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Exact time–integral inversion via Čebyšëv quintic approximations for nonlinear oscillators

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ABSTRACT

The focus of this work is the solution of a fundamental problem that arises in non–dissipative nonlinear oscillators and related applications, namely the rare possibility of explicitly inverting the associated time–integral. Here, the inversion issue is treated by near–minimax approximation of the restoring force via fifth–order Čebyšëv polynomials on a normalised integration interval: this gives rise to a Duffing–type quintic oscillator, whose solutions effectively represent those of the original problem. Indeed, when an odd function describes the restoring force, the elliptic time–integral associated with the quinticate oscillator can be inverted in closed form. This is obtained here, by observing that the integrand involves a quadratic polynomial, built on the quinticate oscillator coefficients, and by studying its discriminant. Based on these findings, we provide a novel solution procedure, implemented within the *Mathematica* scientific environment, that exploits elliptic integrals of the first kind and whose effectiveness is tested on three well–known conservative nonlinear oscillator models.

1. Introduction

In this work, we focus on differential equations governing non–dissipative non–linear oscillators; these arise in different physical models such as the treatment of relativistic oscillators, from the first contribution due to [33] and the further analysis in [36], up to generalizations to Duffing's relativistic oscillators [55]; they also appear in non–relativistic models as that in [49], which deals with cables with an attached midpoint mass, or some harmonic Duffing oscillators discussed in [37,44,51]. From a purely mathematical viewpoint, all these models, further than describing the one–dimensional motion of a particle, share being governed

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by the autonomous problem (4) with odd restoring force f and displacement a, illustrated in the following § 2, and the consequent problem of inverting the associated time–integral; the latter can rarely be solved in explicit terms, excluding the well–known cases of the pendulum and Duffing equations, both unforced.

Here, we tackle the inversion problem by approximating the original differential equation with one yielding an integral equation that is invertible and that admits an exact solution in terms of Jacobi elliptic functions. This also offers the possibility to investigate, albeit through its approximate form, how the displacement a influences the behaviour of the solution; a straightforward solution of the original problem via numerical methods would not allow this.

In this way, we develop a new solution procedure, which revises and refines previous results on exact analytic solutions of quintic oscillators [9,11,15,17,40], as well as on approximate models obtained via Čebyšëv polynomials up to degree five [13,18,27]. Our new approach allows us to study and solve further wave configurations, not considered in the cited papers, and generated by the sign of the discriminant of a certain second–degree polynomial, as described in Theorems 1–2 of this work. A third wave configuration will be presented in a forthcoming communication by the authors [45], as will the results of higher order approximations, also under analysis. Indeed, the particular choice of fifth–order truncation is functional to the inversion of the elliptic integral which describes the time evolution of the oscillatory systems considered in this paper; higher order approximations lead, instead, to the inversion of hyperelliptic integrals [28,39], possibly through Lauricella functions [32,47] i.e. hypergeometric functions of three or more variables, implying a further layer of complexity beyond the scope of the current study.

In this work, our method is applied to the relativistic oscillator proposed by McColl [33] and studied in depth and with different techniques in [7,12,25,36]. More precisely, § 2 details the construction of the normalised quinticate oscillator (12) associated with the original one, whose treatment with our new procedure is then described in § 3, leading to the determination of its exact period and solution. In § 4, the solution process is illustrated on the relativistic oscillator (1); the quality of the results obtained is also validated. In § 5, a similar application to oscillators (2) and (3) confirms both feasibility and robustness of our new solver. Some final comments and indications for future work are reported in the concluding § 6.

Before leaving this section, we briefly mention the most popular techniques in the study of nonlinear oscillatory phenomena: the Lindstedt–Poincaré perturbation and multiple time–scale methods [35,41,42]; the generalised averaging Krylov–Bogoliubov–Mitropolski method [31,35]; the approximate variational or energy–balance method [4,24] to evaluate angular frequencies of nonlinear oscillators; the harmonic–balance method [5,20,21,35,38,54]. A notable source for Duffing oscillators is [30], while [16,30,38] offer an overview of all these methods. The period–amplitude dependence problem was analysed in [46] through the classic thermodynamic equilibrium theory, and an asymptotic period estimate was obtained for the particular case of the predator–prey Volterra–Lotka model, which is a Hamiltonian system, after an appropriate change of variable. The latter approach was extended to a large class of Hamiltonian systems in [19] via the Laplace transform and asymptotic expansions.

2. General quintic oscillator

The differential models examined in this work are the following:

$$\ddot{x} = \frac{-x}{\sqrt{1+x^2}}, \tag{1}$$

$$\ddot{x} = -x - \frac{b\,x}{\sqrt{1+x^2}}, \tag{2}$$

$$\ddot{x} = -x - x^3 - \frac{b\,x}{\sqrt{1+x^2}}. \tag{3}$$

Equation (1) relates to the relativistic oscillator introduced in [33] and analysed in–depth in [36]. It is obtained from $\ddot{x}+(1-\dot{x}^2)^{3/2}\, x=0$ studied in phase–space after a change of variable; details are well–known and reported in several papers, such as [2,6,7,33,36].

Dynamics of cables with an attached midpoint mass are modelled by (2). We highlight the contributions of [3,26,34,37,44,55,56], where classic approximate analytic methods are employed through some algebraic procedures, such as an adapted variant of harmonic–balance.

The Duffing relativistic oscillator (3) is treated in [55] using He's energy–balance method.

