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McKean–Vlasov stochastic equations with Hölder coefficients

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A B S T R A C T

This work revisits the well-posedness of non-degenerate McKean–Vlasov stochastic differential equations with Hölder continuous coefficients, recently established by Chaudru de Raynal. We provide a streamlined and direct proof that leverages standard Gaussian estimates for uniformly parabolic PDEs, bypassing the need for derivatives with respect to the measure argument and extending applicability to hypoelliptic PDEs under weaker assumptions.

1. Introduction

In [1], Chaudru de Raynal recently established the well-posedness of non-degenerate McKean–Vlasov (MKV) stochastic differential equations with Hölder drift. His approach is based on the associated PDE formulated on the domain $[0,T]\times\mathbb{R}^d\times\mathcal{P}^2(\mathbb{R}^d)$, where $T>0$, d is the dimensionality of the system, and $\mathcal{P}^2(\mathbb{R}^d)$ represents the space of probability measures on $\mathbb{R}^d$ with finite second moment.

This note aims to present a more streamlined and direct proof of a more general result, avoiding the use of PDEs incorporating derivatives with respect to the measure variable. Our proof relies solely on standard Gaussian estimates for uniformly parabolic PDEs on $[0,T]\times\mathbb{R}^d$. This approach not only simplifies the argument but also renders the proof adaptable to broader classes of degenerate MKV equations, where the associated PDEs are hypoelliptic, provided that upper Gaussian bounds for their fundamental solutions are available.

Let $\mathcal{P}(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ and $[Z]$ be the law of a random variable Z. We consider the MKV equation

$$X_t = X_0 + \int_0^t B(s, X_s, [X_s])\mathrm{d}s + \int_0^t \Sigma(s, X_s, [X_s])\mathrm{d}W_s, \qquad t\in[0,T], \tag{1.1}$$

where W is a standard d-dimensional Brownian motion defined on a filtered probability space $(\Omega,\mathcal{F},\mathsf{P},\mathcal{F}_t)$ and $T>0$. We consider coefficients

$$B:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\longrightarrow\mathbb{R}^d, \qquad \Sigma:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\longrightarrow\mathbb{R}^{d\times d}, \tag{1.2}$$

of the form

$$B(t,x,\mu)=\int_{\mathbb{R}^d} b(t,x,y)\mu(\mathrm{d}y), \qquad \Sigma(t,x,\mu)=\int_{\mathbb{R}^d}\sigma(t,x,y)\mu(\mathrm{d}y). \tag{1.3}$$

This structural assumption can be significantly weakened to allow for broader forms. For further details and specific examples, refer to Remarks 1.5 and 1.6.

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Assumption 1.1 (*Regularity*). The coefficients $b, \sigma \in L^\infty([0,T]; bC^\alpha(\mathbb{R}^d \times \mathbb{R}^d))$ for some $\alpha \in]0,1]$, where bC^α denotes the space of bounded and α-Hölder continuous functions, equipped with the norm

$$\|f\|_{bC^\alpha} := \sup_z |f(z)| + [f]_{C^\alpha}, \qquad [f]_{C^\alpha} := \sup_{z \neq \zeta} \frac{|f(z) - f(\zeta)|}{|z-\zeta|^\alpha}.$$

Assumption 1.2 (*Non-degeneracy*). The matrix $C := \Sigma\Sigma^*$ is uniformly positive definite, that is

$$\langle C(t,x,\mu)y, y\rangle \geq \lambda |y|^2, \qquad t \in [0,T], \; x, y \in \mathbb{R}^d, \; \mu \in \mathcal{P}(\mathbb{R}^d),$$

for some positive constant λ.

