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Environmental Quality along the Process of Economic Growth: A Theoretical Reappraisal☆

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Abstract

This paper studies the dual interaction between economic growth and environmental quality in an endogenous growth model. We exhibit multiple equilibria and complex local and global dynamics, resulting in potential indeterminacy, hysteresis effects, or long-lasting growth and environmental cycles. From a policy perspective, we reveal that changes in the environmental policy should be handled with care, as they may generate aggregate instability or condemn the economy to an environmental poverty trap associated with a possible irreversibility of environmental degradation. Lastly, our analysis provides a reassessment of pollution taxes, which are found to improve long-run economic growth when the model is well-determined, but reduce it in the presence of indeterminacy.

Keywords: Economic growth; Environmental Quality; Pollution; Poverty Traps; Endogenous Cycles; Pollution Taxes.

JEL Codes: E32, O44, Q50.

1. Introduction

The interaction between economic development and the environment is a long-standing research topic. Despite a considerable and growing empirical literature, there is no conclusive evidence on the shape of the growth-environment relationship (see Pérez-Suárez and López-Menéndez, 2015). Accordingly, the link between polluting emissions and per capita income takes the form of an inverted U-shaped curve, a continuously increasing curve, or non-monotonic shapes such as cyclical patterns.[1]

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[1]In their meta-analysis, Pérez-Suárez and López-Menéndez (2015) find that an inverted U-shaped is supported by 55.7% of the studies. This characterizes the well-known "Environmental Kuznets Curve" (Grossman and Krueger, 1995), suggesting that, at high levels of income, economic growth and pollution emissions are negatively-linked. Instead, 11.5% of the studies report increasing trends between emissions

This lack of consensus can be related to possible statistical pitfalls (see the survey of Stern, 2006), but also to fragile theoretical foundations. In empirical studies, the relationship between economic growth and the environment is often derived as a reduced-form equation with an unidirectional causality running from income to environmental quality. However, natural resources are essential inputs for production in many sectors, such that environmental degradation may reversely affect the process of economic growth, as notably pointed out by Arrow et al. (1995).

From a theoretical standpoint, a number of growth models integrate an environmental module, but without being able to reproduce the diversity of long-run income-environment relationships highlighted by the empirical literature. On the one hand, existing exogenous growth setups, which find multiple (i.e. two) steady states and various environmental and income trajectories, do not—by construction—take into account long-run growth and are based solely on the negative externality of polluting emissions or environmental degradation on households' preferences (see e.g. Antoci et al., 2011, 2021; Bosi and Desmarchelier, 2018a,b). However, it is essential to study the link between growth and the environment from a long-term perspective, since environmental degradation and the increase in polluting emissions are long-term phenomena. On the other hand, existing endogenous growth models generally focus on a single steady state with determined transitional dynamics (see e.g. Bovenberg and Smulders, 1995; Chen et al., 2003; Fullerton and Kim, 2008). A notable exception is Itaya (2008), who shows that (local) indeterminacy can appear, leading to multiple expectation-driven transition paths. However, because of the uniqueness of the steady state, Itaya's model remains silent regarding the possibility of several long-run relationships between economic growth and the environment.

The goal of this paper is precisely to investigate the dual interaction between economic growth and environmental quality in a simple endogenous growth model that can capture the variety of growth-environment relationships identified in the empirical literature, through illustrating complex short- and long-run dynamics. In the spirit of Tahvonen and Kuuluvainen (1991) and Bovenberg and Smulders (1995), we model environmental quality as a stock of natural capital that accumulates due to the Nature's regenerative capacity and depreciates because of pollution, which is seen as the extractive use of natural resources for productive services.

The key elements of our model are the following. First, the natural capital exerts a positive externality on the total factor productivity as in Bovenberg and Smulders (1995) and Fullerton and Kim (2008). Second, in contrast to these authors, we consider an endogenous labor supply. This feature is of major importance. Indeed, a unique well-

and the per capita income, in reference to an "environmental logistic curve". Lastly, 20% of studies suggest that the relationship could be characterized by stronger nonlinearities in the form of cyclical patterns (with no clear pattern for the remaining 12.8% of studies).

defined steady state is obtained in the two aforementioned papers, but at the price of the restrictive hypothesis of fixed exogenous labor supply. With an elastic labor supply, on the contrary, our model exhibits the possibility of multiple steady states. Third, thanks to investment in abatement knowledge (provided by public spending), economic growth and environmental preservation can be compatible in the long run.[2] For example, if pollution damages the very engine of growth, namely the human capital accumulation, a cleaner environment leads to a higher rate of growth by increasing human capital. In this way, abatement can overcome the conflict between environment and economic growth (Van Ewijk and Van Wijnbergen, 1995).

Accounting for the dual interaction between economic growth and environmental quality yields the following results.

(1) There is multiplicity of long-run solutions, as three steady states can appear: a "dark" equilibrium characterized by low economic growth and environmental quality; a "green" equilibrium with high growth and natural capital; and an "intermediate" equilibrium. Intuitively, this multiplicity comes from the reciprocal interaction between economic growth and the environment: low environmental quality generates low factor productivity that impedes economic growth, and, conversely, low growth does not ensure enough public spending for abatement, which leads to high emission flows. These mechanisms, which reverse in the case of a high environmental quality that leads to high economic growth, generate multiple self-fulfilling steady states. Moreover, since the dark and the green equilibria are saddle points, and the intermediate equilibrium can be stable or unstable, our model displays both local and global indeterminacy.[3] Consequently, depending on the initial environmental quality and households' expectations, the economy can be trapped in the dark steady state (i.e. an environmental-poverty trap), converge towards the green steady state, or experience oscillating trajectories through endogenous limit cycles. As such, by unveiling a large variety of paths for environmental quality and economic growth, our analysis contributes to the understanding of cross-country heterogeneities reported by empirical studies.[4]

(2) The growth-environment endogenous interaction yields a hysteresis effect, which may plunge the economy towards an irreversible environmental-poverty trap. Consequently, our analysis reveals a crucial role for environmental taxes: to avoid such an undesirable feature, environmental taxes should not be too low. However, even when

[2] Bovenberg and Smulders (1995) discuss the technological conditions under which continued growth is compatible with sustainable regeneration of environmental resources.

