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Technical Note—Options Portfolio Selection

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We develop a new method to optimize portfolios of options in a market where European calls and puts are available with many exercise prices for each of several potentially correlated underlying assets. We identify the combination of asset-specific option payoffs that maximizes the Sharpe ratio of the overall portfolio: such payoffs form the unique solution to a system of integral equations, which reduces to a linear matrix equation under discrete representations of the underlying probabilities. Even when risk-neutral volatilities are all higher than physical volatilities, it can be optimal to sell options on some assets while buying options on other assets, for which the positive hedging demand outweighs negative demand stemming from asset-specific returns.

Key words: options, portfolio choice, Sharpe ratio, duality, multiple assets

Subject classifications: Portfolio: options, Asset Pricing: incomplete markets, Investment: multiple assets.

Area of review: Financial Engineering.

1. Introduction

Today call and put options are available on virtually every asset class, including stocks, bonds, commodities, currencies – and their indexes. In theory, so many investment opportunities hold the promise of high returns with low risk through broad diversification. In practice, they leave investors with the high-dimensional puzzle of finding the combination of nonlinear payoffs in all assets, as to maximize the risk-return tradeoff. This paper tackles such a puzzle.

We develop a method to find options portfolios that maximize the Sharpe ratio in a market where calls and puts are available with many strike prices and on each of several underlying assets. We adopt a one-period model, which reflects the monthly update of a portfolio of short-term options, while refraining from continuous trading in view of options' significant trading costs. At the beginning of the period, the investor observes the prices of options on each asset for all strikes and constructs the portfolio. At the end of the period, the investor collects the option payoffs; then another period begins.

Our main result characterizes the combined option payoff with maximal Sharpe ratio as the arithmetic average of asset-specific option payoffs, identified as the unique solution to a system of integral equations, depending on (i) the risk-neutral marginal probabilities, determined by options' prices, and (ii) the joint physical probability, which reflects the investor's views. In particular, our result does not prescribe how investors should form their views or which model they should specify for asset price dynamics. Instead, we show how to combine an investor's private views with the public information embedded in option prices to obtain the portfolio of options that maximizes the investor's Sharpe ratio. Thus, our approach is reminiscent of Black and Litterman (1992), who combine an investor's views with market information to obtain optimal portfolios.

The integral equations that identify the optimal payoff do not admit a closed form solution except in trivial cases, but they are solved with arbitrary precision by approximating the joint physical probability either through a mixture of independent distributions or through a discrete density on a grid. Either discretization reduces the system of integral equations to a matrix linear equation that yields a solution in arbitrary dimension. Such approximation is similar in spirit to the techniques of Filipović et al. (2013) and Schneider and Trojani (2018), who employ finite-dimensional parametrizations to approximate state-price densities.

Ironically, the main difficulty of this high-dimensional optimization problem is that the number of investments is *too low*. Hypothetically, if options depending on any number of assets were available (rather than asset-specific options alone), then such option prices would determine uniquely the joint risk-neutral measure, thereby completing the market. Instead, the cross-sections of option prices on individual assets only yield the risk-neutral marginals, leaving infinitely many joint risk-neutral laws that fit such marginals.

Our solution is based on a duality approach, which reduces the pursuit of portfolios that maximize the Sharpe ratio to the search for the state-price density with minimum second moment among the ones that correctly price all traded options. In fact, such a dual minimizer is also a payoff, and any optimal payoff coincides with the dual minimizer up to an affine transformation (because the Sharpe ratio is invariant to translation and scaling).

1.1. Financial Insights

The central contribution of this paper is a new method that combines an investor's views with the information embedded in option prices to construct portfolios of options on each of several underlying assets as to maximize the Sharpe ratio. We bring to bear the insight of Hansen and Jagannathan (1991) that mean-variance efficient portfolios are perfectly correlated with the minimum-variance discount factor, showing that optimal option portfolios are the average of several asset-specific portfolios, which in concrete settings are obtained through the solution of a system of linear equations. Such a representation also leads to additional insights.

First, only when assets are independent, option portfolios can be constructed for each asset separately. In general, the optimal portfolio of options on each asset also depends on the prices of options on other assets. In particular, options with small or zero risk premia are an important hedging tool to reduce total portfolio risk, despite being unattractive in isolation.

Second, even though option strategies that focus on a single underlying asset generally imply negative positions, optimal option portfolios on a cross-section of assets may include both negative and positive positions: hedging demand may be so large to offset the demand for asset-specific returns, making it potentially optimal to buy options with negative expected returns to reduce the risk assumed selling other options.

2. The Problem

This section derives the optimality conditions for option portfolios with maximal Sharpe ratio and describes their usage: the natural starting point for the discussion is the classical duality bound. In a one-period setting with a probability space $(\Omega, \mathcal{F}, P)$, consider a market where traded payoffs (i.e., terminal values of any portfolio and initial investment) form a linear subspace $\mathcal{R} \subset L^2(\Omega, \mathcal{F}, P)$. In particular, the linearity of $\mathcal{R}$ implies that payoffs are available in any positive or negative quantity, which means that both leverage and short sales are permitted. The prices of payoffs are characterized by some stochastic discount factor (SDF) $\hat{M} \in L^2(\Omega, \mathcal{F}, P)$ such that $\hat{M} > 0$ almost surely (henceforth, a.s.) to exclude arbitrage opportunities. The set $\mathcal{M} = \{M \in L^2(\Omega, \mathcal{F}, P), E[RM] = E[R\hat{M}]$ for all $R \in \mathcal{R}\}$ denotes all SDFs for the market, which may or may not be strictly positive.

To ease notation, assume also that a safe asset is available with zero interest rate, so that $E[M] = 1$ for all $M \in \mathcal{M}$. For any *excess return*, defined as a payoff R with price zero (i.e., $E[RM] = 0$, cf. Cochrane (2009)), the definition of covariance $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$ and the fact that correlation is greater or equal than minus one, imply that

$$\mathrm{Cov}(R, M) + E[R]E[M] \geq -\sigma(R)\sigma(M) + E[R], \tag{1}$$

where $\sigma(\cdot)$ denotes the standard deviation operator. Rearranging this inequality, and noting that it holds for any excess return $R \in \mathcal{R}$ and SDF $M \in \mathcal{M}$, the Hansen and Jagannathan (1991) bound follows

$$\sup_{\substack{R \in \mathcal{R} \\ \sigma(R) \neq 0, E[MR]=0}} \frac{E[R]}{\sigma(R)} \leq \inf_{M \in \mathcal{M}} \sigma(M), \tag{2}$$

implying that the maximal Sharpe ratio $E[R]/\sigma(R)$ of any non-trivial (i.e., with non-vanishing variance) excess return is bounded above by the minimum SDF volatility. More importantly, to maximize the Sharpe ratio it is enough to find an excess return R that is perfectly negatively correlated with an SDF M, as for any such return the Cauchy-Schwarz inequality holds as an equality. Furthermore, any SDF for which equality holds must have minimum variance, hence minimum second moment (because $E[M]=1$). In short, maximizing the Sharpe ratio leads to the problem

$$\min_{M\in\mathcal{M}} E[M^2]. \tag{3}$$

If the minimizer M^* of this problem is a payoff, then $R^* = -M^* + E[(M^*)^2]$ is a payoff that, by construction, has both price zero and perfect negative correlation with M^*. As a result, the inequality (1) holds as an equality, hence also (2) does, and R^* is optimal. As the Sharpe ratio is leverage-invariant, any return of the form $R = a(M^* - E[(M^*)^2])$ for any $a<0$ is optimal.[1] Finally, all optimal returns are of this form: otherwise, any other optimal return could be combined to form an even better one through diversification.[2]

2.1. One Asset

In a one-period model with European options on a single asset, the dual problem in (3) is easy to solve because the market is complete and therefore the unique SDF is necessarily the minimizer. Denoting by X the risky asset's price at the end of the period, by $c_X(K)$ the price of a call option on X with strike price K, and by $p_X(x)$ the physical marginal density of X at x, the unique SDF is a function m_X of X, uniquely identified by the condition

$$\int_0^\infty m_X(x)p_X(x)(x-K)^+dx = c_X(K) \qquad \text{for all } K\geq 0. \tag{4}$$

Assuming that the call option price is smooth in the strike, the solution to this problem dates back to Breeden and Litzenberger (1978), Green and Jarrow (1987), and Nachman (1988), who note that the risk-neutral density $q_X(K)$ is identified as $q_X(K) := m_X(K)p_X(K) = c''_X(K)$, which follows by twice differentiating (4) with respect to K. Thus, the unique SDF is the random variable $m_X(X)$, where $m_X(x) = c''_X(x)/p_X(x)$. Provided that the function m_X is regular enough, such a return decomposes as a portfolio of call and put options (Carr and Madan 2001b):

$$m_X(K) = m_X(K_0) + m'_X(K_0)(K-K_0) + \int_0^{K_0} m''_X(\kappa)(\kappa-K)^+d\kappa + \int_{K_0}^\infty m''_X(\kappa)(K-\kappa)^+d\kappa. \tag{5}$$

(The strike K_0 that separates puts from calls is arbitrary, and typically equal to the spot or forward price.) Thus, all payoffs with maximal Sharpe ratio are of the form $R = b + a\ m_X(X)$, where $a < 0$ (cf. note 1). Excess returns are obtained identifying the constant b through the zero-price condition $E[m_X(X)R] = 0$.

2.2. Two Assets

In a market with multiple risky assets, similar arguments would identify the unique SDF, *if* European options were available for all combination of strikes (with two assets, for example, digital options with payoffs $1_{\{X>K,Y>L\}}$ would be required for all K, $L > 0$). In practice, such options are not actively (if at all) traded, but European options on each asset are available for a wide range of (ideally, all) strikes. Such a feature makes an option market with multiple assets incomplete, leading to multiple SDFs and hence to a nontrivial dual problem (3).

