Published for SISSA by Springer

Received: *June 7, 2022*
Revised: *November 14, 2022*
Accepted: *December 18, 2022*
Published: *January 5, 2023*

Quantum field theoretic representation of Wilson surfaces. Part II. Higher topological coadjoint orbit model

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Abstract: This is the second of a series of two papers devoted to the partition function realization of Wilson surfaces in strict higher gauge theory. A higher 2-dimensional counterpart of the topological coadjoint orbit quantum mechanical model computing Wilson lines is presented based on the derived geometric framework, which has shown its usefulness in 4-dimensional higher Chern-Simons theory. Its symmetries are described. Its quantization is analyzed in the functional integral framework. Strong evidence is provided that the model does indeed underlie the partition function realization of Wilson surfaces. The emergence of the vanishing fake curvature condition is explained and homotopy invariance for a flat higher gauge field is shown. The model's Hamiltonian formulation is further furnished highlighting the model's close relationship to the derived Kirillov-Kostant-Souriau theory developed in the companion paper.

Keywords: Differential and Algebraic Geometry, Sigma Models, Topological Field Theories, Wilson, 't Hooft and Polyakov loops

ArXiv ePrint: 2205.12321

https://doi.org/10.1007/JHEP01(2023)016

Contents

1 Introduction

Wilson loops were introduced by Wilson in 1974 [1] as a natural set of gauge invariant variables suitable for the description of the non perturbative regime of quantum chromodynamics. Since then, they have found a wide range of applications in many branches of theoretical physics.

In the loop formulation of gauge theory [2–4], Wilson loops constitute a basis of the Hilbert space of gauge invariant wave functionals on the gauge field configuration space.

Wilson loops are also the fundamental constitutive elements of loop quantum gravity, in particular of the spin network and foam approaches of this latter [5, 6]. Wilson loops are further relevant in condensed matter physics at low energy, specifically in the study of topologically ordered phases of matter described by topological quantum field theories [7–9]. Finally, Wilson loops can be employed to study knot and link topology using basic techniques of quantum field theory in 3-dimensional Chern-Simons (CS) theory [10].

Higher gauge theory is a generalization of ordinary gauge theory where gauge fields are higher degree forms [11, 12] that is relevant in string theory [13], spin foam theory [14] and condensed matter physics [15]. Wilson surfaces [16–20], 2-dimensional counterparts of Wilson loops, enter naturally in field theories with higher gauge fields and are expected to be essential elements in the analysis of important aspects of them for reasons analogous to those for which Wilson loops are.

In 4 spacetime dimensions, fractional braiding statistics is adequately described through the correlation functions of Wilson loops and surfaces in BF type topological quantum field theories [21–23]. Wilson surfaces also should be a basic element of any field theoretic approach to the study of 2-dimensional knot topology [24, 25] through an appropriate 4-dimensional version of CS theory [26–28].

The goal of the present two-part study is constructing a 2-dimensional topological sigma model whose quantum partition function computes a Wilson surface in strict higher gauge theory on the same lines as the 1-dimensional topological sigma model furnishing a Wilson loop in ordinary gauge theory.

The idea of representing a given Wilson loop as the partition function of a suitable quantum mechanical system can be traced back to the work of Balachandran et al. [29]. The approach was subsequently further developed by Alekseev et al. in [30] and Diakonov and Petrov in [31, 32]. It was more recently applied to the canonical quantization of CS theory by Elitzur et al. in [33]. See [34–36] for readable reviews.

The quantum system underlying the partition function realization of a Wilson loop has an explicitly Hamiltonian description. The underlying phase space is a coadjoint orbit. As a symplectic manifold, it is described by Kirillov-Kostant-Souriau (KKS) theory [37] and is quantized using the methods of geometric quantization [38, 39]. The resulting quantum theory can be understood at the light of the Borel-Weil theorem [40, 41]. It can also be reproduced as the functional integral quantization of a 1-dimensional sigma model, the topological coadjoint orbit (TCO) model.

The problem of obtaining a partition function realization of a Wilson surface, the main object of our study, has been tackled previously in the literature from different perspectives [42–44]. Our approach to the topic is firmly framed in higher gauge theory. It aims to formulate a higher KKS theory and to construct a higher version of the TCO model through the derived geometrical framework worked out in refs. [45, 46].

1.1 Plan of the endeavour

The present endeavour is naturally divided in two parts, which we refer to as I and II, of which the present paper is the second.

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In I [47], a higher version of the KKS theory of coadjoint orbits is elaborated based on the derived geometric framework. An original definition of derived coadjoint orbit is proposed. A theory of derived unitary line bundles and Poisson structures on regular derived orbits is built. The proper derived counterpart of the Bohr-Sommerfeld quantization condition is then identified. A version of derived prequantization is put forward. The problems hindering a full quantization are discussed and a possible solution is suggested. The theory worked out and the results derived, mostly of a geometric nature, provide the grounding for the field theoretic constructions of II.

In II, the derived TCO sigma model is presented and studied in depth. Its symmetries are described. Its quantization is analyzed in the functional integral set-up. Substantial evidence is provided that the model does indeed underpins the partition function realization of a Wilson surface. It is shown how the vanishing fake curvature condition arises in this context and homotopy invariance for flat derived gauge field is proven. The model's Hamiltonian formulation is further furnished highlighting the model's close relationship to the derived KKS theory developed in I.

2 Part II: derived TCO model

The present paper, which constitutes part II of our endeavour, is devoted to the derived TCO model. In this section, we provide an introductory overview of the model and an outlook on future developments.

The derived TCO model is a 2-dimensional field theory, which under certain conditions turns out to be a sigma model. The model is most naturally formulated in the derived geometrical framework reviewed in great detail in section 3 of I. The derived set-up makes evident the structural affinity of the derived model to the ordinary one.

The derived TCO model is a higher extension of the ordinary TCO model just as derived KKS theory is a higher analog of ordinary KKS theory. From a formal point of view, however, the formulations of the derived and ordinary models are not as close as those of the derived and ordinary theory as presented in I are. The derived model has novel features and is richer than its ordinary counterpart is several respects. There is nevertheless a characteristic version of the derived model, whose analogy to the ordinary model is particularly evident and which is closely related to the derived geometric orbit theory much as the ordinary model is related to ordinary theory.

2.1 Plan of part II

Paper II is organized in three sections with the content described below. The main results are contained in the last section.

In section 3, we present derived gauge theory, a formulation of higher gauge theory based on the derived field framework of I which brings to light the formal affinity of higher to ordinary gauge theory.

In section 4, we review the ordinary TCO model. The topics covered are the model's formulation as a classical field theory, basic symmetries, sigma model interpretation, functional integral quantization and canonical analysis. The presentation of this subject we provide is intentionally structured in a way that directly suggests the derived extension elaborated later.

In section 5, we finally introduce the derived TCO model. The construction is patterned on that of the ordinary TCO model expounded in section 4. The model's formulation as a classical field theory, basic symmetries, interpretation as a sigma model, functional integral quantization and canonical analysis are so studied in depth. Relevant original traits that distinguish the derived model from the ordinary one, in particular the existence of gauge background preserving gauge symmetry are highlighted. We present a number of arguments pointing to the conclusion that the partition function of the derived model is to be identified with a Wilson surface depending on the model's data. We show in particular how the vanishing fake curvature condition emerges and homotopy invariance for a flat higher gauge field is proven. Finally, by means of a Hamiltonian formulation, we highlight the close relationship of the derived TCO model to the derived KKS theory developed in I. We also call attention to limitations of our analysis related to the non rigorous nature of the formal functional integral techniques used and the lack of a in-depth analysis of gauge fixing.

2.2 Overview of the derived TCO model

As we anticipated above, the formulation of the derived TCO model hinges on the derived geometrical set up introduced and described in I. As our remarks about this model have been merely qualitative up to this point, in the rest of this subsection we provide a somewhat more formal introduction to it. A complete more rigorous analysis of the material surveyed below is available in the main body of the paper according to the plan stated in subsection 2.1 above. To justify and provide motivation for the derived approach followed by us, we shall initially review briefly the derived CS theory worked out in ref. [28].

Higher gauge symmetry is described by crossed modules. A Lie group crossed module M consists of a source and target Lie group, E and G, together with a target and an action structure map, τ and μ, relating them with certain properties [48, 49] In the derived set-up, with a Lie group crossed module M there is associated a derived Lie group $\mathrm{D}\mathsf{M}$ with derived Lie algebra $\mathrm{D}\mathfrak{m}$.

Derived gauge fields on a manifold X are just $\mathrm{D}\mathsf{M}$- or $\mathrm{D}\mathfrak{m}$-valued fields. Integration is expressed by means of the Berezinian ϱ_X of X. Upon equipping M with an invariant pairing $\langle\cdot,\cdot\rangle$, it is possible to construct a degree 1 graded symmetric pairing $(\cdot,\cdot)$ of derived fields.

A higher gauge theory with gauge crossed module M features a higher gauge field Ω with 1- and 2-form components ω, $\mathit{\Omega}$ valued respectively in the Lie algebras $\mathfrak{g}$, $\mathfrak{e}$ of G, E and its higher gauge curvature Φ with 2-and 3-form components ϕ, $\mathit{\Phi}$ valued in the same way and expressed in terms of ω, $\mathit{\Omega}$ in a definite way [50, 51]. A gauge transformation U comprises a G-valued component u and an $\mathfrak{e}$-valued 1-form component U acting on the gauge field components ω, $\mathit{\Omega}$ according to a precise rule. In the derived formulation of higher gauge theory, a higher gauge field Ω is just a $\mathrm{D}\mathfrak{m}[1]$-valued derived field. The curvature Φ of Ω is the $\mathrm{D}\mathfrak{m}[2]$-valued derived field given in terms of Ω by the usual gauge theoretic relation, $\Phi = \mathrm{d}\Omega + \frac{1}{2}[\Omega,\Omega]$. Similarly, a higher gauge transformation U is a $\mathrm{D}\mathsf{M}$-valued derived field acting on Ω and consequently Φ in the familiar manner, viz $\Omega^{\mathrm{U}} = \mathrm{Ad}\,\mathrm{U}(\Omega) + \mathrm{U}^{-1}\mathrm{d}\mathrm{U}$, $\Phi^{\mathrm{U}} = \mathrm{Ad}\,\mathrm{U}(\Phi)$. The components of Ω, Φ as well as U as derived fields correspond precisely

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to the components of the fields Ω, Φ and U stand for in higher gauge theory. Further, the above relations, once expressed in components, take precisely the same form as they do in higher gauge theory.

The higher 4-dimensional CS theory worked out in ref. [28] is most naturally formulated in the derived field framework and so it is referred to as derived CS theory. The derived CS model is characterized by two basic data: a crossed module M encoding the model's symmetry equipped with an invariant pairing and a base 4-fold M. The model's field content consists of a derived gauge field Ω. The derived CS action reads as

$$CS(\Omega) = \frac{k}{4\pi} \int_{T[1]M} \varrho_M \left(\Omega, \mathrm{d}\Omega + \frac{1}{3}[\Omega, \Omega]\right), \tag{2.1}$$

where k is a constant, the CS level. In spite of evident similarities, the derived CS model differs from the ordinary one in important respects. It is 4-dimensional as claimed because of the 1 unit of degree provided by the pairing $(\cdot, \cdot)$. Further, it is fully gauge invariant if the 4-fold M has no boundary. When it does, it is gauge variant, the gauge variation of the action reducing to a boundary term. Level quantization can occur only when special boundary conditions are obeyed by the derived gauge fields and gauge transformations. Finally, the models' canonical formulation on a 3-fold with boundary exhibits a rich edge field mode structure.

In the present paper, we proceed along the same lines as CS theory and employ all the main elements of the derived field set-up to formulate the higher 2-dimensional TCO model relevant for Wilson surface partition function realization directly as a derived TCO model.

The derived TCO model we have in mind is therefore a 2-dimensional topological sigma model with a derived homogeneous space as target space (cf. subsection 5.2 of I). As derived CS theory has been built on the assumption of a formal correspondence to ordinary CS theory, it is reasonable to hypothesize that derived TCO theory may be built by invoking a similar formal accordance to ordinary TCO theory. Proceeding in this way leads to the derived field theory outlined next. The data of the model are a crossed module M with invariant pairing, a base 2-fold N, a target manifold M and an embedding ς of N into M. The field content consists of a $\mathrm{D}\mathsf{M}$-valued derived TCO field G. The model's action is

$$S(\mathrm{G}; \Omega) = \int_{T[1]N} \varrho_N(\mathrm{K}, \operatorname{Ad} \mathrm{G}^{-1}(\varsigma^* \Omega) + \mathrm{G}^{-1}\mathrm{d}\mathrm{G}), \tag{2.2}$$

where K is a fixed degree 0 $\mathrm{D}\mathfrak{m}$-valued level current such that $\mathrm{d}\mathrm{K} = 0$ and Ω is a background target space derived gauge field. The structural congruity of S with the 1-dimensional TCO action (1.2.8) of I is manifest. Derived TCO theory is however 2-dimensional again because of the 1 unit of degree supplied by the pairing $(\cdot, \cdot)$. Further, when formulated through the components of G in conventional terms, it features a G-valued field g and a $\mathfrak{e}$-valued 1-form field G.

There are non trivial conditions which the level current K must obey in order the TCO model to be a sigma model having a derived homogeneous space as target space as desired. In the absence of these restrictions, (2.2) describes only a sigma model with $\mathrm{D}\mathsf{M}$ as target space. A $\mathrm{D}\mathsf{M}$-valued map Υ is called a level preserving gauge transformation if $\mathrm{Ad}\,\Upsilon(\mathrm{K}) = \mathrm{K}$. Under right multiplication of the derived TCO fields G by any such gauge transformation Υ, the action S is invariant up to an additive term depending only on Υ and homotopically invariant. If the level current K satisfies an appropriate integrality condition, S is invariant mod 2π. As a consequence, at the quantum level, the TCO model enjoys a level preserving gauge symmetry. Under mild conditions, the level preserving gauge transformations Υ are precisely those which are $\mathrm{D}\mathsf{M}_{\mathrm{K}}$-valued, where M_{K} is a certain crossed submodule of M depending on K. In this way, the model is a sigma model having the derived homogeneous space $\mathrm{D}\mathsf{M}/\,\mathrm{D}\mathsf{M}_{\mathrm{K}}$ as its target space.

In close analogy to the CS model, the derived TCO model has a number of features with no analogue in the ordinary one. The field equations are integrable only if a certain integrability condition is satisfied, which reduces to the fake flatness of Ω well-known in higher gauge theory. It has further a novel gauge background preserving gauge symmetry associated with the special gauge transformations leaving the gauge background $\varsigma^*\Omega$ invariant.

A detailed functional integral analysis indicates that the quantum partition function $Z(\Omega)$ of the derived TCO model enjoys the properties which a Wilson surface $W_R(N)$ is supposed to do. In fact, it is properly defined only when the background derived gauge field Ω is fake flat. As a functional of Ω, $Z(\Omega)$ is gauge invariant. Furthermore, when Ω is flat, $Z(\Omega)$ is also invariant under smooth variations of the embedding ς by virtue of certain Schwinger-Dyson relations under mild assumptions on the symmetry crossed module M. All this provides strong evidence that $Z(\Omega)$ does indeed constitute a functional integral realization of an underlying Wilson surface.

For a special choice of the level current K specified by an element Λ of the source Lie algebra $\mathfrak{e}$, the derived TCO sigma model takes a special form, which we dub 'characteristic'. The model's action, expressed explicitly in components, takes the form

$$\begin{aligned} S(g,G;\omega,\Omega) = & \int_{T[1]N} \varrho_N \left\langle \dot{\tau}(\Lambda), \dot{\mu}\left(g^{-1}, \varsigma^*\Omega + \dot{\mu}(\varsigma^*\omega, G) + dG + \frac{1}{2}[G,G]\right)\right\rangle \\ & - \int_{T[1]\partial N} \varrho_{\partial N} \left\langle \mathrm{Ad}\, g^{-1}\left(\varsigma^*\omega + dgg^{-1} + \dot{\tau}(G)\right), \Lambda\right\rangle. \end{aligned} \tag{2.3}$$

The characteristic model so features an ordinary TCO field, g, coupling to a kind of gauge field, G. Under rather general assumptions on the crossed module M, its pairing $\langle\cdot,\cdot\rangle$ and the datum Λ, the model is in fact a sigma model. Its target space is the derived coadjoint orbit $\mathcal{O}_\Lambda$ of Λ, precisely the kind of derived homogeneous space studied by the derived KKS theory elaborated in I. The relationship of the model to this latter is in this way established, showing convincingly that its quantization must definitely proceed in the realm of 2-dimensional quantum field theory.

2.3 Outlook

To summarize, in this paper we work out a 2-dimensional topological quantum field theory, the derived TCO model, whose partition function provides a candidate field theoretic expression of a corresponding Wilson surface. We present mounting evidence for such an identification based on the matching of several relevant properties. The data specifying the derived TCO model encode those defining the Wilson surface. Equally significantly,

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the relationship of the characteristic version of model and derived KKS theory sheds light on the quantization of this latter.

Though we have made considerable progress toward the goal we set, a few basic issues remain unsolved. The derived TCO model has an extra gauge background preserving gauge symmetry with no analog in the ordinary model. This symmetry has to be appropriately taken care of by fixing the gauge and introducing ghost and antighost fields. Given the geometric nature of the model, this problem is likely to be amenable by the quantization scheme of Batalin and Vilkovsky [52, 53] especially in the formulation of this elaborated by Alexandrov et al. [54]. Finding a viable gauge fixing condition or equivalently a suitable gauge fermion may however turn out to be a hard problem.

Detailed calculations on specific input data would be desirable to further test the model. Further, following the path set long ago in refs. [24, 25], one may try to use the model in 4-dimensional CS theory to study 2-dimensional knot invariants with the basic techniques of quantum field theory along the same lines as ordinary CS theory [10].

3 Higher gauge theory in the derived formulation

In this section, we present derived gauge theory, a formulation of higher gauge theory based on the derived framework of section 3 of I. The derived field formalism has the virtue of bringing to light the close relationship of higher to ordinary gauge theory and allows so to import many ideas and techniques of the latter to the former. The benefits of this approach, which showed themselves previously in the construction of 4d CS theory in ref. [28], will become evident in the elaboration of the derived TCO model in section 5 below.

The topics covered include derived gauge fields and gauge transformations and special derived gauge symmetry. The latter reflects a gauge for gauge symmetry of fake flat gauge fields in higher gauge theory and underlies a basic gauge symmetry of the derived TCO model with no ordinary counterpart.

3.1 Derived gauge fields and transformations

In this subsection, we introduce derived gauge theory. Based on the derived field formalism of I, this constitutes an equivalent reformulation of higher gauge theory as ordinary gauge theory with exotic gauge group, the derived group of the relevant symmetry crossed module.

In the derived field formalism, fields on a base manifold X are valued either in the derived Lie group $\mathrm{D}\mathsf{M}$ of a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ or in the derived Lie algebra $\mathrm{D}\mathfrak{m}$ of the associated Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot{\tau}, \dot{\mu})$. $\mathrm{D}\mathsf{M}$-valued derived fields are elements of the mapping space $\mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$. If $\mathrm{W} \in \mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$ is one such field, then

$$\mathrm{W}(\alpha) = \mathrm{e}^{\alpha W}\, w \tag{3.1}$$

with $\alpha \in \mathbb{R}[-1]$, where $w \in \mathrm{Map}(T[1]X, \mathsf{G})$, $W \in \mathrm{Map}(T[1]X, \mathfrak{e}[1])$. w, W are called the components of W. Similarly, $\mathrm{D}\mathfrak{m}$-valued derived fields are elements of the mapping space $\mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$ for some integer p. If $\Psi \in \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$ is a field of this kind, then

$$\Psi(\alpha) = \psi + (-1)^p \alpha \mathit{\Psi}, \tag{3.2}$$

where $\psi \in \mathrm{Map}(T[1]X, \mathfrak{g}[p])$, $\Psi \in \mathrm{Map}(T[1]X, \mathfrak{e}[p+1])$, the components of Ψ. A more comprehensive review of the derived field formalism is provided in subsection 3.3 of I. Here, we employ the ordinary non internal version of the formalism.

In higher gauge theory, a Lie group crossed module M is assigned and a higher principal M-bundle P on a base manifold X is given. Higher gauge fields and gauge transformations consist in collections of local Lie valued map and form data organized respectively as non Abelian differential cocycles and cocycle morphisms [50, 51]. As the analysis provided below is of a local nature, this level of generality is not necessary. We thus restrict ourselves to the case where P is the trivial M-bundle for which gauge fields and gauge transformations turn out to be maps and forms globally defined on X.

The basic field of higher gauge theory is the derived gauge field, that is a map $\Omega \in \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}[1])$. In components, this reads as

$$\Omega(\alpha) = \omega - \alpha \mathit{\Omega}, \tag{3.3}$$

where $\omega \in \mathrm{Map}(T[1]X, \mathfrak{g}[1])$, $\mathit{\Omega} \in \mathrm{Map}(T[1]X, \mathfrak{e}[2])$ (cf. eq. (3.3.5) of I). ω, $\mathit{\Omega}$ are nothing but the familiar 1- and 2-form gauge fields of higher gauge theory.

The derived gauge field Ω is characterized by its curvature Φ defined by

$$\Phi = \mathrm{d}\Omega + \frac{1}{2}[\Omega, \Omega], \tag{3.4}$$

where the Lie bracket $[\cdot, \cdot]$ and the differential d are defined by (3.3.6) and (3.3.10) of I, respectively. The expression of Φ is otherwise formally identical to that of the curvature of a gauge field in ordinary gauge theory. By construction, $\Phi \in \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}[2])$. Expressed in components, Φ reads as

$$\Phi(\alpha) = \phi + \alpha \mathit{\Phi}, \tag{3.5}$$

where $\phi \in \mathrm{Map}(T[1]X, \mathfrak{g}[2])$, $\mathit{\Phi} \in \mathrm{Map}(T[1]X, \mathfrak{e}[3])$. ϕ, $\mathit{\Phi}$ are just the usual higher gauge theoretic 2- and 3-form curvatures. They are expressible in terms of ω, $\mathit{\Omega}$ through the familiar relations

$$\phi = d\omega + \frac{1}{2}[\omega, \omega] - \dot{\tau}(\mathit{\Omega}), \tag{3.6}$$

$$\mathit{\Phi} = d\mathit{\Omega} + \dot{}\mu\dot{}(\omega, \mathit{\Omega}). \tag{3.7}$$

The derived curvature Φ satisfies the derived Bianchi identity

$$\mathrm{d}\Phi + [\Omega, \Phi] = 0, \tag{3.8}$$

which follows from (3.4) in the usual way. This turns into a pair of Bianchi identities for the curvature components ϕ, $\mathit{\Phi}$, viz

$$d\phi + [\omega, \phi] + \dot{\tau}(\mathit{\Phi}) = 0, \tag{3.9}$$

$$d\mathit{\Phi} + \dot{}\mu\dot{}(\omega, \mathit{\Phi}) - \dot{}\mu\dot{}(\phi, \mathit{\Omega}) = 0. \tag{3.10}$$

A derived gauge transformation is codified in a derived Lie group valued map $\mathrm{U} \in \mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$. U acts on the derived gauge field Ω as

$$\Omega^{\mathrm{U}} = \mathrm{Ad}\,\mathrm{U}^{-1}(\Omega) + \mathrm{U}^{-1}\mathrm{d}\mathrm{U}, \tag{3.11}$$

where the adjoint action and pulled-back Maurer-Cartan form of U in the right hand side are defined in eqs. (3.3.8) and (3.3.14) of I, respectively. Again, in the derived formulation the derived gauge transformation action is formally identical to that of ordinary gauge theory. The derived curvature transforms as

$$\Phi^{\mathrm{U}} = \mathrm{Ad}\,\mathrm{U}^{-1}(\Phi), \tag{3.12}$$

as expected. The gauge transformation U can be expressed in components as

$$\mathrm{U}(\alpha) = \mathrm{e}^{\alpha U}\, u \tag{3.13}$$

with $u \in \mathrm{Map}(T[1]X, \mathsf{G})$, $U \in \mathrm{Map}(T[1]X, \mathfrak{e}[1])$ according to (3.3.1) of I. In terms of these, using systematically relations (3.3.8), (3.3.14) of I, it is possible to write down the gauge transforms of the derived gauge field components ω, $\varOmega$,

$$\omega^{u,U} = \mathrm{Ad}\,u^{-1}\big(\omega + duu^{-1} + \dot\tau(U)\big)\,, \tag{3.14}$$

$$\varOmega^{u,U} = \mu^{\cdot}\Big(u^{-1}, \varOmega + {}^{\cdot}\mu^{\cdot}(\omega, U) + dU + \frac{1}{2}[U,U]\Big)\,, \tag{3.15}$$

as well as those of the derived curvature components ϕ, $\varPhi$,

$$\phi^{u,U} = \mathrm{Ad}\,u^{-1}(\phi)\,, \tag{3.16}$$

$$\varPhi^{u,U} = \mu^{\cdot}\big(u^{-1}, \varPhi + {}^{\cdot}\mu^{\cdot}(\phi, U)\big)\,. \tag{3.17}$$

These relations are the well-known expressions of the gauge transforms of the 1- and 2-form gauge field and 2- and 3-form curvature components of higher gauge theory.