Our main result is the following

Theorem 1.3 (*Weak well-posedness*). *Under Assumptions 1.1 and 1.2, for any $\bar{\mu}_0 \in \mathcal{P}(\mathbb{R}^d)$ there exists a unique weak solution of (1.1) such that $[X_0] = \bar{\mu}_0$.*

A direct consequence of Theorem 1.3 is the following

Corollary 1.4 (*Strong well-posedness*). *Under Assumptions 1.1 and 1.2, if $\sigma = \sigma(t,x,y)$ is also Lipschitz continuous in x uniformly in $(t,y) \in [0,T] \times \mathbb{R}^d$ then (1.1) admits a unique strong solution.*

Remark 1.5. Theorem 1.3 remains valid even if the coefficients B, Σ in (1.2) are not necessarily of the form (1.3), provided they are bounded, satisfy Assumption 1.2 and
the Hölder condition

$$|B(t,x,\mu) - B(t,y,\nu)| + |\Sigma(t,x,\mu) - \Sigma(t,y,\nu)| \leq c(|x-y|^\alpha + \mathfrak{m}_\alpha(\mu,\nu)). \tag{1.4}$$

Here $\mathfrak{m}_\alpha$ is the metric on $\mathcal{P}(\mathbb{R}^d)$ defined by

$$\mathfrak{m}_\alpha(\mu,\nu) := \sup_{\|f\|_{bC^\alpha} \leq 1} \int_{\mathbb{R}^d} f(x)\,(\mu - \nu)\,(\mathrm{d}x). \tag{1.5}$$

As (1.4) ensures the validity of estimate (3.2) in Section 3, the proof of Theorem 1.3 follows through without requiring significant modifications.

Remark 1.6. Our approach can also easily handle the case where the metric $\mathfrak{m}_\alpha$ in (1.4) is replaced by

$$m_\alpha(\mu,\nu) = \sup_{[f]_{C^\alpha} \leq 1} \int_{\mathbb{R}^d} f(x)(\mu-\nu)(\mathrm{d}x),$$

for some $\alpha \in]0,1]$, provided that the initial distribution belongs to $\mathcal{P}^\alpha(\mathbb{R}^d)$. The result in [1] fits as a special case. Specifically, in [1] the drift coefficient has the form

$$B(t,x,\mu) = \beta\left(t, x, \int_{\mathbb{R}^d} \varphi(y)\mu(\mathrm{d}y)\right)$$

where $\beta = \beta(t,x,w)$ is defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$, is bounded, Hölder continuous in x uniformly in (t,w), and is differentiable in w with bounded derivative, uniformly in (t,x); additionally, $\varphi \in C^\alpha(\mathbb{R}^d)$ and the initial distribution is required to belong to $\mathcal{P}^2(\mathbb{R}^d)$.

As mentioned earlier, our approach is naturally suited for generalization to broader classes of equations. Recently, in collaboration with A.Y. Veretennikov [2], we studied the weak well-posedness for class of *degenerate* MKV equations whose prototype is the kinetic system

$$\begin{cases} dX_t = V_t dt, \\ dV_t = B(t, X_t, V_t, [(X_t, V_t)])dt + \Sigma(t, X_t, V_t, [(X_t, V_t)])dW_t. \end{cases} \tag{1.6}$$

The classical Langevin model [3] serves as a particular instance of (1.6), where the solution is a $2d$-dimensional process (X_t, V_t) representing the dynamics of a system of d particles in phase space, with X_t and V_t denoting the position and velocity at time t, respectively. Despite the failure of the non-degeneracy Assumption 1.2 in this context and the associated Kolmogorov PDEs being hypoelliptic rather than uniformly parabolic, the existence and Gaussian estimates for the fundamental solution were established by the first author and collaborators in [4,5]. This case seems attainable and will be the subject of future work, where we aim to generalize the results of this paper to degenerate MKV equations.

The remainder of the paper is structured as follows. Section 2 introduces the analytical tools necessary to frame a fixed-point problem in the space of continuous flows of marginals. In particular, we prove a crucial lemma for the inversion of forward and backward transport operators. Section 3 is dedicated to the proof of Theorem 1.3. Finally, the Appendix compiles some result on the completeness of spaces of measures utilized throughout the paper.