[3] Starting from given initial conditions, local indeterminacy refers to an infinity of possible paths towards a given equilibrium, and global indeterminacy refers to several possible paths towards different equilibria.

[4] Our model displays an Environmental Kuznets Curve (EKC) as a particular case; however, the EKC appears as a long-lasting feature and our policy message differs from existing interpretations, as we will see.

environmental tax are raised to average levels, the economy displays fluctuations both in the short and the long run. A more appealing scenario is observed when taxes are large enough: in this case, high economic growth rates and good environmental quality can go hand in hand. Naturally, imposing—from a policy standpoint—such a "big push" in environmental taxes is most likely not an easy task, all the more in the post-Covid times characterized by a fairly high fiscal pressure that adds to the well-known issues inherent to the management of global environmental goods.

(3) Finally, we provide a new policy perspective over the impact of indeterminacy on economic growth and environmental quality. Relative to the tradeoff between determinacy and high growth revealed by existing work, our findings are more optimistic: environmental taxes support economic growth when the equilibrium is determinate and unique. Instead, we find that it is because of too low levels of environmental taxes that the economy may experience the undesirable feature of indeterminacy.

From an economic standpoint, our results emerge thanks to two ingredients: (i) the externality of natural capital on total factor productivity, and (ii) the endogenous labor supply. The intuition is as follows. Assume that households initially expect high environmental quality in the long run. Due to the externality, this implies that output and the expected net return of capital will be high. Thus, at the initial time, households increase their saving, which reduces the initial consumption-to-capital ratio. With endogenous labor supply, a low consumption ratio increases the marginal gain of hours worked such that households are induced to devote more time to working activities. As a result, output will be high, generating large abatement public spending that will increase environmental quality in the future: the economy evolves on the trajectory that converges towards the green steady state. Since—by the same logic—low expected environmental quality is self-fulfilling and makes the economy converge towards the dark steady state, multiple (self-fulfilling) equilibria coexist, with crucial implications for the dynamics of the economy.

In sum, accounting for the endogenous forces related to the dual interaction between economic growth and environmental quality can lead—even in our simple setup—to unexpected consequences, including multiplicity, local & global indeterminacy, and oscillating dynamics, which exacerbate the role of households' expectations and of the environmental policies. Quantitatively, a calibration exercise based on industrialized countries shows that these different patterns occur for empirically-plausible values of the parameters.

The paper is organized as follows. Section 2 discusses the novelty of our analysis with respect to existing studies, section 3 presents the model, section 4 defines the equilibria and computes the long-run solutions, section 5 provides a quantitative assessment of the model, section 6 studies global dynamics and environmental policies, and section 7 delivers some concluding remarks.

2. Related literature

Our analysis is related to growth models incorporating environmental externalities that exhibit indeterminacy. In exogenous growth setups, the main channel of indeterminacy is the environmental externality in the households' utility function (see e.g. Antoci et al., 2011, 2021; Fernández et al., 2012; Bosi and Desmarchelier, 2018a,b).[5] For example, Bosi and Desmarchelier (2018b) and Antoci et al. (2021) show that two reachable steady states are present if consumption and natural resource are complements in households' preferences. This channel equally matters in endogenous growth setups. Itaya (2008) finds that negative pollution externalities in households' preferences can make the steady-state (locally) indeterminate.

Compared to these papers, in our setup local and global indeterminacy appear without the need of externalities in the households' utility function, which is assumed to be separable between consumption and leisure. Indeterminacy is driven by the productive externality and the environmental policy through government's abatement spending. In our model, high expected pollution leads to a low expected return on capital and low economic growth. As a result, households reduce their savings, which reduces future output and fiscal resources available for abatement expenditures. This leads to high future (self-fulfilling) pollution emissions. In this way, our results amend Itaya (2008)'s statement that "*the presence of public abatement activities makes it more difficult for indeterminacy to emerge.*" Specifically, we show that indeterminacy (both local and global) can arise in models *à la* Bovenberg and Smulders (1995) and Fullerton and Kim (2008) with public abatement activities.

Our model also contributes to a strand of research that focuses on irreversibility in the pollution-growth nexus. Most growth models—including ours—assume that Nature assimilates pollutants at a constant rate. Several authors (see in particular Dasgupta, 1982) challenge this view and consider that high pollution may drastically alter Nature's assimilation capacity. In this vein, Tahvonen and Salo (1996), Toman and Withagen (2000), and Prieur (2009), among others, use a decay function with an exogenous critical pollution level beyond which pollution accumulation is irreversible; in this case, the assimilative capacity of the environment may eventually be exhausted by pollution accumulation. Additional work (see e.g. Mäler et al., 2003; Heijdra and Heijnen, 2013) considers a smooth assimilation function according to a shallow-lake dynamic. Such functions generate a non-convexity in the optimization problem that generally produces multiple equilibria, associated (or not) with irreversible pollution. In contrast, in our model multiple equilibria and hysteresis arise without the need of an exogenous critical pollution level in the decay function, or complex shallow-lake dynamics: irreversibility in

[5] Alternatively, other channels have been studied, including those based on productive public spending *à la* Barro (Pérez and Ruiz, 2007), or endogenous discounting (Yanase, 2011).

the growth-environment relationship arises with a smooth quadratic function of Nature, as in traditional environmental growth models (see e.g. Bovenberg and Smulders, 1995; Smulders, 2000; or Fullerton and Kim, 2008).

On the policy side, our work relates to several endogenous growth models showing that environmental fiscal policy can promote economic growth via two mechanisms. The first is based on productive externalities: environmental taxation improves the quality of the environment, which positively affects total factor productivity, thereby promoting economic growth (see e.g. Van Ewijk and Van Wijnbergen, 1995; Bovenberg and Smulders, 1995; Bovenberg and de Mooij, 1997). The second mechanism relies on endogenous leisure-labor choices (see e.g. Hettich, 1998; Chen et al., 2003): a tax-financed public abatement expenditure reduces the capital stock in the private sector, which in turn induces households to reduce their consumption. Since consumption and leisure are complements in the utility function, households increase their labor supply, which boosts output and economic growth.