An infinitesimal derived gauge transformation is a derived Lie algebra valued map $\Theta \in \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m})$. The gauge variation of the derived gauge field Ω is

$$\delta_\Theta \Omega = \mathrm{d}\Theta + [\Omega, \Theta], \tag{3.18}$$

where as before the Lie bracket $[\cdot,\cdot]$ and the differential d are given by (3.3.6) and (3.3.10) of I, respectively. In the derived formulation, the infinitesimal derived gauge transformation action is again formally identical to that of ordinary gauge theory, in particular it is the linearized form of its finite counterpart. As expected, so, the gauge variation of the derived curvature Φ reads as

$$\delta_\Theta \Phi = [\Phi, \Theta]. \tag{3.19}$$

The gauge transformation Θ can be expressed in components as

$$\Theta(\alpha) = \theta + \alpha\varTheta, \tag{3.20}$$

where $\theta \in \mathrm{Map}(T[1]X, \mathfrak{g})$, $\varTheta \in \mathrm{Map}(T[1]X, \mathfrak{e}[1])$ according to (3.3.1) of I. In terms of these, exploiting relations (3.3.8), (3.3.14) of I, we can write down the gauge variations of the derived gauge field components ω, $\varOmega$,

$$\delta_{\theta,\varTheta}\omega = d\theta + [\omega, \theta] + \dot\tau(\varTheta), \tag{3.21}$$

$$\delta_{\theta,\varTheta}\varOmega = d\varTheta + {}^{\cdot}\mu^{\cdot}(\omega, \varTheta) - {}^{\cdot}\mu^{\cdot}(\theta, \varOmega)\,, \tag{3.22}$$

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as well as those of the derived curvature components ϕ, $\varPhi$,

$$\delta_{\theta,\varTheta}\phi = [\phi, \theta], \tag{3.23}$$

$$\delta_{\theta,\varTheta}\varPhi = \dot{\mu}\dot{}\,(\phi, \varTheta) - \dot{\mu}\dot{}\,(\theta, \varPhi)\,. \tag{3.24}$$

Once more, these are the well-known expressions of the gauge variations of the 1- and 2-form gauge fields and the 2- and 3-form curvatures in higher gauge theory.

A gauge transformation T is special if its components t, T have the form

$$t = \tau(A), \tag{3.25}$$

$$T = -dAA^{-1} - \dot{\mu}(\omega, A), \tag{3.26}$$

where $A \in \mathrm{Map}(T[1]X, \mathsf{E})$. Note the dependence on an underlying derived gauge field Ω. By (3.14), (3.15), its action on the gauge field components ω, $\varOmega$ is

$$\omega^{t,T} = \omega, \tag{3.27}$$

$$\varOmega^{t,T} = \varOmega + \dot{\mu}(\phi, A^{-1}). \tag{3.28}$$

ω is so invariant. $\varOmega$ is not except for when the curvature component ϕ vanishes. The requirement that $\phi = 0$ is known in higher gauge theory as zero fake curvature condition. A derived gauge field Ω with this property is called fake flat. Fake flat gauge fields play an fundamental role in higher gauge theory.

An infinitesimal special gauge transformation Ξ has components

$$\xi = \dot{\tau}(\varPi), \tag{3.29}$$

$$\varXi = -d\varPi - \dot{\mu}\dot{}\,(\omega, \varPi)\,, \tag{3.30}$$

where $\varPi \in \mathrm{Map}(T[1]X, \mathfrak{e})$. In keeping with (3.27), (3.28), the corresponding variations of the derived gauge field components ω, $\varOmega$ are

$$\delta_{\xi,\varXi}\omega = 0, \tag{3.31}$$

$$\delta_{\xi,\varXi}\varOmega = -\dot{\mu}\dot{}\,(\phi, \varPi), \tag{3.32}$$

with $\varOmega$ invariant if the fake flatness condition $\phi = 0$ obtains.

We conclude this subsection introducing some notation that will be used frequently in the following. The field space of pure higher gauge theory on the manifold X is precisely the derived gauge field manifold $\mathcal{C}_{\mathsf{M}}(X) = \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}[1])$. Finite derived gauge transformations are organized in an infinite dimensional Lie group, the derived gauge transformation group $\mathcal{G}_{\mathsf{M}}(X) = \mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$, acting on $\mathcal{C}_{\mathsf{M}}(X)$ according to (3.11). Infinitesimal derived gauge transformations are similarly structured in an infinite dimensional Lie algebra, the derived gauge transformation algebra $\mathfrak{g}_{\mathsf{M}}(X) = \mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m})$, acting variationally on $\mathcal{C}_{\mathsf{M}}(X)$ through (3.18). Their operations are pointwise Lie group multiplication and inversion and bracketing, respectively. $\mathfrak{g}_{\mathsf{M}}(X)$ is further the virtual Lie algebra of $\mathcal{G}_{\mathsf{M}}(X)$.

Special gauge transformations constitute in their finite and infinitesimal form a Lie subgroup $\mathcal{G}_{\mathsf{M},\Omega}(X)$ of $\mathcal{G}_{\mathsf{M}}(X)$ and a Lie subalgebra $\mathfrak{g}_{\mathsf{M},\Omega}(X)$ of $\mathfrak{g}_{\mathsf{M}}(X)$ depending on an assigned gauge field Ω.

3.2 Distinguished features of special gauge symmetry

Special gauge symmetry is a manifestation of a gauge for gauge symmetry of higher gauge theory: special gauge transformations are non trivial gauge transformations acting trivially on fake flat gauge fields. In the derived TCO model elaborated in section 5, special gauge symmetry is associated with a basic gauge background preserving gauge symmetry. Given the importance that this latter holds in the model, we analyze its distinctive features in greater depth in this subsection.

Relations (3.25), (3.26) define a surjective map $\mathcal{T}_\Omega : \mathcal{H}_{\mathsf{M}}(X) \to \mathcal{G}_{\mathsf{M},\Omega}(X)$ depending on Ω, where $\mathcal{H}_{\mathsf{M}}(X) = \mathrm{Map}(T[1]X, \mathsf{E})$. $\mathcal{T}_\Omega$ is a Lie group morphism, as is straightforward to demonstrate. The kernel of $\mathcal{T}_\Omega$, $\mathcal{K}_{\mathsf{M},\Omega}(X)$, is a noteworthy subgroup of $\mathcal{H}_{\mathsf{M}}(X)$, as it consists of the elements $A \in \mathcal{H}_{\mathsf{M}}(X)$ whose associated special gauge transformations are trivial. Explicitly, $\mathcal{K}_{\mathsf{M},\Omega}(X)$ is formed by the maps $A \in \mathrm{Map}(T[1]X, \ker\tau)$ obeying the equation

$$dAA^{-1} + \dot{}\mu(\omega, A) = 0. \tag{3.33}$$

This equation is of a form analogous to that of the equation obeyed by the gauge transformations leaving a given gauge field invariant. We thus expect that $\mathcal{K}_{\mathsf{M},\Omega}(X)$, unlike $\mathcal{H}_{\mathsf{M}}(X)$, is generally finite dimensional. Infinitesimally, we have a surjective map $\dot{\mathcal{T}}_\Omega : \mathfrak{h}_{\mathsf{M}}(X) \to \mathfrak{g}_{\mathsf{M},\Omega}(X)$ defined by relations (3.29), (3.30), where $\mathfrak{h}_{\mathsf{M}}(X) = \mathrm{Map}(T[1]X, \mathfrak{e})$ is the Lie algebra of $\mathcal{H}_{\mathsf{M}}(X)$. $\dot{\mathcal{T}}_\Omega$ is a Lie algebra morphism whose kernel is the Lie subalgebra $\mathfrak{k}_{\mathsf{M},\Omega}(X)$ of $\mathfrak{h}_{\mathsf{M}}(X)$ formed by the elements $\mathit{\Pi} \in \mathfrak{h}_{\mathsf{M}}(X)$ with trivial associated infinitesimal special gauge transformations. $\mathfrak{k}_{\mathsf{M},\Omega}(X)$ consists so of the maps $\mathit{\Pi} \in \mathrm{Map}(T[1]X, \ker\dot{\tau})$ such that

$$d\mathit{\Pi} + \dot{}\dot{\mu}(\omega, \mathit{\Pi}) = 0 \tag{3.34}$$

and is generally finite dimensional.

Via $\mathcal{T}_\Omega$, the group $\mathcal{H}_{\mathsf{M}}(X)$ can be considered as a parameter space for the special gauge transformation group $\mathcal{G}_{\mathsf{M},\Omega}(X)$. The parametrization however is generally not one-to-one because of the kernel $\mathcal{K}_{\mathsf{M},\Omega}(X)$. Similarly, the algebra $\mathfrak{h}_{\mathsf{M}}(X)$ is a parameter space for the infinitesimal special gauge transformation algebra $\mathfrak{g}_{\mathsf{M},\Omega}(X)$ which again is generally not one-to-one because of the kernel $\mathfrak{k}_{\mathsf{M},\Omega}(X)$.

For a fixed derived gauge field $\Omega \in \mathcal{C}_{\mathsf{M}}(X)$, denote by $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ the set of all gauge transformations $\mathrm{T} \in \mathcal{G}_{\mathsf{M}}(X)$ with the property that $\Omega^{\mathrm{T}} = \Omega$. $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ is a subgroup of $\mathcal{G}_{\mathsf{M}}(X)$, the invariance subgroup of Ω. When Ω is fake flat, as we shall assume, the structure of $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ can be analyzed precisely. In this case, as is not difficult to check from (3.25), (3.26), the special gauge transformation group $\mathcal{G}_{\mathsf{M},\Omega}(X)$ is contained in $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ as a normal subgroup. The essential invariance group of Ω, the quotient group

$$\mathcal{I}_{\mathsf{M},\Omega}(X) = \mathcal{G}^*{}_{\mathsf{M},\Omega}(X)/\mathcal{G}_{\mathsf{M},\Omega}(X) \tag{3.35}$$

is then defined. Intuitively, $\mathcal{I}_{\mathsf{M},\Omega}(X)$ describes the gauge transformations leaving Ω invariant not reducible to 'trivial' special gauge transformations.

There are indications that $\mathcal{I}_{\mathsf{M},\Omega}(X)$ is a finite dimensional Lie group. We illustrate this for the simplest choice of Ω, namely $\Omega = 0$. In this case, a gauge transformation T belongs to $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ if and only if

$$dtt^{-1} + \dot{\tau}(T) = 0, \qquad dT + \frac{1}{2}[T,T] = 0 \tag{3.36}$$

by (3.14), (3.15). From (3.29), (3.30), further, two gauge transformations $\mathrm{T}, \mathrm{T}' \in \mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ are equivalent mod $\mathcal{G}_{\mathsf{M},\Omega}(X)$ if and only if

$$t' = \tau(A)t, \qquad T' = \operatorname{Ad} A(T) - dAA^{-1} \tag{3.37}$$

for some $A \in \mathcal{H}_{\mathsf{M}}(X)$. By (3.36), when $\mathrm{T} \in \mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$, T is a flat ordinary E-gauge field and t is a G-valued function gauge trivializing the associated G-gauge field $\dot{\tau}(T)$, so that T has $\ker\tau$, hence central, holonomy. By (3.37), then, $\mathcal{I}_{\mathsf{M},\Omega}(X)$ classifies such pairs (t,T) modulo E-gauge transformations. Under rather mild assumptions, that is that X is connected with torsion free first homology group, it is possible to show that $\mathcal{I}_{\mathsf{M},\Omega}(X) \simeq \mathsf{G} \times H^1(X, \ker\tau)$, where the two factors correspond respectively to the value of t at a base point of X and the holonomy of T. $\mathcal{I}_{\mathsf{M},\Omega}(X)$ is thus a finite dimensional Lie group. As $\Omega = 0$ is a non generic gauge field with a large invariance subgroup, we expect that for a generic fake flat gauge field Ω with a smaller invariance group $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$ the Lie group $\mathcal{I}_{\mathsf{M},\Omega}(X)$ is still finite dimensional. The infinite dimensional special gauge transformation group $\mathcal{G}_{\mathsf{M},\Omega}(X)$ so essentially exhausts $\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)$.

The space of fake flat gauge fields, $\mathcal{C}_{\mathsf{ffM}}(X)$, gets naturally subdivided into strata based on the isomorphism class of the essential invariance group of the gauge fields: two fake flat gauge fields $\Omega, \Omega' \in \mathcal{C}_{\mathsf{ffM}}(X)$ belong to the same stratum if and only if $\mathcal{I}_{\mathsf{M},\Omega'}(X) \simeq \mathcal{I}_{\mathsf{M},\Omega}(X)$. Each stratum is invariant under the gauge transformation action. Indeed, if $\mathrm{U} \in \mathcal{G}_{\mathsf{M}}(X)$ is a gauge transformation and Ω is a fake flat gauge field, then Ω^{U} also is by (3.16). Furthermore, we have that $\mathcal{G}^*{}_{\mathsf{M},\Omega^{\mathrm{U}}}(X) = \mathrm{U}^{-1}\mathcal{G}^*{}_{\mathsf{M},\Omega}(X)\mathrm{U}$ and by virtue of (3.25), (3.26) also that $\mathcal{G}_{\mathsf{M},\Omega^{\mathrm{U}}}(X) = \mathrm{U}^{-1}\mathcal{G}_{\mathsf{M},\Omega}(X)\mathrm{U}$, as is straightforward to demonstrate. The conjugate group $\mathrm{U}^{-1}\mathcal{I}_{\mathsf{M},\Omega}(X)\mathrm{U}$ of $\mathcal{I}_{\mathsf{M},\Omega}(X)$ is hence defined and we have in addition that $\mathcal{I}_{\mathsf{M},\Omega^{\mathrm{U}}}(X) = \mathrm{U}^{-1}\mathcal{I}_{\mathsf{M},\Omega}(X)\mathrm{U}$. Thus, $\mathcal{I}_{\mathsf{M},\Omega^{\mathrm{U}}}(X) \simeq \mathcal{I}_{\mathsf{M},\Omega}(X)$ and so Ω, Ω^{U} belong to the same stratum. The stratum is made in general of several gauge orbits.

Each stratum $\mathcal{S} \subset \mathcal{C}_{\mathsf{ffM}}(X)$ is thus characterized by a group $\mathcal{I}_{\mathsf{M}\mathcal{S}}$ defined up to isomorphism by the property that $\mathcal{I}_{\mathsf{M},\Omega}(X) \simeq \mathcal{I}_{\mathsf{M}\mathcal{S}}$ for $\Omega \in \mathcal{S}$. Strata $\mathcal{S}$ with a higher dimensional essential invariance group $\mathcal{I}_{\mathsf{M}\mathcal{S}}$ are higher codimensional, since they are constituted by more symmetric and thus more constrained gauge fields. In general, it is expected that there exist a leading 0-codimensional stratum $\mathcal{S}_0$ whose associated group $\mathcal{I}_{\mathsf{M}\mathcal{S}_0}$ is trivial together with subleading higher codimensional strata $\mathcal{S}$ with non trivial groups $\mathcal{I}_{\mathsf{M}\mathcal{S}}$.

3.3 Ordinary gauge theory from a derived perspective

In subsection 3.4 of I, we showed that the ordinary geometric framework is in fact a special case of the derived geometric framework. This important property allows one to view ordinary gauge theory from a derived perspective as a special case of derived gauge theory. We devote this final subsection to a brief illustration of this point.

In subsection 3.4 of I, we showed that a Lie group G is fully codified by the unique Lie group crossed module, also denoted as G, with trivial source group 1 and target group G.

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Its Lie algebra $\mathfrak{g}$ is similarly fully codified by the unique Lie algebra crossed module denoted as $\mathfrak{g}$ too with trivial source algebra 0 and target algebra $\mathfrak{g}$. The derived Lie group $\mathrm{D}\mathsf{G}$ of G is just G itself and similarly the derived Lie algebra $\mathrm{D}\mathfrak{g}$ of $\mathfrak{g}$ is $\mathfrak{g}$. Thus, the mapping spaces $\mathrm{Map}(T[1]X, \mathrm{D}\mathsf{G})$ and $\mathrm{Map}(T[1]X, \mathsf{G})$ as well as the mapping spaces $\mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{g})$ and $\mathrm{Map}(T[1]X, \mathfrak{g})$ and their degree shifted versions can be identified.

Since the source Lie algebra is the trivial algebra 0 for the crossed module $\mathfrak{g}$, a derived gauge field Ω of G has a necessarily vanishing degree 2 component $\mathit{\Omega}$ and thus reduces to the degree 1 component ω. Accordingly, the derived curvature Φ of Ω has the degree 2 component ϕ as its only non vanishing component, the degree 3 component $\mathit{\Phi}$ vanishing identically by virtue of (3.7). Since $\mathit{\Omega} = 0$, ϕ is given from (3.6) by the familiar expression of ordinary gauge theory. Further, since $\mathit{\Phi} = 0$, ϕ satisfies by (3.9) the usual Bianchi identity.

A derived gauge transformation U of G has similarly a necessarily vanishing degree 1 component U and thus reduces to the degree 0 component u. The component expression of the gauge transform of a derived gauge field Ω, eqs. (3.14), (3.15), reproduce the ordinary expression for the degree 1 component ω and leave the degree 2 component $\mathit{\Omega}$ vanishing as required.

The only special gauge transformations T of G is the trivial one, as again the source Lie group of the Lie group crossed module G is the trivial group 1. For a derived gauge field Ω, so, the invariance subgroup ${\mathcal{G}^*}_{\mathsf{M},\Omega}(X)$ of Ω is non trivial if its component ω has an accidental symmetry. The essential invariance subgroup $\mathcal{I}_{\mathsf{M},\Omega}(X)$ reduces then to ${\mathcal{G}^*}_{\mathsf{M},\Omega}(X)$ itself.

4 Review of the ordinary TCO model

In this section, we shall review the ordinary TCO model. The topics covered are the model's formulation as a classical field theory, basic symmetries, sigma model interpretation, functional integral quantization and canonical analysis. Our exposition of this subject is purposefully patterned in a way that plainly alludes to the derived extension presented in section 5, even though the latter has significant novel features. Alternative detailed exposition can be found in refs. [35, 36]. We refer the reader to subsection 3.3 above for the gauge theoretic framework used.

4.1 Ordinary TCO model

In this subsection, we shall illustrate the ordinary TCO model as a classical field theory. The model's basic elements are the following.

1. The base manifold: an oriented compact connected 1-dimensional manifold N perhaps with boundary.

So, N is either the circle $\mathbb{S}^1$ or the closed interval $\mathbb{I}$.

2. The ambient manifold: an oriented manifold M.

The case where M is a 3-fold is the most relevant for knot theoretic analyses. We however shall not impose such a restriction here.

3. The embedded curve: a base to ambient manifold embedding $\varsigma : N \to M$.

The embedding provides a homeomorphic image of N in M.

4. The symmetry structure: a Lie group G and the associated Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ is equipped with an invariant pairing $(\cdot, \cdot)$.

$(\cdot, \cdot)$ allows one to systematically identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ as vector spaces.

5. The ordinary TCO dynamical field space: $\mathcal{F}_{\mathsf{G}}(N) := \mathrm{Map}(T[1]N, \mathsf{G})$, the space of ordinary Lie group valued fields g.

6. The ordinary TCO background field space: $\mathcal{C}_{\mathsf{G}}(M)$, the space of ordinary gauge fields ω.

The model is further characterized by a parameter: an element $\kappa \in \mathfrak{g}$ which we shall refer to as the model's level.

The action of the ordinary TCO model reads

$$S(g; \omega) = \int_{T[1]N} \varrho_N(\kappa, \mathrm{Ad}\, g^{-1}(\varsigma^* \omega) + g^{-1} dg). \tag{4.1}$$

Formally, so, the Lagrangian is the component along κ of the gauge transform of ω by g. The action (4.1) defines a Schwarz type 1-dimensional topological field theory or topological quantum mechanics.

Provided appropriate boundary conditions are imposed on g when $\partial N \neq \varnothing$, the field equations read

$$[\mathrm{Ad}\, g^{-1}(\varsigma^* \omega) + g^{-1} dg, \kappa] = 0. \tag{4.2}$$

Since $\dim N = 1$, (4.2) is automatically integrable. Eq. (4.2) constrains the combination $\omega^g := \mathrm{Ad}\, g^{-1}(\varsigma^* \omega) + g^{-1} dg$ to lie in the centralizer of the level κ in $\mathfrak{g}$. A more specific analysis can be made in the case where the Lie group G is compact, as is often assumed. If κ is a regular element of $\mathfrak{g}$, then (4.2) demands that ω^g belongs to $\mathrm{Map}(T[1]N, \mathfrak{t}[1])$, where $\mathfrak{t}$ is the maximal toral subalgebra of $\mathfrak{g}$ in which κ lies. If κ is not regular, then (4.2) allows ω^g to vary in $\mathrm{Map}(T[1]N, \mathfrak{h}[1])$ for a larger subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{t}$. In the extreme non regular case where $\kappa = 0$, ω^g is completely unconstrained by (4.2).

4.2 Symmetries of the ordinary TCO model

In this subsection, we shall describe the symmetries of ordinary TCO theory: the level preserving gauge symmetry and the accidental gauge background preserving symmetry, the latter mainly for the importance its counterpart holds in the derived model. We shall also consider the model's background gauge symmetry.

The first distinguished symmetry of the TCO model is the level preserving gauge symmetry. The associated gauge transformation group, $\mathcal{G}_{\mathsf{G},\kappa}(N)$, is the subgroup of the full gauge transformation group $\mathcal{G}_{\mathsf{G}}(N)$ formed by the elements $\upsilon \in \mathcal{G}_{\mathsf{G}}(N)$ whose adjoint action leaves the model's level κ invariant,

$$\mathrm{Ad}\, \upsilon(\kappa) = \kappa. \tag{4.3}$$

$\mathcal{G}_{\mathsf{G},\kappa}(N)$ is therefore just the gauge transformation group $\mathcal{G}_{\mathsf{G}_\kappa}(N)$, where G_κ denotes the invariance subgroup of κ under the adjoint action of G. $\mathcal{G}_{\mathsf{G},\kappa}(N)$ acts on the TCO field space $\mathcal{F}_{\mathsf{G}}(N)$; the action is given by

$$g^{\upsilon} = g\upsilon \tag{4.4}$$

with $\upsilon \in \mathcal{G}_{\mathsf{G},\kappa}(N)$ and $g \in \mathcal{F}_{\mathsf{G}}(N)$. $\mathcal{G}_{\mathsf{G},\kappa}(N)$ is instead inert on the TCO background gauge field space $\mathcal{C}_{\mathsf{G}}(M)$. The TCO action S of eq. (4.1) is not level preserving gauge invariant. The gauge variation it suffers under a gauge transformation υ however depends on υ but not on g; we have indeed

$$S(g^{\upsilon};\omega) = S(g;\omega) + A(\upsilon), \tag{4.5}$$

where the (classical) gauge anomaly A is given by

$$A(\upsilon) = \int_{T[1]N} \varrho_N(\kappa, \upsilon^{-1}d\upsilon). \tag{4.6}$$

We shall discuss in the next subsection the reasons why the non invariance of S does not compromise the level preserving gauge invariance of TCO theory both at the classical and quantum level.

The gauge background preserving symmetry is the second symmetry of the ordinary TCO model. It is rarely mentioned in the literature on the subject as it is merely accidental, depending as it does on the gauge invariance properties of the pull-backed gauge field $\varsigma^*\omega$, which are generically trivial. It is a rigid symmetry. The associated symmetry group, $\mathcal{G}^*{}_{\mathsf{G},\varsigma^*\omega}(N)$, is the is the subgroup of the full gauge transformation group $\mathcal{G}_{\mathsf{G}}(N)$ of the elements $t \in \mathcal{G}_{\mathsf{G}}(N)$ such that

$$\varsigma^*\omega^t = \varsigma^*\omega. \tag{4.7}$$

$\mathcal{G}^*{}_{\mathsf{G},\varsigma^*\omega}(N)$ is a finite dimensional group, which is non trivial only when $\varsigma^*\omega$ belongs to a finite codimensional subspace of the base manifold gauge field space $\mathcal{C}_{\mathsf{G}}(N)$. $\mathcal{G}^*{}_{\mathsf{G},\varsigma^*\omega}(N)$ acts on the TCO field space $\mathcal{F}_{\mathsf{G}}(N)$: the action reads as

$$g^t = t^{-1}g. \tag{4.8}$$

for $t \in \mathcal{G}^*{}_{\mathsf{G},\varsigma^*\omega}(N)$ and $g \in \mathcal{F}_{\mathsf{G}}(N)$. $\mathcal{G}^*{}_{\mathsf{G},\varsigma^*\omega}(N)$ is instead inert on the background gauge field space $\mathcal{C}_{\mathsf{G}}(M)$ consistently with the invariance condition (4.7). Eq. (4.7) itself immediately implies that

$$S(g^t;\omega) = S(g;\omega). \tag{4.9}$$

The background preserving gauge symmetry of the TCO model can be interpreted as the residual unbroken gauge symmetry left by the breaking of the full left G-gauge symmetry of TCO theory acting according to (4.8) by the gauge background ω.