To ease notation, consider a market with two risky assets with respective prices X and Y, two random variables with joint physical density $p(x,y)$. For each asset, the above argument of Breeden and Litzenberger (1978) identifies uniquely the risk-neutral marginal densities $q_X(x)$ and $q_Y(y)$ implied by any SDF. Considering SDFs of the form $M = m(X,Y)$, the marginal restrictions are:[3]

$$E[m(X,Y)|X=x] = \int_0^\infty m(x,y)\frac{p(x,y)}{p_X(x)}dy = \frac{q_X(x)}{p_X(x)} \qquad \text{for all } x>0 \text{ with } p_X(x)>0, \tag{6}$$

$$E[m(X,Y)|Y=y] = \int_0^\infty m(x,y)\frac{p(x,y)}{p_Y(y)}dx = \frac{q_Y(y)}{p_Y(y)} \qquad \text{for all } y>0 \text{ with } p_Y(y)>0. \tag{7}$$

Thus, the dual problem (3) consists in finding a function $m(x,y)$ that minimizes $E[M^2] = \int_0^\infty \int_0^\infty m(x,y)^2 p(x,y)dxdy$, subject to the constraints (6) and (7), and is reminiscent of the class of optimization problems in Lasserre (2010, Chapter 7.3), with the key difference that our objective is nonlinear rather than linear. To tackle this problem, introduce as (infinite-dimensional) Lagrange multipliers the functions $\Phi_X(x)$ and $\Phi_Y(y)$, reflecting that (6) and (7) hold for all values of x and y. The resulting unconstrained problem is $\min_{m\in\mathcal{M}} F(m)$, where

$$\begin{aligned} F(m) := \Bigg[\int_0^\infty \int_0^\infty m(x,y)^2 p(x,y)dxdy &- \int_0^\infty \Phi_X(x)\left(\int_0^\infty m(x,y)p(x,y)dy - q_X(x)\right)dx \\ &- \int_0^\infty \Phi_Y(y)\left(\int_0^\infty m(x,y)p(x,y)dx - q_Y(y)\right)dy\Bigg], \end{aligned}$$

and the function $m(x,y)$ is now free to vary in the class $\int_0^\infty \int_0^\infty m^2(x,y)p(x,y)dxdy < \infty$.

First, note that the functional $m \mapsto F(m)$ is strictly convex, hence its minimizer m^* is unique if it exists. To identify such m^*, consider a small (square-integrable) perturbation from $m^*(x,y)$ to $m^*(x,y) + \varepsilon g(x,y)$ for $\varepsilon > 0$. Optimality implies that $F(m^*) \geq F(m^* + \varepsilon g)$. Subtracting the right-hand side from the left-hand side, dividing by ε, and passing to the limit, from the definition of $F(m)$ it follows that $\int_0^\infty \int_0^\infty (2m^*(x,y) - \Phi_X(x) - \Phi_Y(y))\, g(x,y)p(x,y)dxdy \geq 0$. As the inequality holds for any perturbation $g(x,y)$ (hence also for $-g(x,y)$), the left hand side is in fact zero, whence

$$m^*(x,y) = \frac{1}{2}(\Phi_X(x) + \Phi_Y(y)). \tag{8}$$

This equation stipulates that the minimum-variance SDF must be additively separable in the cross section. In retrospect, such a decomposition is natural: if an SDF is to close the duality gap in (2), it has to be perfectly correlated with a portfolio of options on X and options on Y. In view of the Carr-Madan representation (5), a portfolio of options on X represents any (regular) payoff $\Phi_X(X)$, and a portfolio of options of Y any payoff $\Phi_Y(Y)$. Thus, (8) is the only representation with the potential to solve both the dual (pricing) and the primal (investment) problems.

To identify the functions Φ_X and Φ_Y, substitute (8) into (6)-(7), which yields the system of integral equations

$$\frac{1}{2}\Phi_X(x)p_X(x) + \frac{1}{2}\int_0^\infty \Phi_Y(y)p(x,y)dy = q_X(x), \qquad x > 0, \tag{9}$$

$$\frac{1}{2}\int_0^\infty \Phi_X(x)p(x,y)dx + \frac{1}{2}\Phi_Y(y)p_Y(y) = q_Y(y), \qquad y > 0. \tag{10}$$

Note that Φ_X and Φ_Y are determined up to an additive constant C because, setting $\Phi'_X = \Phi_X + C$ and $\Phi'_Y = \Phi_Y - C$, equation (8) implies that $m^*(x,y) = \frac{1}{2}(\Phi'_X(x) + \Phi'_Y(y))$. Put differently, the SDF in (8) resulting from the sum of two asset-specific payoffs does not change if a cash position shifts from Φ_X to Φ_Y or vice versa. To eliminate such spurious degree of freedom, we impose the condition $E[\Phi_X(X)] = E[\Phi_Y(Y)]$, i.e.,[4]

$$\int_0^\infty \Phi_X(x)p_X(x)dx = \int_0^\infty \Phi_Y(y)p_Y(y)dy. \tag{11}$$

These conditions have a clear interpretation, as they prescribe that both payoffs $\Phi_X(X)$ and $\Phi_Y(Y)$ have the same expected value. As each function Φ_X, Φ_Y is the payoff of a contingent claim at time T on the respective asset, if sufficiently regular it also admits a representation as a portfolio of call and put options through formula (5).

As shown in Theorem 1 below for several assets, equations (9)-(11) indeed admit a solution that identifies both the minimum SDF and the optimal option payoff. But, before discussing the general result, it is instructive to examine some special cases.

Example 1. If no option carries reward for its risk, it should be optimal to neither buy nor sell options. Indeed, for any joint law of X and Y, suppose that the marginal discount factors satisfy $q_X/p_X = q_Y/p_Y = 1$, which means that the risk-neutral and physical marginals coincide – all risk-premia are null. Then, $\Phi_X = \Phi_Y = 1$ solve the system (9)-(11), hence the minimizing SDF is $m^* = \frac{1}{2}(\Phi_X + \Phi_Y) = 1$, and therefore the maximum Sharpe ratio is $\sigma(m^*) = 0$, which is achieved by any excess return with positive variance, as risk premia are zero.

Example 2. If the assets are independent ($p(x,y) = p_X(x)p_Y(y)$), then the optimal investment problem separates across assets: the optimal position in each family of asset-specific options is insensitive to the presence of options on other assets. Indeed, assuming that $p_X, p_Y > 0$, equations (9) and (10) reduce to

$$\Phi_X(x) + \int_0^\infty \Phi_Y(y)p_Y(y)dy = 2\frac{q_X(x)}{p_X(x)}, \qquad x > 0, \tag{12}$$

$$\int_0^\infty \Phi_X(x)p_X(x)dx + \Phi_Y(y) = 2\frac{q_Y(y)}{p_Y(y)}, \qquad y > 0, \tag{13}$$

hence $\Phi_X(x) = 2\frac{q_X(x)}{p_X(x)} + C_X$ and $\Phi_Y(y) = 2\frac{q_Y(y)}{p_Y(y)} + C_Y$, where the constants C_X and C_Y are, by (11), $C_X = C_Y = -1$. Thus, the minimal SDF is $m^*(x,y) = \frac{1}{2}(\Phi_X(x) + \Phi_Y(y)) = \frac{q_X(x)}{p_X(x)} + \frac{q_Y(y)}{p_Y(y)} - 1$, which is the sum of two option payoffs depending only on the pairs of marginals p_X, q_X and p_Y, q_Y.

3. Numerical Methods

3.1. Mixture Distributions

Mixture distributions are an important application of the above methodology because (i) they can approximate virtually any distribution, (ii) are highly tractable, as the integral equations (9)-(10)

reduce to a linear matrix equation, and (iii) they offer a flexible parametrization that is immune to the curse of dimensionality, in that the dimension of the resulting system of equations increases linearly with the number of underlying assets – not exponentially.

Let $(p_X^i)_{1\le i\le k}$, $(p_Y^i)_{1\le i\le k}$, be strictly positive probability densities on some measurable sets $I^x, I^y \subset \mathbb{R}$ and define the joint probability density as $p(x,y) := \frac{1}{k}\sum_{i=1}^k p_X^i(x)p_Y^i(y)$. Denoting by q_X, q_Y the risk-neutral marginals, the integral equations (9)-(10) reduce to the system of linear equations

$$\frac{1}{2}p_X\Phi_X = q_X - \sum_{i=1}^k c_Y^i p_X^i, \qquad \frac{1}{2}p_Y\Phi_Y = q_Y - \sum_{i=1}^k c_X^i p_Y^i, \tag{14}$$

where the $2k$ constants $(c_X^i)_{1\le i\le k}, (c_Y^i)_{1\le i\le k}$ represent the integrals arising in (9)-(10), i.e., $c_X^i = \frac{1}{2k}\int_0^\infty \Phi_X(x)p_X^i(x)dx$ and $c_Y^i = \frac{1}{2k}\int_0^\infty \Phi_Y(y)p_Y^i(y)dy$. It remains to identify the $2k$ linear equations that uniquely determine these $2k$ unknowns. Substituting (14) into the first integral equations (9) and comparing the coefficients of p_X^i yields the k equations

$$c_Y^i = \frac{1}{k}\int_0^\infty q_Y(y)\frac{p_Y^i(y)}{p_Y(y)}dy - \frac{1}{k}\sum_{j=1}^k c_X^j \int_0^\infty \frac{p_Y(y)^j p_Y^i(y)}{p_Y(y)}dy, \qquad 1\le i\le k. \tag{15}$$

Similarly, another k equations follow from (10):

$$c_X^i = \frac{1}{k}\int_0^\infty q_X(x)\frac{p_X^i(x)}{p_X(x)}dx - \frac{1}{k}\sum_{j=1}^k c_Y^j \int_0^\infty \frac{p_X^j(x) p_X^i(x)}{p_X(x)}dx, \qquad 1\le i\le k. \tag{16}$$

However, the resulting $2k$ equations are not enough to identify the constants c_X^i, c_Y^i without the constraint (11). Indeed, summing either (15) or (16) yields the same equation $\sum_{i=1}^k c_X^i + \sum_{i=1}^k c_Y^i = 1$, which reflects that one of the $2k$ equations is redundant. To obtain a system of linearly independent equations, it suffices to replace one of the equations (15) and (16) with the constraint (11), i.e., $\sum_{i=1}^k c_X^i - \sum_{i=1}^k c_Y^i = 0$.

3.2. Discrete Densities

An alternative approach to approximating a distribution is through a piecewise constant density with a finite number of values. This approximating strategy is also highly tractable, as it reduces the integral equations above to a linear system. As for mixture distributions, discrete densities also lead to a linear system whose size increases linearly in the number of assets, though the representation of the joint probability has a dimension that increases exponentially in the number of underlying assets (holding the grid size constant). For this reason, such a representation is more attractive for a small number of assets. Consider two increasing finite sequences $(x_i)_{0\le i\le k}$ and $(y_j)_{0\le j\le l}$, and assume that:

(i) $P(X\in[x_0,x_k),Y\in[y_0,y_l))=Q(X\in[x_0,x_k),Y\in[y_0,y_l))=1$;

(ii) the joint probability density p is constant and strictly positive on each rectangle $I_i^x\times I_j^y$, where $I_i^x=[x_{i-1},x_i)$, $1\le i\le k$, and $I_j^y=[y_{j-1},y_j)$, $1\le j\le l$.

