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New Characterizations of Bivariate Discrete Schur-constant Models

Nikolai Kolev[1] and Sabrina Mulinacci[2]

[1] Institute of Mathematics and Statistics, University of Sao Paulo, Brazil

[2] Department of Statistics, University of Bologna, Italy

Abstract

We present two characterizations of bivariate discrete Schur-constant models corresponding to continuous case statements.

Keywords: 2-monotone function, Laplace transform.

This note is motivated by Kozlova and Salminen (2004) who characterize the bivariate continuous Schur-constant models in terms of the bivariate Laplace transform as follows.

Lemma 0.1. *The continuous random vector* (X, Y) *is Schur-constant if and only if there exists a positive random variable* V *such that*

$$\mathbb{E}\left(e^{-a\,X-b\,Y}\right)=\frac{1}{a-b}\int_b^a \mathbb{E}\left(e^{-t\,V}\right)\,dt \quad \textit{for} \quad a>0,\, b>0,\, a\neq b, \tag{1}$$

when $V \stackrel{d}{=} X+Y$.

In the proof, Kozlova and Salminen (2004) used the fact that the continuous Schur-constant random vector (X, Y) can be specified by the relation

$$(X,Y) \stackrel{d}{=} (U\,V, (1-U)\,V)\,, \tag{2}$$

where the random variable U is uniformly distributed on $[0, 1]$, to be denoted by $U \sim \mathcal{U}(0, 1)$, and V is an arbitrary positive random variable independent of U. An alternative proof of Lemma 0.1 can be found in Ta and Van (2017). Relation (2) can be rewritten as $(X, Y) \stackrel{d}{=}$

$(U_1, V - U_1)$, where $U_1 \sim \mathcal{U}(0, V)$ and V is an arbitrary positive random variable. Notice that in both the stochastic representations the random variable V shares the same distribution as that of the sum $X + Y$.

Recently, the properties of multivariate Schur-constant discrete distributions and their applications have been investigated extensively by several authors, e.g., see Castaner et al. (2015) and Lefevre at al. (2018). In what follows, we will present two characterizations of bivariate discrete Schur-constant models corresponding to continuous case relations (2) and (1).

Let us introduce the discrete counterpart of the stochastic representation (2), considering Schur-constant survival models for a discrete random vector (X, Y) whose marginals take values in the set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Denote by $S_D(\cdot, \cdot)$ the joint survival function of the non-negative integer-valued random vector (X, Y), i.e., $S_D(x, y) = \mathbb{P}(X \geq x, Y \geq y)$ for $x, y \in \mathbb{N}_0$. According to Proposition 2.2 in Castaner et al. (2015), the bivariate discrete Schur-constant distribution is specified by the assumption

$$S_D(x, y) = G_D(x + y), \quad x, y \in \mathbb{N}_0 \tag{3}$$

for a 2-monotone on $\mathbb{N}_0$ univariate survival function $G_D(.)$. We remind that a function $f : \mathbb{N}_0 \to \mathbb{R}$ is 2-monotone if $f(x) \geq 0$, $f(x+1) \leq f(x)$ and $f(x+2) - 2f(x+1) + f(x) \geq 0$ for all $x \in \mathbb{N}_0$.

Set $W = X + Y$ and define a discrete random variable U_D as

$$U_D \stackrel{d}{=} \begin{cases} \frac{X}{W}, & \text{if} \quad (X, Y) \neq (0, 0); \\ 0, & \text{if} \quad X = Y = 0. \end{cases} \tag{4}$$

The range of values of U_D is the set of rational numbers in $[0, 1]$ and we obtain the stochastic representation

$$(X, Y) \stackrel{d}{=} (U_D W, (1 - U_D) W), \tag{5}$$

which can be considered as the discrete version of (2).

It follows our first characterization of a discrete bivariate Schur-constant random vector.