The differential equations (1)–(3) are all of the form $\ddot{x}=f(x)$, where $f:[-a\,,a]\to\mathbb{R}$ is an odd continuous function. Assuming motion starts from rest, i.e. $\dot{x}(0)=0$, and choosing an initial displacement $a>0$ so that $f(x)\neq 0\ \forall x\in]0\,,a]$, the resulting motion is periodic and the particle satisfies $-a\le x(t)\le a\,,\ \forall\, t\in\mathbb{R}$. In other words, we study an initial value problem (IVP) of the form:

$$\begin{cases}\ddot{x}=f(x)\,,\\ x(0)=a\,,\quad \dot{x}(0)=0\,.\end{cases}\tag{4}$$

Without loss of generality, we can assume $f(a)<0$. Now, consider the even function:

$$\Phi(x):=-2\int_x^a f(s)\,\mathrm{d}s\,,\tag{5}$$

where $\Phi(\pm a)=0$, both roots being simple zeroes in the cases of our interest. Moreover:

$$\mathbb{T}=2\int_{-a}^{a}\frac{1}{\sqrt{\Phi(s)}}\,\mathrm{d}s\tag{6}$$

is the period of the solution to (4). This solution is implicitly defined for $|x|\le a$ by the time–integral:

$$t=\Psi(x)\,,\qquad\qquad \Psi(x):=\int_x^a\frac{1}{\sqrt{\Phi(s)}}\,\mathrm{d}s\,.\tag{7}$$

From a theoretical point of view, problem (4) is solved. But, in practice, the integral in (7) can rarely be first evaluated in closed form and then inverted to yield $x=\Psi^{-1}(t)$, since its inversion often involves unknown functions, the knowledge of which would be indispensable for the explicit description of the motion.

To arrive at a time–integral that can be exactly inverted, we describe f in terms of a near–minimax approximation given by its fifth–order Čebyšëv polynomial of the first kind.

As it is well–known, Čebyšëv's are a numerable family of polynomials, orthogonal with respect to the weight function $w(u)=1/\sqrt{1-u^2}$ and defined for $-1\le u\le 1$ by:

$$T_n(u)=\cos\Big(n\ \ \arccos(u)\Big)=\ {}_2\mathrm{F}_1\left(\begin{matrix}n\,,-n\\ \frac{1}{2}\end{matrix}\,\middle|\,\frac{1}{2}\,(1-u)\right)\qquad\qquad n\in\mathbb{N}\,,$$

where ${}_2\mathrm{F}_1$ denotes the Gauss hypergeometric function [23].

Čebyšëv polynomials form a complete orthogonal set on $[-1\,,1]$ in the appropriate Sobolev space, thus a function g can be expressed on its domain $[-1\,,1]$ via the expansion $g(u)=\tilde{g}(u)+E_r(u)$, where:

$$\tilde{g}(u):=\frac{1}{2}\,\alpha_0\,T_0(u)+\sum_{n=1}^{r}\alpha_n\,T_n(u)\,,\qquad E_r(u):=\sum_{n=r+1}^{\infty}\alpha_n\,T_n(u)\,.\tag{8}$$

If g is Lipschitz continuous on $[-1\,,1]$, then it has a unique representation as the infinite Čebyšëv series (8), which is absolutely and uniformly convergent, with coefficients defined using the weighted inner product [50]:

$$\alpha_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(s)}{\sqrt{1-s^2}}\,T_0(s)\,\mathrm{d}s\,, \quad \alpha_n = \frac{2}{\pi}\int_{-1}^{1} \frac{g(s)}{\sqrt{1-s^2}}\,T_n(s)\,\mathrm{d}s \quad \text{for } n \geq 1\,.$$

Recall that $|T_n(u)| \leq 1 \;\; \forall u \in [-1\,,1\,]$. Moreover, T_n has n distinct real roots in $]-1\,,1\,[$ and $n+1$ extrema in $[-1\,,1\,]$ at which it takes alternating values ± 1. Thus, if the coefficients α_n decrease in magnitude sufficiently rapidly (which depends on the regularity of g), then $E_r(u) \simeq \alpha_{r+1}\; T_{r+1}(u)$ equioscillates $r+2$ times on $[-1\,,1\,]$, implying that $\tilde{g}$ is a near–minimax approximant for g [43].

Coefficients α_n can be determined explicitly for some functions, otherwise they need discretisation via quadrature formulae. Even so, among methods yielding minimax or near–minimax approximations, Čebyšëv series is effective and easy to handle.

To apply Čebyšëv's approximation to the nonlinear oscillators under investigation, the displacement a is normalised to the interval $[-1\,,1\,]$ via a change of dependent variable $u = x/a$, and the following equivalent IVP is considered in place of (4):

$$\begin{cases} \ddot{u} = f_a(u)\,, \\ u(0) = 1\,, \quad \dot{u}(0) = 0\,. \end{cases} \qquad f_a(u) := \frac{1}{a}\; f(a\,u)\,. \tag{9}$$

The normalised force f_a, which is an odd function, can now be described in terms of polynomials T_n. For our purposes, f_a is expanded in Čebyšëv series truncated (or projected) at fifth–order:

$$f_a(u) \;\simeq\; \tilde{f}_a(u) := \alpha_1\, T_1(u) + \alpha_3\, T_3(u) + \alpha_5\, T_5(u)\,, \tag{10}$$

where $T_1(u) = u\,,\;\; T_3(u) = -3\,u + 4\,u^3\,,\;\; T_5(u) = 5\,u - 20\,u^3 + 16\,u^5$, and:

$$\alpha_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f_a(s)}{\sqrt{1-s^2}}\; T_n(s)\,\mathrm{d}s\,, \qquad\qquad n = 1\,,3\,,5\,. \tag{11}$$

Expressing the projected force $\tilde{f}_a$ in the monomial base, a new IVP replaces (9):

$$\begin{cases} \ddot{u} = -(c_1\; u + c_3\; u^3 + c_5\; u^5)\,, \\ u(0) = 1\,, \quad \dot{u}(0) = 0\,, \end{cases} \tag{12}$$

where, setting $\mathcal{C} = -2^5/\pi$:

$$\begin{aligned} c_1 &= -(\alpha_1 - 3\alpha_3 + 5\alpha_5) = \mathcal{C}\int_{-1}^{1} \frac{1}{\sqrt{1-s^2}}\Big(\frac{35}{16}s - 7s^3 + 5s^5\Big)\;\; f_a(s)\,\mathrm{d}s\,, \\ c_3 &= -4(\alpha_3 - 5\alpha_5) = \mathcal{C}\int_{-1}^{1} \frac{1}{\sqrt{1-s^2}}(-7s + 26s^3 - 20s^5) f_a(s)\,\mathrm{d}s, \\ c_5 &= -16\alpha_5 = \mathcal{C}\int_{-1}^{1} \frac{1}{\sqrt{1-s^2}}(5s - 20s^3 + 16s^5)\;\; f_a(s)\,\mathrm{d}s\,. \end{aligned} \tag{13}$$