Denote by $\tilde p^{ij}=P(X\in I_i^x,Y\in I_j^y)$ and their marginal counterparts by $\tilde p_X^i=P(X\in I_i^x)$, $\tilde p_Y^j=P(Y\in I_j^y)$, and $\tilde q_X^i=Q(X\in I_i^x)$, $\tilde q_Y^j=Q(Y\in I_j^y)$, $1\le i\le k, 1\le j\le l$.

By inspection of equation (9), any solution Φ_X,Φ_Y needs to be piecewise constant on $(I_i^x)_{1\le i\le k}$ and $(I_j^y)_{1\le j\le l}$ respectively. Thus, denote by $\Phi_X^i=\Phi_X(x_{i-1})$ and $\Phi_Y^j=\Phi_Y(x_{j-1})$. Integrating equation (9) on I_i^x and (10) on I_j^y, they reduce to:

$$\Phi_X^i\tilde p_X^i+\sum_{j=1}^{l}\Phi_Y^j\tilde p^{ij}=2\tilde q_X^i,\qquad 1\le i\le k, \tag{17}$$

$$\Phi_Y^j\tilde p_Y^j+\sum_{i=1}^{k}\Phi_X^i\tilde p^{ij}=2\tilde q_Y^j,\qquad 1\le j\le l. \tag{18}$$

As in the previous example, one of these $k+l$ equations is redundant (summing the first one over i and the second one over j yields the same equation). The system is completed by replacing one of such equations with the constraint (11), i.e.,

$$\sum_{i=1}^{k}\Phi_X^i\tilde p_X^i-\sum_{j=1}^{l}\Phi_Y^j\tilde p_Y^j=0, \tag{19}$$

thereby obtaining an invertible system of $k+l$ equations in as many unknowns.

4. Main Result

The rigorous statement of the main result requires some additional notation. Recall that $(\Omega,\mathcal{F},P)$ denoted the probability space, denote by $X=(X_1,\dots,X_n)\subset L^2(\Omega,\mathcal{F},P)$ the risky assets' prices, and by $\mathcal{G} = \sigma(X_1, \dots, X_n) \subset \mathcal{F}$ the sigma algebra that they generate. Each asset price takes values in some Lebesgue-measurable subset $I_i \subset \mathbb{R}$. Let $\mathcal{D}_i^c := \prod_{j\neq i} I_i$, $\mathcal{D} := \prod_{i=1}^n I_i$. Furthermore, assume that X admits a strictly positive probability density p on $\mathcal{D}$.

For $\xi \in \mathbb{R}^n$, denote by $\xi_i^c \in \mathbb{R}^{n-1}$ the vector obtained from ξ by omitting the i-th coordinate, that is $\xi_i^c := (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n)$. Similarly, $\xi_{i,j}^c \in \mathbb{R}^{n-2}$ omits the i-th and j-th coordinates. Let $(q_i)_{1\le i\le n}$ be probability densities such that q_i is supported on I_i for all $1 \le i \le n$. Define also the projection (i.e., marginal density) p_i on the i-th component and the complementary projection p_i^c on the remaining $n-1$ components as $p_i(\xi_i) := \int_{\mathcal{D}_i^c} p(\xi) d\xi_i^c = P(X_i \in d\xi_i)/dt, p_i^c(\xi_i^c) := \int_{I_i} p(\xi) d\xi_i = P(X_i^c \in d\xi_i^c)/dt$, $1 \le i \le n$. As p is strictly positive, p_i and p_i^c are also strictly positive.

Decompose any $M \in L^2(\Omega, \mathcal{F}, P)$ as $M = m(X) + (M - m(X))$, where $m(X) = E[M|X]$. The set of square-integrable SDFs is defined as[5]

$$\mathcal{M} := \left\{ M \in L^2(\Omega, \mathcal{F}, P) \middle| E[M|X_i] = E[m(X)|X_i] = \frac{q_i(X_i)}{p_i(X_i)}, 1 \le i \le n \right\} \tag{20}$$

and an element $M^* \in \mathcal{M}$ is minimal if $M^* = \operatorname{argmin}_{M\in\mathcal{M}} E[M^2]$.

THEOREM 1. *Assume that $\mathcal{M} \neq \varnothing$ and that $E\left[\left(\frac{p_i(X)p_i^c(X)}{p(X)}\right)^2\right] < \infty$, $1 \le i \le n$. Then:*

(i) (Existence and Uniqueness) There exists a unique minimal SDF $M^* \in \mathcal{M}$.

(ii) (Linearity) There exists $\Phi := (\Phi_1, \dots, \Phi_n)$, where each $\Phi_i(X_i) \in L^1$ for $1 \le i \le n$, such that the minimal SDF is of the form $M^* = m^*(X)$, where $m^*(\xi) = \frac{1}{n}\sum_{i=1}^n \Phi_i(\xi_i)$.

(iii) (Identification) $\Phi = (\Phi_1, \dots, \Phi_n)$ solves the integral equations $E[m^*(X)|X_i] = q_i(X_i)/p_i(X_i)$, i.e.,

$$\Phi_i(\xi_i) + \sum_{j\neq i} \int_{\mathcal{D}_i^c} \Phi_j(\xi_j) \frac{p(\xi)}{p_i(\xi_i)} d\xi_i^c = n \frac{q_i(\xi_i)}{p_i(\xi_i)}, \qquad 1 \le i \le n \tag{21}$$

subject to the constraints $E[\Phi_i(X_i)] = 1$, i.e.,

$$\int_{I_i} \Phi_i(\xi_i) p_i(\xi_i) d\xi_i = 1, \qquad 1 \le i \le n. \tag{22}$$

(iv) (Performance) Optimal excess returns are of the form $a(m^*(X) - E[(m^*(X))^2])$ for $a < 0$, and their common maximum Sharpe ratio is $SR = \sqrt{\frac{1}{n}\sum_{i=1}^n \int_{I_i} \Phi_i(\xi_i) q_i(\xi_i) d\xi_i - 1}$.

(v) (Regularity) Let $(q_i)_{i=1}^n \subset C^k(\mathbb{R})$ *with* $k \geq 0$. *Denoting the continuous partial derivatives by* $\partial^\beta_{\xi_i} p(\xi)$, $0 \leq \beta \leq k$, *if for any* $R > 0$ *there exists* $\alpha \in (1/2, 1]$ *such that* $\sup_{\xi:\|\xi_i\|\leq R} \left| \frac{\partial^\beta_{\xi_i} p(\xi)}{(p_i^c(\xi_i^c))^\alpha} \right| < \infty$ *and* $\int_{\mathcal{D}_i^c} (p_i^c(\xi_i^c))^{2\alpha-1} d\xi_i^c < \infty$, *then* $m^*(\xi) = \frac{1}{n}\sum_{i=1}^n \Phi_i(\xi_i)$ *is also in* $C^k(\mathbb{R})$.

Proof. See Online Appendix.

As in the two-asset case in the previous section, if the assets' returns are independent, the minimal SDF separates across assets. In other words, if $p(\xi) = \Pi_{i=1}^n p_i(\xi_i)$ then $\Phi_i(\xi_i) = nq_i(\xi_i)/p_i(\xi_i) - (n-1)$ and therefore $m^*(\xi) = \sum_{i=1}^n \frac{q_i(\xi_i)}{p_i(\xi_i)} - (n-1)$.

Note that Theorem 1 requires that the class $\mathcal{M}$ of SDFs is nonempty, so that the problem is well-defined. To verify that this condition holds, it suffices to check that some explicit (possibly sub-optimal) SDF M is square-integrable, i.e. $E[M^2] < \infty$. A natural candidate that admits an explicit solution in terms of the problem's inputs is the SDF associated with risk-neutral independence, i.e., $M = m(X)$, where $m(\xi) = \frac{\Pi_{i=1}^n q_i(\xi_i)}{p(\xi)}$.

5. Numerical Results and Discussion

This section explores the quantitative implications of the above results by calculating optimal option payoffs in a concrete setting with two assets, namely the Variance Gamma model of Madan and Seneta (1990), which offers a parsimonious, arbitrage-free parametrization of asset prices with distinct physical and risk-neutral variances (see the online Appendix for details). Because optimal option payoffs separate across assets when the returns are independent, the central problem is to understand how assets' interdependence affects asset-specific payoffs.

The top row in Figure 1 displays optimal option payoffs in a market where options on asset X (left panel) carry a risk premium, in that physical and risk-neutral marginal densities differ, while options on the second asset carry no risk premium. If the assets are uncorrelated (black curves), then the optimal option payoff is concave in X, which entails short option positions at all strikes in view of (5). If, however, the assets are positively correlated, the option position in X becomes even more concave, reflecting an even more short portfolio, while the payoff in Y becomes convex, i.e. long options of all strikes. The intuition is that even though options in Y are unattractive in isolation, they become interesting hedges for the option position in X. The availability of such hedging instrument enables the investor to take an even larger position of options on X, as its risk is reduced through options on Y.

The third row in Figure 1 considers two assets with significantly positive – but also significantly different – option risk premia. When returns are uncorrelated, optimal payoffs are independent of each other, and entail short positions in options of all strikes in both assets, as payoffs are concave. Notably, even though it would be optimal to short each option individually, high correlation gradually turns the payoff in X convex, while accentuating the concavity of the payoff in Y. The explanation is that high correlation makes options on X more attractive as hedging instruments for options on Y than as investments on their own right. Thus, it is optimal for an investor to sacrifice the risk premium in X in order to reduce the risk of the position in Y, as an even larger exposure to Y, hedged by X, is more attractive than a diversified short position in options on X and Y. In particular, this observation suggests that naïve diversification in the sense of DeMiguel et al. (2007) is unlikely to be optimal for options.

Endnotes

1. The Cauchy-Schwarz inequality implies that $1 = E[M^*] \le E[(M^*)^2]^{1/2}$, whence $E[(M^*)^2] \ge 1$ and thus $E[M^* - (M^*)^2] \le 0$, which implies that the excess return $R = a(M^* - E[(M^*)^2])$ is positive only if a is negative. Positive values of a span the "inefficient" frontier.

2. Suppose that R_1 and R_2 are two excess returns with the same maximum Sharpe ratio. Because rescaling any of them preserves the Sharpe ratio, we may assume without loss of generality that they share the same standard deviation $\bar{\sigma} := \sigma(R_1) = \sigma(R_2)$, hence also the same mean $\bar{\mu} := E[R_1] = E[R_2]$. Let $\lambda \in (0,1)$ be small enough such that the convex combination $R = \lambda R_1 + (1-\lambda)R_2$ has non-zero variance. Then it is also an excess return with the same mean $\bar{\mu}$, and its standard deviation satisfies $0 < \sigma(R) \le \bar{\sigma}$, with the equality holding only if R_1 and R_2 have perfect positive correlation. In any other case, $E[R]/\sigma(R) > \bar{\mu}/\bar{\sigma}$, contradicting the optimality of R_1 and R_2.

