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Abstract

In this paper we consider non-autonomous quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda |x|^\delta f(u) & \text{in } B_1(0) \\ u = 0 & \text{in } \partial B_1(0), \end{cases}$$

where $f : \boldsymbol{R} \to [0, \infty)$ is a nonnegative C^1-function with $f(0) = 0$, $f(U) = 0$ for some $U > 0$, and f is subcritical in 0 and in U with respect to the critical exponent. Assuming some hypotheses at the zeros and at infinity we study existence and multiplicity of positive radial solutions with respect to the parameter λ. In addition, we study the bifurcation diagrams with respect to the maximum over the eventual solutions and the parameter $\lambda > 0$.

1 Introduction

This article is devoted to the study of positive radial solutions of the following semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda |x|^\delta f(u) & \text{in } B_1(0) \\ u = 0 & \text{in } \partial B_1(0), \end{cases} \tag{1}$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^n$, $n > 2$, $\delta > -2$, B_1 is the unit ball, and f is a C^1 function such that $f(0) = 0$, $f(U) = 0$ for some $U > 0$, and it is either linear or super-linear in u for small values. In the case $f(u) = |u|^{p-1}u$, where $p > 1$ equation (1) is traditionally called the Hénon (resp., Hardy or Lane-Emden) equation for $\delta > 0$ (resp., $\delta < 0$, $\delta = 0$).

Concerning the $\delta > 0$ case, in 1973, M. Hénon [19] introduced the equation (1) in the context of a concentric shell model used to investigate numerically the stability of spherical steady states of stellar systems with respect to spherical disturbances. Since then the equation is called in the literature as Hénon equation. Later, in 1982, W. M. Ni [28] wrote the first rigorous study, where he showed that the presence of the radial weight $|x|^\delta$ affects the critical exponent. As a matter of fact, it modifies the Pohozaev identity and produces the new critical exponent $\frac{2(n+\delta)}{n-2}$.

Problems with superlinear nonlinearities which have different behaviors at the origin and at infinity have been extensively studied. For the Laplacian, see for example [1, 7, 8, 23, 24, 6, 16]. For the $p-$Laplacian, see for example [2, 30, 17, 14, 15, 13]. In most of these works, the nonlinearity is strictly

positive for $u > 0$; however, the characteristics of the problem are quite different when the nonlinearity has a positive zero. In the nice work [26], this type of problems is considered for the Laplacian operator and a nonlinearity f that is independent of x, satisfying $f(0) \geq 0$, $f(U) = 0$, and which is positive and superlinear for $u > U > 0$. Using topological degree arguments and under additional technical conditions which ensure a priori bounds, it is shown that there exist two positive solutions of Problem (1). It is further shown that one solution lies strictly below U, while the other has a maximum greater than U. This type of problems was also studied in [27], where again the existence of two positive solutions of Problem (1) was shown. One solution was obtained as a minimal positive solution, while the other was obtained as the limit of a gradient flow whose starting point is properly chosen. This strategy allows showing that certain technical hypotheses given in [26] can be weakened; moreover, a better insight on the behavior with respect to λ of the minimal solution of [26] can be obtained. In this direction, some progress have been made, we cite the works [20, 22] for the $p-$Laplacian case, and [3, 18] for the semilinear case, who have observed that the behavior near the zeros of the nonlinearity is relevant to construct solutions for large λ.

Here, by considering a dynamical system approach, we study the existence and multiplicity of radial solutions in the non-autonomous case and when the nonlinearity f is positive, but it is null in 0 and U. In addition, we study the asymptotic behavior of the solutions with respect to the parameter λ. As far as we know, this is the first attempt to obtain multiplicity results in the weighted case, with a nonlinearity which have a positive zero, compare with [3, 18, 20, 21] and references therein.

Since we just deal with radial solutions we will indeed consider the following singular O.D.E.

$$(u'(r)r^{n-1})' + \lambda f(u)r^{\delta+n-1} = 0\,, \tag{2}$$

where, abusing the notation, we have set $u(r) = u(x)$ for $|x| = r$, and $'$ denotes differentiation with respect to r. We are interested in classical solutions i.e. in solutions $u(r, d)$ of (2) satisfying the following initial condition:

$$u(0, d) = d \geq 0 \qquad u'(0, d) = 0 \tag{3}$$

together with the border condition $u(1, d) = 0$. The prototype of non-linearity f we are interested in is

$$f(u) = u^{q-1}|1 - u|^a \tag{4}$$

where $q > 2$, and either $2 \leq a < 2^*(\delta) = \frac{2(n+\delta)}{n-2}$ or $q + a < 2^*(\delta) = \frac{2(n+\delta)}{n-2}$.

We collect here the main assumptions used in the paper:

A There are $\sigma > 0$, $U > 0$ such that $f(U) = 0$, and $f(u) > 0$ for $u \in (0, U + \sigma]\setminus \{U\}$.

B $f(u) > 0$ for $0 < u < U$.

F0 There are $2 \leq q_s < 2^*(\delta)$, $\hbar_0 > 0$, $c_s > 0$ such that

$$f'(u) = (q_s - 1)c_s u^{q_s-2} + o(u^{q_s-2+\hbar_0})\,, \quad \text{as } u \to 0. \tag{5}$$

F1 There are $2 \leq Q_s < 2^*(\delta)$, $\hbar_1 > 0$, $C_s > 0$ such that

$$f'(U + u) = (Q_s - 1)C_s u^{Q_s-2} + o(u^{Q_s-2+\hbar_1})\,, \quad \text{as } u \to 0. \tag{6}$$

F2 There are $2 < q_u < 2^*(\delta)$, $\hbar_2 > 0$, $c_u > 0$ such that such that

$$f'(u) = (q_u - 1)c_u u^{q_u-2} + o(u^{q_u-2-\hbar_2})\,, \quad \text{as } u \to \infty. \tag{7}$$

As a consequence of our main results we obtain the following.

Theorem 1.1. *Assume* ***A, F0, F1****, then there is $\lambda^* > 0$ such that (1) admits at least 3 radial positive solutions for $\lambda > \lambda^*$.*

Theorem 1.2. *Assume* ***B, F0, F2****, then there is $\lambda^* > 0$ such that (1) admits at least 3 radial positive solution for $\lambda > \lambda^*$, at least 2 radial positive solutions for $\lambda = \lambda^*$, at least 1 radial positive solution for $0 < \lambda < \lambda^*$.*

Theorem 1.3. *Assume f satisfies (5) with $q_s = 2$,* ***A, F1****. Then there is $\lambda^* > 0$ such that (1) admits at least 2 radial positive solution for $\lambda > \lambda^*$.*

Assume f satisfies (5) with $q_s = 2$, ***B, F2****. Then there is $\lambda^* > 0$ such that (1) admits at least 2 radial positive solution for $\lambda > \lambda^*$ and at least 1 radial positive solution for $0 < \lambda < \lambda^*$.*

Remark 1.4. Notice that if f is of type (4) then $\boldsymbol{B}$ is satisfied and consequently $\boldsymbol{A}$ too. Further $\boldsymbol{F0}$, $\boldsymbol{F1}$, and $\boldsymbol{F2}$ are satisfied respectively if $q_s = q \in]2, 2^*(\delta)[$, $Q_s = a \in [2, 2^*(\delta)[$, and $q_u = q + a \in]2, 2^*(\delta)[$.

Corollary 1.5. *The solutions found by Theorems 1.1 and 1.2 have the following properties. For λ large (i.e $\lambda > \lambda^*$) we have 3 positive solutions say $u(r, d_i)$ for $i = 1, 2, 3$. We have $d_i = d_i(\lambda)$, $d_1(\lambda) < d_2(\lambda) < U < d_3(\lambda)$ and $d_1(\lambda) \to 0$, $d_2(\lambda) \to U^-$ as $\lambda \to +\infty$. Further $\frac{\partial}{\partial r} u(r, d_i) < 0$, for any $0 < r \leq 1$, for $i = 1, 2$.*

Moreover if we are in the assumption of Theorem 1.1 then $d_3(\lambda) \to U^+$ as $\lambda \to +\infty$ and $\frac{\partial}{\partial r} u(r, d_i) < 0$. If we are in the assumption of Theorem 1.2 we have $d_1 \to d_2$ as $\lambda \to \lambda^*$, so they become a unique solution for $\lambda = \lambda^*$, and they do not exist for $0 < \lambda < \lambda^*$.

Via Theorem 1.3 we just find the solutions $u(r, d_i)$ for $i = 2, 3$, where $d_2(\lambda) < U < d_3(\lambda)$, which have the properties described above.

In fact the whole discussion is generalized to embrace the more general case of p-Laplace equation, i.e.

$$\operatorname{div}(r^\ell \nabla u |\nabla u|^{p-2}) + \lambda r^{\ell+\delta} f(u) = 0 \tag{8}$$

where $n + \ell > p$, $\delta > -p$, and we also need to assume $1 < p \leq 2$ in order to avoid cumbersome technicalities. Notice that for $\ell = 0$ we obtain the Δ_p operator. Again we are interested in radial positive solutions of (8) of the Dirichlet problem in the ball of radius 1. So we in fact consider the following ODE

$$(u'(r)|u'(r)|^{p-2} r^{\ell+n-1})' + \lambda f(u) r^{\delta+\ell+n-1} = 0 \tag{9}$$

Using the concept of natural dimension introduced in [31] and performing the change of variables introduced in [31, §2, Remark (i)], see also [13, Appendix B] and in particular Remark B.1, we pass from (9) to the following

$$(u'(r)|u'(r)|^{p-2} r^{N-1})' + \lambda f(u) r^{N-1} = 0 \tag{10}$$

where $N = p\frac{\delta+\ell+n}{\delta+p} > 1$ is not anymore an integer and is called *natural dimension*. Obviously (10) can be regarded as the equation for radial solutions of $\Delta_p u + \lambda f(u) = 0$, but asking for x to be in $\mathbb{R}^N$ (where however N is not necessarily a natural number). Going back to the equation (10) we simply have a shift in the values of the critical exponents so that the Sobolev critical exponent p^*, as in the previous section, is replaced by $p^*(\delta) := \frac{p(n+\ell+\delta)}{n+\ell-p}$, which reduces to usual one if $\delta = \ell = 0$.