2. Inversion lemma

The following notations for integrals will be used interchangeably throughout this section:

$$\int_{\mathbb{R}^d} f\,\mathrm{d}\nu = \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) = \int_{\mathbb{R}^d} \nu(\mathrm{d}x) f(x).$$

For a fixed flow of marginals $\mu = (\mu_t)_{t\in[0,T]} \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ and an initial distribution $\bar{\mu}_0 \in \mathcal{P}(\mathbb{R}^d)$, we consider the "linearized" version of (1.1)

$$\mathrm{d}X_t = B(t, X_t, \mu_t)\mathrm{d}t + \Sigma(t, X_t, \mu_t)\mathrm{d}W_t, \qquad [X_0] = \bar{\mu}_0, \tag{2.1}$$

which is a standard (i.e. non-MKV) SDE and admits a unique weak solution denoted by X^μ.

The solution X^μ is a Markov process with infinitesimal generator

$$\mathcal{A}^\mu_{t,x} := \frac{1}{2}\sum_{i,j=1}^d C^\mu_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^d B^\mu_i(t,x)\partial_{x_i}$$

where

$$C^\mu_{ij}(t,x) := (\Sigma\Sigma^*)_{ij}(t,x,\mu_t), \quad B^\mu_i(t,x) := B_i(t,x,\mu_t), \qquad (t,x)\in[0,T]\times\mathbb{R}^d,\ 1\le i,j\le d. \tag{2.2}$$

By the classical theory of uniformly parabolic PDEs with bounded and Hölder continuous coefficients (see [6] or the more recent presentation in [7]), operator $\partial_t + \mathcal{A}^\mu_{t,x}$ has a fundamental solution $p^\mu = p^\mu(t,x;s,y)$, defined for $0\le t<s\le T$ and $x,y\in\mathbb{R}^d$, that is the transition density of X^μ.

The *push-forward* and the *pull-back* operators acting on the distribution $\bar{\mu}_0 \in \mathcal{P}(\mathbb{R}^d)$ are defined as

$$\vec{P}^\mu_{t,s}\bar{\mu}_0(y) := \int_{\mathbb{R}^d} p^\mu(t,x;s,y)\bar{\mu}_0(\mathrm{d}x), \qquad \overleftarrow{P}^\mu_{t,s}\bar{\mu}_0(x) := \int_{\mathbb{R}^d} p^\mu(t,x;s,y)\bar{\mu}_0(\mathrm{d}y),$$

for $0\le t<s\le T$ and $x,y\in\mathbb{R}^d$. Notice that $\vec{P}^\mu_{0,s}\bar{\mu}_0$ is the density of the marginal law $[X^\mu_s]$.

A key element in the proof of Theorem 1.3 is the following inversion lemma, which expresses the push-forward operator in terms of pull-back operators. This result draws inspiration from [8], where a similar formula, equation (28) in [8], is presented, though formulated in the "opposite direction".

Lemma 2.1 (*Inversion*). *Let X^μ, X^ν be the weak solutions of the linearized SDE (2.1) corresponding to the flows of marginals $\mu,\nu \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ respectively, and with the same initial distribution $[X^\mu_0] = [X^\nu_0] = \bar{\mu}_0 \in \mathcal{P}(\mathbb{R}^d)$. Then, for any $f\in bC^\alpha(\mathbb{R}^d)$ and $s\in]0,T]$, we have*

$$I^{\mu,\nu}_s(f) := \int_{\mathbb{R}^d} \mathrm{d}y\, f(y)(\vec{P}^\mu_{0,s} - \vec{P}^\nu_{0,s})\bar{\mu}_0(y) = \int_{\mathbb{R}^d} \bar{\mu}_0(\mathrm{d}x)\int_0^s \mathrm{d}t\, \overleftarrow{P}^\mu_{0,t}\left(\mathcal{A}^\mu_{t,x} - \mathcal{A}^\nu_{t,x}\right)\overleftarrow{P}^\nu_{t,s} f(x). \tag{2.3}$$