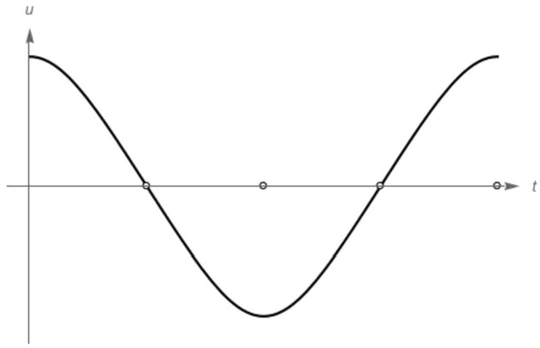

Fig. 1. Plot of the solution to IVP (12) for $c_1 = 1, c_3 = 2, c_5 = 3$, over a period interval.

3. Time–integral

Consider the family of IVPs (12). We highlight several contributions for problems of this type [1,8–11,15,17,29,40]. Application of (5) and (7) to the restoring force in (12) shows that, after an appropriate change of variable, the (squared) solution of this IVP is based on the evaluation of an elliptic integral:

$$t = \sqrt{\frac{3}{2}} \int_{u^2}^{1} \frac{1}{\sqrt{s\ (1-s)\ h_2(s)}} \ \mathrm{d}s \ , \tag{14}$$

with

$$h_2(s) = (6\,c_1 + 3\,c_3 + 2\,c_5) + (3\,c_3 + 2\,c_5)\ \ s + 2\,c_5\,s^2 \ .$$

The discriminant of polynomial h_2 is, discarding a factor of value 3:

$$\Delta = 3\,c_3^2 - 4\,c_5\ (4\,c_1 + c_3 + c_5)\ \ . \tag{15}$$

Given the physical nature of the restoring forces acting in the models of interest, coefficients c_1, c_3, c_5 can be assumed to be such that $h_2(s) > 0\ \forall s \in]0, 1[$. This property is assured if $c_5 > 0$ together with one of the two conditions:

$$\text{(i)}\quad \Delta \le 0\,; \qquad\qquad \text{(ii)}\quad \Delta > 0 \quad \text{and} \quad 6\,c_1 + 3\,c_3 + 2\,c_5 > 0\,. \tag{16}$$

We mention that at least a third quintic scenario exists [45], linked to the sign of Δ and $c_1\,, c_3\,, c_5$, untreated here, as it is not relevant nor necessary for the study of oscillators (1)–(3).

We provide closed–form solution and period for the IVP (12) in Theorems 1 and 2, under conditions (i) or (ii), respectively. Notice that, in the latter case, the roots of h_2, in addition to being real and distinct, are both negative due to Descartes' sign rule. Solution of (12) has a cosine wave behaviour, as Fig. 1 illustrates for the sample case $c_1 = 1$, $c_3 = 2$, $c_5 = 3$.

Theorem 1. *Given the time–integral equation (14), assume $c_5 > 0$ and $\Delta \le 0$ in (15) and define:*

$$A = \frac{\sqrt[4]{6}}{2}\ \frac{1}{\sqrt[4]{\mathcal{P}\ \ \mathcal{Q}}}\,, \qquad B = \frac{1}{6}\ \frac{\mathcal{Q}}{\mathcal{P}}\,, \qquad k^2 = \frac{1}{2} - \frac{\sqrt{6}}{8}\ \frac{\mathcal{K}}{\sqrt{\mathcal{P}\ \ \mathcal{Q}}}\,, \tag{17}$$

with

$$\mathcal{P} = c_1 + c_3 + c_5 \,, \qquad \mathcal{Q} = 6\,c_1 + 3\,c_3 + 2\,c_5 \,, \qquad \mathcal{K} = 4\,c_1 + 3\,c_3 + 2\,c_5 \,. \tag{18}$$

Then, the solution of IVP (12) is:

$$u^2(t) = \frac{\sqrt{B}}{\sqrt{B} + \cot^2\left(\frac{1}{2}\ \mathrm{am}\left(2\,\mathbf{K}(k) - \frac{t}{A}\,,\ k\right)\right)}\,, \tag{19}$$

where cot() *and* am() *are the cotangent and Jacobi amplitude functions. The latter is the inverse of the elliptic integral of the first kind* $F(\varphi\,,k)$*, meaning that* $\varphi = \mathrm{am}(s\,,k)$ *if and only if* $s = F(\varphi\,,k)$*, with:*

$$F(\varphi\,,k) := \int_0^{\sin(\varphi)} \frac{1}{\sqrt{(1-s^2)\,(1-k^2\,s^2)}}\,\mathrm{d}s\,, \qquad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\,, \tag{20}$$

while $\mathbf{K}(k) := F(\frac{\pi}{2}\,,k)$ *is the complete elliptic integral of the first kind, and* k *is the elliptic modulus.*

Solution (19) is periodic, with period:

$$\mathbb{T} = 8\ A\ \mathbf{K}(k)\,, \tag{21}$$

and it is positive for $0 \le t \le \frac{1}{4}\,\mathbb{T}\,, \quad \frac{3}{4}\,\mathbb{T} \le t \le \mathbb{T}$*, negative for* $\frac{1}{4}\,\mathbb{T} < t < \frac{3}{4}\,\mathbb{T}$*.*

Proof. The integral in (14) can be evaluated using entry 3.145–2 in [22], recalled here for ease of reading:

$$\int_\beta^v \frac{1}{\sqrt{(\eta-s)\,(s-\beta)\,\left((s-m)^2+n^2\right)}}\,\mathrm{d}s \;=\; \frac{1}{\sqrt{p\,q}}\,F(\varphi(v)\,,k)\,, \tag{22}$$

where:

$$\beta < v < \eta\,, \qquad p^2 := (m-\eta)^2 + n^2\,, \qquad q^2 := (m-\beta)^2 + n^2\,,$$

and

$$\varphi(v) = 2\,\mathrm{arccot}\sqrt{\frac{q\,(\eta-v)}{p\,(v-\beta)}}\,, \qquad k^2 = \frac{1}{4}\,\frac{(\eta-\beta)^2-(p-q)^2}{p\,q}\,.$$