In this vein, Itaya (2008) shows that an environmental tax stimulates long-run economic growth in an endogenous growth model with elastic labor supply provided that the (unique) long-run equilibrium is locally indeterminate. Our analysis revisits this result by combining the two aforementioned mechanisms. In our setup, the environmental tax improves long-run economic growth when the steady state is (locally and globally) well-determined and can reduce it in the presence of indeterminacy. The economic mechanism is related to the fact that a rise in the environmental tax implies two conflicting effects. First, the price of pollutants increases, so that the firm will use fewer polluting input, which improves environmental quality. Thank to lower emissions, total factor productivity increases, thereby boosting economic growth. Second, a higher environmental tax reduces the return to labor, so that households reduce their labor supply, which has a negative effect on output and economic growth. We show that the first effect always outweighs the second in the long run when the steady state is well-determined; but the opposite can be true when the economy is subject to multiple equilibria and indeterminacy. Therefore, contrary to Itaya (2008), indeterminacy should not be seen as the price to be paid for the environmental policy to be pro-growth, but rather as an undesirable property associated with a too low environmental tax.

3. The model

We consider a closed economy populated by a continuum of representative individuals whose total measure is one, and a government. Each representative agent consists of a household and a competitive firm. All agents are infinitely-lived and have perfect foresights. For each variable, we denote individual quantities by lower case letters (x), and aggregate quantities by corresponding upper case letters (X), with $x = X$ in equilibrium since the continuum of agents has unit measure.

3.1. Environment

Environment is modeled as a renewable resource. By closely following Tahvonen and Kuuluvainen (1991) and Bovenberg and Smulders (1995), the environmental quality or the stock of natural capital (Q_t) accumulates due to the regenerative capacity of Nature and depreciates due to pollution, namely

$$\dot{Q}_t = E(Q_t) - P_t, \tag{1}$$

where a dot over a variable represents its time derivative.

Pollution (P_t) comes from the extractive use of natural resources because production requires pollutant inputs (e.g. pesticides in agriculture, fossil fuels resulting in emissions of carbon, etc.). The mapping $E(\cdot)$ is an environmental regeneration function that reflects the capacity of the environment to absorb pollution. Following Bovenberg and Smulders (1995), Smulders (2000), and Fullerton and Kim (2008), we consider the standard form

$$E(Q) = vQ(\bar{Q} - Q), \tag{2}$$

where $v > 0$ is a scale parameter, and $\bar{Q}$ is the virgin state. This virgin state is the maximal stock of natural resources that can be kept intact by natural regeneration, i.e. $E(\bar{Q}) = P = 0$. However, as we will see, the sustainable steady state ($\dot{Q}_t = 0$) occurs for a strictly positive pollution level in our model, namely $P^* = E(Q^*) > 0$, such as $E(Q^*)$ measures the absorption capacity of the environment. From (2), the absorption capacity initially increases with the environmental quality, but then decreases as the environment is getting closer to the virgin state.

3.2. Firms

Output of the representative firm (y_t) is produced using three inputs: private man-made non-polluting capital (k_t), human capital (h_t), and a polluting input (z_t) that reflects the effective input of "harvested" environmental resources. Our choice for a Cobb-Douglas production function linking the three inputs with the output is rooted in the existent literature, at least for the following reasons.

First, the Cobb-Douglas technology is standard in the macroeconomic literature, and particularly in environmental growth models (see the seminal contributions of Solow, 1974; Stiglitz, 1974), because it can explain the stylized fact that income shares in western economies have been roughly constant during the postwar period. Capitalizing on early statistical work (see e.g. Zellner et al., 1966; Goldberger, 1968), many authors have subsequently shown that the Cobb-Douglas specifications provides a good fit to the data.

Moreover, instead of a Cobb-Douglas function, we may have used a production function with a constant elasticity of substitution (CES). However, it is now rather well recognized that a CES production function generates indeterminacy and multiplicity in growth models (see e.g. Nishimura and Venditti, 2004; Wong and Yip, 2010). In our setup, thanks to the Cobb-Douglas production function, we explicitly neutralize this source of indeterminacy in order to focus on a channel based on two ingredients: the externality of natural capital on total factor productivity and the endogenous labor supply.

Lastly, the consensus in environmental growth models, at least since the seminal work of Dasgupta and Heal (1974), Solow (1974), and Stiglitz (1974), is that output is equal to zero in the absence of pollution (in our model, $p_t = 0 \Rightarrow y_t = 0$), namely the environmental quality is called essential. As pointed out by Solow (1974), the Cobb-Douglas production function displays this important property, which explains why subsequent studies based on growth models with a resource as a stock variable draw upon such a production technology.

Consequently, production writes as

$$y_t = A_t k_t^{\alpha} h_t^{\beta} z_t^{\phi}, \tag{3}$$

where A_t is a productivity factor. Parameters $\alpha \in (0, 1)$, $\beta \in (0, 1)$, and $\phi \in (0, \alpha)$ are the elasticity of output to private capital, human capital, and polluting input, respectively. The condition $\alpha > \phi$ ensures normal factor demand functions.[6]

Four assumptions are made.

First, following Romer (1986), human capital is produced both by raw labor (or training activity) l_t, and by the economy-wide stock of capital K_t, namely $h_t = l_t K_t$.

Second, as Gradus and Smulders (1993), Bovenberg and Smulders (1995), and Fullerton and Kim (2008), we assume that the stock of environmental quality (or natural capital Q_t) exerts a positive externality through the total factor productivity, namely $A_t = A(Q_t)$ where $A' > 0$. This assumption is supported by empirical evidence suggesting that the environmental quality enhances the productivity of inputs by providing non-extractive services (see e.g. Van Ewijk and Van Wijnbergen, 1995).[7] Indeed, clean soil, air, or water provide productive services to economic activities by improving workers' health and productivity, for example. In what follows we use a simple CES function, namely $A(Q_t) = AQ_t^{\delta}$, where $A > 0$ is a scale parameter and $\delta \in (0, 1)$ is the elasticity of productivity to the natural capital.