The prototypical f we are interested in is again (4), where $p^*(\delta)$ replaces $2^*(\delta)$, namely we rephrase $\boldsymbol{F0}$, $\boldsymbol{F1}$, $\boldsymbol{F2}$ as follows.

$\boldsymbol{F0'}$ There are $2 \leq q_s < p^*(\delta)$, $\hbar_0 > 0$, $c_s > 0$ such that (5) holds, but $(p, q_s) \neq (2, 2)$.

$\boldsymbol{F1'}$ There are $2 \leq Q_s < p^*(\delta)$, $\hbar_1 > 0$, $C_s > 0$ such that (6) holds.

$\boldsymbol{F2'}$ There are $2 \leq q_u < p^*(\delta)$, with $(p, q_u) \neq (2, 2)$, $\hbar_2 > 0$, $c_u > 0$ such that (7) holds.

Theorem 1.6. *Assume $1 < p \leq 2$, $\boldsymbol{F0'}, \boldsymbol{F1'}, \boldsymbol{A}$, then there is $\lambda^* > 0$ such that the Dirichlet problem in the ball of radius 1 associated to (8) admits at least 3 radial positive solutions for $\lambda \geq \lambda^*$.*

Theorem 1.7. *Assume $1 < p \leq 2$, $\boldsymbol{F0'}, \boldsymbol{F2'}, \boldsymbol{B}$, then there is $\lambda^* > 0$ such that the Dirichlet problem in the ball of radius 1 associated to (8) admits at least 3 radial positive solutions for $\lambda > \lambda^*$, at least 2 radial positive solutions for $\lambda = \lambda^*$, at least 1 radial positive solution for $0 < \lambda < \lambda^*$.*

Corollary 1.8. *Corollary 1.5 holds in this p-Laplace setting too.*

Remark 1.9. Remark 1.4 and Corollary 1.5 holds in this context too with trivial adaption. In particular if f is of type (4) again we have $q_s = q$, $a = Q_s$ and $q_u = q + a$.

The proofs are developed directly in the more general p-Laplace case, apart from the case where $p = q_s = 2$ which needs a separate discussion.

The outline of the paper goes as follows. In section 2 we introduce Fowler transformation, one of the main tools used in the proofs. In section 3 we turn to consider (2) assuming $\lambda = 1$, and we study the dependence on d of the first zero $R(d)$ of the solution $u(r, d)$ of (2), (3). In fact we aim to prove that $R(d)$ is a graph as sketched in figure 2. Then we study the asymptotic properties of the function $R(d)$ and we look for intersections between such a graph and the level line $R = K > 0$. Finally we conclude with a classical scaling argument: the problem of finding intersections between the graph $R(d)$ and the line $R = K$ for $K > 0$ large is then shown to be equivalent to finding positive radial solutions of (9) in the ball of radius 1 for λ large.

2 Fowler transformation

The main tool of investigation is the Fowler transformation, developed by Fowler in the 30s and extended to the p-Laplace case by Bidaut-Veron in [4] and independently by Franca in [11]. Let us set

$$\begin{array}{lll} \alpha_l = \frac{p}{l-p}, & \beta_l = \frac{(p-1)l}{l-p}, & \gamma_l = \beta_l - (N-1), \\ x_l = u(r)r^{\alpha_l} & y_l = u'(r)|u'(r)|^{p-2}r^{\beta_l} & r = e^t \\ g_l(x_l,t) = f(xe^{-\alpha_l t})e^{\alpha_l(l-1)t} & & \end{array} \tag{11}$$

The new variables x_l, y_l differ from the given ones u, u' in the presence of weight terms, which will help us to determine the asymptotic behaviors. Using (11), we pass from (10) to the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \text{sign}(y)|y|^{1/(p-1)} \\ -g_l(x,t) \end{pmatrix} \tag{12}$$

In the whole paper the dot indicates differentiation with respect to t, and we write $\boldsymbol{x_l}(t,\tau;\boldsymbol{Q}) = (x_l(t,\tau;\boldsymbol{Q}), y_l(t,\tau;\boldsymbol{Q}))$ for a trajectory of (12), evaluated at t and departing from $\boldsymbol{Q} \in \mathbb{R}^2$ at $t = \tau$.

Such a change of variables turns to be particularly useful when $f(u,r) = u^{q-1}$: in this case setting $l = q$ we find $g_l(x_l,t) = x_l^{q-1}$ so (12) is autonomous. Then we denote by M^u the set

$$M^u := \left\{ \boldsymbol{Q} \mid \lim_{t\to-\infty} \|\boldsymbol{x_l}(t,0,\boldsymbol{Q})\| \mathrm{e}^{-\alpha_l t} = c \in \mathbb{R} \right\}$$

From now and on, we assume $1 < p \le 2$ so that (12) is C^1. In fact this assumption may be relaxed but paying the prize of cumbersome technical difficulties, see e.g. [13].

From standard facts in ODE theory we see that M^u is a 1-dimensional C^1 manifold, see e.g. [5, § 13].

The behavior of positive solutions for this equation undergoes to several bifurcation due to the presence of critical exponents, such as the Sobolev critical exponent p^* introduced above, and $p_* := \frac{p(N-1)}{N-p}$.

If $p_* < l < p^*$ the origin is a saddle, so it admits an unstable manifold, M^u, and a stable manifold M^s, i.e. $\boldsymbol{Q} \in M^s$ iff $\boldsymbol{x_l}(t,0,\boldsymbol{Q}) \to (0,0)$ as $t \to +\infty$. If $l = p_*$ the origin has a central direction but M^u is anyway the unstable manifold. If $p < l < p_*$ and $q \ge 2$ the origin is an unstable node and M^u is the strongly unstable manifold, see again [5, §13], and figure 1. Using elementary facts from ODE theory we find that trajectories of (12) correspond to regular solutions of (2), see e.g. [12].

Remark 2.1. Assume $f(u) = cu^{q-1}$, $c > 0$, so that setting $l = q$ we get $g_q(x,t) \equiv cx^{q-1}$; let $q > p$, $q \ge 2$. Regular solutions $u(r)$ of Eq. (12) correspond to trajectories $\boldsymbol{x}_l(s)$ of system (12) departing from points in M^u and viceversa.

Further, using the invariance for t-translations of (12) and the fact that M^u is the graph of three trajectories (one corresponding to $u(r,d)$ where $d > 0$, one corresponding to $u(r,d)$ where $d < 0$ and the origin), we get the following known result.

Remark 2.2. Assume $g_l(x,t) \equiv cx^{q-1}$, where $q > p$, $q \ge 2$, $c > 0$. Fix $\boldsymbol{Q} \in M^u$ and let $u(r,d(\tau))$ be the regular solution of (2) corresponding to $\boldsymbol{x}_l(t,\tau;\boldsymbol{Q})$ of (12). Then $d(\tau)$ is continuous, $d(\tau) \to +\infty$ as $\tau \to 0$ and $d(\tau) \to 0$ as $\tau \to -\infty$.

Proof. Let $\boldsymbol{Q} \in M^u$, $\tau \in \mathbb{R}$ and let $u(r,d(0))$, $u(r,d(\tau))$ be the solution of (10) corresponding to $\boldsymbol{x}_l(t,0,\boldsymbol{Q})$ and $\boldsymbol{x}_l(t,\tau,\boldsymbol{Q})$ respectively. Notice that $\boldsymbol{x}_l(t,0,\boldsymbol{Q}) = \boldsymbol{x}_l(t+\tau,\tau,\boldsymbol{Q})$ for any $t \in \mathbb{R}$, therefore

$$\begin{aligned} d(\tau) &= \lim_{r\to 0} u(r,d(\tau)) = \lim_{t\to-\infty} x(t+\tau,\tau,\boldsymbol{Q})\mathrm{e}^{-\alpha_l(t+\tau)} \\ &= \lim_{t\to-\infty} x(t,0,\boldsymbol{Q})\mathrm{e}^{-\alpha_l(t+\tau)} = \lim_{r\to 0} u(r,d(0))\mathrm{e}^{-\alpha_l\tau} = d(0)\mathrm{e}^{-\alpha_l\tau}. \end{aligned}$$

So the remark follows. □

Using the Pohozaev identity it is easy to show that the phase portrait is as depicted in figure 1, see e.g. [12, Theorem 1], [25] see also [10, 29].

Thus in particular if $\boldsymbol{Q} \in M^u$, there is $T(\boldsymbol{Q})$ such that $y_l(T(\boldsymbol{Q}),0;\boldsymbol{Q}) < 0 = x_l(T(\boldsymbol{Q}),0;\boldsymbol{Q})$. Using Remark 2.1 and 2.2 we get the following well known result.

Remark 2.3. Assume $g_l(x,t) \equiv cx^{q-1}$, where $q \ge 2$, $c > 0$ and $p < q < p^*(\delta)$; then all the regular solutions $u(r,d)$ are *crossing solutions*, i.e. there is $R(d) > 0$ such that $u(R(d),d) = 0$ and $u'(R(d),d) < 0$. Further $R(d) \to 0$ as $d \to +\infty$ and $R(d) \to +\infty$ as $d \to 0$.

We briefly recall the well known results which enable us to draw picture 1, and to deduce the structure of (10) when $f(u) = u^{q-1}$.