Theorem 0.1. *The discrete random vector (X, Y) taking values in $\mathbb{N}_0 \times \mathbb{N}_0$ is Schur-constant if and only if there exists a non-negative integer-valued random variable W for which (5) holds with U_D uniformly distributed on the set $\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\}$ if $W \neq 0$ and $U_D = 0$, if $W = 0$.*

Proof. *Necessity:* We assume that (X, Y) is Schur-constant and we set $W = X + Y$. Clearly $\mathbb{P}(U_D = 0|W = 0) = 1$. Thanks to Lemma 3.1 in Lefèvre et al. (2018), we have that

$$\mathbb{P}(X = x, Y = y) = G_D(x + y) - 2G_D(x + y + 1) + G_D(x + y + 2),$$

and therefore

$$\mathbb{P}(W = w) = \sum_{j=0}^{w} \mathbb{P}(X = w - j, Y = j) = (w + 1)\left[G_D(w) - 2G_D(w + 1) + G_D(w + 2)\right].$$

It follows that, by (4), if $w > 0$ and $u = 0, 1, \ldots, w$,

$$\mathbb{P}\left(U_D = \frac{u}{w} \;\Big|\; W = w\right) = \frac{\mathbb{P}(X = u, Y = w - u)}{\mathbb{P}(W = w)} = \frac{1}{w + 1}$$

and (5) holds with U_D uniformly distributed on $\left\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\right\}$.

Sufficiency: Let (5) be true with U_D being uniformly distributed on $\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\}$ when $W > 0$ and $U_D = 0$ when $W = 0$. Then, if $(x, y) \neq (0, 0)$,

$$\begin{aligned} S_D(x, y) &= \sum_{w=x+y}^{\infty} \mathbb{P}\left(U_D \geq \frac{x}{w}, 1 - U_D \geq \frac{y}{w}\Big| W = w\right) \mathbb{P}(W = w) \\ &= \sum_{w=x+y}^{\infty} \sum_{u=x}^{w-y} \mathbb{P}\left(U_D = \frac{u}{w}\Big| W = w\right) \mathbb{P}(W = w). \end{aligned}$$

Since U_D is uniformly distributed on $\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\}$, then

$$S_D(x, y) = \sum_{w=x+y}^{\infty} \left(1 - \frac{x + y}{w + 1}\right) \mathbb{P}(W = w).$$

Under the substitution $t = x + y$, the joint survival function $S_D(\cdot, \cdot)$ can be rewritten as

$$G_D(t) = \mathbb{P}(W \geq t) - t\,\mathbb{E}\left[\frac{1}{W + 1}\mathbf{1}_{\{W \geq t\}}\right], \tag{6}$$

where $\mathbf{1}_{\{.\}}$ is the indicator function, i.e., relation (3) holds. □

From the proof one can conclude that relation (6) provides a characterization of any 2-monotone survival function on $\mathbb{N}_0$. Also, unlike the continuous case, the random variables U_D and W in the stochastic representation (5) are no more independent.

We are ready to prove the discrete version of Lemma 0.1 as follows.

Theorem 0.2. *The discrete random vector (X, Y) taking values in $\mathbb{N}_0 \times \mathbb{N}_0$ is Schur-constant if and only if there exists a nonnegative discrete random variable W with values in $\mathbb{N}_0$ such that the joint Laplace transform $\mathbb{E}[e^{-a\,X - b\,Y}]$ is given as follows:*

$$\mathbb{E}\left[e^{-aX-bY}\right] = \frac{1}{e^{-b} - e^{-a}} \int_b^a \mathbb{E}[e^{-t(W+1)}]dt, \quad \textit{for} \quad a > 0,\, b > 0,\, a \neq b, \tag{7}$$

where $W \stackrel{d}{=} X + Y$.