Third, following Bovenberg and de Mooij (1997) and Fullerton and Kim (2008), the input z_t depicts the "effective emissions" that can be provided either by the use of pollutants

[6] $\alpha > \phi$ is a (unnecessary) sufficient condition to ensure that global externalities are small enough for the factor demand to negatively depend on prices.

[7] According to the World Bank (1992) report, the main channel through which pollution affects productivity is based on the degradation of human health. In addition, low environmental quality can also cause important capital stock losses, as a consequence of extreme meteorological phenomena. Barbera and McConnell (1986), Xepapadeas et al. (2007), and OECD (2015) discuss the importance of the link between the environment and factor productivity.

(p_t) or through the stock of available abatement knowledge (i.e. pollution-augmenting technological progress). Then, the productive content of pollution depends on the available public knowledge about pollution-augmenting (or abatement) techniques, which is supposed to result from the government's effort $z_t = G_t p_t$, where G_t defines abatement public spending (for simplicity, we neglect other forms of government expenditure).[8] Thus, the firm can generate the same output by using intensively the natural resource if the abatement technology is inefficient, or conversely by using few natural resources if the pollution-augmenting technical progress is efficient.

Fourth, we consider that $\phi = 1 - \alpha - \beta$ in order for the production function (3) to exhibit constant returns-to-scale relative to private factors (rival inputs). As we will see, at the aggregate level the knowledge and public spending externalities allow obtaining an endogenous growth path because the social return of capital is constant.

In a perfect-competition decentralized economy, each firm chooses private factors (k_t, l_t, and p_t) to maximize its profit

$$\Pi_t = y_t - r_t k_t - w_t l_t - \pi_t p_t,$$

where w_t is the hourly wage, r_t the real interest rate, and π_t the environmental tax on polluting input; this tax can be assimilated to the price of a permit to pollute. The first-order conditions ensure that the price of factors is given by their marginal returns in production

$$r_t = \alpha \frac{y_t}{k_t}, \tag{4}$$

$$w_t = \beta \frac{y_t}{l_t}, \tag{5}$$

$$\pi_t = (1 - \alpha - \beta) \frac{y_t}{p_t}. \tag{6}$$

3.3. Preferences

The representative household starts at the initial period with a (predetermined) positive stock of capital (k_0), and chooses the path of consumption $\{c_t\}_{t\geq 0}$, hours worked $\{l_t\}_{t\geq 0}$, and capital $\{k_t\}_{t>0}$, such as to maximize the present discounted value of its

[8]The methodological question of how to treat pollution in production has been long debated in the literature (see e.g. Harford and Karp, 1983; Helfand, 1991), and the consensus that emerges is to treat pollution as a by-product of the production process. More specifically, according to Harford and Karp (1983), pollution may be either a by-product of output itself (as in e.g. Ono, 2003), or a by-product of a "dirty" input. Our specification clearly belongs the second proposal. Formally, the Harford and Karp's production writes $y_t = f(k, z_t)$, where z_t is the "dirty input" (i.e. the fuel), and pollution is a positive function of the fuel, namely $p_t = g(z_t)$, with $g' > 0$. We use a similar specification in our setup: indeed, z_t is the "dirty input", and relation $z_t = G_t p_t$ amounts to $p_t = z_t / G_t$.

lifetime utility

$$U = \int_0^{\infty} e^{-\rho t} u(c_t, l_t)\, dt, \tag{7}$$

where $0 < \rho \ll 1$ the subjective discount rate. As previously highlighted, the instantaneous utility is assumed to be separable,[9] namely

$$u(c_t, l_t) = \log(c_t) - l_t. \tag{8}$$

Households use labor income ($w_t l_t$) and capital revenues ($r_t k_t$), to consume (c_t) and invest ($\dot{k}_t$). They pay taxes on the labor income ($\tau_t w_t l_t$, where τ_t is the labor income tax rate), hence the following budget constraint

$$\dot{k}_t = r_t k_t + (1 - \tau_t) w_t l_t - c_t. \tag{9}$$

The solution of the household's programme (see *Appendix A*) gives rise to the Euler rule

$$\frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{10}$$

and to the static relation

$$1 = (1 - \tau_t) w_t / c_t. \tag{11}$$

Eq. (11) states that at each period t the marginal gain of hours worked (i.e. the net real wage $(1 - \tau_t) w_t$, expressed in terms of marginal utility of consumption $1/c_t$) just equals the marginal cost (which is equal to 1). Finally, the optimal path of consumption has to verify the usual transversality condition

$$\lim_{t \to +\infty} \{\exp(-\rho t) k_t / c_t\} = 0. \tag{12}$$

3.4. The government

The government uses taxes on labor income ($\tau_t w_t L_t$) and on polluting activities ($\pi_t P_t$) to provide abatement expenditure (G_t). It balances its budget at each period t, hence the following budget constraint

$$\tau_t w_t L_t + \pi_t P_t = G_t. \tag{13}$$

As this is the case in macroeconomic models with fiscal policy, one of the fiscal variables must adjust to fulfill the government's budget constraint. In our setup, we follow

[9] In the related literature, indeterminacy often relies upon a non-separable utility function (see e.g. Fernández et al., 2012; Bosi et al., 2015; Bosi and Desmarchelier, 2018a,b). We choose a separable utility function because we want to produce a new channel of indeterminacy, in which the endogenous labor supply plays a role, but without resorting to specific restrictions on the utility function.

the seminal contribution of Schmitt-Grohé and Uribe (1997) and subsequent prominent contributions (see e.g. Giannitsarou, 2007), and assume that the tax rate on income (τ_t) provides the adjustments in the government's budget constraint (13). Consequently, it comes that the paths of both public abatement spending (G_t) and the pollution tax rate (π_t) must be specified in order to allow the model to be closed. To keep the model simple, we assume that the government chooses a constant ratio of public spending-to-GDP, namely $g = G_t/Y_t$, with $g \in (0, 1)$, and a constant ratio of pollution tax-to-capital $\pi_k = \pi_t/K_t$. Equilibrium exists under the mild condition $g < 1 - \alpha$, which we assume throughout the paper.[10]