We emphasize that when $g_q(x,t) = cx^{q-1}$ and $p_*(\delta) < q < p^*(\delta)$, then (12) admits a further critical point in $x > 0$, say $\boldsymbol{P} = (P_x, P_y)$ where $P_x = [\gamma_q(\alpha_q)^{p-1}/c]^{1/(q-p)}$ and $P_y = -(\alpha_l P_x)^{p-1}$, which converges to the origin as $l \to p_*$. When $p_* < l < p^*$ trajectories of (12) converging to $\boldsymbol{P}$ as $t \to -\infty$ correspond to singular solution of (10). When $l = p_*$ singular solution of (10) exist, but correspond to trajectories of the central manifold of (12). When $p < l < p_*$ singular solution of (10) again exist and correspond to trajectories converging to the origin as $t \to -\infty$ but not belonging to the strongly unstable manifold M^u, see e.g. [16, §2] for a proof in the Laplace context. However this fact will not be used in this article.

Remark 2.4. Assume $g_l(x,t) \equiv cx^{q-1}$, where $q > p$, $q \geq 2$, $c > 0$. We can find $R > 0$ such that if $\|\boldsymbol{Q}\| \geq R$ then there is $\bar{T}(\boldsymbol{Q}) > 0$ such that $\boldsymbol{x_l}(t, 0, \boldsymbol{Q})$ crosses transversally the y negative semi-axis at $t = \bar{T}(\boldsymbol{Q})$. Further $\bar{T}(\boldsymbol{Q}) \to 0$ as $\|\boldsymbol{Q}\| \to +\infty$

Proof. The result is a consequence of the superlinearity of g_l and it is borrowed from [15]: we sketch the proof for completeness.

Let us set

$$x_l|x_l|^{p-2} = \rho_l \cos(\theta_l) \qquad\qquad y = \rho_l \sin(\theta_l) \tag{13}$$

From a straightforward computation we see that

$$\begin{aligned}\dot{\theta}_l(t, \boldsymbol{Q}) =& (p-n)\sin(\theta_l)\cos(\theta_l) - (p-1)|\sin(\theta_l)|^{\frac{p}{p-1}}|\cos(\theta_l)|^{\frac{p-2}{p-1}} - \\ &- c\,\mathrm{sign}[\cos(\theta)]|\cos(\theta)|^q \rho^{q-p}\end{aligned} \tag{14}$$

Hence $\dot{\theta}_l(t, \boldsymbol{Q})$ becomes unbounded as $x \to +\infty$ and it is negative if $x = 0$. So the Remark follows. □

Let $k > 0$; we introduce the following set:

$$\mathcal{T}(k) := \{(x, y) \mid 0 < kx < |y|\}, \tag{15}$$

We emphasize the following facts, which follow easily from some standard phase plane analysis and from Remark 2.4, see e.g. [16] for a full fledged proof in the Laplace context, or again [12, Theorem 1].

Remark 2.5. Assume $g_l(x,t) = cx^{q-1}$, $c > 0$, $q > 2$, $p < q \leq p_*(\delta)$; then the origin is the unique critical point of the system and it is unstable. Hence all the trajectories rotate clockwise and cross the coordinate axes indefinitely as $t \to +\infty$.

Remark 2.6. Assume $g_l(x,t) = cx^{q-1}$, $c > 0$, $q > 2$, and $p_*(\delta) < q < p^*(\delta)$; then the origin admits a stable manifold which is a heteroclinic connection between the origin and $\boldsymbol{P}$. However there is $\bar{k} > 0$ such that $\mathcal{T}(\bar{k})$ does not intersect M^s and $\boldsymbol{P} \notin \mathcal{T}(\bar{k})$.

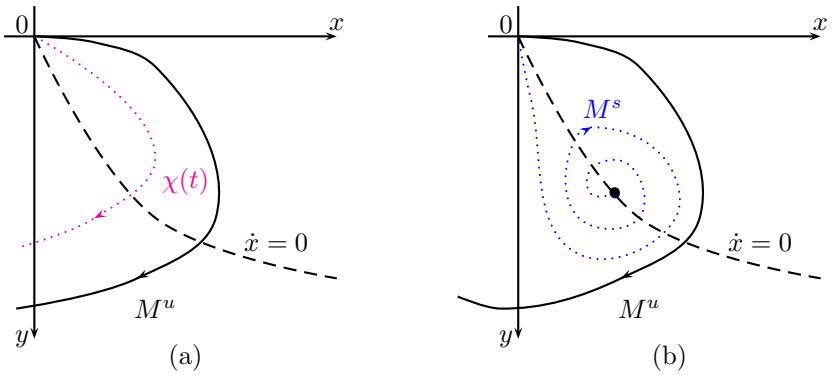

Figure 1: Sketch of the phase portrait of the autonomous system (12) when $g_q(x,t) \equiv x|x|^{q-2}$, $q > 2$, when $p < q \leq p^*(\delta)$ on the left (a) and when $p < q < p^*(\delta)$ on the right (b). The manifold M^u is the solid (black) curve; in fig. a) the dotted (magenta) line denotes a trajectory $\chi(t)$ converging to the origin as $t \to -\infty$ but not staying in the strongly unstable manifold M^u, in fig b) the dashed (blue) line denotes the stable manifold.

2.1 Unstable leaves for non-autonomous systems.

In this subsection, following [14], we combine the results of [5, §13], and [23] to construct the unstable manifolds for (12) when g_l depends on t. Assume ***F0′*** and ***F2′*** and set $\varpi_s = \hbar_0/(2\alpha_{q_s})$, $\varpi_u = \hbar_2/(2\alpha_{q_u})$. Let us consider (12) where we have added the extra variable $z(t) = \mathrm{e}^{\varpi_u t}$ in order to deal with the following 3-dimensional autonomous system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}\alpha_{q_u} & 0 & 0\\ 0 & \gamma_{q_u} & 0\\ 0 & 0 & \varpi_u\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}\mathrm{sign}(y_l)|y_l|^{1/(p-1)}\\ -g_{q_u}(x, \frac{\ln(z)}{\varpi_u})\\ 0\end{pmatrix} \tag{16}$$

System (16) is useful to discuss the behavior of trajectories of (12) as $t \to -\infty$, and correspondingly the behavior of (10) for r small (and u large as we will see below). Similarly if we replace $z(t)$ by $\zeta(t) = \mathrm{e}^{-\varpi_s t}$ we get a system which is is useful to detect the behavior of trajectories of (12) in the future and correspondingly the behavior of (10) for r large (and u small, see below), i.e.:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\zeta}\end{pmatrix} = \begin{pmatrix}\alpha_{q_u} & 0 & 0\\ 0 & \gamma_{q_u} & 0\\ 0 & 0 & -\varpi_s\end{pmatrix}\begin{pmatrix}x\\ y\\ \zeta\end{pmatrix} + \begin{pmatrix}\mathrm{sign}(y_l)|y_l|^{1/(p-1)}\\ -g_{q_s}(x, \frac{\ln(\zeta)}{-\varpi_s})\\ 0\end{pmatrix} \tag{17}$$

Remark 2.7. It is straightforward to check that (16) is C^1 also for $z = 0$ if f satisfies (7) (in particular if ***F2′*** holds), while (17) is C^1 also for $\zeta = 0$ if f satisfies (5) (in particular if ***F0′*** holds).

Proof. Consider (17) and assume (7); notice that

$$g_{q_s}(x,t) = x^{q_s-1}\{c_s + o([x\mathrm{e}^{-\alpha_{q_s}t}]^{\hbar_2})\} = c_s x^{q_s-1} + x^{q_s-1+2\varpi_s} o(\zeta(t)^{2\varpi_s}) \tag{18}$$

as $\zeta(t) \to 0$, i.e. as $t \to +\infty$. Further

$$\frac{\partial g_{q_s}}{\partial x}(x,t) = x^{q_s-2}\{(q_s-1)c_s + o([x\mathrm{e}^{-\alpha_{q_s}t}]^{\hbar_2})\} = (q_s-1)c_s x^{q_s-1} + x^{q_s-1+2\varpi_s} o(\zeta(t)^{2\varpi_s})$$

and it is continuous as $\zeta \to 0$. Moreover, from (18), we see that the derivative with respect to ζ of the second equation in (17) is continuous and converges uniformly to 0 as $\zeta \to 0$ when x is in a compact set. Then it is easy to check that (17) is in fact C^1.

Similarly consider (16) and assume (5); notice that

$$g_{q_u}(x,t) = x^{q_u-1}\{c_u + o([x\mathrm{e}^{-\alpha_{q_u}t}]^{\hbar_0})\} = c_u x^{q_u-1} + x^{q_u-1-2\varpi_u} o(z(t)^{2\varpi_u})$$

as $z(t) \to 0$, i.e. as $t \to -\infty$. Then, reasoning as above, we see that (16) is C^1 too. □

From [6, Remark 2.5] we know that all the solutions of (12) may be continued for any $t \in \mathbb{R}$. Then we see that the α-limit set of the trajectories of (16) is contained in the $z = 0$ plane; moreover such a plane is invariant and the dynamics reduced to the $z = 0$ plane coincides with the one of the autonomous system (12) where $g_{q_u}(x,t) \equiv g_{q_u}(x,-\infty)$. Assume first that $q_u \in [p_*(\delta); p^*(\delta))$; then the origin of (16) admits a 2-dimensional unstable manifold $\mathbf{W^u_{q_u}}$ which is transversal to $z = 0$. Following [15], see also [23, 24, 16] we see that, for any $\tau \in \mathbb{R}$, the sets $W^u_{q_u}(\tau)$ and $W^u_{q_u}(-\infty)$ defined below are C^1 immersed 1-dimensional manifolds, i.e. the graph of C^1 regular curves.

$$W^u_{q_u}(\tau) = \left(\boldsymbol{W^u_{q_u}} \cap \{z = \mathrm{e}^{\varpi_u \tau}\}\right), \qquad W^u_{q_u}(-\infty) = \left(\boldsymbol{W^u_{q_u}} \cap \{z = 0\}\right) \tag{19}$$

Further notice that $W^u_{q_u}(\tau)$ can be characterized as follows:

$$W^u_{q_u}(\tau) := \{\boldsymbol{Q} \mid \boldsymbol{x}_{\boldsymbol{q_u}}(t,\tau,\boldsymbol{Q}) \to (0,0) \ \text{ as } t \to -\infty\} \tag{20}$$

Moreover it depends continuously on τ. More precisely we have the following see e.g. [24], see also [5, §13.4].