3. *A priori*, the restriction to SDFs of this form is intuitively clear, as there are no additional sources of randomness in the model. *A posteriori*, the candidate optimal SDFs is also a payoff

spanned by options, and therefore has minimum variance among all SDFs, including the ones of a different form (cf. Theorem 1 (ii)).

4. In view of (8), the condition $E[\Phi_X(X)] = E[\Phi_Y(Y)]$ is equivalent to $E[\Phi_X(X)] = 1$ or $E[\Phi_Y(Y)] = 1$, because $\frac{1}{2}\int_0^\infty \Phi_X(x)p_X(x)dx + \frac{1}{2}\int_0^\infty \Phi_Y(y)p_Y(y)dy = \int_0^\infty\int_0^\infty m(x,y)p(x,y)dxdy = 1$.

5. Note that, by definition, $E[m(X)] = 1$ because q_i are densities for each $1 \le i \le n$, whence also $E[M] = 1$, because the constant random variable 1 is $\mathcal{G}$-measurable, and $M - m(X)$ is orthogonal to 1, that is $E[M - m(X)] = 0$.

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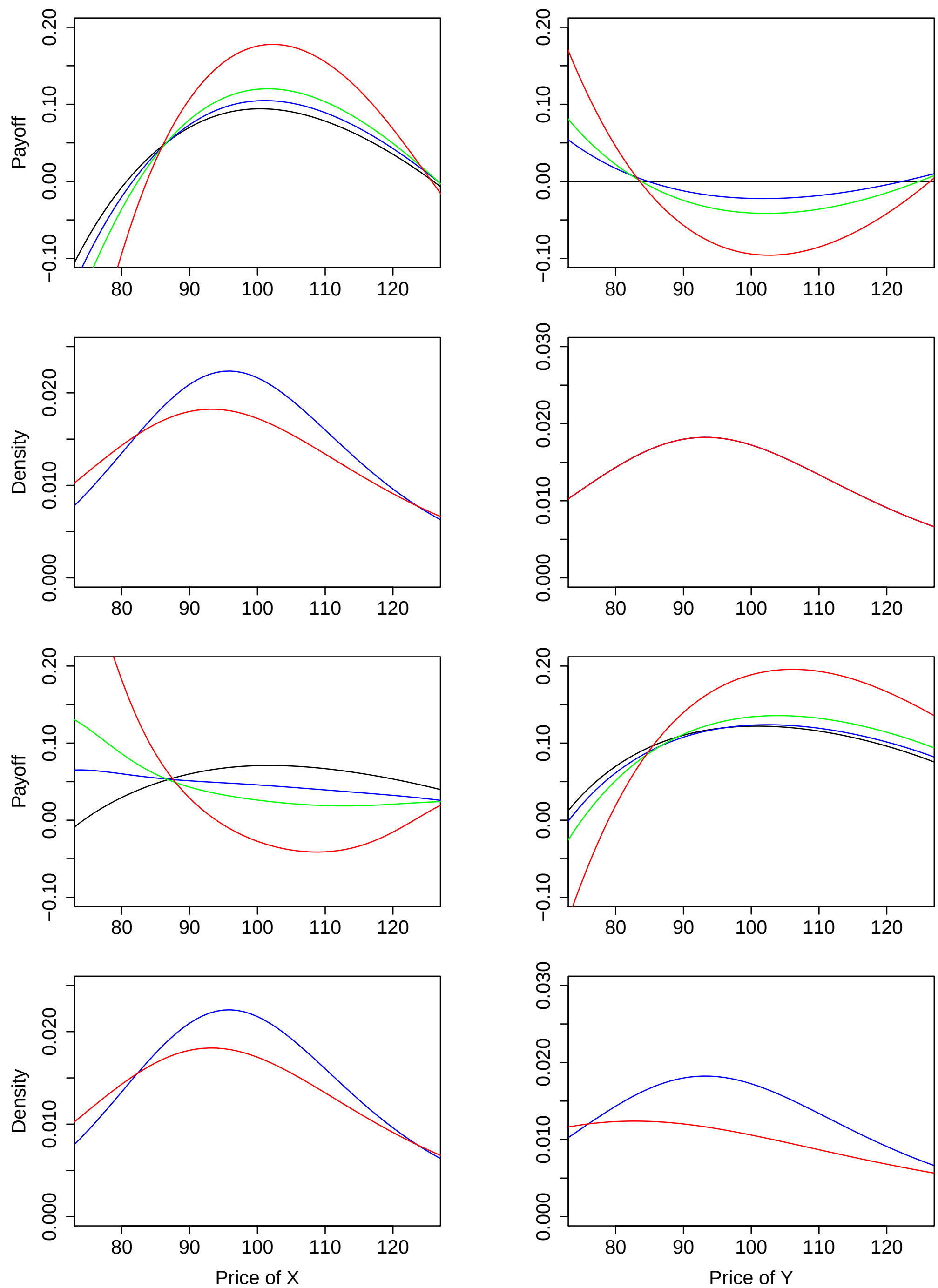

Figure 1 Optimal option payoffs Φ_X and Φ_Y (first and third rows) for two assets X, Y with densities (second and fourth rows) following the Variance Gamma model (physical: blue, risk-neutral: red). In the first two rows, only options on X have a risk premium ($\sigma_X^P = 20\%$, $\sigma_X^Q = \sigma_Y^Q = \sigma_Y^P = 25\%$). In the third and fourth row, both assets have option risk premia, with a higher premium for X than for Y, $\sigma_X^P = 20\%, \sigma_X^Q = 25\%$, $\sigma_Y^P = 25\%, \sigma_Y^Q = 40\%$. Curves in the top panel correspond to correlation $\varrho = 0$ (black), 0.60 (blue), 0.75 (green), 0.90 (red). t-copula has 4 degrees of freedom, $\nu = 0.2$. Optimal option payoffs scaled to zero price and 10% standard deviation. Options' maturity is one year. Asset prices normalized to 100.

Online Appendix

EC.1. Related Literature

In contrast to the voluminous literature on option pricing, optimally investing in options on multiple assets is a far less developed problem, even as empirical work documents large risk premia in options markets (Coval and Shumway 2001, Bakshi and Kapadia 2003, Santa-Clara and Saretto 2009, Schneider and Trojani 2015).

In a complete market, optimal option positions are implied by the condition that marginal utility be proportional to the state-price density. Carr and Madan (2001a) and Carr et al. (2001) show how to compute such optimal payoffs under different beliefs and preferences and one underlying asset. Schneider (2015) links optimal payoffs to the likelihood-ratio swap contract, and shows how to replicate such a contract with a portfolio of options. Guasoni et al. (2011) show how fund managers can use option-writing strategies to create the appearance of outperformance, even in the absence of any ability to predict returns.

In an incomplete market with stochastic investment opportunities and jumps, Liu and Pan (2003) solve in closed form the dynamic portfolio choice problem of an investor trading one stock and one out-of-the-money (OTM) put option on such stock. Eraker (2013) considers combinations of at-the-money (ATM) straddles with OTM calls and puts on the S&P 500 index and finds that they deliver Sharpe ratios close to one. Faias and Santa-Clara (2011) use a simulation approach to find optimal portfolio weights in options on the S&P 500 index and also find significant Sharpe ratios over more than a decade. While these papers focus on options on one underlying asset, Malamud (2014) develops a methodology to find "Greek-efficient" portfolios, identified in terms of higher moments of the underlying assets' returns. Roncoroni and Brik (2017) use integral-equation techniques similar to the one in this paper to obtain an optimal custom hedge for a claim on a price- and a size-linked index.

Our duality approach builds on the intuition that dates back to He and Pearson (1991) and Karatzas et al. (1991), whereby portfolio optimization in an incomplete market is equivalent to portfolio optimization in the *least favorable completion* of such a market – an idea which has proven effective also in tackling portfolio performance evaluation (Haugh et al. 2006) and option pricing (Rogers 2002, Haugh and Kogan 2004). In the evaluation of empirical asset pricing models, such a duality arises in relation to distance and discrepancy measures: for example, the minimization of the square-integral distance of Hansen and Jagannathan (1991, 1997) and the Cressie-Read discrepancy family (Almeida and Garcia 2012, 2016) are the dual counterparts of portfolio optimization with mean-variance and HARA utility, respectively.

EC.2. Proof of Theorem 1

Denote by L_p^2 the space of (equivalence classes of) Lebesgue measurable functions $f:\mathbb{R}^n \mapsto \mathbb{R}$ such that $\|f\|_p^2 := E[f(X)^2] = \int_{\mathcal{D}} |f(u)|^2 p(u)du < \infty$, which is a Hilbert space with inner product $\langle f, g\rangle_p := \int_{\mathcal{D}} f(u)g(u)p(u)du$.

EC.2.1. Existence and Uniqueness

Proof of Theorem 1 (i) Denote by $\Pi : L^2(\Omega,\mathcal{F},P) \to L^2(\Omega,\mathcal{G},P|_{\mathcal{G}})$ the conditional expectation operator. Recall that $\mathcal{M} \subset L^2(\Omega,\mathcal{F},P)$ consists of those discount factors M for which $m = \Pi(M)$ satisfies

$$\int_{\mathcal{D}_i^c} m(\xi)p(\xi)d\xi_i^c = q_i(\xi_i), \quad 1 \le i \le n.$$

Hence, $\mathcal{M}$ is convex, and non-empty by assumption. To check that it is closed, consider a sequence $(M_k)_{k=1}^{\infty} \subset \mathcal{M}$ converging to some $M_0 \in L^2(\Omega,\mathcal{F},P)$. As the sequence M_k converges, the projections $m_k = \Pi(M_k)$ also converge, hence are bounded in $L^2(\Omega,\mathcal{F},P|_{\mathcal{G}})$ norm and thus uniformly integrable. Therefore, the following limit holds in almost sure sense:

$$\int_{\mathcal{D}_i^c} m_0(\xi)p(\xi)d\xi_i^c = \lim_{k\to\infty}\int_{\mathcal{D}_i^c} m_k(\xi)p(\xi)d\xi_i^c = q_i(\xi_i), \quad 1 \le i \le n, \tag{EC.1}$$

which proves that $\mathcal{M}$ is closed in $L^2(\Omega,\mathcal{F},P)$. As any non-empty, closed, convex set in a Hilbert space has a unique element of minimum norm (Rudin 1986, Theorem 4.10), the proof is complete.