Notice that these assumptions are supported by existing studies. First, although the ratio of abatement to GDP is fixed, the level of abatement public spending G_t varies during the transitional dynamics, and also in the long run when they grow with the constant endogenous growth rate of the economy. For comparison, Schmitt-Grohé and Uribe (1997) and Giannitsarou (2007) assume that the level of public spending is exogenous. Considering in our model an exogenous ratio public spending-to-GDP ratio, under which the level of public spending evolves endogenously, is therefore less restricting than assuming a constant public spending level as this is the case in their models. Second, to close the model the government must fix the path of the environmental tax rate (π_t). However, as π_t is a growing variable, to obtain an endogenous balanced-growth path—in which all growing variables grow at the same rate—π_t should grow along this path at the same rate as all the endogenous variables, including e.g. output or capital. To this end, we follow Fullerton and Kim (2008) by considering a constant environmental tax rate per unit of capital, namely $\pi_k := \pi_t/K_t$. Similar to the dynamics of abatement public spending, the environmental tax rate π_t varies both during the transitional dynamics and also in the long run. To sum up, these assumptions—that are consistent with existing studies—allow closing our model and ensuring the existence of a balanced-growth path.

4. Equilibrium

We focus on the equilibrium in a decentralized economy in which all household-firm units behave similarly.

Definition 1.
A competitive equilibrium is a path $\{c_t, C_t, l_t, L_t, k_t, K_t, y_t, Y_t, r_t, \tau_t, w_t, \pi_t, p_t, P_t, Q_t, G_t\}_0^\infty$ that solves Eqs. (1), (2), (3), (4), (5), (6), (10), (11) and (13), the relations $y_t = Y_t$, $k_t = K_t$, $l_t = L_t$, $y_t = Y_t$, $p_t = P_t$, $G_t = gY_t$, satisfies the goods market equilibrium $\dot{K}_t = Y_t - C_t - G_t$, and verifies the transversality condition (12). □

[10] Since α is around 0.3 for a plausible calibration, this corresponds to the mild condition that government abatement expenditure are below 70% of GDP.

From the aggregate perspective, using Eq. (11) the labor supply is $L_t = (1-\tau_t)\beta Y_t/C_t$. Using the government's budget constraint (13), we obtain in equilibrium $(1-\tau_t)\beta = 1-g-\alpha$, and further

$$L_t = (1-g-\alpha)\frac{Y_t}{C_t}. \tag{14}$$

Thus, from (6), the level of pollutants is

$$P_t = \left(\frac{1-\alpha-\beta}{\pi_t}\right) Y_t, \tag{15}$$

and further $Z_t = G_t P_t = g(1-\alpha-\beta)Y_t^2/\pi_t$. From Eq. (3), the aggregate production function writes

$$Y_t = A(Q_t)K_t^{\alpha+\beta}\left[(1-g-\alpha)\frac{Y_t}{C_t}\right]^{\beta} Z_t^{1-\alpha-\beta}. \tag{16}$$

Definition 2. A steady state i is a competitive equilibrium where all growing variables grow at the common (endogenous) rate γ^i, and the environmental quality is constant. At this steady state i, the economy is thus characterized by a balanced-growth path (BGP), namely $\gamma^i := \dot{C}_t/C_t = \dot{K}_t/K_t = \dot{Y}_t/Y_t = \dot{\pi}_t/\pi_t$, and $\dot{Q} = 0$. □

As previously discussed, to obtain an endogenous BGP the environmental tax (π_t) should grow at the same rate as output along that path; consistent with Fullerton and Kim (2008), we consider a constant environmental tax rate per unit of capital, i.e. $\pi_k := \pi_t/K_t$. Similarly, to obtain long-run stationary ratios, we deflate all growing variables by the capital stock (we henceforth omit time indexes), namely: $c_k := C/K$ and $y_k := Y/K$.

Using (15) and (16), we find

$$y_k = \lambda\left[A(Q)c_k^{-\beta}\right]^{1/\varepsilon} =: y_k(Q, c_k), \tag{17}$$

where $\lambda := \left[(1-g-\alpha)^{\beta}\left(g(1-\alpha-\beta)/\pi_k\right)^{1-\alpha-\beta}\right]^{1/\varepsilon} > 0$, and $\varepsilon := 1-\beta-2(1-\alpha-\beta) > 0$ because $\alpha > 1-\alpha-\beta$.

Eq. (17) depicts an inverse relationship between the consumption and the output ratios, whose intuition comes from the labor market equilibrium (11). Following an increase in the consumption ratio, the marginal utility of consumption decreases, inducing households to substitute leisure for working hours (recall that leisure and consumption are complement in equilibrium). As a result, the equilibrium labor supply and output are reduced.

We obtain the reduced-form of the model using Eqs. (1), (4), (10), (15), and (17)

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = (\alpha + g - 1)y_k(Q, c_k) + c_k - \rho, & (a) \\ \\ \dot{Q} = E(Q) - \dfrac{(1-\alpha-\beta)y_k(Q, c_k)}{\pi_k}. & (b) \end{cases} \tag{18}$$

Steady-states solutions are computed by setting $\dot{c}_k = \dot{Q} = 0$ in system (18).

4.1. Existence

The following proposition establishes the existence of three regimes according to the value of the discount rate ρ.

Proposition 1. *If $\varepsilon < \delta$, there are two critical values of the discount rate (ρ_1 and ρ_2, with $0 < \rho_1 < \rho_2 \ll 1$), such that the long-run equilibria are characterized by the following regimes:*

- *Regime $\mathcal{R}_1$: $\rho < \rho_1$. There is only one steady state: a "dark" equilibrium (point D), characterized by low economic growth and environmental quality.*
- *Regime $\mathcal{R}_2$: $\rho > \rho_2$. There is only one steady state: a "green" equilibrium (point G), characterized by high economic growth and environmental quality.*
- *Regime $\mathcal{R}_3$: $\rho_1 < \rho < \rho_2$. There are three steady states: the dark equilibrium (D), the green equilibrium (G), and an intermediate equilibrium (point M).*

Proof. The steady states are given by the crossing-points of two relations between c_k and Q that are depicted in Figure 1. The first relation is the $\dot{c}_k = 0$ locus, denoted by $Q = \Psi(c_k)$. Based on Eq. (18a), this relation describes an increasing continuous curve in the (c_k, Q)-plan. The second relation is the $\dot{Q} = 0$ locus, denoted by $c_k = \Phi(Q)$. Based on Eq. (18b), if $\varepsilon < \delta$, this relation also describes an increasing continuous curve, with an inflexion-point reflecting a change of concavity (see *Appendix B*).