Remark 2.8. Let either $\tau_0 \in \mathbb{R}$ or $\tau_0 = -\infty$, and assume that $W^u_{q_u}(\tau_0)$ intersects transversally a segment L in a point denoted by $\boldsymbol{Q}(\tau_0)$. Then there is a neighborhood U of τ_0 such that $W^u_{q_u}(\tau)$ still intersects L transversally in a point $\boldsymbol{Q}(\tau)$ for $\tau \in U$; moreover $\boldsymbol{Q}(\tau)$ is as smooth as (16).

Assume $\boldsymbol{F2'}$; using standard tools of invariant manifold theory, see e.g. [5, §13.4], we see that if $q_u \in [p_*(\delta); p^*(\delta))$, then $W^u_{q_u}(\tau)$ can be equivalently characterized as follows:

$$W^u_{q_u}(\tau) := \{\boldsymbol{Q} \mid \|\boldsymbol{x_{q_u}}(t,\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha_{q_u}t} \to c \in \mathbb{R} \ \text{ as } t \to -\infty\} \tag{21}$$

If $p < q_u < p_*(\delta)$ the origin has a 3 dimensional unstable manifold (an open set), but we can define a 2-dimensional strongly unstable manifold. Set

$$\boldsymbol{W^u_{q_u}} = \{(\boldsymbol{Q}, \mathrm{e}^{\varpi\tau}) \mid \|\boldsymbol{x_{q_u}}(t,\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha_{q_u}t} \to c \in \mathbb{R} \text{ as } t \to -\infty\}$$

then $\boldsymbol{W^u_{q_u}}$ is an invariant manifold and the manifolds $W^u_{q_u}(\tau)$ and $W^u_{q_u}(-\infty)$ defined as in (19) or as in (21) (but not as in (20)) satisfy Remark 2.8, see again [5, §13.4]. In this case we have also trajectories converging to the origin at a slower exponential rate, corresponding to singular solution of (10); however this fact will not be used in this article.

It is easy to check that, if $\boldsymbol{Q} \in W^u_{q_u}(\tau)$, there is $d > 0$ such that $\boldsymbol{x_{q_u}}(t,\tau,\boldsymbol{Q})\mathrm{e}^{-\alpha_{q_u}t} \to (d, 0)$. Hence the corresponding solution of (2) is a regular solution, i.e. we have the following.

Remark 2.9. Assume $\boldsymbol{F2'}$ with $q_u > p$, then Remark 2.1 still holds.

Further, we get the following generalization of the second part of Remark 2.2, see [6, Lemma 2.10] for a detailed proof.

Remark 2.10. Fix $\tau \in \mathbb{R}$ and consider the manifold $W^u_{q_u}(\tau)$, the trajectory $\boldsymbol{x_{q_u}}(t,\tau;\boldsymbol{Q})$ of (12), where $\boldsymbol{Q} \in W^u_{q_u}(\tau)$ and the corresponding solution $u(r,d)$ of (2), so that $d(\boldsymbol{Q})$ is an invertible function. Follow $W^u_{q_u}(\tau)$ from the origin towards $x > 0$, then d increases as we go further from the origin, and $d(\boldsymbol{Q}) \to 0$ as $\boldsymbol{Q} \to (0,0)$.

With analogous reasoning if $\boldsymbol{F0'}$ holds and $p_*(\delta) < q_s < p^*(\delta)$, we see that (17) admits a two dimensional invariant stable manifold $\boldsymbol{W^s_{q_s}}$. Further

$$W^s_{q_s}(\tau) = \left(\boldsymbol{W^s_{q_s}} \cap \{\zeta = \mathrm{e}^{-\varpi_s\tau}\}\right), \qquad W^s_{q_s}(+\infty) = \left(\boldsymbol{W^s_{q_s}} \cap \{\zeta = 0\}\right) \tag{22}$$

are C^1 immersed 1-dimensional manifold depending in a C^1 way from τ, and $W^s_{q_s}(\tau) \to W^s_{q_s}(+\infty)$ as $\tau \to +\infty$ in the sense specified in Remark 2.8. Moreover

$$W^s_{q_s}(\tau) := \{\boldsymbol{Q} \mid \boldsymbol{x_{q_s}}(t,\tau,\boldsymbol{Q}) \to (0,0) \ \text{ as } t \to +\infty\},$$

and $W^s_{q_s}(+\infty)$ coincide with the stable manifold M^s of the autonomous system (12) where $g_l = c_s x^{q_s-1}$. From now on, since we are just interested in positive solutions, abusing the notation *for $W^u_{q_u}(\tau)$ (respectively $W^s_{q_s}(\tau)$) we mean just the branch of the manifold leaving from the origin towards $x > 0$ and corresponding to solutions $u(r)$ of (10) which are positive for r small (respectively for r large).*

Sometimes it will be useful to switch between different values of l in system (12), e.g. to pass from $l = q_u$ to $l = q_s$. It is straightforward to notice that if $\boldsymbol{x_{q_u}}(t,\tau,\boldsymbol{Q})$ and $\boldsymbol{x_{q_s}}(t,\tau,\boldsymbol{R})$ correspond to the same solution $u(r)$ of (2), then

$$\begin{aligned} &x_{q_s}(t,\tau,\boldsymbol{R}) = x_{q_u}(t,\tau,\boldsymbol{Q})\mathrm{e}^{(\alpha_{q_s}-\alpha_{q_u})t} \\ &y_{q_s}(t,\tau,\boldsymbol{R}) = y_{q_u}(t,\tau,\boldsymbol{Q})\mathrm{e}^{\frac{(\alpha_{q_s}-\alpha_{q_u})t}{p-1}} \\ &R_x = Q_x\mathrm{e}^{(\alpha_{q_s}-\alpha_{q_u})\tau}, \quad R_y = Q_y\mathrm{e}^{\frac{(\alpha_{q_s}-\alpha_{q_u})\tau}{p-1}} \end{aligned} \tag{23}$$

where we used (11) and the fact that $\beta_{q_s} - \beta_{q_u} = \frac{\alpha_{q_s}-\alpha_{q_u}}{p-1}$. It follows that the curves of the form $y = kx|x|^{p-2}$ remain invariant when we pass from $l = q_u$ to $l = q_s$ at a fixed $\tau \in \mathbb{R}$, for any fixed $k \in \mathbb{R}$. In fact the whole portrait is subject either to a dilatation if $(\alpha_{q_s} - \alpha_{q_u})\tau > 0$ or to a contraction if $(\alpha_{q_s} - \alpha_{q_u})\tau < 0$.

Let $\tau \in \mathbb{R}$ and denote by

$$W^u_{q_s}(\tau) := \{(R_x\mathrm{e}^{(\alpha_{q_s}-\alpha_{q_u})\tau}, R_y\mathrm{e}^{(\beta_{q_s}-\beta_{q_u})\tau}) \mid (R_x, R_y) \in W^u_{q_u}(\tau)\}$$

Observe that $W^u_{q_s}(\tau)$ is diffeomorphic to $W^u_{q_u}(\tau)$, hence $W^u_{q_s}(\tau)$ is 1 dimensional too, and inherits the transversal smoothness property described in Remark 2.8.

In fact the unstable leaves $W^u_{q_u}(\tau)$ may be constructed through the invariant manifold theory for non-autonomous systems, simply requiring that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{\partial g_{q_u}}{\partial x_l}(x_l, t) < \varepsilon$ for $|x| \le \delta$ for any $t \le \tau$ (we stress that such an assumption is satisfied if $\boldsymbol{F2'}$ holds), cf [5, §13.4] and in particular Theorems 4.1, 4.3, 4.4. Since the linearization of (12) in the origin has t independent eigenvalues and eigenvectors, from [5, Theorem 4.2, §13.4] we get the following.

Remark 2.11. Assume $\boldsymbol{F2'}$, then for any $\tau \in \mathbb{R}$ the manifold $W^u_{q_u}(\tau)$ is tangent in the origin to the $y=0$ axis. So, thanks to (23), $W^u_{q_s}(\tau)$ is tangent in the origin to the $y=0$ axis, too. Similarly, if $\boldsymbol{F0'}$ holds then $W^s_{q_s}(\tau)$ is tangent to the y-negative semi-axis if $1<p<2$ and to the line $y=-(n-2)x$ if $p=2$.