Proof. Under Assumption 1.1, 1.2 and if the coefficients are also continuous with respect to the time variable[1], it is well-known that the transition density $p^\mu(\cdot,\cdot;s,y)\in C^{1,2}([0,s[\times\mathbb{R}^d)$ for any $(s,y)\in]0,T]\times\mathbb{R}^d$, and is a classical solution of the backward Kolmogorov PDE

$$(\partial_t + \mathcal{A}^\mu_{t,x})p^\mu(t,x;s,y) = 0, \qquad (t,x)\in[0,s[\times\mathbb{R}^d;$$

moreover, for any $(t,x)\in[0,T[\times\mathbb{R}^d$, $p^\mu(t,x;\cdot,\cdot)$ is a distributional solution of the forward Kolmogorov PDE $(\partial_s - \mathcal{A}^{\mu,*}_{s,y})p^\mu(t,x;s,y)=0$, that is

$$\int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, p^\mu(t,x;s,y)(\partial_s + \mathcal{A}^\mu_{s,y})\varphi(s,y) = 0,$$

for any test function $\varphi \in C^\infty_0 \in]t,T]\times\mathbb{R}^d$.

We have

$$I^{\mu,\nu}_s(f) = \int_{\mathbb{R}^d}\mathrm{d}y\, f(y)\int_{\mathbb{R}^d}\bar{\mu}_0(\mathrm{d}x)(p^\mu(0,x;s,y) - p^\nu(0,x;s,y)) =$$

(by Fubini's theorem and standard Gaussian estimates for p^μ and p^ν; see Chapter 1, Section 6 in [6] or Theorem 20.2.5 in [7])

$$= \int_{\mathbb{R}^d}\bar{\mu}_0(\mathrm{d}x)\int_{\mathbb{R}^d}\mathrm{d}y\, f(y)(p^\mu(0,x;s,y) - p^\nu(0,x;s,y))$$

$$= \int_{\mathbb{R}^d}\bar{\mu}_0(\mathrm{d}x)\int_{\mathbb{R}^d}\mathrm{d}y\, f(y)\int_0^s \mathrm{d}t\frac{d}{dt}\int_{\mathbb{R}^d}\mathrm{d}z\, p^\mu(0,x;t,z)p^\nu(t,z;s,y) =$$

[1] This assumption simplifies the presentation without imposing significant restrictions: even if the coefficients are merely measurable in t, $p^\mu(\cdot,x;s,y)$ remains absolutely continuous and the proof proceeds in a similar manner.

(by classical potential estimates; see Chapter 1, Section 3 in [6] or Proposition 20.3.9 in [7])

$$= \int_{\mathbb{R}^d} \bar{\mu}_0(\mathrm{d}x) \int_{\mathbb{R}^d} \mathrm{d}y\, f(y) \int_0^s \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}z \left((\partial_t p^\mu(0,x;t,z)) p^\nu(t,z;s,y) + p^\mu(0,x;t,z) \partial_t p^\nu(t,z;s,y) \right) =$$

(by the forward and backward Kolmogorov equations)

$$= \int_{\mathbb{R}^d} \bar{\mu}_0(\mathrm{d}x) \int_{\mathbb{R}^d} \mathrm{d}y\, f(y) \int_0^s \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}z\, p^\mu(0,x;t,z) \left(\mathcal{A}^\mu_{t,z} - \mathcal{A}^\nu_{t,z} \right) p^\nu(t,z;s,y) =$$

(using the potential estimates once again)

$$= \int_{\mathbb{R}^d} \bar{\mu}_0(\mathrm{d}x) \int_0^s \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}z\, p^\mu(0,x;t,z) \left(\mathcal{A}^\mu_{t,z} - \mathcal{A}^\nu_{t,z} \right) \int_{\mathbb{R}^d} \mathrm{d}y\, f(y) p^\nu(t,z;s,y)$$
$$= \int_{\mathbb{R}^d} \bar{\mu}_0(\mathrm{d}x) \int_0^s \mathrm{d}t\, \overleftarrow{P}^\mu_{0,t} \left(\mathcal{A}^\mu_{t,x} - \mathcal{A}^\nu_{t,x} \right) \overleftarrow{P}^\nu_{t,s} f(x). \qquad \square$$