Notice that the discount rate ρ is present only in the relationship $\Psi(\cdot)$. Thus, it is convenient to characterise the number of steady states by varying this parameter. As ρ gets higher, the curve $Q = \Psi(c_k)$ translates upwards: any increase in the discount rate reduces economic growth in the steady state, and leads to a higher consumption ratio for a given environmental quality. Consequently, given the shape of $\Phi(\cdot)$, there are two critical values of ρ, denoted by ρ_1 and ρ_2, corresponding to the two tangency points between $\Phi(\cdot)$ and $\Psi(\cdot)$. Hence, we obtain three configurations: (i) if $\rho < \rho_1$ there is only one crossing-point, the dark equilibrium (point D, in Figure 1); (ii) if $\rho > \rho_2$ there is only one crossing-point, the green equilibrium (point G), and (iii) if $\rho_1 < \rho < \rho_2$ there are three crossing-points: D, G, and an intermediate equilibrium (point M). Consequently, for $\rho_1 < \rho < \rho_2$ there is a corridor that produces multiplicity (see Figure 1). ■

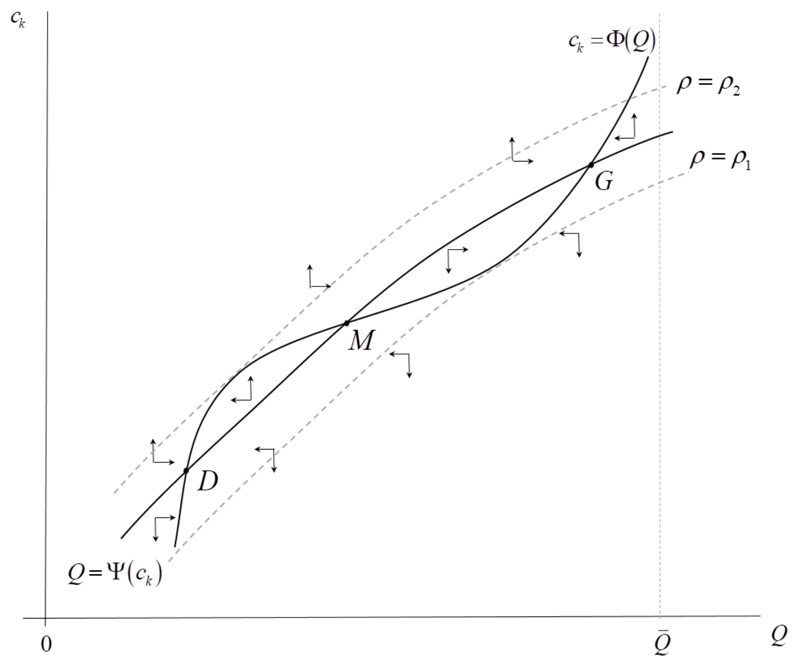


Figure 1: The phase portrait and multiplicity of steady states

Fundamentally, the presence of one or three equilibria depends on the value of the discount rate ρ. At the steady state, the BGP requires that $\gamma_c = \gamma_k$. If the discount rate rises, the incentive to save is reduced, and γ_c falls. Given γ_k, to restore the incentive to save, the productivity of capital must rise; hence, the environmental quality must improve. Therefore, for high values of ρ (i.e. $\rho > \rho_2$), the only steady state that appears is associated with high environmental quality (green equilibrium). Conversely, for low values of ρ (i.e. $\rho < \rho_1$), only the steady state with low environmental quality arises (dark equilibrium). Lastly, for intermediate values of the discount rate, the three steady states can be obtained.

4.2. Some intuition

The multiplicity that we obtain is caused by the interaction between the two increasing relationships linking environmental quality and the consumption ratio.

The first relationship arises from the presence of balanced-growth in the long run (namely $\dot{K}/K = \dot{C}/C = \gamma$). According to the Euler equation (10) and the goods market equilibrium, this condition amounts to $(1-g)y_k - c_k = \alpha y_k - \rho$, or

$$c_k - \rho = (1 - g - \alpha) y_k(c_k, Q). \tag{19}$$

As we have seen, the environmental quality generates a positive externality on the output ratio y_k. Then, the relation between c_k and Q crucially depends on the sign of $1 - g - \alpha$. This condition is intuitive. As the environmental quality increases, the growth rates of consumption ($\dot{C}/C$) and private capital ($\dot{K}/K$) increase, through a positive effect on the output ratio. The impact of the output ratio on the growth rate of consumption depends on the return of capital (α) in Eq. (10), while its impact on the growth rate of capital depends on public spending ($1-g$). As $\alpha < 1-g$, the consumption ratio c_k must rise in response to an increase in Q, in order to restore the equality $\dot{K}/K = \dot{C}/C$ along the BGP; hence, the increasing function $\Psi(Q)$.

The second relationship is related to the environmental regeneration function. In the steady state, sustainable growth implies that pollution emissions are absorbed by the process of ecological regeneration, namely $P = E(Q)$. Pollution emissions positively depend on the output ratio (y_k), which is increasing in the environmental quality (Q) and decreasing in the consumption ratio (c_k). Thus, the link between the consumption ratio and environmental quality crucially depends on the relative effect of Q on pollution emissions and on the environment absorption capacity. If $\varepsilon < \delta$, the former effect always dominates, and Q and c_k are positively associated; hence, the increasing function $\Phi(Q)$.[11]

Multiple equilibria results from the reciprocal interaction between economic growth and the environment. A low environmental quality generates low factor productivity that impedes economic growth. Conversely, low economic growth means low public spending for abatement, and leads to high emission flows relative to the natural regeneration capacity. These mechanisms reverse in the case of high environmental quality, and lead to high economic growth. Therefore, multiple self-fulfilling steady states coexist, as this is the case in previous environmental growth models (e.g. Prieur, 2009; Bosi and Desmarchelier, 2018b).