EC.2.2. Linearity

Proof of Theorem 1 (ii) For any $\psi \in L^\infty(\mathbb{R}^n)$ define

$$\widetilde{\psi}(\xi) := \psi(\xi) - \sum_{i=1}^n \underbrace{\left(\frac{p_i^c(\xi_i^c)}{p(\xi)} \int_{\mathcal{D}_i^c} \psi(\xi)p(\xi)d\xi_i^c\right)}_{=:\widetilde{\psi}_i} + (n-1)\int_{\mathcal{D}} \psi(\eta)p(\eta)d\eta \tag{EC.2}$$

and observe that $\widetilde{\psi} \in \mathcal{N}$, defined as

$$\mathcal{N} := \left\{ \widetilde{\psi} \in L_p^2 \,\middle|\, \int_{\mathcal{D}_i^c} \widetilde{\psi}(\xi)p(\xi)d\xi_i^c \equiv 0, \quad 1 \le i \le n \right\}. \tag{EC.3}$$

Indeed, a repeated application of Fubini's theorem yields

$$\begin{aligned}\int_{\mathcal{D}_i^c} \widetilde{\psi}(\xi)p(\xi)d\xi_i^c = &\int_{\mathcal{D}_i^c} \left(\psi(\xi)p(\xi) - \widetilde{\psi}_i(\xi)p(\xi)\right) d\xi_i^c \\ &- \left(\sum_{j\neq i} \int_{\mathcal{D}_i^c} \widetilde{\psi}_j(\xi)p(\xi)d\xi_i^c - (n-1)p_i(\xi_i)\int_{\mathcal{D}} \psi(\xi)p(\xi)d\xi\right) = 0.\end{aligned}$$

Furthermore, as the first and last terms in the definition of $\widetilde{\psi}$ are in L_p^2, it suffices to show that $\widetilde{\psi}_i \in L_p^2$, for $1 \le i \le n$ to conclude that $\widetilde{\psi} \in L_p^2$. Indeed, by Jensen's inequality,

$$\|\widetilde{\psi}_i\|_p^2 = \int_{\mathcal{D}} |\widetilde{\psi}_i(\xi)|^2 p(\xi)d\xi = \int_{\mathcal{D}} \frac{(p_i^c(\xi_i^c))^2}{p(\xi)^2} \left(\int_{\mathcal{D}_i^c} \psi(\xi_i^c,\xi_i)p(\xi_i^c,\xi_i)d\xi_i^c\right)^2 p(\xi)d\xi \tag{EC.4}$$

$$\le \int_{\mathcal{D}} \frac{(p_i^c(\xi_i^c))^2 p_i^2(\xi_i)}{p(\xi)} \left(\int_{\mathcal{D}_i^c} |\psi(\xi_i^c,\xi_i)|^2 \frac{p(\xi_i^c,\xi_i)}{p_i(\xi_i)} d\xi_i^c\right) d\xi \le \|\psi\|_\infty^2 \left\|\frac{p_i\, p_i^c}{p}\right\|_p^2 < \infty. \tag{EC.5}$$

Let $M^* = m^*(X)$ be the solution of Theorem 1 (i). By minimality, for any $\varepsilon > 0$ and $\psi^* \in \mathcal{N}$,

$$\int_{\mathcal{D}} |m^*(\xi) + \varepsilon\psi^*(\xi)|^2 p(\xi)d\xi - \int_{\mathcal{D}} |m^*(\xi)|^2 p(\xi)d\xi \le 0.$$

Dividing by ε and passing to the limit $\varepsilon \to 0$, and observing the same argument holds for both ε and $-\varepsilon$, the first order condition holds, i.e.,

$$\langle m^*, \psi^* \rangle_p := \int_{\mathcal{D}} \psi^*(\xi)m^*(\xi)p(\xi)d\xi = 0. \tag{EC.6}$$

Hence, for any $\psi \in L^\infty(\mathbb{R}^n)$, setting $\psi^* = \widetilde{\psi}$ it follows that

$$\langle m^*, \widetilde{\psi} \rangle_p = \left\langle \overbrace{m^* - \left(\sum_{i=1}^n \int_{\mathcal{D}_i^c} m^*(\xi)p_i^c(\xi_i^c)d\xi_i^c\right) + (n-1)}^{=:\Psi}, \psi \right\rangle_p = 0. \tag{EC.7}$$

Now, set

$$\Phi_i(\xi_i) = n\int_{\mathcal{D}_i^c} m^*(\xi)p_i^c(\xi_i^c)d\xi_i^c - (n-1), \quad 1\le i\le n.$$

Note that $\Phi_i \in L_p^1$, because

$$\int_{\mathcal{D}}\left|\frac{\Phi_i(\xi_i)+(n-1)}{n}\right| p(\xi)d\xi = \int_{\mathcal{D}}\left|\int_{\mathcal{D}_i^c} m^*(\xi_i,\eta_i^c)p_i^c(\eta_i^c)d\eta_i^c\right| p(\xi_i,\xi_i^c)d\xi_i d\xi_i^c$$
$$\le \int_{\mathcal{D}} |m^*(\xi)|p_i(\xi_i)p_i^c(\xi_i^c)d\xi = \int_{\mathcal{D}} |m^*(\xi)|\sqrt{p(\xi)}\frac{p_i(\xi_i)p_i^c(\xi_i^c)}{\sqrt{p(\xi)}}d\xi \le \|m^*\|_p^2 \left\|\frac{p_i\, p_i^c}{p}\right\|_p^2 < \infty,$$

where the second inequality follows by Cauchy-Schwarz's, and the last one by the integrability assumption and the square integrability of m^*.

Because $m^* \in L_p^2 \subset L_p^1$, it follows that $\Psi \in L_p^1$. As $L^\infty(\mathbb{R})$ is in duality with L_p^1, and (EC.7) holds for all $\psi \in L^\infty(\mathbb{R})$, it follows that $\Psi = 0$ a.s. in p, that is

$$m^*(\xi) = \frac{1}{n}\sum_{i=1}^n \Phi_i(\xi_i) \quad p\text{-a.s.} \tag{EC.8}$$

EC.2.3. Identification

Theorem 1 (i) and (ii) show that a unique minimizing discount factor exists, and its linear structure allows an interpretation as option portfolios. However, the existence proof was not constructive. This section characterizes the solution in terms of a constrained, vector-valued integral equation, which is conveniently solved by discretization.

Proof of Theorem 1 (iii) By the proof of Theorem 1 (ii), the unique minimal discount factor $M^* = m^*(X)$, where $m^* \in L_p^2$ is of the form (EC.8). Therefore, the n constraints in (20) imply the validity of the n equations (21). It remains to establish the constraints (22). Note that

$$\int_{I_i}\Phi_i(\xi_i)p_i(\xi_i)d\xi_i = n\int_{I_i}\left(\int_{\mathcal{D}_i^c} m^*(\xi)\, p_i^c(\xi_i^c)d\xi_i^c\right)p_i(\xi_i)d\xi_i - (n-1)$$
$$= \int_{I_i}\left(\int_{\mathcal{D}_i^c}\left(\sum_{j=1}^n \Phi_j(\xi_j)\right)p_i^c(\xi_i^c)d\xi_i^c\right)p_i(\xi_i)d\xi_i - (n-1) \qquad 1\le i\le n,$$

which simplifies to a system of n linear equations for $\vartheta_j := \int_{I_i}\Phi_j(\xi_j)p_j(\xi_j)d\xi_j$, $1\le j\le n$, i.e.,

$$\sum_{j:j\ne i}\vartheta_j = (n-1), \qquad 1\le i\le n,$$

which has the unique solution $\vartheta_j = 1$ for all $1\le j\le n$.

EC.2.4. Performance

Proof of Theorem 1 (iv) The efficient frontier is spanned by excess returns of the form $a(M^* - E^Q[M^*])$, with $a<0$ (cf. note 1). Because $M^* \in \mathcal{R}$ and $M^* \in \mathcal{M}$, $E[(M^*)^2]$ represents the price of M^* and thus the sum of the prices of $\Phi_i(X_i)$ divided by n. As a result, because the risk-neutral density for options on X_i is q_i,

$$E[(M^*)^2] = \frac{1}{n}\sum_{i=1}^n \int_{I_i} \Phi_i(\xi_i) q_i(\xi) d\xi_i,$$

and, as $E[M^*]=1$, the maximal Sharpe ratio is

$$\frac{|E[M^*] - E^Q[M^*]|}{\sqrt{\mathrm{Var}^Q(M^*)}} = \sqrt{E^Q[M^*]-1} = \sqrt{\frac{1}{n}\sum_{i=1}^n \int_{I_i} \Phi_i(\xi_i) q_i(\xi_i) d\xi_i - 1}.$$

EC.2.5. Regularity

The above arguments prove that, $\mathcal{M} \neq \varnothing$, then a unique minimal discount factor $M^* = m^*(X)$ exists, but they are silent on the regularity of the map $x \mapsto m^*(x)$. This section shows that, if the physical and risk-neutral densities are regular, so are the asset-specific payoffs Φ_i and hence the optimal payoff m^*.

Proof of Theorem 1 (v) As all $(p_i)_{1\le i\le n}$ are strictly positive, the integral equation for Φ_i is equivalent to

$$\Phi_i(\xi_i) = \frac{nq_i(\xi_i)}{p_i(\xi_i)} - \sum_{j\neq i} \frac{1}{p_i(\xi_i)} \int_{\mathcal{D}_i^c} \Phi_j(\xi_j) p(\xi) d\xi_i^c. \tag{EC.9}$$

By assumption, $q_i(\xi_i)$ is continuously differentiable. It suffices to show that the same regularity holds for the functions

$$g_{i,j} : \xi_i \mapsto \int_{\mathcal{D}_i^c} g(\xi_j) p(\xi) d\xi_i^c, \qquad \text{for all } j \neq i.$$

where the integrand $g \in L^2_p$. Once this is shown, then the claim follows by setting $g=1$ to obtain the regularity of the marginal densities p_i, and setting $g=\Phi_j$, $j\neq i$, to obtain the regularity of the sum in (EC.9). Note that, by assumption, there exists $\alpha \in (1/2, 1]$ such that locally in ξ_i, the integrand

$$|g(\xi_j)\partial^\beta_{\xi_i} p(\xi)| \le C |g(\xi_j)| (p_i^c)^\alpha$$

admits an upper bound, independent of ξ_i and integrable, in view of the Cauchy-Schwarz inequality,

$$\int_{\mathcal{D}_i^c} |g(\xi_j)|(p_i^c)^\alpha d\xi_i^c \leq \int_{\mathcal{D}_i^c} |g(\xi_j)|^2 p_i^c d\xi_i^c \times \int_{\mathcal{D}_i^c} p_i^c(\xi_i^c)^{2\alpha-1} d\xi_i^c < \infty.$$

Hence, by dominated convergence and the continuity of the integrand, also $g_{i,j}$ is k times continuously differentiable.