In the case of (14), it is $\beta = 0$, $\eta = 1$, $v = u^2$, and polynomial h_2 is rearranged as follows:

$$\frac{1}{2\,c_5}\,h_2(s) \;=\; \left(s + \left(\frac{3\,c_3}{4\,c_5} + \frac{1}{2}\right)\right)^2 + \left(\frac{3\,c_1}{c_5} - \left(\frac{3\,c_3}{4\,c_5}\right)^2 + \frac{3\,c_3}{4\,c_5} + \frac{3}{4}\right).$$

To apply formula (22), the integral in (14) must be rewritten further as the difference of integrals of the same integrand on intervals $[0\,,1]$ and $[0\,,u^2]$. Equation (14) thus becomes:

$$t = 2\ A\ \mathbf{K}(k) - A\ F\left(2\,\mathrm{arccot}\left(\sqrt[4]{B}\ \sqrt{\Upsilon(u^2)}\right)\,,\ k\right)\,, \tag{23}$$

where $\Upsilon(v) = (1-v)/v$.

Due to the invertibility of the elliptic integral of the first kind, inversion of the time–integral equation (23) is possible and yields solution (19). In a similar way, (21) can be proved to supply the motion period. □

Theorem 2. *Given the time–integral (14), assume* $c_5 > 0$ *and condition (ii) in (16), that is* $h_2(s) = (s - s_1)(s - s_2)$ *with* $s_1 < s_2 < 0$. *Now, define:*

$$k^2 = \frac{s_2 - s_1}{s_1\,(s_2 - 1)}\,. \tag{24}$$

Then, the solution of IVP (12) is:

$$u^2(t) = s_1 + \frac{s_1\,(s_1 - 1)}{\operatorname{sn}^2\left(\dfrac{\sqrt{c_5\, s_1\,(s_2 - 1)}}{\sqrt{3}}\;t\,,\;k\right) - s_1}\,, \tag{25}$$

sn() *being the Jacobi sine amplitude function, i.e.* $\operatorname{sn}(s, k) = \sin(\varphi)$ *with* $\varphi = \operatorname{am}(s, k)$.

Solution (25) is periodic, with period:

$$\mathbb{T} = \frac{4\sqrt{3}}{\sqrt{c_5\, s_1\,(s_2 - 1)}}\;\mathbf{K}(k)\,, \tag{26}$$

and it is positive for $0 \le t \le \frac{1}{4}\mathbb{T}\,,\ \frac{3}{4}\mathbb{T} \le t \le \mathbb{T}$, *negative for* $\frac{1}{4}\mathbb{T} < t < \frac{3}{4}\mathbb{T}$.

Proof. To evaluate the integral in (14), entry 3.147–7 of [22] is used, recalled below:

$$\int_v^\eta \frac{\mathrm{d}s}{\sqrt{(\eta - s)\,(s - \beta)\,(s - \gamma)\,(s - \delta)}} = \frac{2}{\sqrt{(\eta - \gamma)\,(\beta - \delta)}}\;F(\varphi(v), k)\,,$$

where $\delta < \gamma < \beta \le v < \eta$ and:

$$\varphi(v) = \arcsin\sqrt{\frac{(\beta - \delta)(\eta - v)}{(\eta - \beta)(v - \delta)}}\,, \qquad k^2 = \frac{(\eta - \beta)(\gamma - \delta)}{(\eta - \gamma)(\beta - \delta)}\,.$$

In the case of (14), it is $\beta = 0\,, \eta = 1\,,\ v = u^2\,,\ \gamma = s_2\,,\ \delta = s_1$. Therefore, the motion period is given by (26), while (25) provides the solution after the relevant computations, not reported here as they are similar to those performed in the proof of Theorem 1. □

We observe that, if $\Delta = 0$, integral (14) degenerates into an elliptic integral of the third kind, which is tabulated as entry 3.138–6 of [22]. Here, the related computations are omitted for two reasons. First of all, condition $\Delta = 0$ is linked to a very particular value of the initial displacement a. Secondly, even though the occurrence of elliptic integrals of the third kind makes it impossible to invert the time–integral and compute the solution explicitly, thanks to the continuous dependence on data, the relevant solution can be approximated at arbitrary precision with the solutions obtained in Theorems 1 and 2.

4. Application to the relativistic oscillator

In the case of the relativistic oscillator ruled by (1), the normalised equation is:

$$\begin{cases} \ddot{u} = -\dfrac{u}{\sqrt{1 + a^2\;u^2}}\,, \\ u(0) = 1\,, \quad \dot{u}(0) = 0\,. \end{cases} \tag{27}$$

Here, functions Φ, Ψ, defined in (5), (7), respectively, become as follows, after some algebraic adjustments and with, obviously, $0 < \frac{1}{\sqrt{1+a^2}} < \sqrt{\frac{1+a^2\ u^2}{1+a^2}} < 1$:

$$\Phi(u) = \frac{2}{a^2}\left(\sqrt{1+a^2} - \sqrt{1+a^2\ u^2}\right), \tag{28}$$

$$\begin{aligned}\Psi(u) &= \frac{a}{\sqrt{2}\ \sqrt[4]{1+a^2}} \int_u^1 \frac{1}{\sqrt{1-\sqrt{\frac{1+a^2\ s^2}{1+a^2}}}}\ \mathrm{d}s \\ &= \frac{\sqrt[4]{1+a^2}}{\sqrt{2}} \int_{\sqrt{\frac{1+a^2\ u^2}{1+a^2}}}^1 \frac{z}{\sqrt{(1-z)\left(z^2-\frac{1}{1+a^2}\right)}}\ \mathrm{d}z\,. \end{aligned} \tag{29}$$

The integral in (29) can be expressed in explicit form, for example via entry 3.132–5 of [22], through which the time–integral equation $t = \Psi(u)$ becomes:

$$t = \sqrt{2}\ \left(\mathcal{A}\ \ E(\varphi, k) - \frac{F(\varphi, k)}{\mathcal{A}}\right), \qquad \mathcal{A} = \sqrt{\sqrt{1+a^2}+1}\,, \tag{30}$$

being $F(\varphi, k)$ and $E(\varphi, k)$ elliptic integrals of the first and second kind, with:

$$\varphi = \varphi(u^2) = \arcsin\sqrt{\frac{\sqrt{1+a^2}-\sqrt{1+a^2\ u^2}}{\sqrt{1+a^2}-1}}\,, \qquad k = \frac{\sqrt{1+a^2}-1}{a}\,. \tag{31}$$

Note that, for any $a > 0$, the elliptic modulus k satisfies the requirement $k < 1$.