Proof. Necessity: Assume that the discrete random vector (X, Y) with values in $\mathbb{N}_0 \times \mathbb{N}_0$ is Schur-constant and let $a, b > 0$. By Theorem 0.1 there exists a non-negative integer-valued random variable W for which (5) holds with U_D uniformly distributed on the set $\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\}$ if $W \neq 0$ and $U_D = 0$, if $W = 0$. Then

$$\mathbb{E}[e^{-aX-bY}] = \mathbb{E}\left[e^{-W[(a-b)U_D+b]}\right] = \sum_{w=0}^{\infty} \mathbb{E}\left[e^{-w[(a-b)U_D+b]}|W = w\right] \mathbb{P}(W = w). \tag{8}$$

Taking into account that U_D is uniformly distributed on $\{0, \frac{1}{W}, \ldots, \frac{W-1}{W}, 1\}$ yields

$$\mathbb{E}[e^{-aX-bY}] = \sum_{w=0}^{\infty} \sum_{u=0}^{w} e^{-[(a-b)u+bw]} \frac{\mathbb{P}(W = w)}{w+1} = \sum_{w=0}^{\infty} \frac{\mathbb{P}(W = w)}{w+1} e^{-wb} \sum_{u=0}^{w} e^{-(a-b)u}.$$

If $a \neq b$,

$$\mathbb{E}[e^{-aX-bY}] = \sum_{w=0}^{\infty} \frac{\mathbb{P}(W = w)e^{-wb}[1 - e^{-(a-b)(w+1)}]}{(w+1)[1 - e^{-(a-b)}]} = \frac{1}{e^{-b} - e^{-a}} \sum_{w=0}^{\infty} \int_b^a \mathbb{P}(W = w)e^{-t(w+1)}dt.$$

However

$$\sum_{w=0}^{\infty} \int_b^a \mathbb{P}(W = w)e^{-t(w+1)}dt = \int_b^a \mathbb{E}[e^{-t(W+1)}]dt$$

and hence relation (7) is established for $a \neq b$.

If $a = b$ then, from (8) we get that

$$\mathbb{E}[e^{-b(X+Y)}] = \mathbb{E}\left[e^{-bW}\right]$$

and $X + Y \stackrel{d}{=} W$.

Sufficiency: Vice versa, if (7) is true with $W \stackrel{d}{=} X + Y$, let U_D be a random variable uniformly distributed on $\left\{0, \frac{1}{W}, \ldots, 1 - \frac{1}{W}, 1\right\}$ if $W > 0$ and $U_D = 0$ when $W = 0$. Repeating the same computations showed in the necessity part of the proof, one obtains that the Laplace transform of the vector $(U_D W, (1 - U_D)W)$ coincides with that of the vector (X, Y) and thus, taking into account the uniqueness of a distribution by its Laplace transform, (5) holds and the discrete vector (X, Y) is Schur-constant. □

The necessity part of Theorem 0.1 helps to find the distribution of the marginal random variable X as a function of the distribution of the sum $W \stackrel{d}{=} X + Y$.

Corollary 0.3. *If the discrete vector (X, Y) is Schur-constant and $W \stackrel{d}{=} X+Y$, the probability mass function of the marginal distribution of X is given by*

$$\mathbb{P}(X = x) = \sum_{w=x}^{\infty} \frac{\mathbb{P}(W = w)}{w + 1} = \mathbb{E}\left[\frac{1}{W + 1}\mathbf{1}_{\{W \geq x\}}\right] \quad \textit{for} \quad x \in \mathbb{N}_0,$$

where $\mathbf{1}_{\{.\}}$ is the indicator function. Moreover,

$$S_D(x, y) = \mathbb{P}(X \geq x, Y \geq y) = \mathbb{P}(W \geq x + y) - (x + y)\mathbb{P}(X = x + y).$$

Finally note that if (X, Y) is a discrete Schur-constant vector and the Laplace transform of the marginal X is known, one can easily compute the joint Laplace transform $\mathbb{E}[e^{-a\,X-b\,Y}]$. Indeed, denoting $\mathbb{E}[e^{-a\,X}] =: L(a)$ and letting b tend to zero in (7), we get

$$\int_0^a \mathbb{E}[e^{-t(W+1)}]dt = (1 - e^{-a})L(a),$$

i.e. $\int_b^a \mathbb{E}[e^{-t(W+1)}]dt = (1 - e^{-a})L(a) - (1 - e^{-b})L(b)$ for $a \neq b$.

Thus, from (7) we obtain

$$\mathbb{E}[e^{-aX-bY}] = \frac{(1 - e^{-a})L(a) - (1 - e^{-b})L(b)}{e^{-b} - e^{-a}}.$$

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