Remark EC.1. Trading options on a larger set of underlying assets generates at least the same Sharpe ratio as trading options on a smaller set of assets because any option strategy on the smaller set retains the same performance in the larger market. The Sharpe ratio in the larger market is strictly larger if and only if the options in the larger market are not priced correctly by the minimal SDF in the smaller market.

To see this point, denote by M_k^* and M_{k+1}^* the minimal SDFs in the markets with the first k and $k+1$ assets, respectively. Because M_{k+1}^* must satisfy one extra marginal constraint, its second moment is minimal in a smaller class, hence greater or equal than that of M_k^*, i.e., $E[(M_{k+1}^*)^2] \geq E[(M_k^*)^2]$. By Theorem 1, such second moments represent the squared maximal Sharpe ratios in the respective markets. If they are equal, then $M_{k+1}^* = M_k^*$ by the uniqueness of the minimal SDF, as M_{k+1}^* is (trivially) an SDF for the first k assets. Vice versa, if $E[(M_{k+1}^*)^2] > E[(M_k^*)^2]$, then denote by $N = M_{k+1}^* - M_k^*$, which is a payoff involving all $k+1$ assets, and has price $E[NM_{k+1}^*]$. Note also that $E[NM_k^*] = 0$ because M_k^* is minimal among all SDF that price the first k assets. Thus, it follows that

$$\begin{aligned} E[NM_{k+1}^*] - E[NM_k^*] =& E[N^2] = E[N(M_{k+1}^* - M_k^*)] = E[NM_{k+1}^*] \\ =& E[(M_{k+1}^*)^2] - E[M_{k+1}^* M_k^*] = E[(M_{k+1}^*)^2] - E[(M_k^*)^2] > 0, \end{aligned}$$

which means that M_k^* prices N incorrectly.

EC.3. Proofs of Statements in Section EC.4

Proof of Proposition EC.1 Rearrange the two integral equations (12)–(13) as

$$\frac{\Phi_X(x)}{2} = -\frac{1}{2}\int_{I_Y} \Phi_Y(y)\frac{p(x,y)}{p_X(x)}dy + \frac{q_X(x)}{p_X(x)}, \quad \frac{\Phi_Y(y)}{2} = -\frac{1}{2}\int_{I_X} \Phi_X(x)\frac{p(x,y)}{p_Y(y)}dx + \frac{q_Y(y)}{p_Y(y)}$$

and note that

$$m^*(x,y)=\frac{\Phi_X(x)+\Phi_Y(y)}{2}=-\frac{1}{2}\int_{I_Y}\Phi_Y(y)\frac{p(x,y)}{p_X(x)}dy+\frac{q_X(x)}{p_X(x)}-\frac{1}{2}\int_{I_X}\Phi_X(x)\frac{p(x,y)}{p_Y(y)}dx+\frac{q_Y(y)}{p_Y(y)}$$
$$\geq -\frac{\gamma}{2}\int_{I_Y}\Phi_Y(y)p_Y(y)dy-\frac{\gamma}{2}\int_{I_X}\Phi_X(x)p_X(x)dx+\alpha/2+\beta/2=-\gamma+\alpha/2+\beta/2>0, \quad \text{a.e.}$$

as the last equality follows from the conditions $\int \Phi_Y(y)p_Y(y)dy=\int \Phi_X(x)p_X(x)dx=1$.

Proof of Proposition EC.2 Let $M\in\mathcal{M}_+$. By non-negativity of M,

$$E[M\hat{m}(X,Y)]\geq E[M(\Phi_X(X)+\Phi_Y(Y))/2]=E[\hat{m}(X,Y)(\Phi_X(X)+\Phi_Y(Y))/2]=E[\hat{m}^2(X,Y)],$$

where, for the first equality, note that both $\hat{m}(X,Y)$ and M price the individual options on each underlying asset. It follows that

$$E[(M-\hat{m}(X,Y))\hat{m}(X,Y)]\geq 0,$$

and therefore

$$E[M^2]=E[\hat{m}^2(X,Y)]+2E[(M-\hat{m}(X,Y))\hat{m}(X,Y)]+E[(M-\hat{m}(X,Y))^2]\geq E[\hat{m}^2(X,Y)].$$

To prove the second part, note that $M\in\mathcal{M}_+\subset\mathcal{M}$, whence $\mathcal{M}\neq\varnothing$. Thus, Theorem 1 implies that a minimal discount factor $M^*\in\mathcal{M}$ uniquely exists and is of the "linear" form $(\Phi_X^*(X)+\Phi_Y^*(Y))/2$. Hence $\hat{m}(X,Y)\neq M^*$, unless $P[\Phi_X(X)+\Phi_Y(Y)\geq 0]=1$.

Proof of Theorem EC.1 The uniqueness of Φ^ε follows as in the case of Theorem 1 (ii). Furthermore, direct substitution confirms that $f=(f_X,f_Y)$ solves the unperturbed system of integral equations, (9)-(11). Because f_X, $f_Y\in L_p^2$ by assumption, it follows that $\hat{M}\in\mathcal{M}$. By Theorem 1, the unique minimal SDF M^* exists and has the form $M^*=\frac{\Phi_X(X)+\Phi_Y(Y)}{2}$, where $\Phi=(\Phi_X,\Phi_Y)$ solves the unperturbed system of integral equations, (9)-(11). Again, by Theorem 1, the solution to these equations is unique, whence $f_X=\Phi_X$, $f_Y=\Phi_Y$ almost everywhere. Strict positivity follows because $\hat{M}=(1-\varepsilon)M_\varepsilon^*+\varepsilon\geq\varepsilon>0$.

EC.4. No Arbitrage and Positive SDFs

The appeal of the minimal SDF obtained in Theorem 1 is that it is the average of separate functions of the underlying assets, and that it is identified by a system of linear equations. The drawback of such simplicity is that such SDF is not guaranteed to be strictly positive, potentially leading to arbitrage opportunities if used to price contingent claims with nonlinear dependence on multiple assets. This section discusses two approaches to obtain a strictly positive SDF: (a) formulating sufficient conditions for the minimal discount factor in Theorem 1 to be strictly positive; and (b) adding a positivity constraint to the optimization problem and obtain a strictly positive SDF by perturbation. To ease notation, the bulk of this section concentrates on two underlying assets, the multivariate setting being analogous.

EC.4.1. Criteria for Positivity

First, as the SDF in Theorem 1 separates in the cross section, its strict positivity is straightforward to check, because

$$\inf_{\xi} m^*(\xi) = \inf_{\xi} \frac{1}{n} \sum_{i=1}^{n} \Phi_i(\xi_i) = \frac{1}{n} \sum_{i=1}^{n} \inf_{\xi_i} \Phi_i(\xi_i). \tag{EC.10}$$

Thus, the strict positivity of the SDF is equivalent to the strict positivity of the sum of the infima of its additive components.

Equation (EC.10) is sufficient to determine whether the infimum of the SDF is positive from the solution of the integral equations. The following, more restrictive, criterion guarantees the same conclusion a priori, i.e., without solving the equations.

Proposition EC.1. *Let the assumptions of Theorem 1 hold, and assume that for some* $1 \le \gamma < (\alpha+\beta)/2$, *with* $0 \le \alpha, \beta \le 2$

$$\frac{q_X(x)}{p_X(x)} \ge \alpha/2, \quad \frac{q_Y(y)}{p_Y(y)} \ge \beta/2, \quad \textit{and} \quad \frac{p(x,y)}{p_X(x)p_Y(y)} \le \gamma \quad \textit{for a.e. } x,y. \tag{EC.11}$$

Then $m^*(x,y) > 0$ *almost everywhere.*

The interpretation of this result is that when marginal risk premia are large enough and the assets are not too dependent, absence of arbitrage follows. (Note that independence fulfills the assumption with $\gamma = 1$ and risk neutrality with $\alpha = \beta = 2$.) In general, the condition requires that marginal state price densities, which represent the price of Arrow-Debreu securities concentrated in a small interval, are uniformly bounded from below, which means that no state of nature is "too cheap" – risk premia are high. Vice versa, the second part of the condition is equivalent to $P(Y \in dy|X \in dx) = p(x,y)/p_X(x) \leq \gamma p_Y(y) = \gamma P(Y \in dy)$, which means that information on the value of X cannot increase the density of Y by a factor higher than γ, i.e. dependence is not too strong. (Of course, the same reasoning applies if X and Y are exchanged.)

EC.4.2. The Hansen-Jagannathan approach

To obtain positive discount factors, Hansen and Jagannathan (1991) propose to consider SDFs obtained as positive parts of portfolios, which are interpreted as call options on several assets. As they acknowledge, a limit of this approach is that such SDFs may not be unique and may also vanish with positive probability, thereby leading to potential arbitrage opportunities (if used to price claims on both assets). In the present setting, this observation is made precise with the following statement:

Proposition EC.2. *Under the assumptions of Theorem 1. Any SDF of the form* $\hat{m}(X,Y) = \frac{1}{2}\left(\Phi_X(X) + \Phi_Y(Y)\right)_+$ *is minimal in* $\mathcal{M}_+ := \{m \in \mathcal{M} \mid m \geq 0\} \subseteq \mathcal{M}$*. Only if* $\Phi_X(X) + \Phi_Y(Y) \geq 0$ *a.s. such an SDF is minimal in* $\mathcal{M}$ *and closes the duality gap.*

We describe a perturbation argument that yields a strictly positive SDF starting from an SDF that is merely positive. This is the case, for example, for the minimal SDF M_+ in the constrained set $\mathcal{M}_+$. (Indeed, M_+ exists by the same argument as in the proof of Theorem 1. If it were strictly positive, then the positivity constraint would not be binding, and therefore – by uniqueness – it would coincide with the unconstrained M^* in Theorem 1. Thus, whenever positivity is binding,

$P(M_+ = 0) > 0$.) Then, a strictly positive SDF may be obtained through the following perturbation argument. For $\varepsilon \in (0,1)$ define

$$q_X^\varepsilon := \frac{1}{1-\varepsilon}(q_X - \varepsilon p_X), \quad q_Y^\varepsilon := \frac{1}{1-\varepsilon}(q_Y - \varepsilon p_Y),$$

and consider the perturbed integral equations with the usual constraint

$$\frac{1}{2} f_X(x) p_X(x) + \frac{1}{2}\int_{I_Y} f_Y(y) p(x,y) dy = q_X^\varepsilon(x), \tag{EC.12}$$

$$\frac{1}{2} f_Y(y) p_Y(y) + \frac{1}{2}\int_{I_X} f_X(x) p(x,y) dx = q_Y^\varepsilon(y), \tag{EC.13}$$

$$\int_{I_X} f_X(x) p_X(x) dx = \int_{I_Y} f_Y(y) p_Y(y) dy = 1. \tag{EC.14}$$

Theorem EC.1. *Under the assumptions of Theorem 1, and assume that for some $\varepsilon > 0$ the integral equations* (EC.12)-(EC.14) *have a solution $\Phi^\varepsilon := (\Phi_X^\varepsilon, \Phi_Y^\varepsilon) \subset L_p^2$ such that $\Phi_X^\varepsilon, \Phi_Y^\varepsilon \geq 0$. Then Φ^ε is the unique solution, and the SDF defined as*

$$\hat{M} = (1-\varepsilon)\frac{\Phi_X^\varepsilon(X) + \Phi_Y^\varepsilon(Y)}{2} + \varepsilon$$

is greater than $\varepsilon > 0$ and satisfies Theorem 1 (i)-(iv).