The last symmetry of the ordinary TCO model is the background gauge symmetry. The associated gauge transformation group is the ambient manifold gauge transformation group $\mathcal{G}_{\mathsf{G}}(M)$. $\mathcal{G}_{\mathsf{G}}(M)$ acts on the TCO field space $\mathcal{F}_{\mathsf{G}}(N)$ as

$$g^{\mathrm{u}} = \varsigma^* u^{-1} g, \tag{4.10}$$

where $u \in \mathcal{G}_{\mathsf{G}}(M)$ and $g \in \mathcal{F}_{\mathsf{G}}(N)$. $\mathcal{G}_{\mathsf{G}}(M)$ acts similarly on the background gauge field space $\mathcal{C}_{\mathsf{G}}(M)$ by associating with any $\omega \in \mathcal{C}_{\mathsf{G}}(M)$ its gauge transform ω^u as in (3.14). These transformations leave the action invariant,

$$S(g^u; \omega^u) = S(g; \omega). \tag{4.11}$$

4.3 Ordinary TCO sigma model

In this subsection, we explain why the ordinary TCO model is effectively a 1-dimensional sigma model by virtue of the model's level preserving gauge symmetry analyzed in subsection 4.2

Upon modding out the level preserving gauge symmetry, the effective field space of the ordinary TCO model is the $\mathcal{G}_{\mathsf{G},\kappa}(N)$-orbit space

$$\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N) = \mathcal{F}_{\mathsf{G}}(N)/\mathcal{G}_{\mathsf{G},\kappa}(N). \tag{4.12}$$

Since the level preserving gauge transformation group $\mathcal{G}_{\mathsf{G},\kappa}(N)$ is just the group $\mathcal{G}_{\mathsf{G}_\kappa}(N)$ of G_κ-valued gauge transformations, we have

$$\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N) = \mathrm{Map}(T[1]N, \mathsf{G}/\mathsf{G}_\kappa). \tag{4.13}$$

Thus, the TCO model is ultimately a sigma model over the homogeneous space $\mathsf{G}/\mathsf{G}_\kappa$. Viewing it in this way sheds light also on the precise nature of the level preserving gauge symmetry. An objection may be raised against the above conclusion however: by (4.5), the action S is not level preserving gauge invariant. The question must be posed in the appropriate terms classically and quantically.

In classical theory, it is the level preserving gauge covariance of the field equations (4.2) that matters. In order this property to hold, the invariance of the action S up to a field independent additive term A is sufficient and this requirement is indeed satisfied owing to (4.5), (4.6).

In quantum theory, it is the invariance of the exponentiated action e^{iS} entering in the functional integral formulation that matters. From (4.6), the variation of the anomaly $A(\upsilon)$ with respect to υ is $\delta A(\upsilon) = \int_{T[1]\partial N} \varrho_{\partial N}(\kappa, \upsilon^{-1}\delta\upsilon)$. Hence, if either $\partial N = \varnothing$ or we allow only gauge transformations υ belonging to a subgroup of $\mathcal{G}_{\mathsf{G},\kappa}(N)$ of gauge transformations obeying boundary conditions making the above boundary integral vanish, $\delta A = 0$ and A is so a discrete homotopy invariant. Consequently, owing to (4.5), the invariance of e^{iS} is generically ensured provided that an appropriate quantization condition on the model's level κ rendering the exponentiated anomaly e^{iA} trivial is satisfied.

TCO theory is in this fashion akin to and can be in fact viewed as a 1-dimensional analog of 3-dimensional CS theory. The strict non invariance of the action S does not by itself compromise the level preserving gauge invariance of the TCO model: both classically and quantically the action enjoys the appropriate form of level preserving gauge invariance. The TCO model, so, can legitimately be regarded as a sigma model over $\mathsf{G}/\mathsf{G}_\kappa$. As such, the model still features a background gauge field preserving symmetry and a background gauge symmetry, as the ${\mathcal{G}^*}_{\mathsf{G},\varsigma^*\omega}(N)$ and $\mathcal{G}_{\mathsf{G}}(M)$-actions (4.8) and (4.10) commute with the $\mathcal{G}_{\mathsf{G},\kappa}(N)$-action and leave the action S invariant by (4.9) and (4.11).

4.4 Functional integral quantization of the ordinary TCO sigma model

In this subsection, we shall consider the functional integral quantization of the ordinary TCO sigma model. For conciseness, we shall be sketchy about the details of basic computations.

As noticed previously in subsection 4.3, the functional integral quantization of the TCO sigma model requires that the exponentiated action e^{iS} is level preserving gauge invariant or equivalently, by (4.5), that the exponentiated anomaly e^{iA} is trivial. On account of (4.6), this leads to a quantization condition for the TCO level κ. The condition can be stated rather explicitly in the case when the Lie group G is compact, based on the fact that the level preserving gauge transformations are just the G_κ-valued gauge transformations. When κ is regular and G_κ is so a maximal torus T of G, κ is required to belong to the dual $\Lambda_{\mathsf{G}}{}^*$ of the integral lattice Λ_{G} in $\mathfrak{t}$. When conversely κ is not regular, κ is subject to further restrictions.

Whatever the form the quantization condition of the level κ takes, when it is met the exponentiated action e^{iS} constitutes a genuine functional on the TCO sigma model's field space $\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)$ (cf. subsection 4.3) and the model's quantum partition function can be expressed as a functional integral of e^{iS} on $\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)$,

$$Z(\omega) = \int_{\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)} \mathscr{D}g \; \mathrm{e}^{iS(g;\omega)} \,. \tag{4.14}$$

The functional measure $\mathscr{D}g$ of $\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)$ is assumed to be invariant under left $\mathcal{G}_{\mathsf{G}}(N)$ multiplicative shifts. In view of the expected relationship of $Z(\omega)$ to the Wilson line of the embedded curve ς, an analysis of the gauge and homotopy invariance properties of $Z(\omega)$ is in order.

The left $\mathcal{G}_{\mathsf{G}}(N)$-invariance of the measure $\mathscr{D}g$ ensures that the quantum TCO sigma model enjoys all the symmetries acting left multiplicatively on sigma model field space $\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)$ which the classical model does. In particular, the background gauge symmetry of the classical model implies that the partition function $Z(\omega)$ is gauge invariant as a functional of ω. Therefore,

$$Z(\omega^u) = Z(\omega) \tag{4.15}$$

for $u \in \mathcal{G}_{\mathsf{G}}(M)$. The gauge background preserving symmetry has instead no apparent implications for $Z(\omega)$.

We shall analyze next the dependence of the partition function $Z(\omega)$ on the embedded curve ς. Specifically, we shall compute the variation $\delta Z(\omega)$ of $Z(\omega)$ under a variation $\delta\varsigma$ of ς leaving the image $\varsigma(\partial N)$ of the boundary ∂N of N fixed, i.e such that $\delta\varsigma|_{\partial N} = 0$. Formally, expression (4.14) yields

$$\delta Z(\omega) = \int_{\overline{\mathcal{F}}_{\mathsf{G},\kappa}(N)} \mathscr{D}g \; \mathrm{e}^{iS(g;\omega)} \, i\delta S(g;\omega). \tag{4.16}$$

To make the above expression more explicit, a suitable variational framework is required. The embeddings of N into M form the infinite dimensional functional manifold $\mathcal{E}_{N,M} = \mathrm{Emb}(N, M)$. The variational problem we are dealing with is therefore naturally

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framed in the complex consisting of the graded functional algebra $\mathrm{Fun}(T[1]\mathcal{E}_{N,M})$ and the appended variational differential δ. However, given that the embeddings always show up through elements of the function algebra $\mathrm{Fun}(T[1]N)$, it is necessary to enlarge our formal framework to the augmented embedding manifold $\mathcal{E}_{N,M} \times N$ and the associated complex $\mathrm{Fun}(T[1](\mathcal{E}_{N,M} \times N))$, $\delta + d$. As $T[1](\mathcal{E}_{N,M} \times N) = T[1]\mathcal{E}_{N,M} \boxplus T[1]N$,[1] this latter is endowed with an inherent bigrading[2] with δ and d being the bidegree $(1,0)$ and $(0,1)$ contributions of the differential respectively. In this way, the pull-back of an ordinary field $\psi \in \mathrm{Map}(T[1]M, \mathfrak{g}[p])$ by the evaluation map $\epsilon : \mathcal{E}_{N,M} \times N \to M$ yields a map $\epsilon^*\psi \in \mathrm{Map}(T[1](\mathcal{E}_{N,M} \times N), \mathfrak{g}[p])$. $\epsilon^*\psi$ can be decomposed in terms of definite bidegree. In particular, the terms of bidegree $(0,p)$ constitute the component $\epsilon^*\psi_N$ of $\epsilon^*\psi$ along N in $\mathcal{E}_{N,M} \times N$.

The computation of the variation $\delta Z(\omega)$ of $Z(\omega)$ involves first the computation through (4.1) of the variation $\delta S(g;\omega)$ of $S(g;\omega)$ regarded as a degree 0 element of $\mathrm{Fun}(T[1]\mathcal{E}_{N,M})$ depending on g and ω. The expression so obtained is then inserted in (4.16), yielding

$$\delta Z(\omega) = -\int_{\mathcal{F}_{\mathsf{G}}(N)} \mathscr{D}g\ \mathrm{e}^{iS(g;\omega)}\, i \int_{T[1]N} \varrho_N \\ \times \left[\left(\kappa, \mathrm{Ad}\, g^{-1}(\epsilon^*\phi)\right) + \left([\mathrm{Ad}\, g^{-1}(\epsilon^*\omega_N) + g^{-1}dg, \kappa], \mathrm{Ad}\, g^{-1}(\epsilon^*\omega)\right)\right]. \tag{4.17}$$

where $\phi = d\omega + \frac{1}{2}[\omega,\omega]$ is the curvature of ω. The second term within square brackets gives a vanishing contribution to the functional integral under rather general assumptions. Indeed, we have

$$\int_{\mathcal{F}_{\mathsf{G}}(N)} \mathscr{D}g\ \mathrm{e}^{iS(g;\omega)}\, i \Big\{ \mathit{tr}\left(\mathit{Ad}(\mathrm{Ad}\, g^{-1}(\epsilon^*\omega))\right) \\ - \int_{T[1]N} \varrho_N \left([\mathrm{Ad}\, g^{-1}(\epsilon^*\omega_N) + g^{-1}dg, \kappa], \mathrm{Ad}\, g^{-1}(\epsilon^*\omega)\right) \Big\} = 0, \tag{4.18}$$

where tr, Ad denote functional trace and adjoint respectively, as follows from a standard Schwinger-Dyson type argument. Above, $\mathit{tr}\left(\mathit{Ad}(\mathrm{Ad}\, g^{-1}(\epsilon^*\omega))\right)$ contains a factor $\delta_N(0)$, which must be regularized, and a factor that pointwise is of the form $\mathrm{tr}\,\mathrm{ad}\, O$ for some element $O \in \mathfrak{g}[1]$. The latter vanishes if $\mathrm{tr}\,\mathrm{ad}\, x = 0$ for $x \in \mathfrak{g}$, i.e. when the Lie algebra $\mathfrak{g}$ is unimodular. If this property holds, $\mathit{tr}\left(\mathit{Ad}(\mathrm{Ad}\, g^{-1}(\epsilon^*\omega))\right) = 0$. From (4.17) and (4.18), we find then that

$$\delta Z(\omega) = -\int_{\mathcal{F}_{\mathsf{G}}(N)} \mathscr{D}g\ \mathrm{e}^{iS(g;\omega)}\, i \int_{T[1]N} \varrho_N \left(\kappa, \mathrm{Ad}\, g^{-1}(\epsilon^*\phi)\right). \tag{4.19}$$

[1]If E_1, E_2 are vector bundles on distinct manifolds M_1, M_2, their ordinary direct sum cannot be defined whilst their external direct sum can. This is the vector bundle $E_1 \boxplus E_2 = \pi_1{}^*E_1 \oplus \pi_2{}^*E_2$ of base $M_1 \times M_2$, where π_1, π_2 are the standard projection of $M_1 \times M_2$ on its Cartesian factors M_1, M_2. The tangent bundle of a product manifold $X_1 \times X_2$ is the external direct sum $TX_1 \boxplus TX_2$.

[2]There are two conventions for the sign produced by the commutation of two homogeneous elements of a bigraded commutative algebra A. If $x, y \in \mathsf{A}$ have bidegrees (m,p), (n,q), then $xy = (-1)^{(m+p)(n+q)}yx$ according to Bernstein-Leites and $xy = (-1)^{mn+pq}yx$ according to Deligne. It makes no difference which convention is used: any statement can be expressed in principle in any one of them. In this paper, we adhere to the Bernstein-Leites rule as it turns out to be more natural in our construction, although the Deligne rule is more commonly used in the physical literature.

In this way, we have that

$$\delta Z(\omega) = 0 \qquad \text{if } \phi = 0. \tag{4.20}$$

Thus, $Z(\omega)$ is invariant under smooth variations of the embedding ς when the gauge field ω is flat. The background gauge invariance property (4.15) and the homotopy invariance property (4.20) of the partition function $Z(\omega)$ is a clear indication that $Z(\omega)$ computes the trace in a representation R_κ of the holonomy $F_\omega(\varsigma)$ of ω along the parametrized curve ς, viz

$$Z(\omega) = \mathrm{tr}_{R_\kappa}(F_\omega(\varsigma)). \tag{4.21}$$

This relation shows that the TCO sigma model underlies the partition realization of Wilson lines. It has been verified by explicit evaluation of $Z(\omega)$ for specific choices of G and ω in [30–32] and by this reason is by now considered an established result.

4.5 Canonical formulation of the ordinary TCO model

In this subsection, we shall examine the canonical theory of the ordinary TCO model, which will furnish us new insight into TCO theory.

For a canonical formulation of the TCO model, the base manifold N is chosen to be $\mathbb{R}^1$ to be interpreted as the time axis. The action (4.1) reads then as

$$S(g;\omega) = \int_{T[1]\mathbb{R}^1} \varrho_{\mathbb{R}^1}\big(\kappa, \mathrm{Ad}\, g^{-1}({\varsigma^*}_t\omega) + g^{-1}d_t g\big), \tag{4.22}$$

where d_t denotes the ordinary de Rham differential of $\mathbb{R}^1$.

An inspection of the kinetic term of the TCO action S given above, $(\kappa, g^{-1}d_t g)$, reveals that the ambient phase space of the TCO model is the group manifold G. The appropriate form of the presymplectic potential 1-form ϖ of phase space is also indicated by that of the kinetic term, viz

$$\varpi = -(\kappa, g^{-1}dg), \tag{4.23}$$

where g is to be regarded here as a G-valued phase space variable. The associated presymplectic 2-form therefore is

$$\psi = d\varpi = \frac{1}{2}(\kappa, [g^{-1}dg, g^{-1}dg]). \tag{4.24}$$

ψ so depends on the model's level κ.

In subsection 4.3, we found that the ordinary TCO model actually is a 1-dimensional sigma model over the homogeneous space $\mathsf{G}/\mathsf{G}_\kappa$. A basic property such as this should emerge also in the canonical formulation. Indeed it does, as we are going to verify next. The level preserving gauge transformation group appears in the canonical formulation as the subgroup of the group G formed by the elements $\upsilon \in \mathsf{G}$ leaving κ invariant in conformity with (4.3)

$$\mathrm{Ad}\, \upsilon(\kappa) = \kappa. \tag{4.25}$$

It is therefore just the invariance subgroup of κ in G, G_κ. The gauge transformation action on the model's field space $\mathcal{F}_{\mathsf{G}}(N)$ translates in this way as the G_κ-action on the ambient phase space G given by

$$g^\upsilon = g\upsilon \tag{4.26}$$

with $g \in \mathsf{G}$ and $v \in \mathsf{G}_\kappa$, in keeping with (4.4). From (4.25), it follows that the infinitesimal level preserving gauge transformation algebra is the Lie algebra $\mathfrak{g}_\kappa$ of G_κ and that the action (4.26) is implemented infinitesimally for any $z \in \mathfrak{g}_\kappa$ by the vector field $X_z \in \mathrm{Vect}(\mathsf{G})$ such that

$$j_{X_z}(g^{-1}dg) = z, \tag{4.27}$$

where j_V denotes contraction with respect to a vector field $V \in \mathrm{Vect}(\mathsf{G})$. From (4.24) and (4.27), it is immediately checked that

$$j_{X_z}\psi = 0. \tag{4.28}$$

Likewise, any vector field $X \in \mathrm{Vect}(\mathsf{G})$ such that $j_X\psi = 0$ is of the form $X = X_z$ for some $z \in \mathfrak{g}_\kappa$ pointwise in G. The degeneracy of ψ just exhibited and (4.26) together indicate that the phase space of the TCO model is precisely the G_κ-orbit space of the ambient phase space G, that is the homogeneous space $\mathsf{G}/\mathsf{G}_\kappa$. They imply further that ψ induces a symplectic 2-form $\overline{\psi}$ on $\mathsf{G}/\mathsf{G}_\kappa$ and associated with this a Poisson bracket structure $\{\cdot,\cdot\}$ on the function algebra $\mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ of $\mathsf{G}/\mathsf{G}_\kappa$.

We recall next how the Poisson bracket of a pair of functions of $\mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ is computed. Though this prescription can be ascribed to ordinary KKS theory proper with no reference to the TCO model we are examining, we review it because an infinite dimensional version of it will be employed in the canonical formulation of the derived TCO model next section. In practice, it is awkward to operate with orbits as such, while it is relatively more straightforward to work with orbit representatives. With this perspective in mind, we consider the following isomorphisms. The first isomorphism, $\mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa) \simeq \mathrm{Fun}(\mathsf{G})^{\mathsf{G}_\kappa}$, equates the function algebra $\mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ of $\mathsf{G}/\mathsf{G}_\kappa$ and the subalgebra $\mathrm{Fun}(\mathsf{G})^{\mathsf{G}_\kappa}$ of $\mathrm{Fun}(\mathsf{G})$ of the functions invariant under the G_κ-action (4.26). The second isomorphism, $\mathrm{Vect}(\mathsf{G}/\mathsf{G}_\kappa) \simeq \mathrm{W\,Vect}_\kappa(\mathsf{G})$, identifies the vector field Lie algebra $\mathrm{Vect}(\mathsf{G}/\mathsf{G}_\kappa)$ of $\mathsf{G}/\mathsf{G}_\kappa$ and the Weyl Lie algebra $\mathrm{W\,Vect}_\kappa(\mathsf{G})$[3] of the Lie subalgebra $\mathrm{Vect}_\kappa(\mathsf{G})$ of $\mathrm{Vect}(\mathsf{G})$ of vector fields $V \in \mathrm{Vect}(\mathsf{G})$ of the form $V = X_z$ for some $z \in \mathfrak{g}_\kappa$ pointwise in G. So, a function $f \in \mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ is to be thought of as a function $f \in \mathrm{Fun}(\mathsf{G})$ such that $f(g^v) = f(g)$ for $v \in \mathsf{G}_\kappa$ and, similarly, a vector field $V \in \mathrm{Vect}(\mathsf{G}/\mathsf{G}_\kappa)$ as a vector field $V \in \mathrm{Vect}(\mathsf{G})$ defined mod vector fields $V' \in \mathrm{Vect}_\kappa(\mathsf{G})$ and such that $[V, V'] \in \mathrm{Vect}_\kappa(\mathsf{G})$ for any vector field $V' \in \mathrm{Vect}_\kappa(\mathsf{G})$. The computation of the Poisson bracket involving a function $f \in \mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ requires the Hamiltonian vector field $V_f \in \mathrm{Vect}(\mathsf{G}/\mathsf{G}_\kappa)$ of f, which is defined by the property that

$$df + j_{V_f}\psi = 0. \tag{4.29}$$

The Poisson bracket of a pair of functions $f, h \in \mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ is then the function $\{f, h\} \in \mathrm{Fun}(\mathsf{G}/\mathsf{G}_\kappa)$ given by the standard relation

$$\{f, h\} = j_{V_f}dh = -j_{V_h}df. \tag{4.30}$$

[3]Recall that the Weyl Lie algebra of a Lie subalgebra $\mathfrak{s}$ of a Lie algebra $\mathfrak{l}$ is defined as $\mathrm{W}\mathfrak{s} = \mathrm{N}\mathfrak{s}/\mathfrak{s}$, where $\mathrm{N}\mathfrak{s}$ denotes the normalizer of $\mathfrak{s}$ in $\mathfrak{l}$. The normalizer of $\mathrm{N}\mathfrak{s}$ of $\mathfrak{s}$ is the largest subalgebra of $\mathfrak{l}$ containing $\mathfrak{s}$ as an ideal.

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As recalled in section 4 of I, the homogeneous space $\mathsf{G}/\mathsf{G}_\kappa$ is the coadjoint orbit $\mathcal{O}_\kappa$ of κ and the symplectic 2-form $\overline{\psi}$ of $\mathsf{G}/\mathsf{G}_\kappa$ induced by (4.24) is the KKS one. The canonical formulation of the TCO model in this way reproduces the KKS theory of $\mathcal{O}_\kappa$.

In the quantum theory expounded in subsection 4.4, the TCO model's level κ is quantized in a certain way. For such discrete values of κ, the symplectic form $\overline{\psi}$ obeys the Bohr-Sommerfeld quantization condition rendering the geometric quantization of $\mathcal{O}_\kappa$ possible.

5 Derived TCO model

In this section, which is the central one of the present paper, we illustrate the derived TCO model. The construction is patterned on that of the ordinary TCO model of section 4 using systematically the derived geometrical set-up of section 3 of I. The topics covered therefore are the model's formulation as a classical field theory, basic symmetries, interpretation as a sigma model, functional integral quantization and canonical analysis. However, from a formal standpoint, the formulations of the derived and ordinary TCO models are not as close as those of the derived and ordinary KKS theory presented in part I are. The derived TCO model is in many respects richer than its ordinary counterpart.

The derived counterpart of the ordinary TCO level is a closed derived level current. The derived TCO model is a sigma model only if certain restrictions are imposed on the level current and the symmetry crossed module. Further, the relationship of the derived TCO model to derived KKS theory emerges only for a special choice of the level current, which yields the so-called characteristic model.

The derived TCO classical field equations are not automatically integrable as in the ordinary case, but they are only if the gauge background is fake flat. Further, the accidental gauge background preserving symmetry of the ordinary model gets enhanced to a full gauge symmetry in the derived model. These novel features impinge upon the model's quantization and canonical structure.

5.1 Derived TCO model

In this subsection, we shall introduce the derived TCO model and study its most salient properties as a classical field theory. Our formulations parallels to a important extent that of the ordinary model reviewed in subsection 4.1.

The basic elements of the model are the following.

1. The base manifold: an oriented compact connected 2-dimensional manifold N, possibly with boundary.

The topologies of this kind are fully classified and form a denumerable gamut.

2. The ambient manifold: an oriented manifold M.

The case where M is a 4-fold is the only relevant for the study of 2-knot topology. We however shall leave the dimension of M undetermined in what follows.

3. The embedded surface: a base to ambient manifold embedding $\varsigma : N \to M$.

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The embedding provides a homeomorphic image of N in M.

4. The symmetry structure: a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G})$ equipped and the associated Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g})$. $\mathfrak{m}$ is equipped with an invariant pairing $\langle\cdot,\cdot\rangle$.

M is thus balanced as $\langle\cdot,\cdot\rangle$ yields an isomorphism of the vector spaces $\mathfrak{e}$ and $\mathfrak{g}^*$.

5. The derived TCO dynamical field space: $\mathcal{F}_{\mathsf{M}}(N) := \mathrm{Map}(T[1]N, \mathrm{D}\mathsf{M})$, the space of derived Lie group valued fields G.

6. The derived TCO background field space: $\mathcal{C}_{\mathsf{M}}(M)$, the space of derived gauge fields Ω (cf. subsection 3.1)

The derived model is moreover characterized by a parameter: a fixed level current $\mathrm{K} \in \mathrm{Map}'_c(T[1]N, \mathrm{D}\mathfrak{m}[0])$ with compact support in the interior of N obeying

$$\mathrm{dK} = 0. \tag{5.1}$$

A level datum of this kind allows for a broader instantiation than that of the ordinary setting. We notice here that, owing to (3.3.10) of I, condition (5.1) does not reduces to the customary closedness of K, but it is more involved.

The action of the derived TCO model reads as

$$S(\mathrm{G};\Omega) = \int_{T[1]N} \varrho_N(\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}) \tag{5.2}$$

in the derived field formal framework expounded in subsection 3.3 of I. The Lagrangian, formally consisting in the component along K of the gauge transform of Ω by G, is thus totally analogous to the one of the usual TCO model (cf. eq. (4.1)) with derived quantities replacing the ordinary ones. The variation of S with respect to G takes the form

$$\begin{aligned}\delta S(\mathrm{G};\Omega) = &-\int_{T[1]N} \varrho_N(\mathrm{G}^{-1}\delta\mathrm{G}, [\mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}]) \\ &+\int_{T[1]\partial N} \varrho_{\partial N}(\mathrm{K}, \mathrm{G}^{-1}\delta\mathrm{G}).\end{aligned} \tag{5.3}$$

If a suitable boundary condition is imposed on G which makes the boundary term in (5.3) vanish identically, S is differentiable in the sense of refs. [55, 56]. The field equations then take the form

$$[\mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}] = 0. \tag{5.4}$$

As expected by design, the field equations (5.4) are analogous in form to those of the ordinary TCO model (cf. eq. (4.2)). They are so, among other things, also by virtue of the level condition (5.1).