However, the originality of our setup is that three possible steady states can emerge. This novelty comes from the derivative of the $\Phi(Q)$ curve that, provided that $\varepsilon < \delta$, is positive and successively decreasing-increasing. Indeed, along the stationary locus of Q (the $\Phi(Q)$ curve), the effect of a change in Q differs depending on the level of environmental quality. If Q is low, an increase in environmental quality has a major impact on the total factor productivity. If Q is high, an increase in environmental quality sharply reduces the natural absorption capacity (because $E'(Q) < 0$ at high values of Q). In both cases, pollution emissions strongly increase, so that the consumption ratio (c_k) must substantially increase to restore the environmental equilibrium. In contrast, for intermediate values of Q, the effect of an increase in environmental quality on the equilibrium is moderate, because both pollution and natural absorption capacity widen (indeed, $E'(Q) > 0$ at low values of Q); thus, the consumption ratio does not need to vary much to restore the equilibrium. This explains the form of the $\dot{Q} = 0$ locus.

Whether the economy will be dragged down towards the dark steady state, which can

[11] This is the most general case, because it allows exhibiting three steady states. If $\delta < \varepsilon$, $\Phi(Q)$ is depicted by a U-curve. Although the model gives rise to only two steady states, there is still multiplicity and indeterminacy.

be assimilated to an environmental-poverty trap, or will enjoy better environment and growth trajectories in steady states M or G, will depend on the initial environmental stock Q_0 and on households' expectations (since c_{k0} is a jumpable variable). This feature exemplifies the "history versus expectation" scenario of Matsuyama (1991) and Krugman (1991), and opens the door for hysteresis phenomena as we will see.

4.3. Local dynamics

The local dynamics is based on the linearization in the neighborhood of steady state i, $i \in \{D, M, G\}$. The system (18) behaves according to $(\dot{c}_k, \dot{Q}) = \mathbf{J}^i(c_k - c_k^i, Q - Q^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_{k0}) and one pre-determined variable (the environmental quality Q_0). Hence, for steady state i to be well-determined, the Jacobian matrix must contain two opposite-sign eigenvalues. The following proposition establishes the topological behaviour of each steady state.

Proposition 2. *D and G are locally determinate (saddle points), and M is locally indeterminate (stable) or unstable.*

Proof. See *Appendix C*. ■

Since the stability of the steady state M switches (i.e. moving from stable to unstable, or vice versa), a periodic solution can emerge through a Hopf bifurcation, as established in the following corollary.

Corollary 1. *In the neighborhood of $M(c_k^M, Q_k^M)$, there is a critical value $\pi_k^h > 0$, such that a Hopf bifurcation emerges at $\pi_k = \pi_k^h$.*

Proof. See *Appendix C*. ■

Proposition 2 shows that the three steady states can be relevant. Furthermore, from corollary 1, the dynamics in the vicinity of the steady state M can exhibit cyclical properties. It results that further investigations are needed to establish the short- and long-run behavior of the economy from a global dynamics perspective. Beforehand, the following section performs a quantitative analysis showing that the different regimes and the Hopf bifurcation arise for realistic parameters' and variables' values.

5. A quantitative assessment

Although our results on local & global indeterminacy and the existence of a Hopf bifurcation are established analytically, it is not unnecessary to assess their quantitative relevance for empirically-plausible values of parameters. Furthermore, the numerical results will allow us to detect the presence of various type of bifurcations. Studying bifurcations (namely, sudden shifts of the existence and/or the stability of a steady state following a small change of one or several parameters) is of particular importance from an economic policy standpoint, because it enables us to determine the range of parameters such that one or multiple steady states exist and the way they can be reached. Thus, our quantitative results are not only valid for one given calibration of parameters; but the existence, values, and properties of the steady states can be assessed under continuous changes of the different parameters. By varying continuously one (or several) parameters, the topological behaviour of the model (i.e. the number of steady states or their stability) may suddenly shift: these critical values of parameter define the bifurcations.[12]

We look for bifurcations in the vicinity of the following benchmark calibration of the model. Regarding the households' behavior, the discount rate will be scanned over the range $(0.001, 0.03)$, with $\rho = 0.01$ in the baseline calibration—corresponding to the long-run value of the risk-free interest rate used e.g. in Stern (2006).

Regarding the technology, we fix $A = 0.6$ to obtain realistic rates of economic growth; the elasticity of output to physical capital is set to represent the capital-share in GDP, namely $\alpha = 0.3$, and the elasticity to human capital is set to $\beta = 0.65$ in our baseline calibration. The corresponding elasticity relative to the polluted input is $\phi = 0.05$. According to the cost-share theorem, ϕ reflects fiscal revenues from environmental taxes, which do not exceed 5% of GDP in the data in developed countries.

Regarding the government's behavior, the public spending ratio is fixed to its historical average in OECD countries ($g = 0.25$). For this value, our model returns an endogenously-computed tax rate on wages of 0.18, which is comparable to the value observed in the data for developed countries, i.e. 0.16 on average in OECD countries over the period 2000-2019. Moreover, on the balanced-growth path, the value of $\pi_k := \pi_t/K_t$ is the ratio between two growing variables. This parameter will be scanned over a large range of values (from 0.01 to 50) to verify the presence of bifurcations, with $\pi_k = 0.568$ in the baseline calibration, which corresponds to a realistic pollution tax revenue of 5% of GDP.

Regarding the environment block, we fix $v = 0.07$ in the natural regeneration process, close to the value (0.04) used in Nordhaus (1994) and Fullerton and Kim (2008). Finally, $\bar{Q}$ is normalized to unity, and the elasticity of the environmental quality in the production function is fixed at $\delta = 0.65$.

[12]Quite obviously, it must be underlined that indeterminacy and possible bifurcations do not rely on a particular set of parameters, but can be obtained under (possibly very) different vectors of parameters.