3. Proof of Theorem 1.3

The proof of Theorem 1.3 relies on a contraction mapping principle in the space $C([0,T];\mathcal{P}(\mathbb{R}^d))$ of continuous flows of distributions $(\mu_t)_{t\in[0,T]}$, equipped with the metric

$$\mathfrak{M}_\alpha(\mu,\nu) := \max_{t\in[0,T]} \mathfrak{m}_\alpha(\mu_t,\nu_t),$$

where $\mathfrak{m}_\alpha$ is defined in (1.5). Specifically, denoting X^μ the solution of (2.1), we claim that the map

$$\mu \longmapsto [X^\mu_t]_{t\in[0,T]}$$

is a contraction on $C([0,T];\mathcal{P}(\mathbb{R}^d))$, at least for $T > 0$ suitably small. The thesis will readily follow from this assertion. Indeed, by Proposition A.1, $(C([0,T];\mathcal{P}(\mathbb{R}^d)),\mathfrak{M}_\alpha)$ is a complete metric space and therefore there would exist a unique fixed point $\bar{\mu} \in C([0,T];\mathcal{P}(\mathbb{R}^d))$, such that $\bar{\mu} = [X^{\bar{\mu}}]$. Thus, $X^{\bar{\mu}}$ is the unique weak (or strong, under the assumption of Corollary 1.4) solution of the MKV Eq. (1.1).

Recalling the notation (2.2) for the coefficients of the infinitesimal generators, we have the preliminary estimate

$$\|\mathcal{C}^\mu - \mathcal{C}^\nu\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|B^\mu - B^\nu\|_{L^\infty([0,T]\times\mathbb{R}^d)} \le c\,\mathfrak{M}_\alpha(\mu,\nu) \tag{3.1}$$

for some positive constant c which depends only on the norms of b and σ in $L^\infty([0,T];bC^\alpha(\mathbb{R}^d\times\mathbb{R}^d))$. Estimate (3.1) follows from

$$|B^\mu(t,x) - B^\nu(t,x)| = \left| \int_{\mathbb{R}^d} b(t,x,y) \left(\mu_t - \nu_t\right)(\mathrm{d}y) \right| \le \|b(t,x,\cdot)\|_{bC^\alpha(\mathbb{R}^d)} \mathfrak{m}_\alpha(\mu_t,\nu_t) \tag{3.2}$$

and the analogous estimate for $\mathcal{C}$.

Now, we note that

$$\mathfrak{M}_\alpha(\mu,\nu) = \max_{s\in[0,T]} \sup_{\|f\|_{bC^\alpha}\le 1} I^{\mu,\nu}_s(f) \tag{3.3}$$

with $I^{\mu,\nu}_s(f)$ as in (2.3), and we have

$$|I^{\mu,\nu}_s(f)| \le \int_0^s \|\overleftarrow{P}^\mu_{0,t}(\mathcal{A}^\mu_{t,\cdot} - \mathcal{A}^\nu_{t,\cdot})\overleftarrow{P}^\nu_{t,s} f\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}t$$
$$\le \int_0^s \|(\mathcal{A}^\mu_{t,\cdot} - \mathcal{A}^\nu_{t,\cdot})\overleftarrow{P}^\nu_{t,s} f\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}t \le$$

(by (3.1))

$$\le c\,\mathfrak{M}_\alpha(\mu,\nu) \int_0^s \max_{1\le i,j\le d} \left(\|\partial_{x_i x_j} \overleftarrow{P}^\nu_{t,s} f\|_{L^\infty(\mathbb{R}^d)} + \|\partial_{x_i} \overleftarrow{P}^\nu_{t,s} f\|_{L^\infty(\mathbb{R}^d)} \right) \mathrm{d}t \le$$