The discussion of the choice of the appropriate boundary condition is far more involved in the derived TCO model than it is in the ordinary one, as the boundary ∂N of the base manifold N is 1- instead than 0-dimensional. An in-depth analysis of this matter is beyond the scope of the present work and will not be tackled here. In some generality, anyway, the boundary condition takes the following form. Consider the space $\mathcal{F}_{\mathsf{M}}(\partial N) = \mathrm{Map}(T[1]\partial N, \mathrm{D}\mathsf{M})$ of boundary derived TCO fields G_∂ and the associated variational complex $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(\partial N))$, δ_∂. The complex contains a special degree 1 element corresponding to the boundary contribution to the action's variation $\delta S(\mathrm{G};\Omega)$ in (5.3), viz

$$\varPi_\partial = -\int_{T[1]\partial N} \varrho_{\partial N}(\mathrm{K}, \mathrm{G}_\partial{}^{-1}\delta_\partial \mathrm{G}_\partial). \tag{5.5}$$

The boundary condition has then the basic form

$$i_\partial{}^*\mathrm{G} \in \mathcal{L}, \tag{5.6}$$

where $i_\partial : \partial N \to N$ is the canonical injection of $\partial N \subset N$ and $\mathcal{L}$ is a functional submanifold of $\mathcal{F}_{\mathsf{M}}(\partial N)$ such that $\kappa_{\mathcal{L}}{}^*\varPi_\partial = 0$, $\kappa_{\mathcal{L}} : \mathcal{L} \to \mathcal{F}_{\mathsf{M}}(\partial N)$ being the canonical injection of $\mathcal{L} \subset \mathcal{F}_{\mathsf{M}}(\partial N)$. There is very little more specific that can be said about the boundary condition without breaking G into its components g, G.

The integrability of the field equations (5.4) requires that

$$[\mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Phi), \mathrm{K}] \approx 0, \tag{5.7}$$

where Φ is the curvature of the derived gauge field Ω (cf. eq. (3.4)) and $\approx$ denotes equality on shell. Since the base manifold N of the TCO model is 2-dimensional, the degree 3 component $\varsigma^*\varPhi$ of $\varsigma^*\Phi$ vanishes identically. The degree 2 component $\varsigma^*\phi$ of $\varsigma^*\Phi$, conversely, may be non vanishing. Therefore, a broadly general sufficient condition for integrability to obtain is that the pull-back $\varsigma^*\Omega$ of the gauge field Ω by the embedding ς is fake flat

$$\varsigma^*\phi = 0 \tag{5.8}$$

(cf. subsection 3.1). The occurrence of a non trivial integrability condition of the field equations is a novel feature of the derived TCO theory with no analogue in the ordinary one, whose field equations are always integrable (cf. subsection 4.1).

5.2 Symmetries of the derived TCO model

In this subsection, we shall study the symmetries of derived TCO theory. The derived model's main symmetries are essentially the same as those of the ordinary one: the level preserving gauge symmetry, the gauge background preserving symmetry and the background gauge symmetry. The main difference concerns the second, which is an accidental rigid symmetry in the ordinary case while it gets promoted to a gauge symmetry in the derived one with far-reaching implications.

The first symmetry of the derived TCO model is the level preserving gauge symmetry. Its nature and action are totally analogous to those of its counterpart of the ordinary model (cf. subsection 4.2). The associated gauge transformation group, $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$, is the subgroup of the full derived gauge transformation group $\mathcal{G}_{\mathsf{M}}(N)$ formed by the transformations $\Upsilon \in \mathcal{G}_{\mathsf{M}}(N)$ whose adjoint action leaves the model's level current K invariant,

$$\mathrm{Ad}\,\Upsilon(\mathrm{K}) = \mathrm{K} \tag{5.9}$$

(cf. eq. (4.3)). Unlike in the ordinary case, in the derived case there generally does not exist anything like an invariance crossed submodule M_{K} of K, by means of which $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ can be equated to the derived gauge transformation group $\mathcal{G}_{\mathsf{M}_{\mathrm{K}}}(N)$. (This feature of the model will be further discussed in subsection 5.3.) $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ acts on the derived TCO field space $\mathcal{F}_{\mathsf{M}}(N)$; the action reads as

$$\mathrm{G}^{\Upsilon} = \mathrm{G}\Upsilon \tag{5.10}$$

for $\Upsilon \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ and $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(N)$ (cf. eq. (4.4)). $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ is instead inert on the TCO derived background gauge field space $\mathcal{C}_{\mathsf{M}}(M)$. As in the ordinary case, the derived TCO action S is not level preserving gauge invariant. The gauge variation it undergoes by effect of a gauge transformation Υ has again a simple form depending only on Υ but not on G,

$$S(\mathrm{G}^{\Upsilon};\Omega) = S(\mathrm{G};\Omega) + A(\Upsilon), \tag{5.11}$$

where the (classical) gauge anomaly A is given by

$$A(\Upsilon) = \int_{T[1]N} \varrho_N(\mathrm{K}, \Upsilon^{-1}\mathrm{d}\Upsilon). \tag{5.12}$$

(cf. eqs. (4.5), (4.6)). Later, we shall explain the reasons why the non invariance of S does not mar the level preserving gauge invariance of TCO theory.

The derived TCO model is also characterized by a second gauge background preserving symmetry, like the ordinary model (cf. subsection 4.2). The associated symmetry group, $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$, is the invariance subgroup of the pull-backed derived gauge field $\varsigma^*\Omega$, that is the subgroup of the full derived gauge transformation group $\mathcal{G}_{\mathsf{G}}(N)$ formed by the elements $\mathrm{T} \in \mathcal{G}_{\mathsf{M}}(N)$ obeying

$$\varsigma^*\Omega^{\mathrm{T}} = \varsigma^*\Omega \tag{5.13}$$

(cf. eq. (4.7)). In derived gauge theory, $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$ is an infinite dimensional group (cf. subsection 3.2). Indeed, the symmetry is no longer an accidental rigid symmetry, but a full gauge symmetry, as discussed later below. $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$ acts on the TCO field space $\mathcal{F}_{\mathsf{M}}(N)$: for $\mathrm{T} \in \mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$ and $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(N)$, we have

$$\mathrm{G}^{\mathrm{T}} = \mathrm{T}^{-1}\mathrm{G} \tag{5.14}$$

(cf. eq. (4.8)). $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$ is instead inert by design on the background gauge field space $\mathcal{C}_{\mathsf{M}}(M)$. The invariance condition (5.13) implies that

$$S(\mathrm{G}^{\mathrm{T}};\Omega) = S(\mathrm{G};\Omega) \tag{5.15}$$

(cf. eq. (4.9)). Analogously to the ordinary model, this symmetry property can be interpreted to the effect that the TCO field space $\mathcal{F}_{\mathsf{M}}(N)$ possess a $\mathcal{G}_{\mathsf{M}}(N)$-gauge symmetry acting according to (5.14) that the gauge background Ω breaks down to $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$. We shall have more to say about this below.

The third symmetry of the derived TCO model is the background gauge symmetry. Its nature and action are also totally analogous to those of its counterpart of the ordinary model (cf. subsection 4.2). The associated gauge transformation group is the ambient space gauge transformation group $\mathcal{G}_{\mathsf{M}}(M)$. $\mathcal{G}_{\mathsf{M}}(M)$ acts on the TCO field space $\mathcal{F}_{\mathsf{M}}(N)$ as

$$\mathrm{G}^{\mathrm{U}} = \varsigma^{*}\mathrm{U}^{-1}\mathrm{G} \tag{5.16}$$

for $\mathrm{U} \in \mathcal{G}_{\mathsf{M}}(M)$ and $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(N)$ (cf. eq. (4.10)). $\mathcal{G}_{\mathsf{M}}(M)$ acts further on the derived background gauge field space $\mathcal{C}_{\mathsf{M}}(M)$ by associating with any $\Omega \in \mathcal{C}_{\mathsf{M}}(M)$ its gauge transform Ω^{U} given by (3.11). The action is invariant,

$$S(\mathrm{G}^{\mathrm{U}};\Omega^{\mathrm{U}}) = S(\mathrm{G};\Omega) \tag{5.17}$$

(cf. eq. (4.11)).

If $\mathrm{U} \in \mathcal{G}^{*}{}_{\mathsf{M},\Omega}(M)$ is a background gauge transformation of the invariance subgroup of Ω in $\mathcal{G}_{\mathsf{M}}(M)$, then by (5.17)

$$S(\mathrm{G}^{\mathrm{U}};\Omega) = S(\mathrm{G};\Omega), \tag{5.18}$$

since $\Omega^{\mathrm{U}} = \Omega$ in this case. A comparison of (5.18) and (5.15) indicates that the gauge background preserving gauge symmetry extends at the base manifold level the gauge background preserving background gauge symmetry existing at the ambient manifold level, because $\varsigma^{*}\mathcal{G}^{*}{}_{\mathsf{M},\Omega}(M) \subseteq \mathcal{G}^{*}{}_{\mathsf{M},\varsigma^{*}\Omega}(N)$.

The invariance properties of the derived TCO action S under the level preserving and gauge background preserving gauge symmetries hold with no need to impose boundary conditions on either the TCO field $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(N)$ or the gauge transformations $\Upsilon \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$, $\mathrm{T} \in \mathcal{G}^{*}{}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ involved. If however a boundary condition is applied to G to make the variational problem well-defined, the gauge transformations Υ, T generally modify such boundary condition, since Υ, T have boundary restrictions $i_{\partial}{}^{*}\Upsilon$, $i_{\partial}{}^{*}\mathrm{T}$ acting on the boundary TCO field G_{∂} congruently with eqs. (5.10), (5.14) and this boundary action does not generally preserve the functional submanifold $\mathcal{L} \subset \mathcal{F}_{\mathsf{M}}(\partial N)$ that specifies the boundary condition (cf. subsection 5.1).

While the content of the level preserving gauge group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ is as a rule straightforward to find out, the content of the gauge background preserving gauge group $\mathcal{G}^{*}{}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ is hard to determine for a generic gauge field Ω. Fortunately, more about it is known for a gauge field Ω obeying the fake flatness condition (5.8) guaranteeing the fulfillment of the integrability condition (5.7) of the classical field equations. In this important case, as explained in subsection 3.2, $\mathcal{G}^{*}{}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ reduces up to a finite dimensional quotient group $\mathcal{I}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ (cf. eq. (3.35)) to its normal special gauge transformation subgroup $\mathcal{G}_{\mathsf{M},\varsigma^{*}\Omega}(N)$, whose explicit description was provided in subsection 3.1. We submit that only the subgroup $\mathcal{G}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ codifies a genuine gauge symmetry while the group $\mathcal{I}_{\mathsf{M},\varsigma^{*}\Omega}(N)$, if non trivial, reflects an accidental residual rigid symmetry. It is this latter that is the proper counterpart of the gauge background preserving symmetry of the ordinary TCO model (cf. subsection 4.2). For this reason, in what follows we shall generally concentrate on the special subgroup $\mathcal{G}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ of $\mathcal{G}^{*}{}_{\mathsf{M},\varsigma^{*}\Omega}(N)$.

For a fake flat gauge field Ω, the existence of a special gauge background dependent surjective Lie group morphism $T_{\varsigma^{*}\Omega} : \mathcal{H}_{\mathsf{M}}(N) \to \mathcal{G}_{\mathsf{M},\varsigma^{*}\Omega}(N)$, shown in subsection 3.2, allows us to regard the gauge transformation action of $\mathcal{G}_{\mathsf{M},\varsigma^{*}\Omega}(N)$ as one of $\mathcal{H}_{\mathsf{M}}(N)$: by way of relations (3.25), (3.26), $\mathcal{T}_{\varsigma^*\Omega}$ provides the gauge transformations of $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ through which $\mathcal{H}_{\mathsf{M}}(N)$ acts on the TCO sigma model field space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$. Explicitly, for $A \in \mathcal{H}_{\mathsf{M}}(N)$, we have

$$\mathrm{G}^{A} = \mathrm{G}^{\mathcal{T}_{\varsigma^*\Omega}(A)}, \tag{5.19}$$

where G^{T} with $\mathrm{T} \in \mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ is given by (5.14). The $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ and $\mathcal{H}_{\mathsf{M}}(N)$ forms of the symmetry, which we shall call dressed and bare respectively, differ in several respects. On one hand, $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ depends on $\varsigma^*\Omega$ while $\mathcal{H}_{\mathsf{M}}(N)$ does not. On the other, the gauge transformation action of $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ does not depend on $\varsigma^*\Omega$ while that of $\mathcal{H}_{\mathsf{M}}(N)$ does. In addition, the $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$-action is effective, while the $\mathcal{H}_{\mathsf{M}}(N)$ is not because of the non trivial kernel $\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ of $\mathcal{T}_{\varsigma^*\Omega}$. As long as we are concerned with the analysis of the gauge symmetry at the classical level, it makes no difference whether either the dressed or the bare standpoint is adopted. At the quantum level, as we shall explain in subsection 5.4 below, it does: the bare standpoint is the appropriate one.

The special gauge symmetry superficially seems to be of a nature analogous to that of the level preserving gauge symmetry, since by virtue of (5.10), (5.14) both symmetries appear to instantiate a certain form of multiplicative redundancy of the derived TCO field G. A component analysis of their gauge transformation action, however, brings to light their inherent difference, as we now show. From (5.10), for $\Upsilon \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ the components of the gauge transform G^{Υ} of G are $g^{\upsilon,\Upsilon} = g\upsilon$, $G^{\upsilon,\Upsilon} = G + \dot{\mu}(g,\Upsilon)$. Similarly, from (5.14), the components of the bare form gauge transform G^{A} of G with $A \in \mathcal{H}_{\mathsf{M}}(N)$ read as $g^{A} = \tau(A)^{-1}g$, $G^{A} = \mathrm{Ad}\,A^{-1}(G + dAA^{-1} + \dot{\mu}(\varsigma^*\omega, A))$. While the action of the component g is indeed multiplicative in both cases, that on the component G reduces to a shift in the former and an ordinary gauge transformation of G regarded as a kind of gauge field in the latter. The special gauge symmetry, therefore, when properly analyzed through its bare expression, turns out to be akin to an ordinary gauge theoretic symmetry in contrast to the level preserving gauge symmetry. As such it should be treated in the quantum theory of the model, as we shall see in greater detail again in subsection 5.4.

5.3 Derived TCO sigma model

Unlike the ordinary model, the derived TCO sigma model is not automatically a sigma model. In this subsection, we discuss the conditions under which the derived model is effectively a 2-dimensional sigma model in virtue of the model's level preserving gauge symmetry studied in subsection 5.2.

Upon modding out the level preserving gauge symmetry, the derived TCO model can be interpreted as an effective theory of the $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ orbit space

$$\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N) = \mathcal{F}_{\mathsf{M}}(N)/\mathcal{G}_{\mathsf{M},\mathrm{K}}(N) \tag{5.20}$$

just as in the ordinary model (cf. eq. (4.12)). In the derived model, however, $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ is not a genuine field space in general. It is one provided there exists a crossed submodule M_{K} of the symmetry crossed module M such that the level preserving gauge transformation group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ is just the group $\mathcal{G}_{\mathsf{M}_{\mathrm{K}}}(N)$ of $\mathrm{D}\mathsf{M}_{\mathrm{K}}$-valued gauge transformations. In such a case, we have indeed that

$$\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N) = \mathrm{Map}(T[1]N, \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}_{\mathrm{K}}) \tag{5.21}$$

and the model can be conceived as a sigma model over the derived homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}_{\mathrm{K}}$ (cf. eq. (4.13)). Accordingly, a field $\mathrm{G} \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ is to be regarded as a field $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(N)$ defined up to right multiplication by a field $\mathrm{G}' \in \mathcal{F}_{\mathsf{M}_{\mathrm{K}}}(N)$. We do not know general conditions on the crossed module M and the level current K ensuring the existence of the submodule M_{K}, but we shall examine in depth in subsection 5.6 below a special kind of TCO model, the characteristic one, where M_{K} can be shown to exist and explicitly described. In the rest of this subsection, we assume unless otherwise stated that M_{K} does exist and call the resulting model derived TCO sigma model.

As in the ordinary set-up, an apparent problem with the sigma model reinterpretation of the TCO model presented in the previous paragraph is that the action S does not really enjoy the level preserving gauge symmetry by (5.11). The issue is solved here essentially in the same manner as in the ordinary theory (cf. subsection 4.3).

Classically, the basic requirement to be met is the level preserving gauge covariance of the field equations (5.4). The invariance of the action S up to a field independent additive term A attested by (5.11), (5.12) is enough for that property to hold.

Quantically, the provision to be satisfied is the level preserving gauge invariance of the exponentiated action e^{iS} entering in the functional integral formulation. This is secured provided that the exponentiated anomaly $\mathrm{e}^{iA(\Upsilon)} = 1$ for any gauge transformation $\Upsilon \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ or equivalently that

$$A(\Upsilon) \in 2\pi\mathbb{Z}. \tag{5.22}$$

Inspection of expression (5.12) shows that the range of values which $A(\Upsilon)$ can take depends on the symmetry crossed module M, the invariant pairing $\langle\cdot,\cdot\rangle$ of M defining the pairing $(\cdot,\cdot)$ (cf. eq. (3.3.15) of I) and the level current K. We know no general conditions on such data ensuring that the integrality condition (5.22) is satisfied by $A(\Upsilon)$. Standard arguments however indicate that this is a clear possibility. As in the ordinary model, the variation of the anomaly $A(\Upsilon)$ with respect to Υ, given by $\delta A(\Upsilon) = \int_{T[1]\partial N} \varrho_{\partial N}(\mathrm{K}, \Upsilon^{-1}\delta\Upsilon)$ from (5.12), vanishes if either $\partial N = \varnothing$ or the gauge transformations Υ is restricted to belong to a subgroup of $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ of gauge transformations obeying boundary conditions making the boundary integral vanish. Further, the values taken by $A(\Upsilon)$ form a group, as by (5.9) and (5.12) $A(\Upsilon) + A(\Upsilon') = A(\Upsilon\Upsilon')$ and $-A(\Upsilon) = A(\Upsilon^{-1})$ for $\Upsilon, \Upsilon' \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$. Hence, A is likely a discrete homotopy invariant valued in a lattice of $\mathbb{R}$. Eq. (5.22) can so generically be satisfied if the level current K is suitably quantized. Admittedly, all this is conjectural. The characteristic model studied in subsection 5.6 will furnish however a concrete realization of this scenario.

Derived TCO theory is hence akin to 4-dimensional CS theory [28] and indeed a 2-dimensional counterpart of this, analogously to ordinary TCO theory and in agreement with the general expectations of categorification. The rigorous non invariance of the action S does not impinge upon the level preserving gauge invariance of the derived TCO model, since the action enjoys the appropriate form of level preserving gauge invariance both classically and quantically. In this way, when the crossed submodule M_{K} introduced above does actually exist, the derived TCO model can be deemed as a sigma model over $\mathsf{DM}/\mathsf{DM}_{\mathrm{K}}$. The sigma model keeps featuring a gauge background preserving symmetry and a background gauge symmetry, as the $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$- and $\mathcal{G}_{\mathsf{M}}(M)$-actions (5.14) and (5.16) commute with the $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$-action and leave the action S invariant by (5.15) and (5.17).

5.4 Functional integral quantization of the derived TCO sigma model

In this subsection, we shall study the functional integral quantization of the derived TCO sigma model. Our considerations will lead to the identification of the model's partition function and a Wilson surface depending on the models data. The analysis we carry out employs formal functional integral techniques and hinges on certain assumptions concerning the fixing of the special gauge symmetry stated in detail in the text. For these reasons, strictly speaking, it provides only a strong, theoretically well-grounded validation of the identification without being a conclusive proof of it.

The quantum functional integral formulation of the derived model is broadly patterned on that of the ordinary model. The gauge nature of the gauge background preserving symmetry in the derived case requires however a special treatment that is not necessary in the ordinary one.

We posit again that there exists a crossed submodule M_{K} of the symmetry crossed module M such that the level preserving gauge transformation group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ equals the group $\mathcal{G}_{\mathsf{M}_{\mathrm{K}}}(N)$ of DM_{K}-valued gauge transformations. As argued in subsection 5.3, this renders the derived TCO model a sigma model over the derived homogeneous space $\mathsf{DM}/\mathsf{DM}_{\mathrm{K}}$. On condition that the level current K satisfies a suitable quantization condition trivializing the anomaly e^{iA}, the exponentiated action e^{iS} is then a genuine functional on the sigma model's field space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$. In the quantum theory, therefore, functional integration can be performed directly on $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$.

We assume in what follows that the functional measure $\mathscr{D}\mathrm{G}$ of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ is invariant under left $\mathcal{G}_{\mathsf{M}}(N)$ multiplicative shifts. As the exponential action e^{iS} enjoys the gauge background preserving and background gauge symmetries as a functional on $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$, as highlighted in subsection 5.3, all the symmetries of the classical sigma model then extend to the quantum one.

There are convincing indications representing that at quantum level derived TCO field theory is unsound unless the background gauge field Ω satisfies the fake flatness requirement (5.8). In the classical theory, as we saw, (5.8) is a general sufficient condition for the integrability of the field equations. In the quantum theory, the same condition ensures the existence of a semiclassical regime and, with this, the possibility of establishing a perturbative expansion around relevant classical field configurations, a property that any sound quantum field theory presumably should enjoy. For this reason, we assume henceforth that Ω is fake flat.

Since the derived background gauge field Ω is fake flat, the gauge background preserving gauge transformation group $\mathcal{G}^*{}_{\mathsf{M},\varsigma^*\Omega}(N)$ reduces to its normal special gauge transformation subgroup $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ up to a finite dimensional quotient group $\mathcal{I}_{\mathsf{M},\varsigma^*\Omega}(N)$ (cf. subsection 3.2). As we have argued in subsection 5.2, only the subgroup $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ codifies a genuine gauge symmetry of the derived TCO sigma model, while the group $\mathcal{I}_{\mathsf{M},\varsigma^*\Omega}(N)$, if non trivial, reflects an accidental residual rigid symmetry answering to the gauge background preserving symmetry of the ordinary TCO model (cf. subsection 4.2).

From the above considerations, it follows that the quantum partition function of the derived TCO sigma model is given by

$$Z(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)} \int_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)} \mathscr{D}\mathrm{G}\ \mathrm{e}^{iS(\mathrm{G};\Omega)}\,. \tag{5.23}$$

In the right hand side, the volume $\mathrm{vol}\,\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ of the special gauge transformation group $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ has been divided out to turn the integration on the sigma model field space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ into an effective one on the associated $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$-orbit space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)/\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$, as required by the nature of the special gauge symmetry. The above formal expression actually requires further refinement, about which we shall say momentarily.

The background gauge symmetry of the exponentiated action e^{iS} and the left multiplicative shift invariance of the functional measure $\mathscr{D}\mathrm{G}$ imply that the partition function $Z(\Omega)$ is gauge invariant as a functional of Ω. Consequently,

$$Z(\Omega^{\mathrm{U}}) = Z(\Omega) \tag{5.24}$$

for any background gauge transformation $\mathrm{U} \in \mathcal{G}_{\mathsf{M}}(M)$, similarly to ordinary model (cf. eq. (4.15)). The background gauge invariance of $Z(\Omega)$, albeit quite simple, is a salient property of $Z(\Omega)$.

The partition function $Z(\Omega)$ depends on the base to ambient space embedding ς. In view of a possible relationship with Wilson surfaces, it is interesting to study the invariance properties of $Z(\Omega)$ under continuous deformations of ς, analogously to what we did for the ordinary model in subsection 4.4. An adequate analysis of this issue, which is considerably complicated by the special gauge symmetry of the derived model, requires the elaboration of apposite formal tools, as we shall do next.

As a preliminary step in this direction, we necessitate a more accurate form of the functional integral expression of $Z(\Omega)$ in (5.23). As anticipated in subsection 5.2, the existence of a gauge background dependent surjective Lie group morphism $T_{\varsigma^*\Omega} : \mathcal{H}_{\mathsf{M}}(N) \to \mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ allows one to convert the dressed special gauge transformation action of $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ for the bare one of $\mathcal{H}_{\mathsf{M}}(N)$. If we adopt the second viewpoint, which we claim to be the appropriate one for the model's quantization, one should properly divide by the volume $\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)$ of $\mathcal{H}_{\mathsf{M}}(N)$ rather than the volume $\mathrm{vol}\,\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ of $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ in (5.23). Eq. (5.23) is therefore to be replaced by the more precise expression

$$Z(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)} \mathscr{D}\mathrm{G}\ \mathrm{e}^{iS(\mathrm{G};\Omega)}\,. \tag{5.25}$$

Unlike $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$, the gauge transformation group $\mathcal{H}_{\mathsf{M}}(N)$ is independent from the pullback $\varsigma^*\Omega$ of the background gauge field Ω. Such dependence has been turned over to the action of $\mathcal{H}_{\mathsf{M}}(N)$ on the field space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ encoded in the group morphism $T_{\varsigma^*\Omega}$, rendering its analysis more straightforward at least in principle.

We are now going to study the dependence of the partition function $Z(\Omega)$ on the embedding ς by computing the variation $\delta Z(\Omega)$ of $Z(\Omega)$ under a variation $\delta\varsigma$ of ς leaving the image $\varsigma(\partial N)$ of the boundary ∂N of N fixed, i.e such that $\delta\varsigma|_{\partial N} = 0$. Formally, from (5.25) we have

$$\delta Z(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)} \mathscr{D}\mathrm{G}\; \mathrm{e}^{iS(\mathrm{G};\Omega)}\, i\delta S(\mathrm{G};\Omega). \tag{5.26}$$

This naive approach suffers however a potential problem: even if the action S is gauge invariant, its variation δS may fail to be so because of the dependence of the $\mathcal{H}_{\mathsf{M}}(N)$ gauge transformation action on $\varsigma^*\Omega$. While in a gauge theory the formal quotient by the volume of the gauge transformation group can be carried out via the Faddeev-Popov (FP) procedure when the integrand of the functional integral is gauge invariant such as in (5.25), it is not immediately evident whether the same can be done when the integrand is not such as in (5.26). Before proceeding any further, it is therefore necessary to clarify this point.