Lemma 2.12. *Assume* $\boldsymbol{F0'}$ *and let* $\tilde{u}(r)$ *be a solution of* (10) *such that* $\tilde{u}(r)>0$ *for any* $r>0$ *and* $\lim_{r\to\infty}\tilde{u}(r)=0$. *Then the corresponding trajectory* $\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)$ *of* (12) *is bounded.*

Proof. From $\boldsymbol{F0'}$ we see that for any $\varepsilon>0$ there is $\tilde{R}>0$ such that

$$(c_s-\varepsilon)|\tilde{u}(r)|^{q_s-1} < f(\tilde{u}(r)) < (c_s+\varepsilon)|\tilde{u}(r)|^{q_s-1}$$

for any $r\geq\tilde{R}$. Correspondingly we find that

$$(c_s-\varepsilon)|\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)|^{q_s-1} < g_{q_s}(\tilde{x}_{q_s}(t),t) < (c_s+\varepsilon)|\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)|^{q_s-1}$$

for any $t\geq\tilde{\tau}=\ln(\tilde{R})$. Let us introduce polar coordinates as in (13). Assume for contradiction that $\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)$ becomes unbounded as $t\to+\infty$. Then repeating the computation in (14) we see that the angular coordinate $\tilde{\theta}(t)$ of $\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)$ is such that $\dot{\tilde{\theta}}(t)$ is bounded above from a negative constant, i.e. there is $K>0$ such that $\dot{\tilde{\theta}}(t)\leq -K$ for $t\geq\tilde{\tau}$. Hence $\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)$ has to cross the coordinate axes indefinitely and $\tilde{u}(r)$ changes sign, but this is a contradiction, so $\tilde{\boldsymbol{x}}_{\boldsymbol{q_s}}(t)$ is bounded and the Lemma is proved. □

3 Proofs.

Let us observe that the flow of (12) on the y axis rotates clockwise for any $t\in\mathbb{R}$ and the origin is a critical point. Hence for the corresponding solutions $u(r)$ of (10) we see that all the zeroes are non-degenerate.

Let us introduce the following set

$$I := \{d>0 \mid u(r,d) \text{ is a crossing solution}\} \tag{24}$$

i.e. $u(r,d)$ has a (non-degenerate) zero for a certain $r=R(d)>0$ iff $d\in I$. The scheme of the proof is the following: in the first two subsections we set $\lambda=1$ and we consider (12); first we show that I is open, $R(d)$ is continuous. In § 3.1 we fix $\lambda=1$, and we consider the setting of Theorem 1.7 and we aim to draw the graph of $R(d)$ i.e. picture 2b: this is the content of Proposition 3.9. Then in § 3.2 we use the information of § 3.1 to draw the graph of $R(d)$ with $\lambda=1$ in the setting of of Theorem 1.6, see picture 2a and Proposition 3.15. Then we adapt the argument to draw the graph of $R(d)$ in the setting of Theorem 1.3. Finally in § 3.3 we perform a scaling argument to obtain the proof of Theorems 1.7 and 1.6.

Let us begin with the following results developed directly on (10).

Remark 3.1. Assume f satisfies $\boldsymbol{A}$ and (5) with $2\leq q_s<p^*(\delta)$. If $d\notin I$, $d\leq\sigma$, $d\neq U$, then $u(r,d)>0$ for any $r>0$ and either $\lim_{r\to\infty}u(r,d)=0$ or $\lim_{r\to\infty}u(r,d)=U$.

Proof. First of all observe that $u(r,d)$ is positive and strictly decreasing for $r>0$ small enough, see e.g. [13, Lemma 2.1]. Set

$$\rho_d = \sup\left\{R \mid u(r,d)>0,\ \frac{\partial u}{\partial r}(r,d)<0, \quad \text{for any } 0<r<R\right\},$$

and $L(d)=\lim_{r\to\rho_d}u(r,d)$.

It is easy to check that if $\rho_d=+\infty$ then $f(L(d))=0$ so the Remark is proved, so we just need to discuss the case $\rho_d<\infty$.

We claim that *if* $L(d)=0$, $L'(d):=\lim_{r\to\rho_d}u'(r,d)=0$ *then* $\rho_d=+\infty$. In fact let $\boldsymbol{x}(t)$ be the trajectory of (12) corresponding to $u(r)$, then $\boldsymbol{x}(t)\to(0,0)$ as $t\to\ln(\rho_d)$, and the origin is a critical point of (12) so the claim follows.

Analogously *if* $L(d)=U$, $L'(d)=\lim_{r\to\rho_d}u'(r,d)=0$ *then* $\rho_d=+\infty$. In fact let us consider the modified system (12) where g is obtained via (11) but replacing f by $\tilde{f}(u)=f(u-U)$. Let $\tilde{\boldsymbol{x}}(t)$ be the trajectory of the modified system (12) corresponding to $u(r,d)$: it follows that $\boldsymbol{x}(t)\to(0,0)$ as $t\to\ln(\rho_d)$, and the origin is again a critical point so $\rho_d=+\infty$, and the claim in proved.

So we can assume $\rho_d<\infty$, $f(L(d))>0$, and $L'(d)=u'(\rho_d,d)=0$; but from (10) we get

$$u''(\rho_d,d) = -\frac{n-1}{\rho_d}u'(\rho_d,d) - f(u(\rho_d,d))|u'(\rho_d,d)|^{p-2} < 0\,.$$

This contradicts the fact that $u'(r,d)$ is negative for $r<\rho_d$ and null for $r=\rho_d$, so the Remark is proved. □

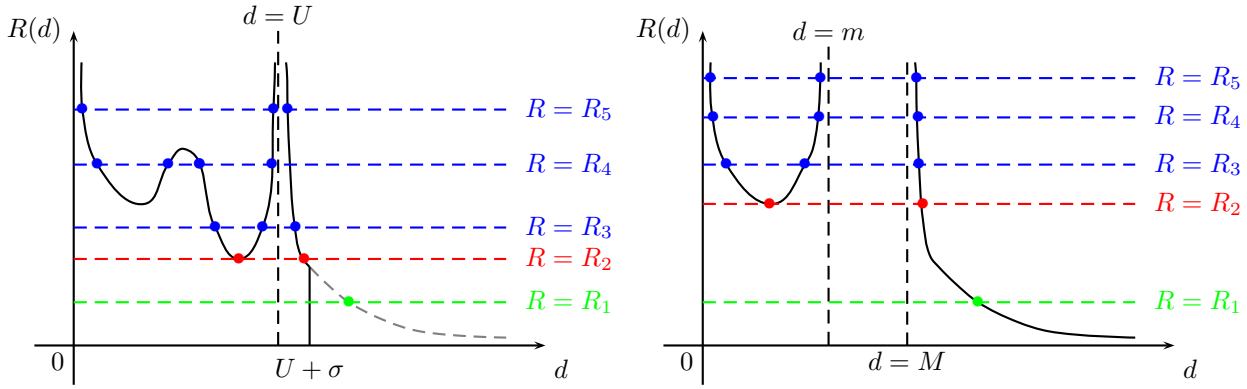

Figure 2: Sketch of the graph of the function $R(d)$ in the setting of Proposition 3.15 on the left, and in the setting of Proposition 3.9 on the right. In picture a) we have drawn with solid line the part of the graph constructed using the original problem, and with the dotted line the part constructed using a modified problem to which Proposition 3.9 could be applied. In both graphs a) and b) we have at least 3 values d_i, such that $R(d_i) = R$ when $R > R_2$; further $0 < d_1 < d_2 < U < d_3$. Moreover in picture a) we have $d_1 \to 0^+$, $d_2 \to U^-$ and $d_3 \to U^+$ as $R \to +\infty$, while in picture b) we have $d_1 \to 0^+$, $d_2 \to m^-$ and $d_3 \to M^+$ as $R \to +\infty$.

Using the part of the proof concerning $\rho_d < \infty$ we easily get the following.

Remark 3.2. Assume f satisfies $\boldsymbol{B}$ and (5) with $2 \le q_s < p^*(\delta)$. If there is $R \ge 0$ such that $u(R) < U$ and $u'(R) \le 0$ then either $u(r)$ has a zero for $r > R$, or $u(r) > 0$ for any $r > 0$ and $\lim_{r\to+\infty} u(r) = 0$. In particular if $u(r) = u(r, d)$ is a regular solution and $d < U$, then either $d \in I$ or $u(r, d) > 0$ for any $r > 0$ and $\lim_{r\to+\infty} u(r, d) = 0$.

Remark 3.3. Assume f satisfies $\boldsymbol{B}$ and (5) with $2 \le q_s < p^*(\delta)$. Then $u(r, d)$ is decreasing as long as it is positive for any $0 < d < U$.

We give a result inspired by [13, Lemma 3.16].

Lemma 3.4. *Assume f satisfies* (5) *with $2 \le q_s < p^*(\delta)$; then I is open and the function $R(d) : I \to (0, +\infty)$ is continuous. Moreover if $u(r, d_*) > 0$ for any $r \ge 0$, and $(d_*, d_* + \varepsilon) \subset I$ and/or $(d_* - \varepsilon, d_*) \subset I$, then $\lim_{d\to d_*^+} R(d) = +\infty$ and/or $\lim_{d\to d_*^-} R(d) = +\infty$.*

Proof. First of all we prove that I is open. Let $D \in I$ we show that there is $\nu > 0$ such that $(D-\nu, D+\nu) \subset I$. Fix $\tau \in \mathbb{R}$ and consider the trajectories $\boldsymbol{x}_{\boldsymbol{q_s}}(t, \tau, \boldsymbol{Q}(D))$ and $\boldsymbol{x}_{\boldsymbol{q_s}}(t, \tau, \boldsymbol{Q}(d))$ of (12) corresponding respectively to $u(r, D)$ and $u(r, d)$, where $|D - d| < \nu$. From Remarks 2.9 and 2.10 we see that for any $\sigma > 0$ we can find $\nu(\sigma) > 0$ such that $\|\boldsymbol{Q}(D) - \boldsymbol{Q}(d)\| < \sigma$. Observe that the flow of (12) on the y negative semi-axis is transversal. Therefore we can find T slightly larger than $\ln(R(D))$ such that $x_{q_s}(T, \tau, \boldsymbol{Q}(D)) < 0$. Using continuous dependence on initial data we see that we can find $\sigma > 0$ small enough so that $x_{q_s}(T, \tau, \boldsymbol{Q}(d)) < 0$. Consequently we see that $u(r, d)$ is a crossing solution if $\nu > 0$ is small enough and its first zero $R(d)$ satisfies $R(d) < \mathrm{e}^T$. Using a continuity argument and the transversality of the flow of (12) on the y negative semi-axis, it is easy to see that $R(d) \to R(D)$ as $d \to D$.