EC.5. Variance Gamma Model

The Variance Gamma model specifies the asset price as $X_t = X_0 e^{\omega t + Z_t(\sigma,\nu,\theta)}$, where the random variable $Z_t(\sigma,\nu,\theta)$ is identified by the characteristic function

$$E[e^{iuZ_t}] = (1 - i\theta\nu u + \frac{\sigma^2}{2}u^2\nu)^{-t/\nu}, \quad u \in \mathbb{R} \tag{EC.15}$$

hence has mean $E[Z_t(\sigma,\nu,\theta)] = \theta t$ and variance $\mathrm{Var}(Z_t(\sigma,\nu,\theta)) = (\theta^2\nu + \sigma^2)t$. Furthermore, this distribution corresponds to the marginal law of a Levy process where the jump size has the density (Madan et al. 1998)

$$k_Z(x) = \frac{e^{\theta x/\sigma^2}}{\nu|x|} e^{-\frac{\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}}{\sigma}|x|}. \tag{EC.16}$$

Henceforth, for the sake of simplicity set $\theta = 0$, which means that positive and negative jumps of the same magnitude are equally likely and that the variance of the log price at time t is simply $\sigma^2 t$.

Such an assumption abstracts from the asymmetry of the volatility "smile" observed in practice, while focusing on the tension between physical and risk-neutral volatility as the main determinant of option positions.

A full specification of the model consists of the joint law of assets' returns under the physical probability P and their marginal distributions under the risk-neutral probability Q. The risk-neutral marginal under Q of the i-th asset's return is a Variance Gamma law with parameters $(\sigma_i^Q)_{i=X,Y}$, for a fixed shape parameter ν, assumed common to all assets to keep the volatility parameter as the main determinant of option prices. In addition, the risk-neutrality condition requires that $\omega_i^Q = -\frac{1}{\nu}\log(1-\nu(\sigma_i^Q)^2/2)$, completing the specification of risk-neutral marginals.

The joint physical law is described through the separate specification of physical marginals and their copula. The P marginals are also assumed of the form (EC.15), with different variance parameters $(\sigma_i^P)_{i=X,Y}$ but with the same shape parameter ν used for the risk-neutral dynamics. Furthermore, assume that assets do not carry risk-premia even under the physical measure, whence $\omega_i^P = -\frac{1}{\nu}\log(1-\nu(\sigma_i^P)^2/2)$. Note that such an assumption is not dictated by absence of arbitrage: instead, its purpose is to ensure that all the demand for options in the model is driven by option risk-premia rather than by the motive to gain exposure to the asset's risk premium through options. Put differently, such a condition removes assets' demand from options' demand.

Finally, the dependence among the returns follows a bivariate t-copula with parameters d (the degrees of freedom in the t distribution) and correlation ϱ. Thus, the joint distribution is

$$P(X \le x, Y \le y) = T_{\varrho,d}(T^{-1}(F_X(x)), T^{-1}(F_Y(y))), \qquad \text{(EC.17)}$$

where $T_{\varrho,d}$ is distribution function of the standard t-copula with parameters ϱ, d, while T is the distribution function of the t-distribution with d degrees of freedom, and F_X, F_Y are the distribution functions of the Variance Gamma marginals X and Y.

EC.5.1. Performance and the Limits of Naïve Optimization

Theorem 1 implies that, in general, the optimal asset-specific option payoff depends on the risk-neutral densities of all assets. Yet, a natural question is whether assets' interdependence (the dependence of each payoff on the risk-neutral densities of other assets) is a second-order effect, which perhaps could be bypassed by a two-stage optimization approach that treats options on each asset in isolation, as follows. First, find the optimal option payoff $\Psi_i(X_i)$ on each asset X_i, as if options on other assets did not exist. Second, construct a portfolio of the form $\sum_{i=1}^n w_i \Psi_i(X_i)$, by choosing the weights w_i that maximize the global Sharpe ratio.

While such a divide-and-conquer approach is intuitively appealing, its performance is significantly worse than the solution to the joint optimization problem formulated in the paper: assets' interdependence is a first-order effect. To see this point clearly, consider the situation described in Figure 1: as options on the asset Y have zero risk-premia, the optimal payoff of options in Y is identically zero. Thus, the two-stage optimization procedure trivially yields the optimal payoff of options on X, while all hedging gains – a prominent feature of global optimization – are lost.

The broader message of this example is that, because the asset-specific option payoffs are entirely determined by their internal risk-return tradeoffs, they are ill-suited for hedging, which is an intrinsically global problem. In addition, two-stage optimization performs most poorly precisely for the cheapest hedging instruments, that is when the risk-premium to be sacrificed in hedging is lowest (zero, in the above example).

Figure EC.1 displays the performance of optimal option strategies against correlation, and shows how high (positive or negative) correlation lifts the Sharpe ratio. This effect is present when options on both asset carry risk premia (right panel) and when options on the second asset have no risk premium (left panel), hence are merely hedging instruments.

For low correlation, the performance of the optimal portfolio of Theorem 1 (black line) is marginally better than an equally weighted combination of the asset-specific portfolios (green), which is optimal with independent assets (Example 2). Indeed, low correlation implies that hedging opportunities are limited and assets' interdependence insignificant. As correlation becomes stronger, the situation reverses: the performance of the global optimizer and the zero-correlation portfolio diverge quickly. When options on both assets have risk premia (right panel), the two-stage

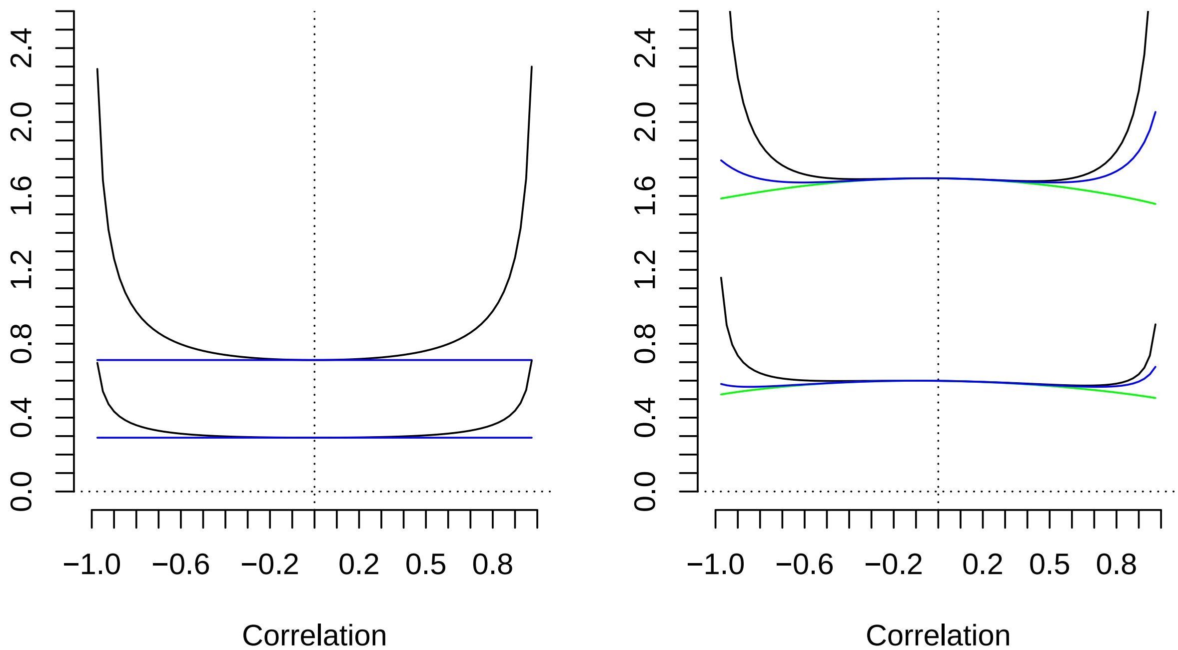

Figure EC.1 Sharpe Ratios (vertical axis) against correlation (horizontal) corresponding to Figure 1, with annual (lower) and monthly (upper) horizon, for the optimal portfolio (black), the optimal combination of asset-specific portfolios for given correlation between the assets' return (blue), and the combination of asset-specific portfolios for zero correlation (green).

optimizer (blue line) performs better than the zero-correlation portfolio, but still significantly worse than the global optimizer.

Importantly, keeping physical and risk-neutral volatilities constant, the optimal performance rises dramatically as the trading frequency increases from annual to monthly, more than doubling the Sharpe ratio across a range of typical positive correlations, with or without risk premia in the second asset. This phenomenon may seem counterintuitive at first, as the processes considered have independent, identically distributed returns, which hint at a constant Sharpe ratio. Upon closer inspection, however, the Sharpe ratio here does not result from exposure to the assets themselves (assets' risk premia are assumed to be zero), but rather from exposure to the nonlinearity in option payoffs (Gamma, in traders' jargon).