In TCO theory, the FP approach is an algorithm employed for computing the formal quotient by the volume $\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)$ of the gauge transformation group $\mathcal{H}_{\mathsf{M}}(N)$ in an expression of the form

$$Z_W(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)} \mathscr{D}\mathrm{G}\; \mathrm{e}^{iS(\mathrm{G};\Omega)}\, W(\mathrm{G};\Omega), \tag{5.27}$$

where W is some gauge invariant functional. In outline, the method works as follows.

The FP algorithm is based on the basic FP identity defining the FP determinant. It involves the choice of a suitable gauge fixing prescription. Here, we shall not provide an explicit one, something in general very hard to do as well-known. We only shall assume as a working hypothesis that one does exist. The gauge fixing condition is specified by means of a suitable functional $F : \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N) \to \mathcal{V}$, where $\mathcal{V}$ is some functional vector space: the gauge fixing amounts to impose the condition $F(\mathrm{G}) = 0$ with $\mathrm{G} \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ defining the gauge slice. F should satisfy certain basic requirements: first, each gauge orbit should intersect the gauge slice and, second, it should do so just once. In other words, Gribov type issues should not occur or be somewhat harmless for the particular gauge choice made. We presently do not have a general proof of the existence of a gauge fixing with the above properties, though it should be possible to get one for the characteristic TCO model of subsection 5.6, which is a more conventional kind of gauge theory. Though a simple Lorenz like gauge fixing prescription might work in that case, a BV formulation [52, 53] of the model in the AKSZ framework [54] may be required. At any rate, this is not a matter that can be solved in the present paper and is left for future work. It is for this reason that, as stated at the beginning of this subsection, our analysis is still to some measure conjectural.

In the TCO sigma model, the FP identity takes the form

$$\det\left((F \circ B_{\varsigma^*\Omega|\mathrm{G}_F})_*(1_N)|_{N_{1_N}\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)}\right) \times \frac{1}{\mathrm{vol}\,\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)} \int_{\mathcal{H}_{\mathsf{M}}(N)} \mathscr{D}A\, \delta_{\mathcal{V}}\left(F \circ B_{\varsigma^*\Omega|\mathrm{G}}(A)\right) = 1. \tag{5.28}$$

In the above relation, $B_{\varsigma^*\Omega|\mathrm{G}} : \mathcal{H}_{\mathsf{M}}(N) \to \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ is the $\mathcal{H}_{\mathsf{M}}(N)$-orbit map of an element $\mathrm{G} \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ defined by $B_{\varsigma^*\Omega|\mathrm{G}}(A) = \mathrm{G}^A$ with $A \in \mathcal{H}_{\mathsf{M}}(N)$, where the A-transform G^A of G is defined in (5.19) through special gauge symmetry dressed to bare action conversion map $T_{\varsigma^*\Omega} : \mathcal{H}_{\mathsf{M}}(N) \to \mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ met earlier. The kernel $\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ of $T_{\varsigma^*\Omega}$ is here the subgroup of $\mathcal{H}_{\mathsf{M}}(N)$ of the elements $A \in \mathcal{H}_{\mathsf{M}}(N)$ such that $B_{\varsigma^*\Omega|\mathrm{G}}(A) = \mathrm{G}$. $(F \circ B_{\varsigma^*\Omega|\mathrm{G}})_*(1_N) : T_{1_N}\mathcal{H}_{\mathsf{M}}(N) \to T_{F(\mathrm{G})}\mathcal{V}$ is the tangent map of the map $F \circ B_{\varsigma^*\Omega|\mathrm{G}} : \mathcal{H}_{\mathsf{M}}(N) \to \mathcal{V}$ at the neutral element $1_N \in \mathcal{H}_{\mathsf{M}}(N)$. $N\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ is the normal subbundle of $\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ in $\mathcal{H}_{\mathsf{M}}(N)$ and $N_{1_N}\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ is the fiber of $N\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ at 1_N.[4] Lastly, $\mathrm{G}_F \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ is the unique element of the $\mathcal{H}_{\mathsf{M}}(N)$-orbit of G obeying $F(\mathrm{G}_F) = 0$. As is well-known, assuming that the measure $\mathscr{D}A$ is invariant under both left and right multiplicative shifts of A, insertion of the left hand side of relation (5.28) into the functional integral in (5.27) allows one to factorize the volume $\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)$ of the gauge transformation group $\mathcal{H}_{\mathsf{M}}(N)$ out of the integral on one hand and restrict the integration to the gauge slice defined by F on the other, furnishing

$$\begin{aligned} Z_W(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)} \int_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)} \mathscr{D}\mathrm{G}\; \mathrm{e}^{iS(\mathrm{G};\Omega)}\, W(\mathrm{G};\Omega) \\ \times \det\Big((F \circ B_{\varsigma^*\Omega|\mathrm{G}})_*(1_N)|_{\mathrm{N}_{1_N}\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)}\Big)\delta_{\mathcal{V}}\left(F(\mathrm{G})\right). \end{aligned} \tag{5.29}$$

Making the above expression fully explicit in Lagrangian quantum field theory requires the introduction of ghost, antighost and Nakanishi-Lautrup fields.

At the formal level, one can give a meaning to expression (5.27) even when W is not gauge invariant again by inserting the left hand side of relation (5.28) into the functional integral. The FP gauge fixed expression obtained in this way is of the form (5.29) with W replaced by its $\mathcal{H}_{\mathsf{M}}(N)$-average

$$\langle W\rangle_{\mathcal{H}_{\mathsf{M}}(N)}(\mathrm{G};\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\mathcal{H}_{\mathsf{M}}(N)} \mathscr{D}A\, W(B_{\varsigma^*\Omega|\mathrm{G}}(A), \Omega). \tag{5.30}$$

Unlike W, $\langle W\rangle_{\mathcal{H}_{\mathsf{M}}(N)}$ is always gauge invariant by the multiplicative shift invariance of the measure $\mathscr{D}A$. Further, when W is gauge invariant, $\langle W\rangle_{\mathcal{H}_{\mathsf{M}}(N)}$ reduces to W. A less formal definition of $\langle W\rangle_{\mathcal{H}_{\mathsf{M}}(N)}$ would however be desirable. However, we can get information useful for our analysis even without one.

The variation $\delta Z(\Omega)$ of $Z(\Omega)$ under a variation $\delta\varsigma$ of ς given by eq. (5.26) can be written in the form (5.27) as

$$\delta Z(\Omega) = Z_{i\delta S}(\Omega). \tag{5.31}$$

It can therefore be cast and analyzed through the FP framework that we have detailed above.

As in the corresponding problem of the ordinary model (cf. subsection 4.4), the variational analysis of the embedding dependence of the partition function $Z(\Omega)$ requires a

[4] The formal definition of the functional measure $\mathcal{D}A$ of $\mathcal{H}_{\mathsf{M}}(N)$ requires endowing $\mathcal{H}_{\mathsf{M}}(N)$ with a Riemannian metric. The normal bundle $N\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ of $\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ is the orthogonal complement of the tangent bundle $T\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ of $\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$ with respect to this metric. More abstractly, we have $N\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N) = T\mathcal{H}_{\mathsf{M}}(N)|_{\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)}/T\mathcal{K}_{\mathsf{M},\varsigma^*\Omega}(N)$.

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suitable variational framework. This is totally analogous to that used for the ordinary model. The variational problem is thus naturally framed in the complex $\mathrm{Fun}(T[1]\mathcal{E}_{N,M})$, δ, where $\mathcal{E}_{N,M} = \mathrm{Emb}(N, M)$ is the infinite dimensional functional manifold of the embeddings of N into M and δ is the variational differential of $\mathcal{E}_{N,M}$ and more broadly in the augmented complex $\mathrm{Fun}(T[1](\mathcal{E}_{N,M} \times N))$, $\delta + d$, d being the de Rham differential of N. Recall that the augmented complex has a natural bigrading induced by the external direct sum decomposition $T[1](\mathcal{E}_{N,M} \times N) = T[1]\mathcal{E}_{N,M} \boxplus T[1]N$ (cf. footnotes 1, 2), with δ, d the bidegree $(1, 0)$, $(0, 1)$ terms of $\delta + d$. Recall moreover that the pull-back of a derived field $\Psi \in \mathrm{Map}(T[1]M, \mathrm{D}\mathfrak{m}[p])$ by the evaluation map $\epsilon : \mathcal{E}_{N,M} \times N \to M$, a derived field $\epsilon^*\Psi \in \mathrm{Map}(T[1](\mathcal{E}_{N,M} \times N), \mathrm{D}\mathfrak{m}[p])$, can be decomposed in terms with definite bidegree, the terms of bidegree $(0, p)$, $(0, p + 1)$ forming the component $\epsilon^*\Psi_N$ of $\epsilon^*\Psi$ along N.

The action $S(\mathrm{G}; \Omega)$ can be regarded as a degree 0 element of $\mathrm{Fun}(T[1]\mathcal{E}_{N,M})$ depending on G and Ω. The variation $\delta S(\mathrm{G}; \Omega)$ can be straightforwardly computed from (5.2). No boundary contributions occur owing to the boundary condition $\delta\varsigma|_{\partial N} = 0$. Inserting the resulting expression in (5.26), we obtain

$$\delta Z(\Omega) = \frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\mathcal{F}_{\mathsf{M}}(N)} \mathscr{D}\mathrm{G}\ \mathrm{e}^{iS(\mathrm{G};\Omega)}\, i \int_{T[1]N} \varrho_N \tag{5.32}$$
$$\times \big[\big(\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Phi)\big) + \big([\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega_N) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}], \mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega)\big)\big].$$

We shall now show that under rather general assumptions the second term within square brackets in the right hand side gives a vanishing contribution. Let $\mathrm{J}, \mathrm{H} \in \mathrm{Map}_c(T[1]N, \mathrm{D}\mathfrak{m}[0])$. Under a variation of the field G such that $\mathrm{G}^{-1}\delta_{\mathrm{H}}\mathrm{G} = \mathrm{H}$, by the left shift invariance of the functional measure $\mathscr{D}\mathrm{G}$, we have

$$\frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\mathcal{F}_{\mathsf{M}}(N)} \mathscr{D}\mathrm{G}\, \delta_{\mathrm{H}} \left(\mathrm{e}^{iS(\mathrm{G};\Omega)} \int_{T[1]N} \varrho_N(\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega), \mathrm{J}) \right) = 0. \tag{5.33}$$

By explicit calculation of the variation of the expression within round brackets, we obtain the Schwinger-Dyson type identity

$$\frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\mathcal{F}_{\mathsf{M}}(N)} \mathscr{D}\mathrm{G}\ \mathrm{e}^{iS(\mathrm{G};\Omega)}\, i \bigg\{ \int_{T[1]N} \varrho_N(\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega), [\mathrm{H}, \mathrm{J}]) \tag{5.34}$$
$$- \int_{T[1]N} \varrho_N\big([\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega_N) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}], \mathrm{H}\big) \int_{T[1]N} \varrho_N(\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega), \mathrm{J}) \bigg\} = 0.$$

From this relation, by the arbitrariness of J, H, it follows that

$$\frac{1}{\mathrm{vol}\,\mathcal{H}_{\mathsf{M}}(N)} \int_{\mathcal{F}_{\mathsf{M}}(N)} \mathscr{D}\mathrm{G}\ \mathrm{e}^{iS(\mathrm{G};\Omega)}\, i \bigg\{ \mathit{tr}\,\big(\mathit{Ad}(\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega))\big) \tag{5.35}$$
$$- \int_{T[1]N} \varrho_N\big([\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega_N) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}], \mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega)\big) \bigg\} = 0,$$

where tr, Ad denote respectively functional trace and adjoint respectively. Above, $\mathit{tr}\,\big(\mathit{Ad}(\mathrm{Ad}\,\mathrm{G}^{-1}(\epsilon^*\Omega))\big)$ contains a factor $\delta_N(0)$, which must be regularized, and a factor that pointwise is of the form $\mathrm{tr}\,\mathrm{ad}\,O$ for some $O \in \mathrm{D}\mathfrak{m}[1]$. On account of (3.3.4) of I, one has that $\operatorname{tr}\operatorname{ad} X = 0$ for every $X \in \mathrm{D}\mathfrak{m}$ if and only if $\operatorname{tr}\operatorname{ad} x = 0$ and $\operatorname{tr}\dot{\mu}(x,\cdot) = 0$ for $x \in \mathfrak{g}$. We call a Lie algebra crossed module $\mathfrak{m}$ with such a property unimodular. If this property holds, $tr\left(Ad(\operatorname{Ad}\mathrm{G}^{-1}(\epsilon^*\Omega))\right) = 0$ and so

$$-\frac{1}{\operatorname{vol}\mathcal{H}_{\mathsf{M}}(N)}\int_{\mathcal{F}_{\mathsf{M}}(N)}\mathscr{D}\mathrm{G}\,\mathrm{e}^{iS(\mathrm{G};\Omega)} \times\int_{T[1]N}\varrho_N\left([\operatorname{Ad}\mathrm{G}^{-1}(\epsilon^*\Omega_N)+\mathrm{G}^{-1}\mathrm{d}\mathrm{G},\mathrm{K}],\operatorname{Ad}\mathrm{G}^{-1}(\epsilon^*\Omega)\right) = 0, \tag{5.36}$$

as anticipated. From (5.32) and (5.36), we find then that

$$\delta Z(\Omega) = \frac{1}{\operatorname{vol}\mathcal{H}_{\mathsf{M}}(N)}\int_{\mathcal{F}_{\mathsf{M}}(N)}\mathscr{D}\mathrm{G}\,\mathrm{e}^{iS(\mathrm{G};\Omega)}\int_{T[1]N}\varrho_N\left(\mathrm{K},\operatorname{Ad}\mathrm{G}^{-1}(\epsilon^*\Phi)\right). \tag{5.37}$$

In this way, we have that

$$\delta Z(\Omega) = 0 \qquad \text{if } \Phi = 0. \tag{5.38}$$

We find in this way that $Z(\Omega)$ is invariant under suitably boundary restricted variations of the embedding ς when the derived gauge field Ω is flat. In this respect, the derived TCO model is completely akin to the ordinary one (cf. eq. (4.20)).

In refs. [18, 19] a general definition of surface knot holonomy is provided. A closed base 2-fold N of genus ℓ is said marked when it is endowed with a choice of a point p_N and 2ℓ closed curves C_{Ni} intersecting at p_N only representing the homology classes of the standard a- and b-cycles. An ambient manifold M is said marked when it is endowed with a choice of a point p_M and 2ℓ closed curves C_{Mi} intersecting at p_M only. A based surface knot of genus ℓ of the marked manifold M is an embedding $\varsigma : N \to M$ that is marking preserving, i.e. such that $\varsigma(p_N) = p_M$ and $\varsigma(C_{Ni}) = C_{Mi}$. In a higher gauge theory with gauge crossed module M, one can associate with any flat derived gauge field Ω and based surface knot ς the surface holonomy $F_\Omega(\varsigma) \in \mathsf{E}$. $F_\Omega(\varsigma)$ is not invariant under the gauge transformations of Ω of the type (3.11) and depends also on the marking of M for a fixed marking of N. It can be shown however that gauge transforming Ω and smoothly deforming the marking and the surface knot ς through ambient isotopy affects $F_\Omega(\varsigma)$ only by a μ-conjugation,

$$F_\Omega(\varsigma) \to \mu(a)(F_\Omega(\varsigma)) \tag{5.39}$$

for some marking dependent element $a \in \mathsf{G}$. Therefore, for given Ω and ς, only the μ-conjugation class of $F_\Omega(\varsigma)$ is uniquely determined in a gauge independent and ambient isotopy invariant fashion. If we wish to extract numerical invariants out of the holonomy of a surface knot ς, we need therefore a μ-trace over E, which we define as a mapping $\operatorname{tr}_\mu : \mathsf{E} \to \mathbb{C}$ invariant under μ-conjugation, viz $\operatorname{tr}_\mu(\mu(a)(A)) = \operatorname{tr}_\mu(A)$ for $a \in \mathsf{G}$ and $A \in \mathsf{E}$. Our claim is that the partition function $Z(\Omega)$ possibly computes a μ-trace of the holonomy of ς

$$Z(\Omega) = \operatorname{tr}_\mu(F_\Omega(\varsigma)). \tag{5.40}$$

This conjecture is supported by the background gauge invariance and homotopy invariance properties (5.24) and (5.38) and the analogy to the ordinary case, see eq. (4.21), where a similar identification can be established for line knot holonomies. However, the arguments expounded above employ formal functional integral techniques and are based on certain assumptions concerning the fixing of the special gauge symmetry, which albeit reasonable, remain to be verified. For these reasons, our analysis does not provided strictly speaking a full proof of the identification but only an argument in favour of it, albeit one with a sound theoretically grounding. To conclusively establish the identification, it would be necessary to settle the gauge fixing issue and check it by carrying out calculations of the partition function for simple choices of the model's data, in particular for the characteristic TCO model of subsection 5.6 whose relation to the derived KKS theory developed in I can be shown.

While the derived TCO model is a topological quantum field theory, the ordinary one is just a topological quantum mechanics. It is to be noticed, moreover, that the gauge fixing issue is specific of the former, which is a gauge theory, and has no analog in the latter, which is not. This explains why the functional integral analysis of the ordinary model reviewed in section 4 culminating with relation (4.21) can be considered a proof of the identification of the model's partition function and a Wilson loop while the corresponding analysis of the derived model leading to relation (5.40), in spite of many formal similarities, can be considered a proof of the identity of the partition function and a Wilson surface somewhat less so. We add to that (4.21) is supported by the calculations presented in refs. [30–32], while similar calculations checking (5.40) are not available at present.

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5.5 Canonical formulation of the derived TCO model

In this subsection, we shall construct the canonical theory of the derived TCO model. The different perspective allowed by the canonical formulation will furnish us new insight into the model.

The canonical formulation of the derived TCO model broadly follows the pattern of that of the ordinary model (cf. subsection 4.5). In the derived case, however, the canonical theory is far richer because of the appearance of first class constraints corresponding to the special gauge symmetry.

To set up the canonical framework, we assume that the model's base manifold N is $\widetilde{L} = \mathbb{R}^1 \times L$, where L is a compact connected 1-fold, so either $\mathbb{S}^1$ or $\mathbb{I}^1$, the circle and the interval. The factor $\mathbb{R}^1$ is just the time axis. The resulting external direct sum decomposition $T[1]\widetilde{L} = T[1]\mathbb{R}^1 \boxplus T[1]L$ (cf. footnote 1) of the degree shifted tangent bundle of $\widetilde{L}$ entails that the function algebra $\mathrm{Fun}(T[1]\widetilde{L})$ is not only graded but also bigraded in compatible fashion (cf. footnote 2). A generic field $\widetilde{\lambda} \in \mathrm{Map}(T[1]\widetilde{L}, \mathbb{R}[p])$ can therefore be expressed as $\widetilde{\lambda} = \lambda_t + \lambda$, where $\lambda_t, \lambda \in \mathrm{Map}(T[1]\widetilde{L}, \mathbb{R}[p])$ have bidegrees $(1, p-1)$, $(0, p)$, respectively.[5] Similarly, the de Rham differential $\widetilde{d}$ of $\widetilde{L}$ splits as $\widetilde{d} = d_t + d$, where the bidegree $(1, 0)$, $(0, 1)$ d_t, d terms are accordingly the de Rham differentials of $\mathbb{R}^1$, L.

In the derived framework, fields are amenable to a similar bidegree analysis. A field $\widetilde{\mathrm{U}} \in \mathrm{Map}(T[1]\widetilde{L}, \mathrm{DM})$ factorizes as

$$\widetilde{\mathrm{U}} = \mathrm{U}_t \mathrm{U} \tag{5.41}$$

[5]For dimensional reasons, $\widetilde{\lambda}$ is non vanishing only for $p = 0, 1, 2$. Furthermore, for $p = 0$, $\lambda_t = 0$; for $p = 2$, $\lambda = 0$.

with $\mathrm{U}_t, \mathrm{U} \in \mathrm{Map}(T[1]\widetilde{L}, \mathrm{D}\mathsf{M})$. The multiplicative field constituents U_t, U are uniquely determined by certain requirements on the their components (cf. eq. (3.3.1) of I), viz that $u_t = 1_{\mathsf{G}}$ and that U_t, U have bidegrees $(1,0)$, $(0,1)$, respectively. In similar fashion, a field $\widetilde{\Psi} \in \mathrm{Map}(T[1]\widetilde{L}, \mathrm{D}\mathfrak{m}[p])$ decomposes as

$$\widetilde{\Psi} = \Psi_t + \Psi, \tag{5.42}$$

where $\Psi_t, \Psi \in \mathrm{Map}(T[1]\widetilde{L}, \mathrm{D}\mathfrak{m}[p])$. The additive field constituents Ψ_t, Ψ are again uniquely determined by conditions on their components, namely that ψ_t, ψ, $\mathit{\Psi}_t$, $\mathit{\Psi}$ have bidegrees $(1, p-1)$, $(0, p)$, $(1, p)$, $(0, p+1)$, respectively. Accordingly, the derived differential $\widetilde{\mathrm{d}}$ of $\widetilde{L}$ can be expressed as

$$\widetilde{\mathrm{d}} = d_t + \mathrm{d}, \tag{5.43}$$

where the bidegree $(1,0)$, $(0,1)$ components d_t, d correspond to the ordinary de Rham differential of $\mathbb{R}^1$ and the derived differential of L.

Relying the above geometrical set-up, we can write the derived TCO model's action S on $N = \widetilde{L}$ in the form required for canonical analysis. The TCO field factorizes as $\widetilde{\mathrm{G}} = \mathrm{G}_t\mathrm{G}$ in accordance with (5.41). The base manifold pull-back of the background derived gauge field splits as $\widetilde{\varsigma}^*\Omega = \varsigma^*{}_t\Omega + \varsigma^*\Omega$ as in (5.42). The level current similarly reads as $\widetilde{\mathrm{K}} = \mathrm{K}_t + \mathrm{K}$. By explicit computation, we find that S gets expressed in terms of the field constituents G_t, G, $\varsigma^*{}_t\Omega$, $\varsigma^*\Omega$ K_t, K as

$$\begin{aligned} S(\widetilde{\mathrm{G}};\Omega) = \int_{T[1]\widetilde{L}} \varrho_{\widetilde{L}}\Big[-\big(\mathrm{Ad}\,\mathrm{G}([\mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}, \mathrm{K}]), \widehat{\mathrm{G}}_t\big) \\ + \big(\mathrm{K}_t, \mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}\big) + \big(\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*{}_t\Omega) + \mathrm{G}^{-1}d_t\mathrm{G} + \mathrm{G}^{-1}\mathrm{dG}\big)\Big], \end{aligned} \tag{5.44}$$

where $\widehat{\mathrm{G}}_t \in \mathrm{Map}(T[1]\widetilde{L}, \mathrm{D}\mathfrak{m}[0])$ is a derived field of components $\widehat{g}_t = 0$, $\widehat{G}_t = G_t$. Inspection of relation (5.44) reveals that $\widehat{\mathrm{G}}_t$ is a non dynamical Lagrange multiplier field enforcing the constraints

$$[\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}] \approx 0. \tag{5.45}$$

As we shall show in due course, when the background gauge field Ω obeys the fake flatness condition (5.8), these constraints reflect the special gauge symmetry of the derived TCO model (cf. subsection 5.2). In canonical derived TCO theory, the gauge symmetry is partially fixed by imposing the condition $\widehat{\mathrm{G}}_t = 0$, that is,

$$\mathrm{G}_t = 1_{\mathrm{D}\mathsf{M}}. \tag{5.46}$$

Eq. (5.46) leaves the special gauge transformations $\widetilde{\mathrm{T}} \in \mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(\widetilde{L})$ such that $\mathrm{T}_t = 1_{\mathrm{D}\mathsf{M}}$ with regard to the factorization (5.41) as the only allowed ones.

In canonical theory, we replace the derived TCO field G with a time independent field, also denoted as G, viewed as a point of an ambient functional phase space $\mathcal{F}_{\mathsf{M}}(L) := \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M})$. The study of the symplectic structure of $\mathcal{F}_{\mathsf{M}}(L)$ is naturally carried out using the variational Cartan calculus of $\mathcal{F}_{\mathsf{M}}(L)$. This features a collection of variational derivations of the functional algebra $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ comprising the degree -1 contractions ι_V and degree 0 Lie derivatives λ_V along the vector fields $V \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$

and the degree 1 differential δ and satisfying graded commutation relations of the form (A.3.1)–(A.3.4) of I. Since the fields appearing in our analysis can be reduced to fields belonging to the graded tensor product algebra $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L)) \otimes \mathrm{Fun}(T[1]L)$, they are naturally bigraded (cf. footnote 2).