Calculation of the integral in (29) also leads to the exact determination of the period of oscillation:

$$\mathbf{T} = 4\sqrt{2}\ \left(\mathcal{A}\ \ \mathbf{E}(k) - \frac{\mathbf{K}(k)}{\mathcal{A}}\right), \qquad \mathcal{A} = \sqrt{\sqrt{1+a^2}+1}\,. \tag{32}$$

$\mathbf{K}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind, whose modulus k is as in (31).

Formula (32) provides, explicitly, the period of the relativistic oscillator, which is useful in itself, and also because it allows a comparison with the period of the approximated quintic oscillator we are to obtain.

As regards the u sought, the solution obtained implicitly in (30) does not allow the inversion in closed form; therefore, we replace the normalised force in (27) by its fifth–order Čebyšëv–projected force, expressed in the monomial base through coefficients c_1, c_3, c_5 given by (13): according to the sign of the discriminant Δ, built on such coefficients as shown in (15), the solution to IVP (12) is given by (19) or by (25).

Application of formulae (13) to problem (27) suggests to introduce three elliptic integrals:

$$J_n(a) = \int_{-1}^1 \frac{s^n}{\sqrt{(1-s^2)(1+a^2\ s^2)}}\ \mathrm{d}s\,, \qquad n = 2\,,4\,,6\,.$$

Due to the form of the normalised force in (27), in fact, coefficients (13) become:

$$\begin{aligned} c_1 &= \mathcal{C}\ \left(\frac{35}{16}\,J_2(a) - 7\ J_4(a)\ +\ 5\,J_6(a)\right), \\ c_3 &= \mathcal{C}\ \left(-7\,J_2(a) + 26\,J_4(a) - 20\,J_6(a)\right)\ , \\ c_5 &= \mathcal{C}\ \left(\ 5\ J_2(a)\ - 20\,J_4(a) + 16\,J_6(a)\right)\ , \end{aligned} \tag{33}$$

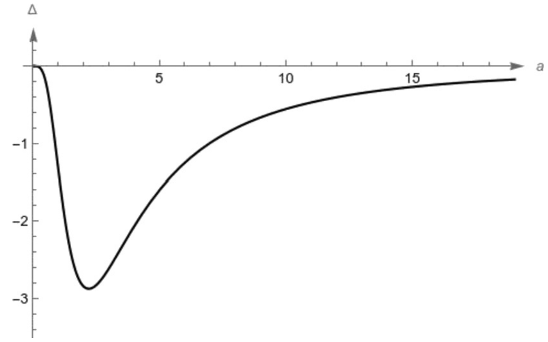

Fig. 2. Discriminant $\Delta = \Delta(a)$ given by (15) and associated to the quinticate form of oscillator (1).

where $\mathcal{C} = 2^5/\pi$. Using entries 236.16 and 331.01–03 of [14], it follows:

$$\begin{aligned} J_2(a) &= \frac{2}{a^2}\left(\mathcal{J}\ \mathbf{E}(h^2) - \frac{1}{\mathcal{J}}\ \mathbf{K}(h^2)\right), \\ J_4(a) &= \frac{2}{3\,a^4}\left(2\,(a^2-1)\ \mathcal{J}\ \mathbf{E}(h^2) - (a^2-2)\ \frac{1}{\mathcal{J}}\ \mathbf{K}(h^2)\right), \\ J_6(a) &= \frac{2}{15\,a^6}\left((8a^4-7a^2+8)\,\mathcal{J}\,\mathbf{E}(h^2) - (4a^4-3a^2+8)\,\frac{1}{\mathcal{J}}\,\mathbf{K}(h^2)\right), \end{aligned} \tag{34}$$

where $\mathcal{J} = \sqrt{1+a^2}$ and the elliptic modulus is given by $h = a/\mathcal{J}$.

In other words, inserting values (34) of integrals $J_2\,, J_4\,, J_6$ into (33), it becomes possible to identify closed–form expressions for coefficients $c_1\,, c_3\,, c_5$ of the quinticate form of oscillator (1).

The consequent expression (15) of the discriminant Δ is complicated, but still tractable using computer algebra systems such as *Mathematica* [52], that provides symbolic and numeric functionalities and their hybrid use [48,53], and graphics tools enabling a visual analysis as that shown in Fig. 2.

It is verified that the assumptions of Theorem 1 are fulfilled, although proving c_5 positive involves computational difficulties which we faced using symbolic calculus.

It is important to note that, to use the procedure presented, what is ultimately needed is to calculate the integrals expressing the orthogonal projection onto the space of Čebyšëv polynomials. From that point on, through the coefficients of the quintic polynomial replacing the normalised restoring force and by Theorem 1 (or Theorem 2, if appropriate), one arrives at the quinticate solution and its period.

To validate the quality of the obtained solution, we study the behaviour of the following differential operator:

$$L\,u = \ddot{u} - f_a(u)\,. \tag{35}$$

Here, it is $f_a(u) = -u/\sqrt{1+a^2\,u^2}$, since we are studying oscillator (1), while $f_a(u) = f_a(u,b) = -u - b\,u/\sqrt{1+a^2\,u^2}$ for oscillator (2) and finally $f_a(u) = f_a(u,b) = -u - a^2\,u^3 - b\,u/\sqrt{1+a^2\,u^2}$ for oscillator (3), as we will see in § 5.1 and § 5.2 respectively.

