ is a polynomial of degree 2, then

$$(-\Delta)^s\big(pu_s\big) \quad \text{is a polynomial of degree 2.}$$

Keeping all this in mind, it is possible to prove that for any $s \in (1, \sqrt{3}+3/2)$ there exists $a_0 = a_0(s) > 1$ such that for any $a > a_0$ there is a sign-changing (in E_a) polynomial $p_a(x,y)$ of degree 2 for which

$$(-\Delta)^s\big(p_a u_s\big) > 0 \qquad \text{in } E_a. \tag{39}$$

The proof is extremely technical and this is why it only deals with polynomials of degree 2: working with polynomials of higher degree should in principle improve the attained range for s (meaning, values of s greater than $\sqrt{3}+3/2$ should fall into the analysis). Also, an asymptotic argument for $a \uparrow \infty$ is run and this is the reason why the result only holds for very eccentric ellipses ($a \gg 1$).

A computer-assisted analysis helps in overcoming the computational difficulties hidden in the problem: see Figure 1.

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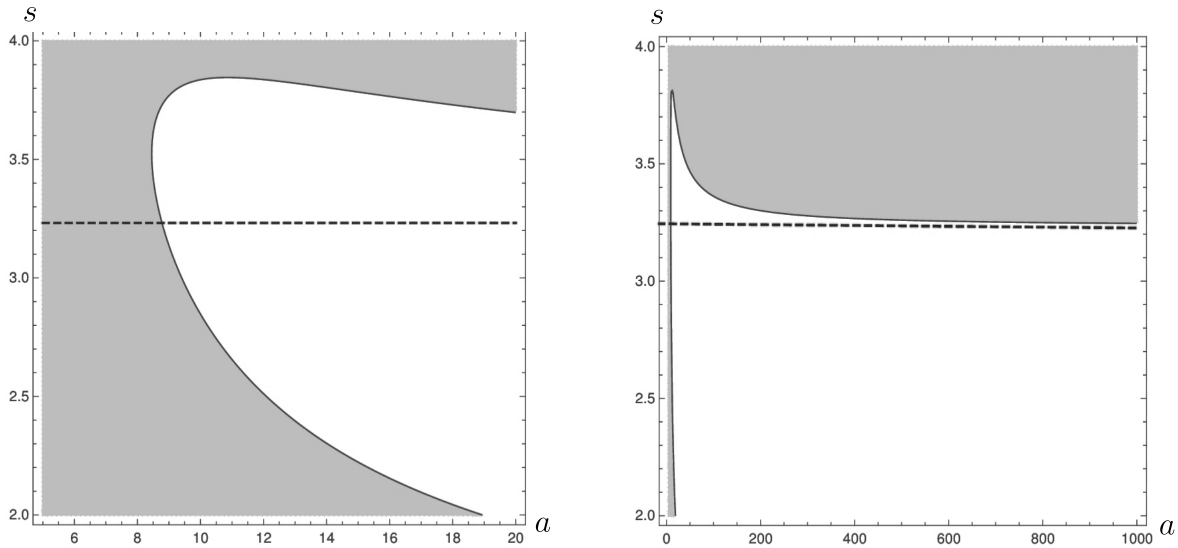

FIGURE 1. A depiction of the region in the (a, s)-plane for which (34) admits a counterexample of the form (39): for (a, s) in the white region it is possible to construct a sign-changing polynomial p_a of degree 2 such that (39) holds. On the left, $s \in (2, 4)$ and $a \in (5, 20)$, whereas, on the right, $s \in (2, 4)$ and $a \in (5, 1000)$. The dashed line represents $s = \sqrt{3} + 3/2$ which is the threshold for the asymptotic argument $a \uparrow \infty$ to work. Note that there is a whole region for $s \in (\sqrt{3} + 3/2, 3.85...)$ which is not captured by this compactness argument.

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