As in the canonical formulation of the ordinary model (cf. subsection 4.5), the appropriate form of the presymplectic potential 1-form $\mathit{\Pi}$ is indicated by that of the kinetic term $(\mathrm{K}, \mathrm{G}^{-1} d_t \mathrm{G})$ appearing in the expression of the TCO action S given in eq. (5.44). It therefore depends on the level current term K. However, the requirement that $\mathit{\Pi}$ and the deriving presymplectic 2-form $\mathit{\Psi} = \delta \mathit{\Pi}$ be time independent necessitates that certain restrictions are imposed on $\widetilde{\mathrm{K}}$. The most obvious and natural one is that

$$\mathrm{K}_t = 0. \tag{5.47}$$

Eq. (5.47) together with condition (5.1), here reading as $\widetilde{\mathrm{d}}\widetilde{\mathrm{K}} = 0$, then yield

$$d_t \mathrm{K} = 0 \quad \text{and} \quad \mathrm{dK} = 0. \tag{5.48}$$

The first relation simply states the time independence of K. The second avoids problematic terms in our analysis. The presymplectic potential 1-form

$$\mathit{\Pi} = \int_{T[1]L} \varrho_L(\mathrm{K}, \mathrm{G}^{-1}\delta\mathrm{G}) \tag{5.49}$$

and the associated presymplectic 2-form

$$\mathit{\Psi} = \delta\mathit{\Pi} = \frac{1}{2}\int_{T[1]L} \varrho_L\left(\mathrm{K}, [\mathrm{G}^{-1}\delta\mathrm{G}, \mathrm{G}^{-1}\delta\mathrm{G}]\right) \tag{5.50}$$

have in this way the required time independence properties. The form of $\mathit{\Pi}$ and $\mathit{\Psi}$ clearly mirrors that of the corresponding objects of the ordinary model's canonical theory (cf. eqs. (4.23), (4.24))

In subsection 5.2, we found that the derived TCO model is an effective theory of the level preserving $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$-orbit space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(N)$ (cf. eq. (5.20)), leading under the assumption of the existence of the crossed submodule M_{K} of M to the reinterpretation of the model as a sigma model. Such primary property of the model is expected to turn up also in the model's canonical theory. In the canonical formulation, the level preserving gauge transformation group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$ is the subgroup of $\mathcal{G}_{\mathsf{M}}(L)$ formed by the elements $\Upsilon \in \mathcal{G}_{\mathsf{M}}(L)$ obeying the condition

$$\mathrm{Ad}\,\Upsilon(\mathrm{K}) = \mathrm{K}, \tag{5.51}$$

consistently with (5.9), and acting on the fields $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(L)$ as

$$\mathrm{G}^{\Upsilon} = \mathrm{G}\Upsilon, \tag{5.52}$$

accordingly with (5.10). By (5.51) and (5.52), the infinitesimal level preserving gauge transformation algebra is the Lie algebra $\mathfrak{g}_{\mathsf{M},\mathrm{K}}(L)$ of $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$ and the action (5.52) is implemented infinitesimally for any $\mathrm{Z} \in \mathfrak{g}_{\mathsf{M},\mathrm{K}}(L)$ by the vector field $X_{\mathrm{Z}} \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ acting as

$$\iota_{X_{\mathrm{Z}}}(\mathrm{G}^{-1}\delta\mathrm{G}) = \mathrm{Z}. \tag{5.53}$$

By (5.50) and (5.53), we have

$$\iota_{X_{\mathrm{Z}}}\Psi = 0. \tag{5.54}$$

Conversely, any vector field $X \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ such that $\iota_X\Psi = 0$ is of the form $X = X_{\mathrm{Z}}$ for some $\mathrm{Z} \in \mathfrak{g}_{\mathsf{M},\mathrm{K}}(L)$ pointwise in $\mathcal{F}_{\mathsf{M}}(L)$. The degeneracy of Ψ highlighted by these properties and its form reveal that the phase space of the model is to be properly identified with the $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$-orbit space of $\mathcal{F}_{\mathsf{M}}(L)$,

$$\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L) := \mathcal{F}_{\mathsf{M}}(L)/\mathcal{G}_{\mathsf{M},\mathrm{K}}(L). \tag{5.55}$$

Ψ indeed induces a symplectic 2-form $\overline{\Psi}$ on $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ together with the associated Poisson bracket structure $\{\cdot,\cdot\}$ on the functional algebra $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$. These findings are in line with the facts recalled at the start of this paragraph. The above analysis closely parallel and is indeed the obvious derived extension of the corresponding analysis for the ordinary model in subsection 4.5 (cf. eqs. (4.25)–(4.28)).

The canonical set-up of the derived TCO model is now fully in place. Expressing the Poisson bracket structure of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ is hardly doable if one works with orbits, while it is apparently handier to do if one relies on orbit representative, much as for the ordinary model in subsection 4.5. We thus compute the Poisson bracket of functionals of $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ relying on two basic isomorphisms. The first isomorphism, $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)) \simeq \mathrm{Fun}(\mathcal{F}_{\mathsf{M}}(L))^{\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)}$, identifies the functional algebra $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ and the subalgebra $\mathrm{Fun}(\mathcal{F}_{\mathsf{M}}(L))^{\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)}$ of $\mathrm{Fun}(\mathcal{F}_{\mathsf{M}}(L))$ of the functionals invariant under the $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$-action (5.52). The second isomorphism, $\mathrm{Vect}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)) \simeq \mathrm{W\,Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$, equates the vector field Lie algebra $\mathrm{Vect}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ and the Weyl Lie algebra $\mathrm{W\,Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$ of the Lie subalgebra $\mathrm{Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$ of $\mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ of vector fields $V \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ of the form $V = X_{\mathrm{Z}}$ for some $\mathrm{Z} \in \mathfrak{g}_{\mathsf{M},\mathrm{K}}(L)$ pointwise in $\mathcal{F}_{\mathsf{M}}(L)$ (cf. footnote 3). In this way, a functional $F \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ will be thought of as a functional $F \in \mathrm{Fun}(\mathcal{F}_{\mathsf{M}}(L))$ such that $F(\mathrm{G}^{\Upsilon}) = F(\mathrm{G})$ for $\Upsilon \in \mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$. Similarly, a vector field $V \in \mathrm{Vect}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ will be regarded as a vector field $V \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ defined mod vector fields $V' \in \mathrm{Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$ and such that $[V,V'] \in \mathrm{Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$ for any vector field $V' \in \mathrm{Vect}_{\mathrm{K}}(\mathcal{F}_{\mathsf{M}}(L))$. The expression of the Poisson bracket including a functional $F \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ involves the Hamiltonian vector field $V_F \in \mathrm{Vect}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ of F defined by the relation

$$\delta F + \iota_{V_F}\Psi = 0 \tag{5.56}$$

(cf. eq. (4.29)). The Poisson bracket of a pair of functionals $F, G \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ is then the functional $\{F,G\} \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ given by the expression

$$\{F,G\} = \iota_{V_F}\delta G = -\iota_{V_G}\delta F \tag{5.57}$$

(cf. eq. (4.30)).

We can now tackle the issue of the canonical characterization of the constraints (5.45). A preliminary observation: since the restriction of the embedding $\tilde{\varsigma}$ of $\tilde{L}$ into M to a time slice of $\{t\} \times L$ of $\tilde{L}$ is an embedding $\{t\} \times L$ into M, the canonical theoretic counterpart of $\tilde{\varsigma}$ is an embedding $\varsigma : L \to M$. Reflecting the structure of (5.45), we now define the functional

$$N(\mathrm{W})(\mathrm{G}; \Omega) = \int_{T[1]L} \varrho_L \left([\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\varsigma^*\Omega) + \mathrm{G}^{-1}\mathrm{dG}], \mathrm{Ad}\,\mathrm{G}^{-1}(\mathrm{W}) \right), \tag{5.58}$$

where $\mathrm{W} \in \mathcal{P}_{\mathsf{M}}(L)$ with $\mathcal{P}_{\mathsf{M}}(L) = \mathrm{Map}(T[1]L, \mathrm{D}\mathfrak{m}[-1])$. Recalling that $\mathrm{dK} = 0$, one verifies that $N(\mathrm{W})$ is $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$-invariant, so that $N(\mathrm{W}) \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$. The Hamiltonian vector field $V_{\mathrm{W}} := V_{N(\mathrm{W})} \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ of $N(\mathrm{W})$ is readily expressed through its action on $\mathcal{F}_{\mathsf{M}}(L)$, which reads as

$$\iota_{V_{\mathrm{W}}}(\delta\mathrm{G}\mathrm{G}^{-1}) = -\mathrm{d}V_{\mathrm{W}} - [\varsigma^*\Omega, V_{\mathrm{W}}] \tag{5.59}$$

for $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(L)$. Using this relation, it is straightforward to compute the Poisson bracket of $N(\mathrm{W})$, $N(\mathrm{W}')$ for $\mathrm{W}, \mathrm{W}' \in \mathcal{P}_{\mathsf{M}}(L)$. We find,

$$\{N(\mathrm{W}), N(\mathrm{W}')\} = N([\mathrm{W}, \mathrm{W}']_*), \tag{5.60}$$

where $[\mathrm{W}, \mathrm{W}']_* \in \mathcal{P}_{\mathsf{M}}(L)$ is given by the expression

$$[\mathrm{W}, \mathrm{W}']_* = \frac{1}{2}\left([\mathrm{W}, \mathrm{dW}' + [\varsigma^*\Omega, \mathrm{W}']] - [\mathrm{W}', \mathrm{dW} + [\varsigma^*\Omega, \mathrm{W}]]\right). \tag{5.61}$$

The bracket $[\cdot,\cdot]_*$ is antisymmetric but does not obey the Jacobi identity. The failure $[\cdot,\cdot]_*$ to be a Lie bracket remains however compatible with the Jacobi property of the Poisson bracket $\{\cdot,\cdot\}$, as is not difficult to verify.

The emergence of the constraints (5.45) in the canonical analysis of the derived TCO model indicates that the constraints

$$N(\mathrm{W}) \approx 0 \tag{5.62}$$

with $\mathrm{W} \in \mathcal{P}_{\mathsf{M}}(L)$ should be imposed. Relation (5.60) shows that the functionals $N(\mathrm{W})$ are first class. Therefore, these constraints are associated with a gauge symmetry. We are now going to see that this is precisely the special symmetry studied in subsection 5.2.

In subsection 5.2, we saw that the special gauge transformation action can take either the dressed $\mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(N)$ or the bare $\mathcal{H}_{\mathsf{M}}(N)$ form. In subsection 5.4, we concluded that the second option is the most natural one for functional integral quantization. Based on these findings, we suppose that $\mathcal{H}_{\mathsf{M}}(L)$ is the appropriate gauge transformation group in the canonical formulation. The transform of a field $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(L)$ by a gauge transformation $A \in \mathcal{H}_{\mathsf{M}}(L)$ is

$$\mathrm{G}^A = \mathrm{G}^{T_{\varsigma^*\Omega}(A)} \tag{5.63}$$

in conformity with (5.19), where the morphism $T_{\varsigma^*\Omega} : \mathcal{H}_{\mathsf{M}}(L) \to \mathcal{G}_{\mathsf{M},\varsigma^*\Omega}(L)$ is defined in subsection 3.2. Correspondingly, the action of an infinitesimal gauge transformation $\Pi \in \mathfrak{h}_{\mathsf{M}}(L)$ on a field $\mathrm{G} \in \mathcal{F}_{\mathsf{M}}(L)$ reads as

$$\delta_\Pi \mathrm{G}\mathrm{G}^{-1} = -\dot{T}_{\varsigma^*\Omega}(\Pi). \tag{5.64}$$

The component w of $\mathrm{W} \in \mathcal{P}_{\mathsf{M}}(L)$ vanishes identically, since w has degree -1 and $T[1]L$ is a non negatively graded manifold. The component W of W, which has degree 0, can conversely be non zero. Therefore, there is an element $\varPi \in \mathrm{Map}(T[1]L, \mathfrak{e}) = \mathfrak{h}_{\mathsf{M}}(L)$ such that $\mathrm{W} = \mathrm{W}_{\varPi}$, where $\mathrm{W}_{\varPi} \in \mathcal{P}_{\mathsf{M}}(L)$ with[6]

$$\mathrm{W}_{\varPi}(\alpha) = \alpha\varPi. \tag{5.65}$$

Set $V_{\varPi} = V_{\mathrm{W}_{\varPi}}$ for brevity. Inserting (5.65) into (5.59), we get

$$\iota_{V_{\varPi}}(\delta \mathrm{G}\mathrm{G}^{-1})(\alpha) = -\dot{\tau}(\varPi) + \alpha\left(d\varPi + \dot{\mu}\dot{}(\omega, \varPi)\right). \tag{5.66}$$

Expressing alternatively $\dot{T}_{\varsigma^*\Omega}(\varPi)$ in (5.64) in terms of $\varPi$ by utilizing (3.29), (3.30), it emerges that

$$\iota_{V_{\varPi}}(\delta \mathrm{G}\mathrm{G}^{-1}) = \delta_{\varPi}\mathrm{G}\mathrm{G}^{-1}. \tag{5.67}$$

$V_{\varPi}$ is thus the vector field enacting the infinitesimal transformation $\varPi \in \mathfrak{h}_{\mathsf{M}}(L)$. The identification of the gauge symmetry associated with the first class constraints (5.62) with the special gauge symmetry is thereby ascertained.

From the above discussion, it follows that

$$N(\varPi) = N(\mathrm{W}_{\varPi}) \tag{5.68}$$

are the Hamiltonian functionals of the infinitesimal special gauge transformations $\varPi \in \mathfrak{h}_{\mathsf{M}}(L)$. From relations (5.60), (5.61), taking (5.65) into account, we find further that for $\varPi, \varPi' \in \mathfrak{h}_{\mathsf{M}}(L)$

$$\{N(\varPi), N(\varPi')\} = N([\varPi, \varPi']). \tag{5.69}$$

The appearance in the right hand side of this relation of the Lie bracket of $\mathfrak{h}_{\mathsf{M}}(L)$ supports by its naturality our assumption that $\mathcal{H}_{\mathsf{M}}(L)$ is the appropriate realization of the special gauge transformation group.

The above analysis indicates that the physical phase space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)$ of the derived TCO sigma model is the subspace of the ambient phase space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ defined through the constraints

$$N(\varPi) \approx 0 \tag{5.70}$$

for $\varPi \in \mathfrak{h}_{\mathsf{M}}(L)$. That $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)$ is indeed a subspace of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ follows from the $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$ invariance of the functionals $N(\varPi)$. The reduced physical phase space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{red}\,\mathrm{phys}}(L)$ is the quotient of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)$ by the special gauge symmetry

$$\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{red}\,\mathrm{phys}}(L) := \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)/\mathcal{H}_{\mathsf{M}}(L). \tag{5.71}$$

Both $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)$ and $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{red}\,\mathrm{phys}}(L)$ depend secretly on the background gauge field Ω, because the Hamiltonian functionals $N(\varPi)$ and the action of $\mathcal{H}_{\mathsf{M}}(L)$ on $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{phys}}(L)$, implemented by the dressed to bare action conversion map $T_{\varsigma^*\Omega}$, do. The Poisson bracket structure of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{red}\,\mathrm{phys}}(L)$ has a standard description. Let $\mathscr{I}_N$ be the ideal of the algebra $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ generated by the functionals $N(\varPi)$ with $\varPi \in \mathfrak{h}_{\mathsf{M}}(L)$. By virtue of the first

[6]Note that under the sign convention (3.3.5) of I, one has $W_{\varPi} = -\varPi$.

classness of the $N(\Pi)$, $\mathscr{I}_N$ is a Poisson subalgebra of $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$. The isomorphism $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}\,\mathrm{red\,phys}}(L)) \simeq \mathrm{W}\mathscr{I}_N$ then holds, where $\mathrm{W}\mathscr{I}_N$ is the Weyl Poisson Lie algebra of $\mathscr{I}_N$ (cf. footnote 3). The isomorphism can be understood as follows. $\mathscr{I}_N$ is constituted by the functionals $F \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ with $F|_{\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,phys}}(L)} = 0$. As any functional of $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,phys}}(L))$ can be extended non uniquely to some functional of $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$, we have an isomorphism $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,phys}}(L)) \simeq \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))/\mathscr{I}_N$. From the first classness of the Hamiltonian generators $N(\Pi)$ again, the infinitesimal special gauge transformation action on $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ descends onto $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,phys}}(L))$ with $\delta_\Pi(F+\mathscr{I}_N) = \{N(\Pi), F\}+\mathscr{I}_N$ for $F \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ and $\Pi \in \mathfrak{h}_{\mathsf{M}}(L)$. A functional $F + \mathscr{I}_N \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,red\,phys}}(L))$ is therefore one such that $\{F', F\} \in \mathscr{I}_N$ for $F' \in \mathscr{I}_N$, i.e. that $F \in \mathrm{N}\mathscr{I}_N$, the Poisson normalizer of $\mathscr{I}_N$. As a consequence, $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,red\,phys}}(L)) \simeq \mathrm{N}\mathscr{I}_N/\mathscr{I}_N = \mathrm{W}\mathscr{I}_N$ as claimed. Concretely, the upshot of this analysis is that a functional $F + \mathscr{I}_N \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,red\,phys}}(L))$ can be viewed as a functional $F \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ defined up to the addition of functionals $F' \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ with $F' \approx 0$ and with the property that $\{F', F\} \approx 0$ for all $F' \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ with $F' \approx 0$. The Poisson bracket of two functionals $F + \mathscr{I}_N, G + \mathscr{I}_N \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,red\,phys}}(L)$ is given now by the natural expression

$$\{F + \mathscr{I}_N\, G + \mathscr{I}_N\} = \{F, G\} + \mathscr{I}_N. \tag{5.72}$$

It can be checked that this Poisson bracket structure is well-defined, in particular that it has $\mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K\,red\,phys}}(L))$ as its range, and has all the required properties.

The derived TCO phase space $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ is characterized by a wider phase space gauge symmetry subsuming the special gauge symmetry. The associated gauge transformation group is the full derived gauge transformation group $\mathcal{G}_{\mathsf{M}}(L)$. The action $\mathcal{G}_{\mathsf{M}}(L)$ on $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$ reads as

$$\mathrm{G}^{\mathrm{T}} = \mathrm{T}^{-1}\mathrm{G} \tag{5.73}$$

with $\mathrm{T} \in \mathcal{G}_{\mathsf{M}}(L)$ and $\mathrm{G} \in \overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$. The action of an infinitesimal gauge transformation $\mathrm{S} \in \mathfrak{g}_{\mathsf{M}}(L)$ is enacted by a vector field $U_{\mathrm{S}} \in \mathrm{Vect}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ such that

$$\iota_{U_{\mathrm{S}}}(\delta\mathrm{G}\mathrm{G}^{-1}) = -\mathrm{S}. \tag{5.74}$$

This symmetry is Hamiltonian with respect to the symplectic structure of $\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L)$: there is a functional $F(\mathrm{S}) \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$ obeying

$$\delta F(\mathrm{S}) + \iota_{U_{\mathrm{S}}}\Psi = 0. \tag{5.75}$$

$F(\mathrm{S})$ is given by the expression

$$F(\mathrm{S})(\mathrm{G}) = \int_{T[1]L} \varrho_L\left(\mathrm{K}, \mathrm{Ad}\,\mathrm{G}^{-1}(\mathrm{S})\right). \tag{5.76}$$

$F(\mathrm{S})$ is $\mathcal{G}_{\mathsf{M},\mathrm{K}}(L)$-invariant and so $F(\mathrm{S}) \in \mathrm{Fun}(\overline{\mathcal{F}}_{\mathsf{M},\mathrm{K}}(L))$, as is straightforward to verify.

The functionals $F(\mathrm{S})$ with $\mathrm{S} \in \mathfrak{g}_{\mathsf{M}}(L)$ constitute a first class set of functionals. Indeed, they satisfy the Poisson bracket algebra

$$\{F(\mathrm{S}), F(\mathrm{S}')\} = F([\mathrm{S}, \mathrm{S}']) \tag{5.77}$$

with $\mathrm{S}, \mathrm{S}' \in \mathfrak{g}_{\mathsf{M}}(L)$. The Hamiltonians $\mathcal{N}(\mathrm{W})$, $\mathrm{W} \in \mathcal{P}_{\mathsf{M}}(L)$, of the special gauge symmetry are simply related to the $\mathcal{F}(\mathrm{S})$,

$$\mathcal{N}(\mathrm{W}) = \mathcal{F}(\mathrm{S})|_{\mathrm{S}=\mathrm{dW}+[\varsigma^*\Omega,\mathrm{W}]}. \tag{5.78}$$

This finding reveals how the special gauge symmetry arises from the wider phase space gauge symmetry. The former is the gauge symmetry left over after the breaking of the latter due the TCO model's coupling to the background gauge field Ω. However, while the special symmetry is dynamical, the phase space symmetry is merely kinematical.

5.6 Characteristic derived TCO sigma model

The characteristic model is a specialization of the derived TCO model elaborated in the preceding subsections based on a specific choice of the level current. Its distinguished features highlight the TCO model's connection to the derived KKS theory worked out in section 5 of I and by this very reason provide the main motivation for studying the model in the first place.

In the characteristic derived TCO model with base 2-fold N, the level current is taken to be the most general current $\mathrm{K} \in \mathrm{Map}'(T[1]N, \mathrm{D}\mathfrak{m}[0])$ obeying $\mathrm{dK} = 0$ with components k and K proportional to the Heaviside and Dirac distributions θ_N and $\delta_{\partial N}$ of N and ∂N, respectively. Recall that $\theta_N \in \mathrm{Map}'(T[1]N, \mathbb{R})$, $\delta_{\partial N} \in \mathrm{Map}'(T[1]N, \mathbb{R}[1])$ are defined by the relations $\int_{T[1]N} \varrho_N \theta_N \varphi = \int_{T[1]N} \varrho_N \varphi$, $\int_{T[1]N} \varrho_N \delta_{\partial N} \varphi = \int_{T[1]\partial N} \varrho_{\partial N} \varphi$ for $\varphi \in \mathrm{Fun}(T[1]N)$ and satisfy by Stokes' theorem the equation $d\theta_N + \delta_{\partial N} = 0$.[7] We have

$$k = \theta_N \dot{\tau}(\Lambda), \tag{5.79}$$

$$K = \delta_{\partial N} \Lambda \tag{5.80}$$

for some $\Lambda \in \mathfrak{e}$, as is immediately checked.

In the characteristic set-up, the derived TCO action $S(\mathrm{G}; \Omega)$ is more usefully expressed in terms of the components g, G and ω, Ω of the TCO field G and the background gauge field Ω. Using (5.79), (5.80), expression (5.2) furnishes

$$\begin{aligned} S(\mathrm{G}; \Omega) = &\int_{T[1]N} \varrho_N \left\langle \dot{\tau}(\Lambda), \dot{\mu}\left(g^{-1}, \varsigma^*\Omega + \dot{\mu}\dot{}(\varsigma^*\omega, G) + dG + \frac{1}{2}[G, G]\right)\right\rangle \\ &- \int_{T[1]\partial N} \varrho_{\partial N} \left\langle \mathrm{Ad}\, g^{-1}\left(\varsigma^*\omega + dgg^{-1} + \dot{\tau}(G)\right), \Lambda\right\rangle. \end{aligned} \tag{5.81}$$

---

[7] The Heaviside distribution θ_N and Dirac distribution $\delta_{\partial N}$ are singular precisely on the boundary ∂N of N. To make the above distributional identities to make sense, we imagine that N is extended to a larger 2-fold N' by attaching an outer collar to each connected component of ∂N. It is N' that rigorously corresponds to the base 2-fold of the model in the sense meant in subsection 5.1 and denoted by N there. For notational simplicity, we shall imagine the collars as 'infinitesimally thin' and shall not distinguish between N and N'.

The characteristic model's field equations can be obtained either from the derived form equation (5.4) or by variation of the action (5.81). Componentwise, they take the form

$$\left[\dot{\mu}\left(g^{-1},\varsigma^*\Omega+\dot{\mu}(\varsigma^*\omega,G)+dG+\frac{1}{2}[G,G]\right),\Lambda\right]=0 \qquad \text{on } N, \tag{5.82}$$

$$\dot{\tau}\left(\dot{\mu}\left(\operatorname{Ad}g^{-1}(\varsigma^*\omega+dgg^{-1}+\dot{\tau}(G)),\Lambda\right)\right)=0 \qquad \text{on } N, \tag{5.83}$$

$$\dot{\mu}\left(\operatorname{Ad}g^{-1}(\varsigma^*\omega+dgg^{-1}+\dot{\tau}(G)),\Lambda\right)=0 \qquad \text{on } \partial N. \tag{5.84}$$

While the first method of derivation is more direct, the second one is more instructive. The variation δS of the action S displays beside a bulk piece supported on the interior of the base 2-fold N also an edge contribution localized on the boundary ∂N of N, that cannot be absorbed into the bulk one by means of Stokes' theorem without generating terms containing derivatives of the variations δg, δG of the component fields g, G not allowed. This indicates that the action functional S is not differentiable in the sense of refs. [55, 56]. In a situation like this, the edge dynamics is added to the bulk one and the field equations split into bulk and edge. Eq. (5.82), (5.83) are bulk equations yielded by variation of S with respect to g, G, respectively. Eq. (5.84) is an edge equation obtained by varying g. There is no edge equation associated with variation with respect to G, because the bulk G variation generates a boundary term via Stokes' theorem that exactly cancels that engendered by the edge G variation. Bulk and edge dynamics must be compatible: the edge equations must imply the restriction to the boundary of the bulk ones. Indeed, (5.84) clearly implies (5.83) at the boundary. The characteristic model's integrability conditions can similarly be obtained either from the derived form condition (5.7) or from the field equations (5.82)–(5.84). Componentwise, they read as

$$\dot{\tau}\left(\dot{\mu}\left(\varsigma^*\phi,\dot{\mu}(g,\Lambda)\right)\right)\approx 0 \qquad \text{on } N, \tag{5.85}$$

$$\dot{\mu}\left(\varsigma^*\phi,\dot{\mu}(g,\Lambda)\right)\approx 0 \qquad \text{on } \partial N. \tag{5.86}$$

A further condition yielded in this way, $[\varsigma^*\Phi+\dot{\mu}(\varsigma^*\phi,G),\dot{\mu}(g,\Lambda)]\approx 0$ on N, is trivially satisfied by dimensional reasons. The conditions are manifestly realized if the fake flatness requirement (5.8) is met.