For the quinticate form of oscillator (1), Fig. 3 reports the graph of the deviation from zero produced in (35) by solution u in the first quarter of the period, and at initial displacements equal to $a = 1$, 2, 3, and $a = 8$, 20, 30; plots are kept separate for effective rendering reasons.

It is reasonable for the quinticate solution behaviour to initially get worse as the displacement a increases, given the normalisation of the integration interval. However, computation of the maximum norm $\|L\,u\|_\infty$ in the first quarter of the period yields results in Table 1, showing that an upper bound is given by the value 0.0375439 reached at $a = 8$, from where a monotonic decrease can be observed.

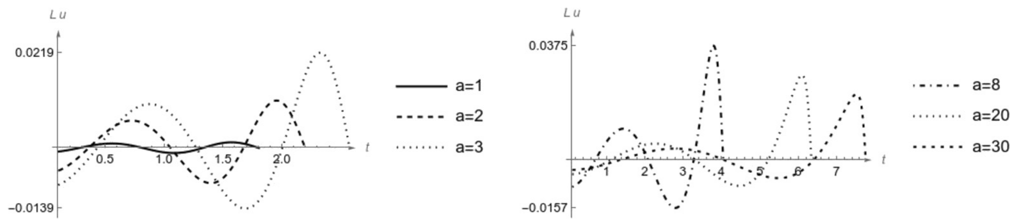

Fig. 3. Behaviour of $L\,u$, where u is solution (19) with coefficients (33), for the quinticate form of oscillator (1).

Table 1
Uniform norm of $L\,u$, where u is solution (19) with coefficients (33), for the quinticate form of oscillator (1).

a	1	2	3	8	20	30
$\|L\,u\|_\infty$	.0013005	.0109030	.0219219	.0375439	.0278857	.0216839

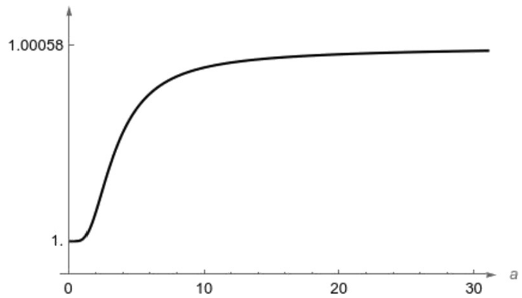

Fig. 4. Period ratio in the case of the quinticate form of oscillator (1).

Here and in the following § 5.1 and § 5.2, all results are obtained working in machine precision, and are displayed rounded to their first seven significant digits. Arbitrary precision was also employed, and showed that the order of magnitude of the quinticate solution remains stable even with increased precision. In particular, we asked for up to fifty significant decimal digits, with which an improvement on the fourth decimal digit was obtained on average; as an example, it is $\|L\,u\|_\infty = 0.0373455$ in the worse case of $a = 8$ and working with precision 50. As already mentioned in § 1, results from orthogonal projections of order higher than five, and their further computational cost, are still under investigation.

The validity of the obtained solution is further testified by the ratio between exact period (32) of the gravitational oscillator and period (21) of its quinticate form: this ratio remains close to 1, and bounded above by the value 1.00058 for large values of a. Fig. 4 illustrates the ratio behaviour for displacements up to $a = 30$.

5. Quinticate oscillators: further results and discussion

The procedure introduced in § 2–3 is now applied to the conservative nonlinear oscillatory system (2) and the Duffing relativistic oscillator (3). Results obtained are wholly analogous to those seen for the relativistic oscillator (1). In particular, it is still possible to determine closed–form expressions for the coefficients of the Čebyšëv quintic approximant, in terms of complete elliptic integrals of the first kind.

5.1. Nonlinear oscillator (2)

Here, the normalised IVP is:

$$\begin{cases} \ddot{u} = -u - \dfrac{b\,u}{\sqrt{1+a^2\ \ u^2}}\ , \\ u(0) = 1\,, \quad \dot{u}(0) = 0\ , \end{cases} \tag{36}$$

thus function Φ defined in (5) becomes:

$$\Phi(u) = 1 - u^2 + \frac{2\,b}{a^2}\left(\sqrt{1+a^2} - \sqrt{1+a^2\,u^2}\right), \tag{37}$$

so that forming $\Psi(u)$ as in (7) requires $2\,b > \sqrt{1+a^2} - \sqrt{1+a^2\,u^2} > 0$, being $a > 0$ and $0 < u < 1$.

In this case, the exact period, obtained using entry 3.148–7 in [22], is:

$$\mathbf{T} = \frac{4}{\sqrt{\mathcal{A}}}\ \left((1+\sqrt{1+a^2})\ \ \mathbf{\Pi}\,(\mathcal{N}\ ,\ \mathcal{K})\ \ -\ \ \mathbf{K}(\mathcal{K})\right), \tag{38}$$

with

$$\mathcal{A} = \sqrt{1+a^2} + b\ , \qquad \mathcal{N} = \frac{1-\sqrt{1+a^2}}{2}\ , \qquad \mathcal{K} = \frac{\mathcal{N}\ \ (b-\mathcal{N})}{\mathcal{A}}\ .$$

$\mathbf{K}(\mathcal{K})$ is the complete elliptic integral of the first kind, while $\mathbf{\Pi}(\mathcal{N}\,,\,\mathcal{K})$ is the elliptic integral of third kind, $\mathcal{N}$ is the elliptic characteristic, and the modulus satisfies $\mathcal{K} < 1$ for $a\,,b > 0$.