Assume now that $d_* \notin I$, but $(d_* - \varepsilon, d_*) \subset I$ as above, so that $x_{q_s}(t, \tau, \boldsymbol{Q}(d_*)) > 0$ for any $t \in \mathbb{R}$. For any $\bar{T} > \tau$ we can find $\sigma > 0$ such that $\boldsymbol{x}_{\boldsymbol{q_s}}(t, \tau, \boldsymbol{Q})$ lies in $x > 0$ for any $\tau \le t \le \bar{T}$, whenever $\|\boldsymbol{Q} - \boldsymbol{Q}(d_*)\| < \sigma$. Consequently we can find $\nu \in (0, \varepsilon)$ such that $u(d, r)$ is positive and decreasing for any $\mathrm{e}^\tau \le r \le \mathrm{e}^{\bar{T}}$, whenever $0 < d_* - d < \nu$. Further, since $u(r, d)$ is a regular solution we can assume that there is $\rho = \rho(d)$ small enough so that $u(r, d)$ is positive for $0 \le r \le \rho(d)$ (see e.g. [13, Lemma 2.1]). Hence if we choose $\tau < \ln(\rho)$ we see that $u(r, d)$ is positive for $0 \le r \le \mathrm{e}^{\bar{T}}$. From the arbitrariness of $\bar{T}$ we easily conclude that $R(d) \to +\infty$ as $d \to d_*$. □

3.1 Theorem 1.7: the graph of $R(d)$ for $\lambda = 1$ fixed.

Lemma 3.5. *Assume $\boldsymbol{F2'}$; then there is $M > 0$ large such that $[M, +\infty) \subset I$ and $R(d) \to 0$ as $d \to +\infty$.*

Proof. The proof is based on the comparison between the original non-autonomous system (12) and the autonomous system obtained at $\tau = -\infty$, i.e. by setting $g_{q_u}(x, t) \equiv c_u x^{q_u - 1}$. Let $u(r, d)$, $\boldsymbol{x}_{\boldsymbol{q_u}}(t, \tau, \boldsymbol{Q})$,

$W^u_{q_u}(\tau)$ denote a regular solution of (10), the corresponding trajectory of (12) and the unstable manifold of the original non-autonomous system, and let $\bar u(r,\bar d)$, $\bar{\boldsymbol{x}}_{\boldsymbol{q_u}}(t,\tau,\bar{\boldsymbol{Q}})$, $W^u_{q_u}(-\infty)$ be the ones of the autonomous system where $g_{q_u}(x,t)\equiv c_u x^{q_u-1}$.

Follow $W^u_{q_u}(-\infty)$ from the origin towards $x>0$: it intersects the $\dot x=0$ isocline in $\bar{\boldsymbol{R}}=(\bar R_x,\bar R_y)$, where $\bar R_y<0<\bar R_x$ and then the y-negative semi-axis transversally in $\bar{\boldsymbol{Q}}=(0,\bar Q_y)$, $\bar Q_y<0$, see figure 1. From Remark 2.8 it follows that there is $K>0$ such that for any $\tau<-K$ the manifold $W^u_{q_u}(\tau)$ intersects the y-negative semi-axis transversally in $\boldsymbol{Q}(\tau)=(0,Q_y(\tau))$, where $Q_y(\tau)<0$ approaches $\bar Q_y$ as $\tau\to-\infty$.

Using a continuity argument we see that

$$\sup\{\|\boldsymbol{x}_{\boldsymbol{q_u}}(t,\tau;\boldsymbol{Q}(\tau))-\bar{\boldsymbol{x}}_{\boldsymbol{q_u}}(t,\tau,\bar{\boldsymbol{Q}})\|\mid t\le\tau\}\to 0\qquad\text{as }\tau\to-\infty.$$

Hence there is $\bar\tau=\bar\tau(\tau)<\tau$ such that $\dot x_{q_u}(\bar\tau(\tau),\tau;\boldsymbol{Q}(\tau))=0$ and $x_{q_u}(\bar\tau,\tau;\boldsymbol{Q}(\tau))>\bar R_x/2$, for $\tau<-K$ (possibly choosing a larger K). Thus we can choose K large enough so that $u(\mathrm{e}^{\bar\tau},d)>\frac{\bar R_x}{2}\mathrm{e}^{-\alpha_{q_u}\tau}=C>0$ is large enough so that $f(u)>0$ for $u\ge C$. Then, using the fact that $u'(\mathrm{e}^{\bar\tau},d)<0$ and arguing as in the proof of Remark convergence, we see that $u'(r,d)<0$ for any $0<r\le\mathrm{e}^{\bar\tau}$.

Therefore $u(r,d(\tau))$ is positive and decreasing for $0\le r<R(d(\tau))=\mathrm{e}^\tau$, it becomes null with negative slope for $r=\mathrm{e}^\tau$ and

$$d(\tau)=u(0,d(\tau))>u(\mathrm{e}^{\bar\tau},d(\tau))=\bar R_x/2\mathrm{e}^{-\alpha_{q_u}\tau}.$$

Hence $d(\tau)\to+\infty$ as $\tau\to-\infty$ and we easily conclude. □

Lemma 3.6. *Assume* ***F0′***, ***F2′*** *with* $2<q_s\le p_*(\delta)$ *and* ***B***. *Let* $u(r)$ *be such that there is* $\tilde R>0$ *at which* $u(\tilde R)<U$ *and* $\frac{\partial u}{\partial r}(\tilde R)\le 0$. *Then* $u(r)$ *has a non-degenerate zero at some* $R>\tilde R$.

Proof. From Remark 3.2 we see that either $u(r)\to 0$ as $r\to+\infty$ or it is a crossing solution and we are done. Assume the former; then from Lemma 2.12 we see that the trajectory $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t)$ corresponding to $u(r)$ is bounded; then from Remark 2.5 and a standard continuity argument we find that $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t)$ crosses the coordinate axes indefinitely so the Lemma is proved. □

We recall that (17) admits a critical point $\boldsymbol{\mathcal{P}^s}=(P^s_x,P^s_y,0)$ such that $P^s_x>0$ iff $p_*(\delta)<q_s<p^*(\delta)$, where $P^s_x=[\gamma_{q_s}(\alpha_{q_s})^{p-1}/c_s]^{1/(q_s-p)}$ and $P^s_y=-(\alpha_l P^s_x)^{p-1}$. In this case there is a unique trajectory, say $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t),\zeta(t))$, which converges to $\boldsymbol{\mathcal{P}^s}$ as $t\to+\infty$. Let $\mathcal{T}(k):=\{(x,y)\mid |y|<kx\}$. Using Remark 2.6 and a standard continuity argument, we obtain the following.

Lemma 3.7. *Assume* ***F0′***, $p_*(\delta)<q_s<p^*(\delta)$. *Then we can find* $\tau_0>0$ *and* $k_0>0$ *such that* $\mathcal{T}(k_0)$ *does not intersect* $W^s_{l_s}(\tau)$ *for any* $\tau\ge\tau_0$, *and* $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t)\notin\mathcal{T}(c_0)$ *for any* $t\ge\tau_0$.

Now we are ready to prove the following.

Lemma 3.8. *Assume* ***B***, ***F0′***, ***F2′***; *then there is* $D>0$ *such that* $(0,D)\subset I$ *and* $R(d)\to+\infty$ *as* $d\to 0$.

Proof. From Remark 3.2 we know that, if $0<d<U$ then either $d\in I$ or $u(r,d)\to 0$ as $r\to+\infty$.

Assume first $p_*(\delta)<q_s<p^*(\delta)$. Let $\tau>\tau_0$ where τ_0 is defined in Lemma 3.7; choose $\rho>0$ small enough so that $W^u_{q_s}(\tau)$ intersects the line $x=\rho$ for any $\tau\ge\tau_0$. Follow $W^u_{q_s}(\tau)$ from the origin towards $x>0$ and denote by $W^u_{q_s}(\tau,\rho)$ the branch of $W^u_{q_s}(\tau)$ between the origin and the first interesection with $x=\rho$. Since $W^u_{q_s}(\tau)$ is tangent to the x axis in the origin, possibly choosing a smaller $\rho>0$, we can assume that $W^u_{q_s}(\tau,\rho)$ is a graph on the x axis and $W^u_{q_s}(\tau,\rho)\subset\mathcal{T}(k_0)$. Further we can find $D\in(0,U)$ such that, if $\boldsymbol{x}_{\boldsymbol{q_s}}(t,\tau,\boldsymbol{Q})$ corresponds to a solution $u(r,d)$ of (10) then $\boldsymbol{Q}\in W^u_{q_s}(\tau,\rho)$, for any $0<d<D$: this is an easy consequence of Remarks 2.2 and 2.8. Assume for contradiction that there is $0<d<D$ such that $d\notin I$ and let $\boldsymbol{x}_{\boldsymbol{q_s}}(t,\tau,\boldsymbol{Q}(d))$ be the corresponding trajectory of (12). Then $u(r,d)\to 0$ as $r\to+\infty$, so, from Lemma 2.12, $\boldsymbol{x}_{\boldsymbol{q_s}}(t,\tau,\boldsymbol{Q}(d))$ is bounded: hence either $\boldsymbol{x}_{\boldsymbol{q_s}}(t,\tau,\boldsymbol{Q}(d))$ converges to the origin or it coincides with $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t)$ and converges to to (P^s_x,P^s_y) as $t\to+\infty$. In the former case we have $\boldsymbol{Q}(d)\in W^s_{q_s}(\tau)$, in the latter $\boldsymbol{Q}(d)=\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(\tau)$ but this contradicts Lemma 3.7. So $d\in I$ and there is $T(\boldsymbol{Q}(d))>\tau$ such that $\boldsymbol{x}_{\boldsymbol{q_s}}(t,\tau,\boldsymbol{Q}(d))$ intersects transversally the y negative semi-axis at $t=T(\boldsymbol{Q}(d))$.