(by the Gaussian estimates for p^ν, adjusting the constant c as needed)

$$\le c\,\mathfrak{M}_\alpha(\mu,\nu) \int_0^s \frac{1}{(s-t)^{1-\frac{\alpha}{2}}} \mathrm{d}t \le c\,T^{\frac{\alpha}{2}} \mathfrak{M}_\alpha(\mu,\nu). \tag{3.4}$$

In conclusion, combining (3.3) and (3.4), we obtain

$$\mathfrak{M}_\alpha(\mu,\nu) \le c\,T^{\frac{\alpha}{2}} \mathfrak{M}_\alpha(\mu,\nu)$$

which proves the thesis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

We provide the proof of routine results for which we could not find an appropriate reference. We recall the notations

$$\mathfrak{M}_\alpha(\mu,\nu) = \max_{t\in[0,T]} \mathfrak{m}_\alpha(\mu_t,\nu_t), \qquad \mathfrak{m}_\alpha(\mu_t,\nu_t) = \sup_{\|f\|_{bC^\alpha}\le 1} \int_{\mathbb{R}^d} f(x)\left(\mu_t-\nu_t\right)(\mathrm{d}x),$$

$$M_\alpha(\mu,\nu) = \max_{t\in[0,T]} m_\alpha(\mu_t,\nu_t), \qquad m_\alpha(\mu_t,\nu_t) = \sup_{[f]_{C^\alpha}\le 1} \int_{\mathbb{R}^d} f(x)(\mu_t-\nu_t)(\mathrm{d}x),$$

Proposition A.1. *For any $\alpha\in\,]0,1]$, $\left(C([0,T];\mathcal{P}(\mathbb{R}^d)),\mathfrak{M}_\alpha\right)$ and $\left(C([0,T];\mathcal{P}^\alpha(\mathbb{R}^d)),M_\alpha\right)$ are complete metric spaces.*

Proof. It suffices to show that $(\mathcal{P}(\mathbb{R}^d),\mathfrak{m}_\alpha)$ and $(\mathcal{P}^\alpha(\mathbb{R}^d),m_\alpha)$ are complete metric spaces. We use the fact that $(\mathbb{R}^d,d_\alpha)$, where $d_\alpha(x,y):=|x-y|^\alpha$, is a Polish space. Then, we notice that $\mathfrak{m}_\alpha$ is the bounded Lipschitz distance with respect to d_α, that is

$$\mathfrak{m}_\alpha(\mu,\nu) = \sup_{\|f\|_{b\mathrm{Lip}(d_\alpha)}\le 1} \int_{\mathbb{R}^d} f(x)(\mu-\nu)(\mathrm{d}x) =: \|\mu-\nu\|^*_{\mathrm{BL}(d_\alpha)}, \qquad \mu,\nu\in\mathcal{P}(\mathbb{R}^d).$$

By Theorem 8.10.43 in [9], $(\mathcal{P}(\mathbb{R}^d),\|\cdot\|^*_{\mathrm{BL}(d_\alpha)}) = (\mathcal{P}(\mathbb{R}^d),\mathfrak{m}_\alpha)$ is complete. This proves the first part of the statement.

Analogously, m_α is the 1-Wasserstein metric on $\mathcal{P}^1(\mathbb{R}^d)$, where $\mathbb{R}^d$ is equipped with d_α: indeed, we have

$$m_\alpha(\mu,\nu) = \sup_{[f]_{\mathrm{Lip}(d_\alpha)}\le 1} \int_{\mathbb{R}^d} f(x)(\mu-\nu)(\mathrm{d}x) =: W_{1,d_\alpha}(\mu,\nu), \qquad \mu,\nu\in\mathcal{P}^1(\mathbb{R}^d,d_\alpha).$$

By Theorem 6.18 in [10], $(\mathcal{P}^1(\mathbb{R}^d,d_\alpha),W_{1,d_\alpha}) = (\mathcal{P}^\alpha(\mathbb{R}^d),m_\alpha)$ is a complete metric space. This concludes the proof. □

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