As for the relativistic oscillator, also for system (36) the three coefficients of the quinticate force can be computed explicitly, using again (13) recalled here for reading convenience:

$$c_1 = -(\alpha_1 - 3\,\alpha_3 + 5\,\alpha_5)\ , \qquad c_3 = -4\ (\alpha_3 - 5\,\alpha_5)\ , \qquad c_5 = -16\ \alpha_5\ ,$$

where, in this case:

$$\begin{aligned} \alpha_1 &= \quad \mathcal{C}\left(w_2 + b\,J_2(a)\right), \\ \alpha_3 &= -3\,\mathcal{C}\left(w_2 + b\,J_2(a)\right) +\ 4\,\mathcal{C}\left(w_4 + b\,J_4(a)\right), \\ \alpha_5 &= \quad 5\,\mathcal{C}\left(w_2 + b\,J_2(a)\right) - 20\,\mathcal{C}\left(w_4 + b\,J_4(a)\right) + 16\,\mathcal{C}\left(w_6 + b\,J_6(a)\right), \end{aligned} \tag{39}$$

having set $\mathcal{C} = -2/\pi$. The form of the normalised force in (36), in fact, suggests to use again the exactly computable $J_2(a)\,,J_4(a)\,,J_6(a)$ in (34), and to introduce the following integrals that can also be computed exactly:

$$w_n = \int_{-1}^{1} \frac{s^n}{\sqrt{1-s^2}}\ , \qquad n = 2\,,4\,,6\,,8\ , \tag{40}$$

that is $w_2 = \pi/2\ ,\ w_4 = 3\pi/8\ ,\ w_6 = 5\pi/16\ ,\ w_8 = 35\pi/128$, the last one being needed in § 5.2.

Fig. 5 depicts discriminant $\Delta = \Delta(a\,,b)$ and coefficient $c_5 = c_5(a\,,b)$ for varying $a > 0$ and $0 < b \leq 1$, showing that condition (i) of (16) is verified. Theorem 1 thus applies.

The qualitative and quantitative behaviour of solution and period for the quinticate approximant to oscillator (2) is analogous to that seen in § 4 for the quinticate relativistic oscillator. The period ratio stays close to 1, as shown in Fig. 6 (left). As regards the differential operator (35), results were obtained with parameters $a > 0$ and $0 < b \leq 1$ and are not reported here, since they exhibit, precisely, same behaviour and equal order of magnitude as those summarised in Fig. 3 and Table 1 for the relativistic oscillator.

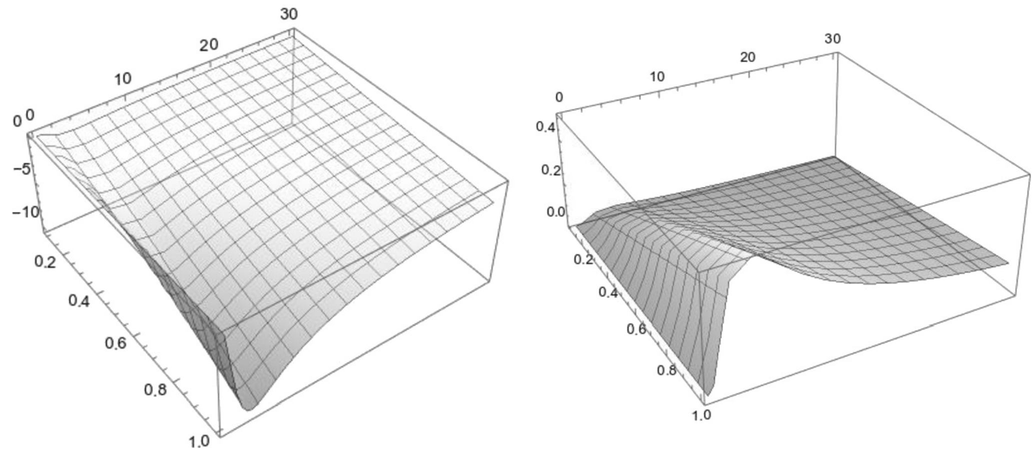

Fig. 5. Discriminant $\Delta = \Delta(a\,,b) \le 0$ (left) and coefficient $c_5 = c_5(a\,,b) > 0$ (right) for the quinticate form of oscillator (2) for $(a\,,b) \in]0\,,30] \times]0\,,1]$.

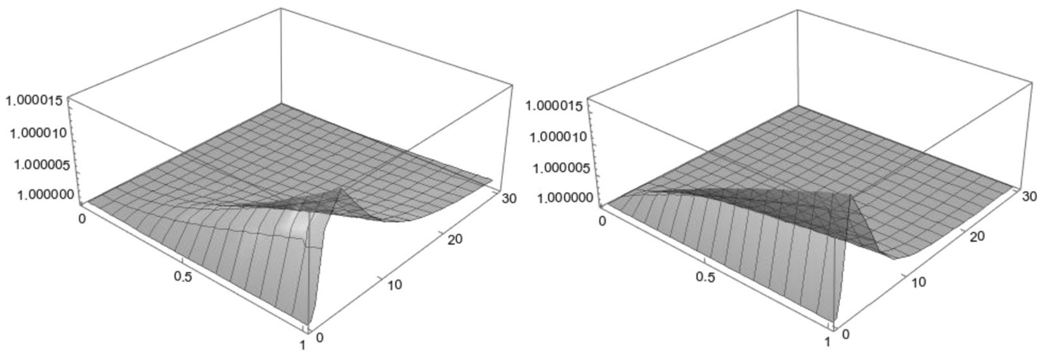

Fig. 6. Period ratio in the case of the quinticate form of oscillator (2) (left) and oscillator (3) (right).

5.2. Duffing relativistic oscillator (3)

The normalised IVP, in this case, is:

$$\begin{cases} \ddot{u} = -u - a^2\, u^3 - \dfrac{b\,u}{\sqrt{1+a^2\;\;u^2}}\,, \\ u(0) = 1\,, \quad \dot{u}(0) = 0\,, \end{cases} \tag{41}$$

so that function Φ defined in (5) becomes:

$$\Phi(u) = \frac{1}{2}\,(1-u^2)\,(2+a^2+a^2\,u^2) + \frac{2\,b}{a^2}\left(\sqrt{1+a^2} - \sqrt{1+a^2\,u^2}\right), \tag{42}$$

and forming Ψ in (7) requires $4b > (2+a^2+a^2u^2)(\sqrt{1+a^2} - \sqrt{1+a^2u^2}) > 0$, being $a > 0$ and $0 < u < 1$.