A lower trading frequency (i.e., a longer investment period) reduces an investor's ability to gain nonlinear exposure because, even if a portfolio of options is optimal at the beginning of the period,

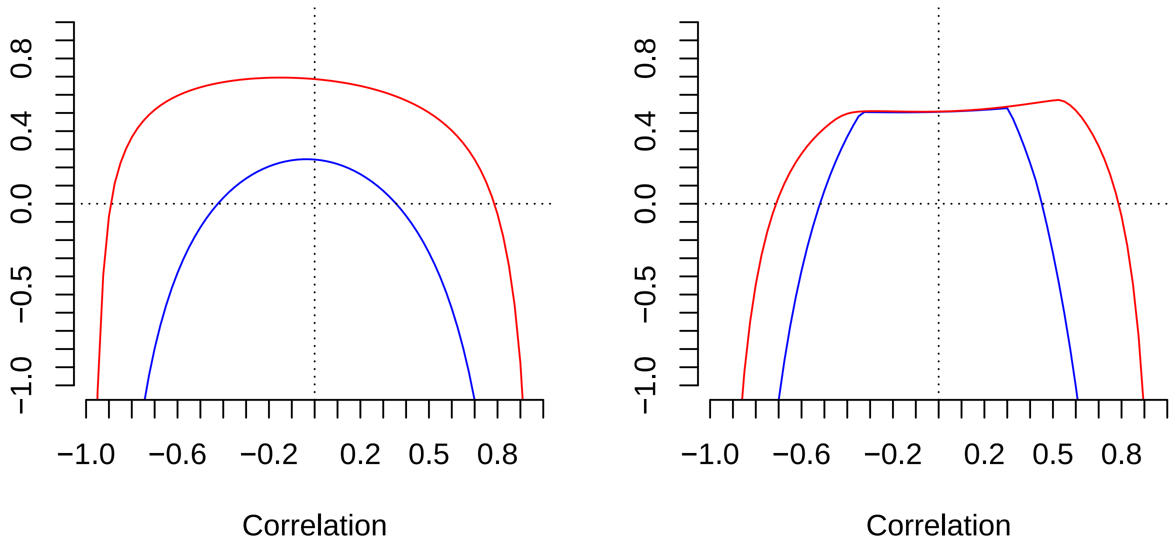

Figure EC.2 Minimum value of the minimal SDF (vertical axis) against correlation (horizontal) corresponding to Figure 1, with monthly (lower, blue) and annual (upper, red) horizon.

once the asset price moves in either direction, most of the options become either firmly in the money or firmly out of the money, thereby losing much of their nonlinearity for the rest of the period. In addition, once the asset price moves, the option portfolio gains nonzero exposure to the asset itself (i.e., Delta), which has zero risk premium, thereby adding idiosyncratic risk that reduces the Sharpe ratio. Vice versa, with a higher trading frequency the investor can reset the option payoff after typically smaller variations in asset prices, restoring the rewarded nonlinear exposure that generates return while neutralizing linear exposure (Delta hedging) that only adds unrewarded risk – ultimately boosting the Sharpe ratio.

This phenomenon arises for two reasons: First, as the option maturity grows, the second moment of the state-price density increases at varying rates, and the trading frequency scales the average rate for a given maturity to the unit interval. (The ideal trading frequency depends on the model and the parameters considered.) Second, as the trading frequency increases, so does the hedging frequency (with nonzero correlation), reducing risk even further.

The overall message is that access to options on several assets significantly increases the overall performance of an option portfolio, even when some of these options are unattractive *per se*, because they play a critical role as hedging instruments. Performance gains are especially significant at short horizons and for options on highly correlated assets.

EC.5.2. SDF Positivity and Arbitrage

To understand when the SDF implied by Theorem 1 is consistent with the absence of arbitrage, Figure EC.2 examines the minimal value of the SDF in equation (EC.10) as asset correlation varies. The SDF is positive when correlation is weak: for the left panel, corresponding to Figure 1, the absence of arbitrage follows from the absence of risk premia on the second asset ($\frac{q_Y}{p_Y} = 1$). Indeed, the joint specification in (EC.17) implies that zero correlation corresponds to independence, and in this case the minimal SDF reduces to $m^*(x,y) = \frac{q_X(x)}{p_X(x)} + \frac{q_Y(y)}{p_Y(y)} - 1 = \frac{q_X(x)}{p_X(x)} > 0$.

As correlation becomes stronger, the minimum value of the SDF drops below zero first with an annual horizon and eventually even with a monthly horizon for near-perfect correlation. To understand this phenomenon, consider the limit case of perfect (positive or negative) correlation: in this case, the assets' returns are linked by an affine transformation, which in turn identifies one marginal (e.g., Y) in terms of the other (e.g., X). However, if the pre-specified risk-neutral marginal of Y is not compatible with any such transformation (and in general, it is not), then an arbitrage opportunity arises by selling a claim on Y for the price implied by its risk-neutral marginal, while hedging it perfectly for the price implied by the risk neutral marginal of X (or vice versa, depending on which price is higher). Put differently, while two marginals can always be joined through independence, they may not be compatible with an arbitrarily high correlation value. (This is true even for lognormals, as pointed out by Embrechts et al. (2002).) When the prescribed marginals are incompatible with the prescribed correlation, any kernel that joins them cannot be strictly positive, otherwise it would be a joint probability law.

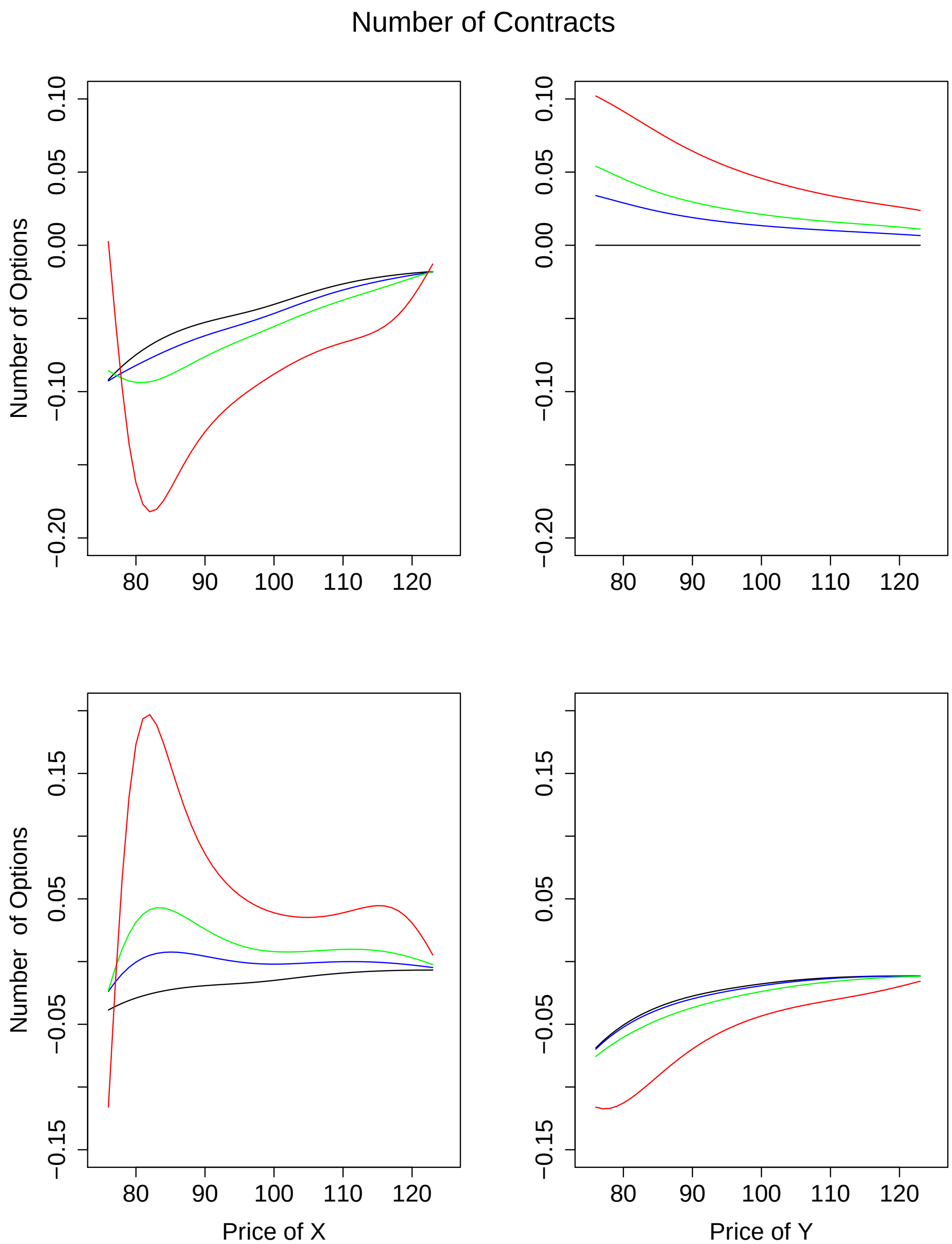

Figure EC.3 Number of contracts (vertical axis) for each strike price (horizontal) in the first (left) and the second (right) asset, corresponding to the parameters of Figure 1. The number of contracts is the same for calls and puts, interchangeably, in view of put-call parity.

EC.5.3. Number of Contracts

A legitimate concern for the implementation of the trading strategies implied by the main result is whether the number of contracts that is required for each strike is actually available in the market.

Figure EC.3 displays the number of contracts (determined up to an arbitrary multiplicative factor, as for optimal payoffs) corresponding to the parameter values in Figure 1. (Note that the number of contracts for a fixed strike is the same for calls and puts, in view of put-call parity.) In most settings of interest, and consistent with the qualitative features of optimal payoffs, the number of contracts moderately increases in absolute value as the strike declines, both for the short positions designed to generate returns, and for their hedging long positions.

The exception to this rule of thumb is the extreme configuration that combines two highly correlated assets (red line, corresponding to 90% correlation) with sharply different risk premia for option prices (in the bottom panels $\sigma_X^Q - \sigma_X^P = 5\%$ while $\sigma_Y^Q - \sigma_Y^P = 15\%$): in such unrealistic scenario, the number of contracts would be hump-shaped, with the number of contracts peaking around 15% below the money, less than one standard deviation from the initial asset price, in the annual horizon considered.

EC.6. Conclusion

We have introduced a method to compute the combination of options, each of them written on one of many assets with several available strikes, as to maximize the Sharpe ratio of the option portfolio. The method identifies optimal payoffs as solutions to a system of integral equations that, under appropriate discretizations of physical and risk-neutral probabilities, reduces to a matrix equation. In a concrete model with correlated assets, optimal payoffs display significant interactions, with the global optimal option portfolio departing significantly from optimal asset-specific payoffs, and sometimes even entailing reverse positions.

This paper focuses on investors who trade options on several underlying assets with a common expiration. An important future development is to extend our analysis to include multiple expirations. The framework in this paper may also be adapted to investigate the Ross (2015) recovery of the physical measure from option prices, subject to additional information on either preferences or market moments, in the spirit of Schneider (2015) and Schneider and Trojani (2018).

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