We examine next the symmetries of the characteristic model starting with the level preserving gauge symmetry (cf. subsection 5.2). Since the level current is completely determined by the source Lie algebra datum Λ, we shall denote the level preserving gauge transformation group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ as $\mathcal{G}_{\mathsf{M},\Lambda}(N)$. From (5.9), the components υ, Υ of a gauge transformation $\Upsilon\in\mathcal{G}_{\mathsf{M},\Lambda}(N)$ obey

$$[\Upsilon,\Lambda]=0 \qquad \text{on } N, \tag{5.87}$$

$$\dot{\tau}(\dot{\mu}(\upsilon,\Lambda)-\Lambda)=0 \qquad \text{on } N, \tag{5.88}$$

$$\dot{\mu}(\upsilon,\Lambda)-\Lambda=0 \qquad \text{on } \partial N \tag{5.89}$$

by the characteristic form (5.79), (5.80) of the level current. Eq. (5.89) implies (5.88) at the base boundary making the bulk and edge properties compatible. The gauge transform

G^{Υ} of the TCO field G reads in components as

$$g^{\upsilon,\Upsilon} = g\upsilon, \tag{5.90}$$
$$G^{\upsilon,\Upsilon} = G + \dot{\mu}(g,\Upsilon). \tag{5.91}$$

In the characteristic TCO model, the classical anomaly $A(\Upsilon)$ of a gauge transformation $\Upsilon \in \mathcal{G}_{\mathsf{M},\Lambda}(N)$ given in eq. (5.12) takes the form

$$A(\upsilon,\Upsilon) = -\int_{T[1]\partial N} \varrho_{\partial N}\langle d\upsilon\upsilon^{-1},\Lambda\rangle. \tag{5.92}$$

The remarkable property of A is its being a pure boundary term. This makes the derived TCO model akin to the 4-dimensional CS action [28].[8]

The special gauge symmetry has in the characteristic TCO model no particular features which distinguish it from the general model. We report here the bare form component expression of the gauge transform G^{A} of the TCO field G by a special gauge transformation $A \in \mathcal{H}_{\mathsf{M}}(N)$,

$$g^{A} = \tau(A)^{-1}g, \tag{5.93}$$
$$G^{A} = \operatorname{Ad} A^{-1}(G + dAA^{-1} + \dot{\mu}(\varsigma^{*}\omega, A)) \tag{5.94}$$

(cf. eq. (5.19)).

The gauge background gauge symmetry has too no special features in the characteristic TCO model. The component expression of the transform G^{U} of the TCO field G under a background gauge transformation $\mathrm{U} \in \mathcal{G}_{\mathsf{M}}(M)$ reads as

$$g^{u,U} = \varsigma^{*}u^{-1}g, \tag{5.95}$$
$$G^{u,U} = \dot{\mu}(\varsigma^{*}u^{-1}, G - \varsigma^{*}U) \tag{5.96}$$

(cf. eq. (5.16)).

The natural question arises about whether the characteristic model is a sigma model and, having in mind a possible relationship to derived KKS theory it is one over the derived coadjoint orbit $\mathcal{O}_{\Lambda}$ of Λ (cf. subsects. 5.1 and 5.9 of I). In subsection 5.3, we found that the general derived TCO model is a sigma model if there exists a crossed submodule M_{K} of M such that the level preserving gauge transformation group $\mathcal{G}_{\mathsf{M},\mathrm{K}}(N)$ turns out to equal the $\mathrm{D}\mathsf{M}_{\mathrm{K}}$-valued gauge transformation group $\mathcal{G}_{\mathsf{M}_{\mathrm{K}}}(N)$. Inspection of eqs. (5.87)–(5.89) stating the conditions which a level preserving gauge transformation $\Upsilon \in \mathcal{G}_{\mathsf{M},\Lambda}(N)$ must fulfill reveals that there are three groups relevant for settling the issue for the characteristic model: the subgroup $\mu\,\mathsf{ZE}_{\Lambda}$ of G of the elements $a \in \mathsf{G}$ such that $\dot{\mu}(a,\Lambda) = \Lambda$, the subgroup $\mu\,\mathsf{Z}^{*}\mathsf{E}_{\Lambda}$ of G of the elements $a \in \mathsf{G}$ such that $\dot{\mu}(a,\Lambda) = \Lambda \bmod \ker\dot{\tau}$ and the subgroup ZE_{Λ} of E of the elements $A \in \mathsf{E}$ such that $\operatorname{Ad} A(\Lambda) = \Lambda$. By (5.87)–(5.89), a level preserving gauge

[8] It is not difficult to understand how this relation comes about. By comparing (5.2) and (5.12), it is apparent that $A(\upsilon,\Upsilon) = S(\upsilon,\Upsilon;0,0)$. On account of conditions (5.88), (5.87) and the invariance of the pairing, all the terms of the form $\mu(\upsilon,\cdot)$ and $[\Upsilon,\cdot]$ drop out in the expression of $S(\upsilon,\Upsilon;0,0)$ furnished by (5.81) leaving only the boundary contribution shown in (5.92).

transformation $\Upsilon \in \mathcal{G}_{\mathsf{M},\Lambda}(N)$ is one such that its components υ, Υ are valued in $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$, $\mathrm{Z}\mathsf{E}_\Lambda$ in the base manifold interior N and in $\mu\,\mathrm{Z}\mathsf{E}_\Lambda$, $\mathrm{Z}\mathsf{E}_\Lambda$ in the base manifold boundary ∂N. $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ and $\mathrm{Z}\mathsf{E}_\Lambda$ are the source and target groups of a crossed submodule $\mathrm{Z}^*\mathsf{M}_\Lambda$ of M. Similarly, $\mu\,\mathrm{Z}\mathsf{E}_\Lambda$ and $\mathrm{Z}\mathsf{E}_\Lambda$ are the source and target groups of a crossed submodule $\mathrm{Z}\mathsf{M}_\Lambda$ of M, in fact the centralizer crossed submodule of Λ (cf. subsection 5.1 of I). $\mathrm{Z}\mathsf{M}_\Lambda$ is evidently a crossed submodule of $\mathrm{Z}^*\mathsf{M}_\Lambda$. Since $\mathcal{O}_\Lambda = \mathrm{D}\mathsf{M}/\mathrm{DZ}\mathsf{M}_\Lambda$, the answer to the question we posed at the beginning of this paragraph is positive provided that $\mathrm{Z}^*\mathsf{M}_\Lambda = \mathrm{Z}\mathsf{M}_\Lambda$.

In general, $\mathrm{Z}\mathsf{M}_\Lambda$ is strictly smaller that $\mathrm{Z}^*\mathsf{M}_\Lambda$. A straightforward way of arranging that $\mathrm{Z}^*\mathsf{M}_\Lambda = \mathrm{Z}\mathsf{M}_\Lambda$ is by requiring that $\ker\dot{\tau} = 0$. A crossed module M with this property is called quasi injective. A restriction such as this one seems however to be exceedingly severe. The identity $\mathrm{Z}^*\mathsf{M}_\Lambda = \mathrm{Z}\mathsf{M}_\Lambda$ can be ensured with less drastic limitations on M as follows.

We recall that the action $\mu\dot{}$ of G on $\mathfrak{e}$ leaves the subspace $\ker\dot{\tau} \subseteq \mathfrak{e}$ invariant. The crossed module M is said to be inert on the target kernel if the induced action $\mu\dot{}$ of G on $\ker\dot{\tau}$ is trivial. Note that every quasi injective crossed module is inert on the target kernel, but the converse is false in general. Therefore, target kernel inert crossed modules constitute a broader set of crossed modules.

Suppose that M is a compact crossed module (cf. subsection 5.1 of I) inert on the target kernel. The map $C : \mu\,\mathrm{Z}^*\mathsf{E}_\Lambda \to \ker\dot{\tau}$ defined by $C(a) = \mu\dot{}(a,\Lambda) - \Lambda$ for $a \in \mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ satisfies the relation $C(ab) = C(a) + C(b)$ for all $a, b \in \mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ because of the triviality of the G-action $\mu\dot{}$ on $\ker\dot{\tau}$. C is so a 1-cocycle of the group $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ with coefficients in the trivial $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$-module $\ker\dot{\tau}$. From the compactness of $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ ensuing from that of G, it follows that the 1-cocycle C vanishes identically.[9] So, $\mu\,\mathrm{Z}^*\mathsf{E}_\Lambda = \mu\,\mathrm{Z}\mathsf{E}_\Lambda$. Hence, $\mathrm{Z}^*\mathsf{M}_\Lambda = \mathrm{Z}\mathsf{M}_\Lambda$ as desired.

In subsection 5.1 of I, we examined a number of Lie group crossed module models to illustrate basic notions of derived KKS theory. We can use the same models to check whether the range of crossed modules which are inert on the target kernel is large enough. The crossed module $\mathrm{INN}_{\mathsf{G}}\,\mathsf{N} = (\mathsf{N}, \mathsf{G}, \iota, \kappa)$ associated with a pair of a Lie group G and a normal subgroup N of G, where ι is the inclusion map of N into G and κ is the left conjugation action of G on N, is the first such model we consider. $\mathrm{INN}_{\mathsf{G}}\,\mathsf{N}$ is quasi injective and so trivially inert to the target kernel. The second model we scrutinize consists in the crossed module $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G}) = (\mathsf{Q}, \mathsf{G}, \pi, \alpha)$ stemming from a central extension $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \mathsf{Q} \xrightarrow{\pi} \mathsf{G} \longrightarrow 1$ of Lie groups, where the action α is given by $\alpha(a, A) = \sigma(a)A\sigma(a)^{-1}$ for $a \in \mathsf{G}$, $A \in \mathsf{Q}$ with $\sigma : \mathsf{G} \to \mathsf{Q}$ a section of the projection π. $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ is not quasi injective in general, but it is always inert on the target kernel since $\ker\pi$ is central in Q. As final model, we examine the crossed module $\mathsf{D}(\rho) = (\mathsf{V}, \mathsf{G}, 1_{\mathsf{G}}, \rho)$ of a Lie group G, a vector space V regarded as an Abelian group, the trivial morphism 1_{G} of V into G and a representation ρ of G in V. This crossed module is inert on the target kernel only if the representation ρ is trivial.

As the crossed module entering as basic symmetry datum of a derived TCO model is

[9] Since $\mathrm{Z}^*\mathsf{E}_\Lambda$ is a compact group, $C(\mathrm{Z}^*\mathsf{E}_\Lambda)$ is a compact subset of $\ker\dot{\tau}$ by virtue of the continuity of C. The topology of $\ker\dot{\tau}$ can by described by means of a norm and every compact set of $\ker\dot{\tau}$ turns out to be bounded with respect to that norm. In particular, $C(\mathrm{Z}^*\mathsf{E}_\Lambda)$ is bounded. C being a 1-cocycle entails that for any $a \in \mu\,\mathrm{Z}^*\mathsf{E}_\Lambda$ and any positive integer n one has $C(a^n) = nC(a)$. This property is compatible with the boundedness of $C(\mathrm{Z}^*\mathsf{E}_\Lambda)$ only if $C(a) = 0$.

always equipped with an invariant pairing, it is balanced (cf. subsection 3.1 of I). This, however, involves only an illusory loss of generality. In fact, every crossed module M can always be trivially extended to a balanced crossed module M^c (see footnote 7 of I for an explicit description of this latter). M^c can then be employed instead of M without this making any difference as far as the symmetry properties of the resulting TCO model are concerned. It is not difficult to show that, if M is trivial on the target kernel, then M^c also is. The target kernel trivial crossed modules considered in the previous paragraph are generally non balanced. This is however no problem, for the reasons just explained.

In this way, the analysis of subsection 5.3 and the above considerations lead us to the conclusion that when the symmetry crossed module M is compact and target kernel inert, as we assume henceforth, the derived characteristic TCO model field space is $\overline{\mathcal{F}}_{\mathsf{M},\varLambda}(N) = \mathrm{Map}(T[1]N, \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{ZM}_\varLambda)$ and that so the model is truly a sigma model over the derived coadjoint orbit $\mathcal{O}_\varLambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{ZM}_\varLambda$. In this respect, the close relationship of the characteristic to the ordinary TCO model is quite evident.

The quantization condition (5.22) of the anomaly $\mathsf{A}(\Upsilon)$ required by the consistency of the quantum theory (cf. subsects. 5.3, 5.4) reads as

$$\int_{T[1]\partial N} \varrho_{\partial N} \langle dvv^{-1}, \varLambda\rangle \in 2\pi\mathbb{Z}. \tag{5.97}$$

The following conditions on $\varLambda$ suffice to guarantee that this requirement is met. First, the centralizer crossed module $\mathsf{ZM}_\varLambda$ of $\mathsf{M}_\varLambda$ must be a maximal toral crossed submodule $\mathsf{J} = (\mathsf{H}, \mathsf{T})$ of M and, second, the map $\xi_\varLambda : \mathsf{T} \to \mathsf{U}(1)$ defined by $\xi_\varLambda(\mathrm{e}^x) = \mathrm{e}^{i\langle x,\varLambda\rangle}$ with $x \in \mathfrak{t}$ must be a character of T. Such prerequisites are the same as those imposed in subsection 5.9 of I. The former entails that $\varLambda$ is a regular element of $\mathfrak{e}$ and that the derived coadjoint orbit of $\varLambda$ is thus $\mathcal{O}_\varLambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$. The latter implies that the restriction of the mapping $x \to \langle x, \varLambda\rangle/2\pi$ to the integer lattice Λ_G of T lies in the dual integral lattice $\Lambda_\mathsf{G}{}^*$ of Λ_G. Again, by another independent route, a direct contact with derived KKS theory is established.

The canonical formulation of subsection 5.5 can be readily adapted to the characteristic TCO model. The canonical set-up has the virtue of revealing other important features not immediately evident in the Lagrangian approach. In canonical theory, where $N = \widetilde{L} = \mathbb{R}^1 \times L$ with either $L = \mathbb{S}^1$ or $L = \mathbb{I}^1$, the derived TCO action (5.44) reads as

$$\begin{aligned} S(\mathrm{G};\Omega) = \int_{T[1]\tilde{L}} \varrho_{\tilde{L}} \Big\langle \dot{\tau}(\varLambda), \dot{\mu}\big(g^{-1}, \varsigma^*{}_t\varOmega + d_tG \\ + \dot{\mu}(\varsigma^*{}_t\omega, G) + \dot{\mu}(\varsigma^*\omega + dgg^{-1} + \dot{\tau}(G), G_t)\big)\Big\rangle \\ - \int_{T[1]\partial\tilde{L}} \varrho_{\partial\tilde{L}} \Big\langle \mathrm{Ad}\, g^{-1}(\varsigma^*{}_t\omega + d_tgg^{-1}), \varLambda\Big\rangle \end{aligned} \tag{5.98}$$

after some straightforward rearrangements. The notation used in (5.98) is defined at the beginning of subsection 5.5. Eq. (5.98) shows clearly that G_t is a non dynamical Lagrange multiplier field as expected from the general analysis carried out in subsection 5.5. The constraint it enforces upon variation, corresponding to (5.45), takes the form

$$\dot{\tau}(\dot{\mu}(\mathrm{Ad}\, g^{-1}(\varsigma^*\omega + dgg^{-1} + \dot{\tau}(G)), \varLambda)) \approx 0. \tag{5.99}$$

G_t is done away with by imposing the gauge fixing condition

$$G_t = 0 \tag{5.100}$$

equivalent to (5.46).

The ambient phase space of the characteristic TCO model consists of all TCO fields G on L and so is just $\mathcal{F}_{\mathsf{M}}(L) := \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M})$. The model's phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ is obtained by modding out the level preserving gauge transformation action (5.90), (5.91) in accordance with the definition (5.55). If M is compact and inert on the target kernel, then $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L) = \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M}/\mathrm{DZ}\mathsf{M}_\Lambda)$, as suited for the sigma model over the derived coadjoint orbit $\mathcal{O}_\Lambda = \mathrm{D}\mathsf{M}/\mathrm{DZ}\mathsf{M}_\Lambda$ encountered in the Lagrangian analysis.

When $N = \widetilde{L}$, the constituents K_t, K of the level current $\widetilde{\mathrm{K}}$ given in eq. (5.79) obey the requirement (5.47) and the relations (5.48). We further have

$$k = \theta_L \dot{\tau}(\Lambda), \tag{5.101}$$

$$K = \delta_{\partial L} \Lambda, \tag{5.102}$$

where θ_L and $\delta_{\partial L}$ are the Heaviside and Dirac distributions of L and ∂L satisfying $d\theta_L + \delta_{\partial L} = 0$. For the characteristic TCO model, the presymplectic 2-form Ψ of eq. (5.50) exhibits by virtue of the special form of the current constituent K of eqs. (5.101), (5.102) a bulk and an edge contribution[10]

$$\begin{aligned}\Psi_\Lambda = &\int_{T[1]L} \varrho_L \left\langle \dot{\tau}(\Lambda), \dot{}\mu\dot{}\left(g^{-1}\delta g, \dot{\mu}(g^{-1}, \delta G)\right)\right\rangle \\ &+ \frac{1}{2}\int_{T[1]\partial L} \varrho_{\partial L} \left\langle [g^{-1}\delta g, g^{-1}\delta g], \Lambda \right\rangle.\end{aligned} \tag{5.103}$$

The presymplectic 2-form Ψ_Λ induces a symplectic 2-form and so a Poisson bracket structure on the phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ yielded by modding out the level preserving gauge symmetry as detailed in subsection 5.5.

The symmetries of the characteristic model are described in canonical theory in a way that exactly parallels that in which they are in the Lagrangian set-up through the appropriate analogs of eqs. (5.88)–(5.91), (5.93), (5.94), (5.95), (5.96) with the field space $\mathcal{F}_{\mathsf{M}}(N)$ replaced by the ambient phase space $\mathcal{F}_{\mathsf{M}}(L)$ and the gauge transformations groups $\mathcal{G}_{\mathsf{M}}(N)$, $\mathcal{G}_{\mathsf{M},\Lambda}(N)$, $\mathcal{H}_{\mathsf{M}}(N)$ by their phase space counterparts $\mathcal{G}_{\mathsf{M}}(L)$, $\mathcal{G}_{\mathsf{M},\Lambda}(L)$, $\mathcal{H}_{\mathsf{M}}(L)$.

The first class Hamiltonians $\mathsf{N}(\Pi)$ of the infinitesimal gauge background preserving gauge transformations $\Pi \in \mathfrak{h}_{\mathsf{M}}(L)$, defined in (5.58) and (5.68), take in the characteristic model the form

$$\mathsf{N}(\Pi)(\mathrm{G};\Omega) = \int_{T[1]L} \varrho_L \left\langle \dot{\tau}(\dot{}\mu\dot{}(\mathrm{Ad}\, g^{-1}(\varsigma^*\omega + dgg^{-1} + \dot{\tau}(G)), \Lambda)), \dot{\mu}(g^{-1}, \Pi)\right\rangle. \tag{5.104}$$

The weak vanishing of the $\mathsf{N}(\Pi)$, in accordance with (5.70), singles out the physical phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda\,\mathrm{phys}}(L)$ of the characteristic model within the phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$.

[10] Since ∂L, if non empty, consists of just two points, here and in similar relations below the notation $\int_{T[1]\partial L} \varrho_{\partial L} \ldots$ indicates in a somewhat overwritten manner summation over those points with signs determined by the orientation of L.

The symplectic 2-form Ψ_Λ of the phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ exhibits a bulk and an edge term which have a formal structure analogous to that of the two components of the derived symplectic structure $-i\mathrm{B}_\Lambda$ of the regular derived coadjoint orbit $\mathcal{O}_\Lambda$ (cf. subsection 5.9 and eqs. (5.9.4), (5.9.5) of I). This is not accidental. The relationship between the two symplectic structures will be elucidated more formally in an appropriate transgressional framework in subsection 5.7 below.

In the above canonical analysis, in particular in the last paragraph, we did not mention the integrality condition that the source Lie algebra datum Λ must satisfy in the quantum theory of the characteristic model as discussed above and that was also assumed in our treatment of a regular derived coadjoint orbit in subsection 5.9 of I. In the model's canonical theory, the quantization of Λ is expected to stem from requiring that the symplectic 2-form Ψ_Λ of the phase space $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ equals $-i$ times the curvature of a prequantum line bundle $\mathscr{L}_\Lambda$ on $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ whose properly normalized sections form the model's prequantum Hilbert space in the spirit of geometric quantization. Since $\overline{\mathcal{F}}_{\mathsf{M},\Lambda}(L)$ is an infinite dimensional graded manifold, this would inevitably lead us to an infinite dimensional graded geometric setting which, as we have remarked in I, is very difficult to deal with and for this reason we want to avoid for the time being.

The problems with the geometric quantization of the characteristic derived TCO model pointed out in the previous paragraph pair with the difficulties to work out a full prequantization of derived KKS theory noticed in subsection 5.8 of I. The point is that the only viable quantization scheme of the TCO model and the KKS theory that secretly informs it presently at our disposal, notwithstanding its shortcomings, is the functional integral one of subsection 5.4. Though in an indirect way and with certain underlying assumptions, this should provide the quantization of both the characteristic TCO canonical set-up and derived KKS theory.

5.7 Canonical formulation of the TCO model and derived KKS theory

In this subsection, we unveil the relationship between the regular case derived KKS theory worked out in subsection 5.9 of I and the canonical formulation of the characteristic derived TCO model studied in subsection 5.6.

Transgression is a procedure used in geometry and topology for transferring cohomology classes from one space to another in the absence of a morphism relating them. Below we shall work out a version of transgression as a chain map turning forms of $\mathrm{D}\mathsf{M}$ into forms on the TCO model phase space $\mathcal{F}_{\mathsf{M}}(L)$, where the former are viewed as functions on $T[1]\,\mathrm{D}\mathsf{M}$ and the latter as functionals on $T[1]\mathcal{F}_{\mathsf{M}}(L)$ as usual.

In simple terms, the transgression of a function $F \in \mathrm{F{\scriptsize UN}}(T[1]\,\mathrm{D}\mathsf{M})$ involves two steps: *i* generating a functional $\mathrm{ev}^*F \in \mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L) \boxplus T[1]L)$ by means of the pull-back of the evaluation map $\mathrm{ev} : T[1]\mathcal{F}_{\mathsf{M}}(L) \times T[1]L \to T[1]\,\mathrm{D}\mathsf{M}$ and *ii* constructing an element $\mathcal{T}(F) \in \mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ by integration of ev^*F on a cycle of L. We shall examine such steps individually.

The analysis of transgression requires setting up a suitable Cartan calculus for the space $\mathcal{F}_{\mathsf{M}}(L) \times L$ (cf. appendix A.3 of I). We shall do that by separately constructing appropriate

Cartan calculi for the two Cartesian factors $\mathcal{F}_{\mathsf{M}}(L)$, L roughly proceeding along the lines followed for the space $\mathrm{D}\mathsf{M}$ is subsection 5.2 of I.

With the above in mind, we have to appraise which range of vector fields of $\mathcal{F}_{\mathsf{M}}(L)$, L are the most appropriate for contractions and Lie derivatives. The full vector field Lie algebras $\mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$, $\mathrm{Vect}(L)$ are too large. Working with more restricted algebras is enough. To this end, we reconsider the Cartan calculus of $\mathrm{D}\mathsf{M}$ from a more general viewpoint. Let $V \in \mathrm{VECT}(\mathrm{D}\mathsf{M})$ be an internal vector field. For the base coordinate Γ of $T[1]\,\mathrm{D}\mathsf{M}$ (cf. subsection 5.2 of I), we have

$$\Gamma^{-1} j_V \Gamma = 0, \tag{5.105}$$

$$\Gamma^{-1} l_V \Gamma = \mathrm{N}_V(\Gamma) \tag{5.106}$$

on general grounds, where j_V, l_V denote contraction and Lie derivation with respect to V and $\mathrm{N}_V \in \mathrm{MAP}(\mathrm{D}\mathsf{M}, \mathrm{D}\mathfrak{m}[0])$ is an internal map depending on V. $\mathrm{N}_V(\Gamma)$ is nothing but the coordinate expression of V. For instance, if S_{LH} is the vertical vector field of left target kernel $\mathrm{D}\mathsf{M}_\tau$-action of the derived Lie algebra element $\mathrm{H} \in \mathrm{D}\mathfrak{m}_\tau$ (cf. subsection 5.3 of I), then $\mathrm{N}_{S_{\mathrm{LH}}}(\Gamma) = -\mathrm{Ad}\,\Gamma(\mathrm{H})$. From (5.2.1) of I, it then follows that the fiber coordinate Σ of $T[1]\,\mathrm{D}\mathsf{M}$ obeys

$$j_V \Sigma = \mathrm{N}_V(\Gamma), \tag{5.107}$$

$$l_V \Sigma = \mathrm{d}\mathrm{N}_V(\Gamma) + [\Sigma, \mathrm{N}_V(\Gamma)]. \tag{5.108}$$

In fact, it is possible to show that, under the assumption of the validity of relations (5.2.1) of I and (5.105), (5.106) for any $V \in \mathrm{VECT}(\mathrm{D}\mathsf{M})$, the basic Cartan calculus relations (A.3.1)–(A.3.4) of I hold if and only if (5.2.2) of I holds, (5.107), (5.108) hold true for any $V \in \mathrm{VECT}(\mathrm{D}\mathsf{M})$ and moreover the relation

$$\mathrm{N}_{[V,W]}(\Gamma) = l_V \mathrm{N}_W(\Gamma) - l_W \mathrm{N}_V(\Gamma) + [\mathrm{N}_V(\Gamma), \mathrm{N}_W(\Gamma)] \tag{5.109}$$

is satisfied for $V, W \in \mathrm{VECT}(\mathrm{D}\mathsf{M})$. We let $\mathrm{VECT}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$ be the set of all vector fields $V \in \mathrm{VECT}(\mathrm{D}\mathsf{M})$ such that N_V is an ordinary non internal map so that $\mathrm{N}_V \in \mathrm{Map}(\mathrm{D}\mathsf{M}, \mathrm{D}\mathfrak{m}[0])$. From (5.109), $\mathrm{VECT}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$ is a Lie subalgebra of $\mathrm{VECT}(\mathrm{D}\mathsf{M})$, the restricted subalgebra. It is shown below that it is possible associate with every vector field $V \in \mathrm{VECT}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$ vector fields $\varUpsilon(V) \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$, $Y(V) \in \mathrm{Vect}(L)$, called the phase space and space manifold transplants of V, with natural properties. The vector fields yielded in this fashion will be the ones considered in the Cartan calculi of $\mathcal{F}_{\mathsf{M}}(L)$, L in what follows.