Setting $\mathcal{C} = -2/\pi$ and exploiting formulae (34) and (40), we have:

$$\begin{aligned} \alpha_1 &= \quad \mathcal{C}\,\left(w_2 + a^2\,w_4 + b\,J_2(a)\right)\,, \\ \alpha_3 &= -3\,\mathcal{C}\left(w_2 + a^2\,w_4 + b\,J_2(a)\right) + 4\,\mathcal{C}\left(w_4 + a^2\,w_6 + b\,J_4(a)\right) \\ \alpha_5 &= \quad 5\,\mathcal{C}\left(w_2 + a^2\,w_4 + b\,J_2(a)\right) - 20\,\mathcal{C}\left(w_4 + a^2\,w_6 + b\,J_4(a)\right) + 16\,\mathcal{C}\left(w_6 + a^2\,w_8 + b\,J_6(a)\right), \end{aligned} \tag{43}$$

on which $c_1 = -(\alpha_1 - 3\,\alpha_3 + 5\,\alpha_5)$, $c_3 = -4\,(\alpha_3 - 5\,\alpha_5)$ and $c_5 = -16\,\alpha_5$ are built.

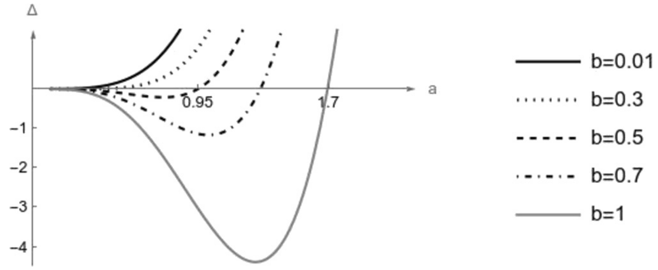

Fig. 7. Discriminant $\Delta = \Delta(a\,,b)$ for the quinticate form of oscillator (3) for various couples $(a\,,b)$.

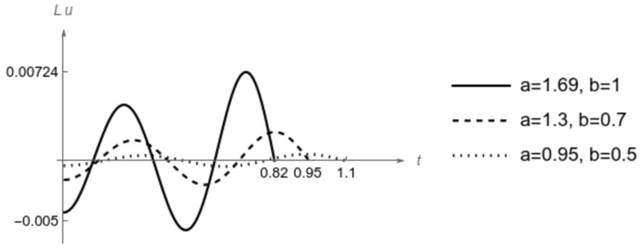

Fig. 8. $L\,u$, where u is solution (19) with coefficients built on (43), for the quinticate form of oscillator (3).

Table 2
Uniform norm of $L\,u$, with u solution (19) and coefficients built on (43), for the quinticate form of oscillator (3).

a	0.95	1.3	1.69
b	0.5	0.7	1
$\Vert L\,u\Vert_\infty$	0.000487249	0.00229373	0.00724625

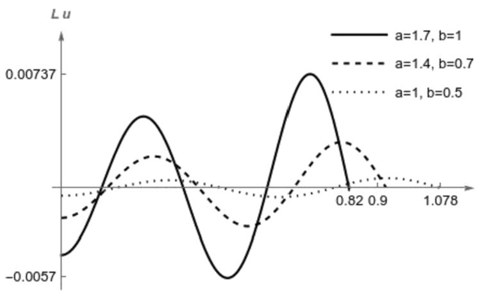

Fig. 9. $L\,u$, where u is solution (25) with coefficients built on (43), for the quinticate form of oscillator (3).

Fig. 7 depicts the discriminant $\Delta = \Delta(a\,,b)$ for varying $a\,,b > 0$ and illustrates how the values of a and b that satisfy either condition (i) or (ii) of (16) are closely related. As an example, it is $\Delta(a\,,0.5) \le 0$ and $c_5(a\,,0.5) > 0$ for $a \lessapprox 0.95$; similarly, $\Delta(a\,,1) \le 0$ and $c_5(a\,,1) > 0$ for $a \lessapprox 1.7$; therefore, in both of these cases, Theorem 1 applies. In particular, we observed that condition (i) starts being significantly verified when $b \ge 0.4$, where $b \approx 0.4$ requires $a \lessapprox 0.7$. Conversely, for $b \lessapprox 0.4$ and any a, condition (ii) is verified and Theorem 2 comes into play.

The qualitative and quantitative behaviour of solution and period for the quinticate approximant to oscillator (3) is analogous to those commented in § 4 and § 5.1. The period ratio stays close to 1, as shown in Fig. 6 (right). As for the differential operator (35), Fig. 8 and Table 2 report results achieved with couples $(a\,,b)$ requiring the solution u given by Theorem 1, while Fig. 9 and Table 3 present the outcome related to couples $(a\,,b)$ for which the u defined in Theorem 2 must be used; in both cases, we attain the

Table 3
Uniform norm of $L\,u$, with u solution (25) and coefficients built on (43), for the quinticate form of oscillator (3).

a	1	1.4	1.7
b	0.5	0.7	1
$\|L\,u\|_\infty$	0.00064411	0.00298016	0.00737777

same behaviour and equal, or improved, order of magnitude as in the previously studied quinticate forms of oscillators (1) and (2).

6. Conclusions and future work

In this work we exploit Čebyšëv's fifth-order approximations by applying them to three popular nonlinear oscillator models, which share the fact that the integrals obtained in the projection are expressible in closed form by means of elliptic integrals. The quinticate systems obtained, which by their nature constitute very good approximations of the considered models, are, in turn, explicitly solved in terms of Jacobi elliptic functions.

The procedure outlined in this work is new, and accounts for wave configurations that have not been considered in the related literature. The quality of the solutions obtained is confirmed in terms of the norm of the deviation of the solution, and in terms of the ratio between the periods of the quinticate systems and the periods of the correspondent non–approximate systems solutions, which admit a closed–form expression in two out of the three cases considered. All simulations are performed in machine precision and within the scientific environment of *Mathematica*, Version 13; arbitrary precision arithmetics were also employed, and confirmed the results obtained in machine precision.

More wave configurations are currently under investigation, together with a generalisation to approximations of order higher than five, which require the inversion of hyperelliptic integrals, and related functionalities, not yet available and under development.

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Declaration of competing interest

The authors declare no conflict of interest.

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