Since we can choose τ arbitrarily large the Lemma is proved by observing that $R(d)=\exp[T(\boldsymbol{Q}(d))]>\mathrm{e}^\tau$.

Assume now $2\le q_s\le p_*(\delta)$ (but $(p,q_s)\ne(2,2)$). In this case from Lemma 3.6 we see directly that $u(r,d)$ is a crossing solution for any $0<d<U$. Notice that the origin is again a saddle, so we can construct $W^u_{q_s}(\tau,\rho)$ and it is still tangent to the x axis in the origin (for any τ). Hence repeating the last lines of the previous case we easily see that $R(d)\to+\infty$ as $d\to 0$.

We emphasize that if $(p,q_s)=(2,2)$ the argument fails because we cannot apply anymore Fowler transformation which indeed requires $l=q_s>p$. □

Notice that in Lemma 3.8 assumption ***F2′*** is needed just to construct the unstable manifold.

Now, putting together Lemmas 3.4, 3.5, 3.8 and the fact that $U \notin I$ if $\boldsymbol{B}$ holds, we get the following result, which is summarized in figure 2b.

Proposition 3.9. *Consider (10) where $\lambda = 1$. Assume **F2′**, **F0′** and **B**, then I is open. Further there is $M \geq U$ such that $(0, U) \cup (M, +\infty) \subset I$, but $U, M \notin I$. Let $R(d)$ be the first zero of $u(r, d)$, then $R(d)$ is continuous in I, $\lim_{d\to+\infty} R(d) = 0$, while $\lim_{d\to 0^+} R(d) = \lim_{d\to U^-} R(d) = \lim_{d\to M^+} R(d) = +\infty$.*

Now we turn to consider the case where (5) holds but $p = q_s = 2$: some changes are needed. First of all notice that we cannot set $l = q_s$ in (11) so we cannot construct system (17). On the other hand if ***F2*** holds we can set $l = q_u > 2$ to construct the unstable manifolds $W^u_{q_u}(-\infty)$ and $W^u_{q_u}(\tau)$, and Remarks 2.8, 2.9, 2.10 hold with no changes. However, since the linearization of (12) in the origin does not have constant eigenvalues and eigenvectors, we can just say that $W^u_{q_u}(-\infty)$ is tangent to the x axis in the origin, but the tangent to $W^u_{q_u}(\tau)$ may (and usually will) change with τ. Hence Remark 2.11 does not hold in this context.

Further notice that Remark 3.1 and Lemma 3.4 hold (with no changes in the proof). However we have to replace Lemma 3.6 by the following.

Lemma 3.10. *Assume **F0**, **F2** with $p = q_s = 2$, $\delta > -2$ and **B**; then we get the same conclusions as in Lemma 3.6. In particular $(0, U) \subset I$.*

Proof. This result follows from some standard result in oscillation theory, see, e.g., [9, Theorem 3.1.4]; however we give a full fledged proof for completeness.

Again from Remark 3.2 we see that either $u(r) \to 0$ as $r \to +\infty$ or it is a crossing solution and we are done, so we assume the former. Hence for any $\varepsilon > 0$ we can find $\tilde{R}_1 > \tilde{R}$ such that $u(r) < \varepsilon$ and $(n-1)/r < \varepsilon$ if $r \geq \tilde{R}_1$. Then we can choose $\varepsilon > 0$ small enough and $\tilde{R}_1$ large enough so that $\frac{c_s}{2} u(r) < f(u(r)) < 2c_s u(r)$ for $r \geq \tilde{R}_1$. Let us consider the equations

$$\begin{array}{c} u'' + \varepsilon u' + \frac{c_s}{2} u = 0 \\ u'' + 2c_s u = 0 \end{array} \tag{25}$$

and denote respectively by $\overline{u}(r)$ and $\underline{u}(r)$ the solutions of the former and the latter equation in (25) such that $\overline{u}(\tilde{R}_1) = u(\tilde{R}_1) = \underline{u}(\tilde{R}_1)$ and $\overline{u}'(\tilde{R}_1) = u'(\tilde{R}_1) = \underline{u}'(\tilde{R}_1)$. Notice that there are $\underline{R} > \overline{R} > \tilde{R}_1$ such that $\underline{u}(r)$ and $\overline{u}(r)$ are positive and decreasing respectively for $\tilde{R}_1 \leq r < \underline{R}$, and for $\tilde{R}_1 \leq r < \overline{R}$ and they become null with nonnegative slope at $r = \underline{R}$ and at $r = \overline{R}$.

We claim that there is $R \in (\overline{R}, \underline{R})$ such that $u(r)$ is positive and decreasing for $\tilde{R}_1 < r < R$ and $u(R) = 0 > u'(R)$, so that the Lemma is proved. To prove the claim consider the phase plane u, u' and draw the curves

$$\overline{\Gamma} := \{(\overline{u}(r), \overline{u}'(r)) \mid \tilde{R}_1 \leq r \leq \overline{R}\}, \quad \underline{\Gamma} := \{(\underline{u}(r), \underline{u}'(r)) \mid \tilde{R}_1 \leq r \leq \underline{R}\},$$

and denote by E the compact set enclosed by $\overline{\Gamma}$, $\underline{\Gamma}$ and the line $u = 0$. Notice that the flow of (2) on $\overline{\Gamma} \cup \underline{\Gamma}$ points towards the interior of E for any $r \geq \tilde{R}_1$. So there is $R \in (\overline{R}, \underline{R})$ such that $(u(r), u'(r)) \in E$ for any $\tilde{R}_1 \leq r \leq R$ and it crosses transversally the line $u = 0$ for $r = R$, and the claim is proved. □

Remark 3.11. We emphasize that in the assumption of Lemma 3.10 we lose control of $R(d)$ as $d \to 0^+$. In fact it might be shown that $R(d) \to R(0) := \lambda_1/c_s > 0$ as $d \to 0^+$ where λ_1 is the first eigenvalue of the Laplacian in the ball of radius 1. We do not give a proof of the result which is beyond the purpose of this paper, however see [26].

Now we are ready to state the counterpart of Proposition 3.9 for the case $p = q_s = 2$, see 3.

Proposition 3.12. *Consider (10) where $\lambda = 1$. Assume **F0** and **B**, and that f satisfies (5) with $p = q = 2$. Then I is open and there is $M \geq U$ such that $(0, U) \cup (M, +\infty) \subset I$, but $U, M \notin I$. Further $R(d)$ is continuous in I, and $\lim_{d\to U^-} R(d) = \lim_{d\to M^+} R(d) = +\infty$.*

3.2 Theorem 1.6: the graph of $R(d)$ for $\lambda = 1$ fixed.

In this subsection we assume the hypotheses of Theorem 1.6.

Let us introduce now some auxiliary functions which will allow us to construct the unstable manifold. Set

$$f^m(u) = \begin{cases} f(u) & \text{if} \quad 0 \leq u \leq U+1 \\ \phi(u) & \text{if} \quad U+\sigma \leq u \leq U+\sigma+1 \\ \frac{f(U+\sigma+1)}{(U+\sigma+1)^{q_u-1}} u^{q_u-1} & \text{if} \quad u \geq U+\sigma+1 \end{cases}, \tag{26}$$

and $f^h(u) = f^m(u-U)$; here $\phi(u)$ is a positive function such that $f^m(u)$ is C^1 and $2 < q_u < p^*(\delta)$.

We introduce the following notation: we denote by $g_l^m(x,t)$ and by $g_l^h(x,t)$ the functions g_l of (11) where f is replaced by f^m and f^h respectively. Similarly we refer to system (12) where g_l is replaced by g_l^m and g_l^h as to (12m) and (12h). Observe that both $f^m(u)$ and $f^h(u)$ satisfy $\boldsymbol{F2'}$ so, for any $\tau \in \mathbb{R}$, (12m) and (12h) admit an unstable manifold denoted by $W^{u,m}_{q_u}(\tau)$ and by $W^{u,h}_{q_u}(\tau)$ respectively. Analogously we denote with a "m" and "h" all the quantities of (12m) and (12h) respectively to distinguish them from the ones of (12). We use the same notation for $u^m(r,d)$, $u^h(r,d)$ and $u(r,d)$. Notice in particular that $u^h(r,d) = u^m(r,d+U)$.