We now illustrate the Cartan calculus of the phase space $\mathcal{F}_{\mathsf{M}}(L)$ and its main properties. The action of the calculus' variational derivations on the functional algebra $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ of $T[1]\mathcal{F}_{\mathsf{M}}(L)$ is specified by that on suitable field coordinates of $T[1]\mathcal{F}_{\mathsf{M}}(L)$. As $\mathcal{F}_{\mathsf{M}}(L) = \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M})$, $\mathcal{F}_{\mathsf{M}}(L)$ is a group manifold. By the isomorphism $T[1]\mathcal{F}_{\mathsf{M}}(L) \simeq \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M}) \times \mathrm{Map}(T[1]L, \mathrm{D}\mathfrak{m})[1]$, it is natural to use coordinates adapted to the Cartesian factors $\mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M})$, $\mathrm{Map}(T[1]L, \mathrm{D}\mathfrak{m})[1]$ which we may think of as base and fiber coordinates of $T[1]\mathcal{F}_{\mathsf{M}}(L)$, hence field variables $\mathrm{G} \in \mathrm{Map}(T[1]L, \mathrm{D}\mathsf{M})$, $\mathrm{S} \in \mathrm{Map}(T[1]L, \mathrm{D}\mathfrak{m})[1]$.

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There is a vector field $\Upsilon(V) \in \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ associated with each vector field $V \in \mathrm{V\small{ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$, whose coordinate expression $\mathrm{N}_{\Upsilon(V)} \in \mathrm{Map}(\mathcal{F}_{\mathsf{M}}(L), \mathrm{D}\mathfrak{m}[0])$ is

$$\mathrm{N}_{\Upsilon(V)}(\mathrm{G}) = \mathrm{N}_V \circ \mathrm{G}. \tag{5.110}$$

$\Upsilon(V)$ is the phase space transplant of V. On account of property (5.109), the map $\Upsilon : \mathrm{V\small{ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M}) \to \mathrm{Vect}(\mathcal{F}_{\mathsf{M}}(L))$ is a Lie algebra morphism.

The way the variational derivations of the Cartan calculus of $\mathcal{F}_{\mathsf{M}}(L)$ act on the field coordinates is determined by the interpretation of these and consistency. The variational differential δ acts on the field coordinate G according to

$$\mathrm{G}^{-1}\delta\mathrm{G} = \mathrm{S}, \tag{5.111}$$

identifying S as the variational Maurer-Cartan field form associated with G. The action of δ on the field coordinate S in turn is mandated by the requirement of nilpotence of δ and reduces to the variational Maurer-Cartan equation,

$$\delta\mathrm{S} = -\frac{1}{2}[\mathrm{S}, \mathrm{S}]. \tag{5.112}$$

The variational contraction and Lie derivative along the vector fields $\Upsilon(V)$ with $V \in \mathrm{V\small{ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$, $\iota_{\Upsilon(V)}$ and $\lambda_{\Upsilon(V)}$, act on G according to

$$\mathrm{G}^{-1}\iota_{\Upsilon(V)}\mathrm{G} = 0, \tag{5.113}$$

$$\mathrm{G}^{-1}\lambda_{\Upsilon(V)}\mathrm{G} = \mathrm{N}_{\Upsilon(V)}(\mathrm{G}) \tag{5.114}$$

on general grounds in analogy to (5.105), (5.106). From relation (5.111), it then follows that they act on S as

$$\iota_{\Upsilon(V)}\mathrm{S} = \mathrm{N}_{\Upsilon(V)}(\mathrm{G}), \tag{5.115}$$

$$\lambda_{\Upsilon(V)}\mathrm{S} = \delta\mathrm{N}_{\Upsilon(V)}(\mathrm{G}) + [\mathrm{S}, \mathrm{N}_{\Upsilon(V)}(\mathrm{G})] \tag{5.116}$$

in analogy to (5.107), (5.108). The variational derivations δ, ι and λ of $\mathcal{F}_{\mathsf{M}}(L)$ obey the basic Cartan relations analogously to the derivations d, j and l of $\mathrm{D}\mathsf{M}$.

We outline next the Cartan calculus of the space manifold L illustrating its main properties. The action of the calculus derivations on the function algebra $\mathrm{Fun}(T[1]L)$ is specified by that on suitable coordinates on $T[1]L$. As $L = \mathbb{S}^1$ or $\mathbb{I}^1$, $T[1]L \simeq L \times \mathbb{R}[1]$. So, we can choose the variables $r \in L$, $z \in \mathbb{R}[1]$ as natural coordinates.

It is reasonable to assume that no action on L results from any action on $\mathrm{D}\mathsf{M}$ via transgression. Therefore, for a vector field $V \in \mathrm{V\small{ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$ we take its space manifold transplant $Y(V) \in \mathrm{Vect}(L)$ to vanish identically, $Y(V) = 0$. The map $Y : \mathrm{V\small{ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M}) \to \mathrm{Vect}(L)$ is then trivially a Lie algebra morphism.

The expressions of the derivations of the Cartan calculus of L are elementary. The action of the de Rham differential d of L on r, z reads as usual as

$$dr = z, \qquad dz = 0. \tag{5.117}$$

The vanishing of the transplant $Y(V)$ of a vector field $V \in \mathrm{V\scriptstyle ECT}_{\mathsf{r}}(D\mathsf{M})$ implies that the contraction and Lie derivative along $Y(V)$, $j_{Y(V)}$ and $l_{Y(V)}$, act trivially,

$$j_{Y(V)}r = 0, \qquad l_{Y(V)}r = 0, \qquad j_{Y(V)}z = 0, \qquad l_{Y(V)}z = 0. \tag{5.118}$$

The formulation of the Cartan calculi of the phase space $\mathcal{F}_{\mathsf{M}}(L)$ and space manifold L furnishes us readily that of the product space $\mathcal{F}_{\mathsf{M}}(L)\times L$. In fact, the shifted tangent bundle $T[1](\mathcal{F}_{\mathsf{M}}(L)\times L)$ of $\mathcal{F}_{\mathsf{M}}(L)\times L$ is isomorphic the external direct sum $T[1]\mathcal{F}_{\mathsf{M}}(L)\boxplus T[1]L$ (cf. footnote 1) which turns out to be just the product $T[1]\mathcal{F}_{\mathsf{M}}(L)\times T[1]L$ as a manifold. The transplant of a vector field $V \in \mathrm{V\scriptstyle ECT}_{\mathsf{r}}(D\mathsf{M})$ in $\mathcal{F}_{\mathsf{M}}(L)\times L$ is the vector field $\Upsilon(V)+Y(V)$ of $\mathcal{F}_{\mathsf{M}}(L)\times L$ with the transplants $\Upsilon(V)$, $Y(V)$ of V in $\mathcal{F}_{\mathsf{M}}(L)$, L as components. The differential and the contraction and Lie derivative along $\Upsilon(V)+Y(V)$ of $\mathcal{F}_{\mathsf{M}}(L)\times L$ are given by $\delta+d$ and $\iota_{\Upsilon(V)}+j_{Y(V)}$, $\lambda_{\Upsilon(V)}+l_{Y(V)}$ in terms of the differentials δ, d and the contractions and Lie derivatives $\iota_{\Upsilon(V)}$, $j_{Y(V)}$, $\lambda_{\Upsilon(V)}$, $l_{Y(V)}$ along $\Upsilon(V)$, $Y(V)$ of $\mathcal{F}_{\mathsf{M}}(L)$, L, respectively. With this framework available, we can now proceed to working out the relevant transgression map following the two steps outlined at the beginning of the present subsection.

The first step of the construction of the transgression map consists in writing down the expressions of the evaluation map $\mathrm{ev} : T[1]\mathcal{F}_{\mathsf{M}}(L)\times T[1]L \to T[1]\,D\mathsf{M}$ of $T[1]\mathcal{F}_{\mathsf{M}}(L)$ and its pull-back. The map ev is given explicitly by

$$\mathrm{ev}(\mathrm{G},\mathrm{S},r,z) = (\mathrm{G}(r,z),\mathrm{S}(r,z)+\mathrm{G}^{-1}\mathrm{dG}(r,z)). \tag{5.119}$$

The reason why the term $\mathrm{G}^{-1}\mathrm{dG}$ is added to S in the right hand side will be explained momentarily.

The pull-back ev^* of ev generates functionals of $T[1]\mathcal{F}_{\mathsf{M}}(L)\boxplus T[1]L$ from functions of $T[1]\,D\mathsf{M}$. Since our goal is obtaining ordinary non internal functionals of the former space as a result, we have to restrict the range of functions of the latter space on which we act with ev^*. To this end, it is enough to limit ourselves to the restricted subalgebra $\mathrm{F\scriptstyle UN}_{\mathsf{r}}(T[1]\,D\mathsf{M})$ formed by the functions $F \in \mathrm{F\scriptstyle UN}(T[1]\,D\mathsf{M})$ of the form $F = E_F(\Gamma,\Sigma)$, where $E_F \in \mathrm{Fun}(D\mathsf{M}\times D\mathfrak{m}[1])$ is an ordinary function. The evaluation map pull-back map obtained in this way is a degree 0 algebra morphism $\mathrm{ev}^* : \mathrm{F\scriptstyle UN}_{\mathsf{r}}(T[1]\,D\mathsf{M}) \to \mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L)\boxplus T[1]L)$. Explicitly, by (5.119), for any $F \in \mathrm{F\scriptstyle UN}_{\mathsf{r}}(T[1]\,D\mathsf{M})$ we have

$$\mathrm{ev}^*F(\mathrm{G},\mathrm{S},r,z) = E_F(\mathrm{G}(r,z),\mathrm{S}(r,z)+\mathrm{G}^{-1}\mathrm{dG}(r,z)). \tag{5.120}$$

Using relations (5.110)–(5.116) and (5.117), (5.118), it is straightforward to verify that the pull-back map ev^* has the expected properties. In particular,

$$\mathrm{ev}^*\,dF = (\delta+d)\,\mathrm{ev}^*F. \tag{5.121}$$

The validity of this important relation rests crucially on the addition of the key term $\mathrm{G}^{-1}\mathrm{dG}$ to S in the expression of the evaluation map, eq. (5.119). For any vector field $V \in \mathrm{V\scriptstyle ECT}_{\mathsf{r}}(D\mathsf{M})$, we have analogously

$$\mathrm{ev}^*\,j_VF = \iota_{\Upsilon(V)}\,\mathrm{ev}^*F, \tag{5.122}$$

$$\mathrm{ev}^*\,l_VF = \lambda_{\Upsilon(V)}\,\mathrm{ev}^*F. \tag{5.123}$$

The second step of the construction of the transgression map consists in the integration on a cycle of L of the functionals of $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L) \boxplus T[1]L)$ yielded by operating with the pull-back map ev^* on the functions of $\mathrm{F{\small UN}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M})$. This yields functionals of $\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ as a result. The cycle can be conveniently encoded by a current $C \in \mathrm{Fun}'(T[1]L)$ obeying $dC = 0$. The transgression of a function $F \in \mathrm{F{\small UN}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M})$ resulting from combining the two steps described above has therefore the form

$$\mathcal{T}(F) = \int_{T[1]L} \varrho_L \, C \, \mathrm{ev}^* F. \tag{5.124}$$

We assume below for simplicity that the current C is homogeneous. Since L is 1-dimensional, its degree p can take only the values 0, 1. Transgression then furnishes a degree $p-1$ linear map $\mathcal{T} : \mathrm{F{\small UN}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}) \to \mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$. By (5.121) and the fact that $dC = 0$, $\mathcal{T}$ is a chain map,

$$\mathcal{T}(dF) = -(-1)^p \delta \mathcal{T}(F). \tag{5.125}$$

For a vector field $V \in \mathrm{V{\small ECT}}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$, we have similarly

$$\mathcal{T}(j_V F) = -(-1)^p \iota_{\Upsilon(V)} \mathcal{T}(F), \tag{5.126}$$

$$\mathcal{T}(l_V F) = \lambda_{\Upsilon(V)} \mathcal{T}(F). \tag{5.127}$$

by (5.122), (5.123) and the fact that $j_{Y(V)}C = 0$ and $l_{Y(V)}C = 0$ by virtue of (5.118).[11]

The transgression procedure illustrated above admits a number of generalizations. We consider in what follows the extension of transgression to $\mathrm{I{\small NN}}\,\mathbb{R}$-valued maps, where $\mathrm{I{\small NN}}\,\mathbb{R} = (\mathbb{R}, \mathbb{R}, \mathrm{id}_{\mathbb{R}}, o_{\mathbb{R}})$ is the inner derivation crossed module of the Abelian Lie algebra $\mathbb{R}$ (cf. subsection 3.1 of I), where $o_{\mathbb{R}}$ is the trivial action. A map $\mathrm{F} \in \mathrm{M{\small AP}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ (cf. footnote 4 of I for notation) is characterized by its components $f, F \in \mathrm{M{\small AP}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{S}\mathbb{R}) \simeq \mathrm{F{\small UN}}(T[1]\,\mathrm{D}\mathsf{M})$. We let $\mathrm{M{\small AP}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ be the vector subspace of $\mathrm{M{\small AP}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ constituted by all the maps $\mathrm{F} \in \mathrm{M{\small AP}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ with components $f, F \in \mathrm{F{\small UN}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M})$. The evaluation map pull-back is the linear mapping $\mathrm{ev}^* : \mathrm{M{\small AP}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R}) \to \mathrm{Map}(T[1]\mathcal{F}_{\mathsf{M}}(L) \boxplus T[1]L, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ induced by the previously defined pull-back ev^* map acting componentwise. Let $\mathrm{C} \in \mathrm{Map}'(T[1]L, \mathrm{I{\small NN}}\,\mathbb{R})$ be a derived current satisfying $\mathrm{dC} = 0$, where here d denotes the derived differential. C may be regarded as a derived cycle of L. The transgression of a map $\mathrm{F} \in \mathrm{M{\small AP}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ reads as

$$\mathcal{T}(\mathrm{F}) = \int_{T[1]L} \varrho_L \, (\mathrm{C}, \mathrm{ev}^* \mathrm{F})_{\mathbb{R}}. \tag{5.128}$$

In this expression, $(\cdot, \cdot)_{\mathbb{R}}$ is the pairing of $\mathrm{M{\small AP}}(T[1]L, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R})$ associated according to (3.3.15) of I with the canonical invariant pairing $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ of $\mathrm{I{\small NN}}\,\mathbb{R}$ given by $\langle x, X \rangle_{\mathbb{R}} = xX$ with $x \in \mathbb{R}$, $X \in \mathbb{R}$. In this way, a degree 0 linear transgression map $\mathcal{T} : \mathrm{M{\small AP}}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{I{\small NN}}\,\mathbb{R}) \to \mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ is defined. $\mathcal{T}$ obeys relations analogous

[11] The extra -1 factor appearing in (5.125), (5.126) is due to the Berezin measure ϱ_L having degree -1.

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to (5.125)–(5.127). Indeed, using (5.128) and proceeding as in the proof of those relations, one can verify that

$$\mathcal{T}(\mathrm{dF}) = -\delta\mathcal{T}(\mathrm{F}) \tag{5.129}$$

and that for $V \in \mathrm{VECT}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$

$$\mathcal{T}(j_V\mathrm{F}) = -\iota_{\Upsilon(V)}\mathcal{T}(\mathrm{F}), \tag{5.130}$$

$$\mathcal{T}(l_V\mathrm{F}) = \lambda_{\Upsilon(V)}\mathcal{T}(\mathrm{F}). \tag{5.131}$$

We now come to the issue that ultimately motivates the elaborate transgression theory developed up to this point in this subsection: using transgression to generate a Poisson structure of the derived TCO model's phase space $\mathcal{F}_{\mathsf{M}}(L)$ from the Poisson structure of the homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ associated with the curvature B of a connection A of the unitary derived line bundle $\mathcal{L}_\beta$ with J a maximal toral crossed submodule of M and β an assigned character of J in the constructive derived framework of subsection 5.6 of I.

The inner derivation crossed module $\mathrm{INN}\,\mathfrak{u}(1)$ of the Lie algebra $\mathfrak{u}(1)$ is essentially the same as the crossed module $\mathrm{INN}\,\mathbb{R}$ by the identity $\mathfrak{u}(1) = i\mathbb{R}$. Since $\mathrm{MAP}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{INN}\,\mathfrak{u}(1)) = i\,\mathrm{MAP}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{INN}\,\mathbb{R})$ as vector spaces, a restricted space $\mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{INN}\,\mathfrak{u}(1)) = i\,\mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{INN}\,\mathbb{R})$ and a transgression map $\mathcal{T} : \mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{SD}\,\mathrm{INN}\,\mathfrak{u}(1)) \to i\,\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$ can be defined with the same properties as before.

A connection A of $\mathcal{L}_\beta$ is restricted if $\mathrm{A} \in \mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\,\mathrm{INN}\,\mathfrak{u}(1)[1])$. On account of relation (5.5.16) of I, the curvature B of A is then restricted as well, i.e. $\mathrm{B} \in \mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\,\mathrm{INN}\,\mathfrak{u}(1)[2])$. The transgressions $\mathcal{T}(\mathrm{A})$, $\mathcal{T}(\mathrm{B})$ of A, B are so degree 1, 2 elements of $i\,\mathrm{Fun}(T[1]\mathcal{F}_{\mathsf{M}}(L))$, respectively. Further,

$$\mathcal{T}(\mathrm{B}) = -\delta\mathcal{T}(\mathrm{A}) \tag{5.132}$$

by (5.129) owing to (5.5.16) of I and

$$\delta\mathcal{T}(\mathrm{B}) = 0 \tag{5.133}$$

owing to the Bianchi identity (5.5.17) of I. By relations (5.132), (5.133), $-i\mathcal{T}(\mathrm{B})$ can be regarded as a presymplectic form on the derived model's TCO phase space $\mathcal{F}_{\mathsf{M}}(L)$ with presymplectic potential $-i\mathcal{T}(\mathrm{A})$. A Poisson bracket $\{\cdot,\cdot\}_{\mathcal{T}(\mathrm{A})}$ is therefore available for the algebra $\mathrm{Fun}_{\mathrm{A}}(\mathcal{F}_{\mathsf{M}}(L))$ of Hamiltonian phase space functionals on general grounds.

In subsection 5.6 of I, we studied the space $\mathrm{DFNC}_{\mathrm{A}}(\mathrm{D}\mathsf{M})$ of Hamiltonian derived functions of $\mathrm{D}\mathsf{M}$. As $\mathrm{DFNC}_{\mathrm{A}}(\mathrm{D}\mathsf{M}) \subset \mathrm{DFNC}(\mathrm{D}\mathsf{M}) = \mathrm{MAP}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\mathbb{R})$ and $\mathrm{MAP}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\mathbb{R}) = \mathrm{MAP}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\,\mathrm{INN}\,\mathbb{R})$ as vector spaces, it is natural to limit our set-up to the space of restricted derived Hamiltonian functions $\mathrm{DFNC}_{\mathsf{r}\mathrm{A}}(\mathrm{D}\mathsf{M}) = \mathrm{DFNC}_{\mathsf{A}}(\mathrm{D}\mathsf{M}) \cap \mathrm{MAP}_{\mathsf{r}}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\,\mathrm{INN}\,\mathbb{R})$. If $\mathrm{F} \in \mathrm{DFNC}_{\mathsf{r}\mathrm{A}}(\mathrm{D}\mathsf{M})$ is a restricted Hamiltonian function, its Hamiltonian vector field $P_{\mathrm{F}} \in \mathrm{VECT}_{\mathsf{r}}(\mathrm{D}\mathsf{M})$ is also restricted. By (5.6.5) of I and (5.129), (5.130),

$$\delta\mathcal{T}(\mathrm{F}) - i\iota_{\Upsilon(P_{\mathrm{F}})}\mathcal{T}(\mathrm{B}) = 0. \tag{5.134}$$

Hence, the functional $\mathcal{T}(\mathrm{F})$ is Hamiltonian too.

In this fashion, a linear map $\mathcal{T} : \mathrm{DF{\scriptstyle NC}}_{r\mathrm{A}}(\mathsf{DM}) \to \mathrm{Fun}_{\mathrm{A}}(\mathcal{F}_{\mathsf{M}}(L))$ is established. $\mathcal{T}$ has the distinguished property that

$$\{\mathcal{T}(\mathrm{F}), \mathcal{T}(\mathrm{H})\}_{\mathcal{T}(\mathrm{A})} = \mathcal{T}(\{\mathrm{F}, \mathrm{H}\}_{\mathrm{A}}) \tag{5.135}$$

for $\mathrm{F}, \mathrm{H} \in \mathrm{DF{\scriptstyle NC}}_{r\mathrm{A}}(\mathsf{DM})$, as is straightforwardly verified. In spite of its formal appearance, relation (5.134) cannot be interpreted as indicating that $\mathcal{T}$ is a Poisson map. In fact, while $\mathrm{Fun}_{\mathrm{A}}(\mathcal{F}_{\mathsf{M}}(L))$ is a functional algebra and $\{\cdot,\cdot\}_{\mathcal{T}(\mathrm{A})}$ is a genuine Poisson bracket structure on it, $\mathrm{DF{\scriptstyle NC}}_{\mathrm{A}}(\mathsf{DM})$ is a mere vector space and $\{\cdot,\cdot\}_{\mathrm{A}}$ is only a twisted Lie bracket structure thereon reducing to a genuine Poisson structure only upon restriction to the short algebra $\mathrm{DF{\scriptstyle NC}}_{\varpi\mathrm{A}}(\mathsf{DM})$ (cf. subsection 5.6 of I). The twisting of the Lie bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ is compatible with the Jacobi property of $\{\cdot,\cdot\}_{\mathcal{T}(\mathrm{A})}$ since for any function triple $\mathrm{F}, \mathrm{H}, \mathrm{K} \in \mathrm{DF{\scriptstyle NC}}_{r\mathrm{A}}(\mathsf{DM})$

$$\mathcal{T}(\langle \mathrm{F}, \mathrm{H}, \mathrm{K}\rangle_{\mathrm{A}}) = i\delta\iota_{\Upsilon(P_{\mathrm{K}})}\iota_{\Upsilon(P_{\mathrm{H}})}\iota_{\Upsilon(P_{\mathrm{F}})}\mathcal{T}(\mathrm{B}) = 0 \tag{5.136}$$

by grading reasons (cf. eqs. (5.6.7), (5.6.8) of I).

We can now establish a relationship between the regular case derived KKS theory developed in subsection 5.9 of I and the canonical formulation of the characteristic derived TCO model worked out in subsection 5.6. We shall do that using the transgression map $\mathcal{T}$ of eq. (5.128) in the special case where the derived current C has components

$$c = \theta_L, \qquad C = \delta_{\partial L} \tag{5.137}$$

corresponding to the distributional factors of the characteristic model's level current in the canonical formulation (cf. eqs. (5.101), (5.102)). By a straightforward computation, it can be verified that the presymplectic form Ψ_Λ of the characteristic model's ambient phase space $\mathcal{F}_{\mathsf{M}}(L)$ given by eq. (5.103) is precisely the presymplectic form $-i\mathcal{T}(B_\Lambda)$ yielded by transgression of the curvature B_Λ of the unitary connection A_Λ of the derived line bundle $\mathcal{L}_\Lambda$ given componentwise by eqs. (5.9.2), (5.9.3) and (5.9.4), (5.9.5) of I,

$$\Psi_\Lambda = -i\mathcal{T}(\mathrm{B}_\Lambda). \tag{5.138}$$

The observation made at the end of subsection 5.6 that Ψ_Λ exhibits a bulk and an edge term with a formal structure analogous to that of the two components of the derived symplectic structure $-i\mathrm{B}_\Lambda$ of the regular orbit $\mathcal{O}_\Lambda$ now takes a more precise meaning in the light of the above transgressional analysis and add new evidence for the close relationship between the characteristic model and KKS theory.

5.8 Conclusions

In this section, we have studied in detail a 2-dimensional derived TCO model, its symmetries and its sigma model interpretation in both the Lagrangian and canonical perspective and provided significant evidence to support the claim the model furnishes the partition function realization of an underlying Wilson surface upon quantization, which was the ultimate motivation of our endeavour. Most importantly, we have shown that the characteristic version of the model is intimately related to derived KKS theory and may yield important clues on the eventual geometric quantization of this latter, an open problem. The unifying element of the multiple constructions we have carried out is the derived geometric framework of higher gauge theory, whose basicness the present work highlights conclusively.

Acknowledgments

The author thanks E. Meinrenken for correspondence. The author acknowledges financial support from the grant GAST of INFN Research Agency under the provisions of the agreement between Bologna University and INFN.

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