Lemma 3.13. *Assume $\boldsymbol{F1'}$ and $\boldsymbol{A}$. Then there is $\varepsilon > 0$, $0 < \varepsilon < \sigma$, such that for any $U < d < U + \varepsilon$ there is $R_1(d) > 0$ such that $u(R_1(d), d) = U$ and $\frac{\partial u}{\partial r}(R_1(d), d) < 0$.*

Proof. The Lemma is equivalent to say that $u^h(r,d)$ is a crossing solution for $0 < d < \varepsilon$. But this follows simply observing that f^h satisfies $\boldsymbol{F0'}$, $\boldsymbol{F2'}$ and applying Lemma 3.8, if $(p, Q_s) \neq (2,2)$ and applying Lemma 3.10 if $p = Q_s = 2$. □

Lemma 3.14. *Assume $\boldsymbol{F0'}$, $\boldsymbol{F1'}$ and $\boldsymbol{A}$. Then there is $\varepsilon > 0$, $0 < \varepsilon < \sigma$, such that $u(r,d)$ is a crossing solution for any $U < d \leq U + \varepsilon$, and it is decreasing as long as it is positive.*

Proof. Observe that $u(r,U) \equiv u^m(r,U) \equiv U$ for any $r \geq 0$. Let $\tau_0 > 0$ be the constant defined in Lemma 3.7; it follows that $\bar{\boldsymbol{Q}} := (U\mathrm{e}^{\alpha_{q_s}\tau_0}, 0) \in W^{u,m}_{q_s}(\tau_0)$ and $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{q_s}}(t, \tau_0, \bar{\boldsymbol{Q}}) = (U\mathrm{e}^{\alpha_{q_s} t}, 0) \in W^{u,m}_{q_s}(t)$ for any $t \in \mathbb{R}$. Notice that $\bar{\boldsymbol{Q}} \in \mathcal{T}(c_0)$ and $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{q_s}}(t, \tau_0, \bar{\boldsymbol{Q}}) \in \mathcal{T}(c_0)$ for any $t \in \mathbb{R}$, where $\mathcal{T}(c_0)$ is defined in Lemma 3.7. Since $W^{u,m}_{q_s}(\tau_0)$ is a connected manifold there is a small connected branch of $W^{u,m}_{q_s}(\tau_0)$ containing $\bar{\boldsymbol{Q}}$ which is contained in $\mathcal{T}(c_0)$, say $w^{u,m}(\tau_0) \subset \mathcal{T}(c_0)$. Let $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(t)$ be the unique trajectory of (12) converging to the critical point $\boldsymbol{P^s} = (P^s_x, P^s_y)$ as $t \to +\infty$. From Lemma 3.7 we see that $W^{s,m}_{q_s}(\tau_0)$ does not intersect $\mathcal{T}(c_0)$ and $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{q_s}}(\tau_0) \notin \mathcal{T}(c_0)$ too. Hence if $\boldsymbol{Q} \in w^{u,m}(\tau_0)$ then $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{q_s}}(t, \tau_0, \bar{\boldsymbol{Q}})$ cannot converge neither to the origin nor to $\boldsymbol{P^s}$ as $t \to +\infty$. Therefore, using also Lemma 2.12, we see that the corresponding solutions $u^m(r,d)$ of (10m) do not converge to 0. Hence, possibly choosing a smaller $\varepsilon > 0$, we can assume that $u^m(r,d)$ does not converge to 0 for any $U < d < U + \varepsilon$.

Further, reasoning as in the proof of Remark 3.1, it is easy to check that $u^m(r,d)$ is decreasing as long as it is positive. Hence from Lemma 3.13 and Remark 3.1 we see that, for any $U < d < U + \varepsilon$, there is $R(d) > 0$ such that $u^m(r,d)$ is positive and decreasing for $0 \leq r < R(d)$ and becomes null with non-negative slope at $r = R(d)$. Further $u^m(r,d)$ is actually smaller than $U + \sigma$ for any $0 \leq r \leq R(d)$; so it solves the original equation (2), and we have $u^m(r,d) \equiv u(r,d)$ for $0 \leq r \leq R(d)$. □

From Lemma 3.14 we see that in the setting of Theorem 1.6 we can still apply Lemma 3.8. So using also Remark 3.4 we get the following.

Proposition 3.15. *Consider (10) where $\lambda = 1$ Assume $\boldsymbol{F0'}$, $\boldsymbol{F1'}$ and $\boldsymbol{B}$, then I is open and there is $\varepsilon > 0$ such that $(0, U+\varepsilon]\setminus U \subset I$. Further $\lim_{d\to U^+} R(d) = \lim_{d\to U^-} R(d) = \lim_{d\to 0} R(d) = +\infty$.*

Similarly if $p = q_s = 2$, combining Lemma 3.14, Lemma 3.8 we get a result analogous to Proposition 3.12.

Proposition 3.16. *Consider (10) where $\lambda = 1$. Assume f satisfies (5) with $p = q_s = 2$, $\boldsymbol{F1'}$ and $\boldsymbol{B}$, then I is open and there is $\varepsilon > 0$ such that $(0, U+\varepsilon]\setminus U \subset I$. Further $\lim_{d\to U^+} R(d) = \lim_{d\to U^-} R(d) = +\infty$.*

Remark 3.17. In the assumption of Proposition 3.15 $u(r,d)$ is decreasing as long as it is positive for any $0 < d < U + \varepsilon$, $d \neq U$.

Proof. The proof for $0 < d < U$ follows from Remark 3.3, the case $U < d < U + \varepsilon$ follows from the argument of the proof of Lemma 3.14. □

3.3 Scaling argument

From Propositions 3.9 and 3.15 we immediately get the following.

Corollary 3.18. *Either in the assumptions of Proposition 3.9 or in the assumptions of Proposition 3.15 there is $K_0 > 0$ such that the equation in d $R(d) = K$, has at least 3 solutions, say $d_1 < d_2 < d_3$, for any $K \geq K_0$. Further $d_2 < U < d_3$ and as $K \to +\infty$ we have $d_1 \to 0$ and $d_2 \to U^-$, $d_3 \to M^+$ (notice that $M = U$ if $\boldsymbol{A}$ holds).*

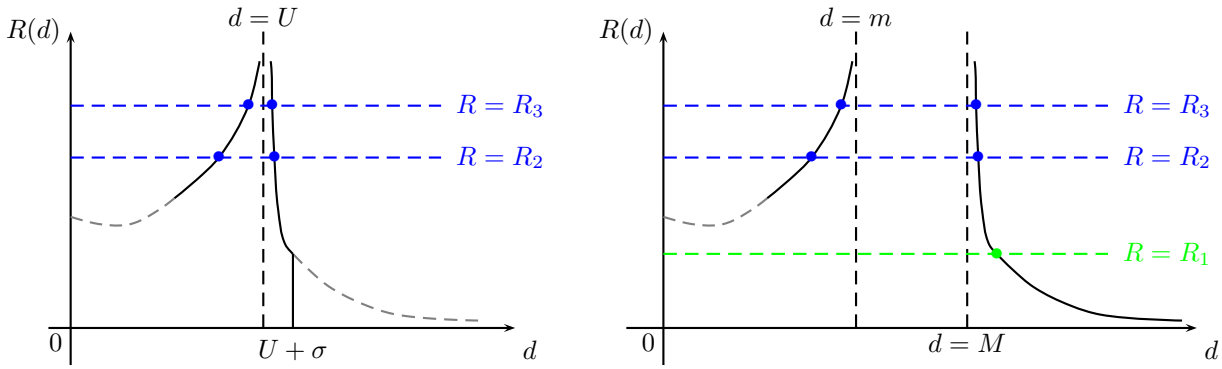

Figure 3: Sketch of the graph of the function $R(d)$ in the setting of Proposition 3.12 on the left, and in the setting of Proposition 3.16 on the right. We have drawn with solid lines the part of the graph which has been constructed through the Propositions and with dotted lines the part of the graph constructed through modified problem (for d large in fig. a)) or which are just conjectured (for d small in both fig. a) and b)). In both graphs a) and b) we have at least 2 values d_i, such that $R(d_i) = R$ when $R > R_2$; further $0 < d_2 < U < d_3$. Moreover in picture a) we have $d_2 \to U^-$ and $d_3 \to U^+$ as $R \to +\infty$, while in picture b) we have $d_2 \to m^-$ and $d_3 \to M^+$ as $R \to +\infty$.

Now the proof of Theorems 1.7, 1.6 easily follows from a standard scaling argument.

Proof of Theorem 1.7 and Theorem 1.6. Assume that we are in the hypotheses either of Theorem 1.7 or of Theorem 1.6. Then the following Dirichlet problem

$$\begin{aligned} &(w'|w'|^{p-2})r^{n-1})' + f(w) = 0\,, \\ &w(0,d) = d > 0\,,\ w(1,d) = 0\,,\ w(r) > 0\,, \text{for } 0 < r < R \end{aligned} \tag{27}$$

admits at least 3 solutions for any $R \geq K_0$.
Now we turn to consider the original equation (10) where $\lambda \neq 1$. Set $u(r) = w(r\lambda^a)$, $a = \frac{1}{p-1} > 0$, $\rho = R\lambda^{-a}$, $\rho^*(\lambda) = K_0\lambda^{-a}$. If $w(r)$ satisfies (27) then $u(r)$ satisfies the following:

$$\begin{aligned} &(u'|u'|^{p-2})r^{n-1})' + \lambda f(u) = 0\,, \\ &u(0,d) = d > 0\,,\ u(\rho,d) = 0\,,\ u(r) > 0\,, \text{for } 0 < r < \rho \end{aligned} \tag{28}$$

Set $\lambda_0^* = K_0^{1/a}$, then the Dirichlet problem (10) in the ball of radius 1 admits at least 3 positive solutions for $\lambda \geq \lambda_0^*$. So the proof of Theorem 1.6 is concluded.

Now restrict to Theorem 1.7, so that Proposition 3.9 holds. Then denote by $m^* = \min\{R(d) \mid 0 < d < U\}$, and set $\lambda^* = [m^*]^{1/a}$, then we easily see that (28) admits at least 3 solutions for $\lambda > \lambda^*$ at least 2 solutions for $\lambda = \lambda^*$, and at least 1 solution for $0 < \lambda < \lambda^*$, so the proof of Theorem 1.7 is concluded. □

With the same scaling argument from Propositions 3.12, 3.16 we obtain Theorem 1.3.

We conclude by noticing that Corollaries 1.5 and 1.8 follows from Corollary 3.18, Remarks 3.3 and 3.17.

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