PARAMETERS		
Households		
ρ	0.01	Discount rate
Technology		
A	0.6	Total productivity parameter
α	0.3	Physical capital elasticity in the production function
β	0.65	Labor elasticity in the production function
ϕ	0.05	Polluting-input elasticity in the production function
Government		
g	0.25	Government spending
π_k	0.568	Pollution tax
Environment		
v	0.07	Natural regeneration process
$\bar{Q}$	1	Maximal environmental level
δ	0.65	Elasticity of the environmental quality in the production function

Table 1: Baseline calibration

As we have seen, our two-dimensional model is highly non-linear and can give birth to three steady states. In this context, various forms of bifurcation can occur.

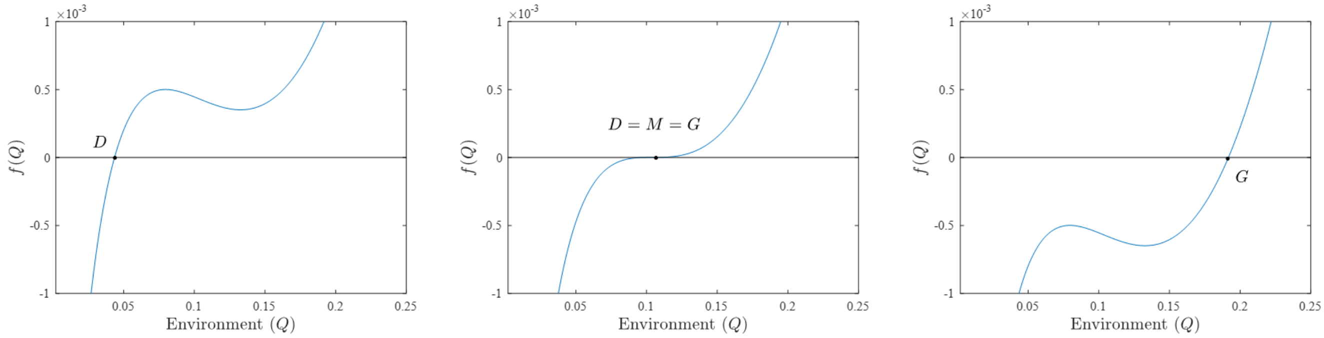
First, our model exhibits two (codim 1)[13] saddle-node bifurcations at $\rho = \rho_1$ and $\rho = \rho_2$, when the two steady states collide. Using our baseline calibration, these bifurcations arise for $\rho_1 \simeq 0.010626$ and $\rho_2 \simeq 0.010717$, respectively. To fix ideas, let us define the function $f(Q) := \Phi(Q) - \Psi^{-1}(Q)$, such that the steady states are obtained when $f(Q) = 0$. As depicted in Figure 2, at $\rho = \rho_1$ points M and G collide (saddle-node bifurcation SN_1, Figure 2d): for closely-lower values of ρ, equilibria M and G disappear and the system switches from regime $\mathcal{R}_3$ to regime $\mathcal{R}_1$ (Figure 2a). At $\rho = \rho_2$ points D and M collide (saddle-node bifurcation SN_2, Figure 3f): for closely-higher values of ρ, equilibria D and M disappear and the system switches from $\mathcal{R}_3$ to $\mathcal{R}_2$ (Figure 2c).

Second, if one authorizes another parameter to vary, say the pollution tax π_k that defines the environmental fiscal policy instrument, we obtain a (codim 2) cusp bifurcation. At the cusp point the two saddle-node bifurcations meet. For nearby parameter values, the system has three equilibria which collide and disappear pairwise via the saddle-node bifurcations. This bifurcation is obtained when points M, D, and G collide, namely at

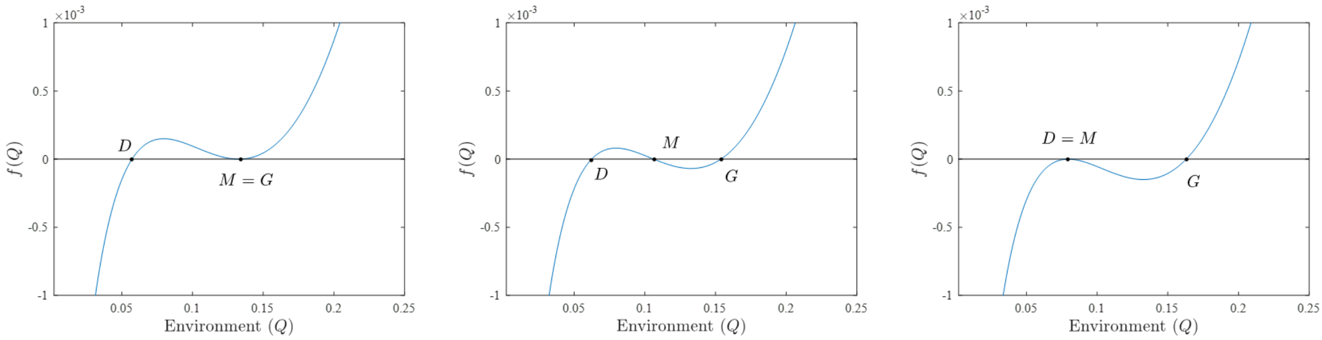
[13] The codimension of a bifurcation is the number of parameters that must be varied to generate the bifurcation. For example, with a codimension of 1 (codim 1), one parameter alone can generate bifurcations.

$(\rho, \pi_k) \simeq (0.011, 0.556)$ in our benchmark calibration. In Figure 2b, the cusp bifurcation appears when $f(Q) = 0$ at the inflexion-point. The cusp point is of great usefulness because it allows organizing the dynamics: as shown by Figure 2, regime $\mathcal{R}_2$ characterizes a situation where the unique equilibrium (G) is located above the inflexion-point of $f(Q)$, while regime $\mathcal{R}_1$ characterizes a situation where the unique equilibrium (D) is located below this inflexion-point.

Altogether, the calibration exercise shows that the various steady states and complex dynamics that our model unveils arise for realistic values of parameters and of the endogenous variables. Consequently, we look in the following at two important economic policy questions: the ranking of the various steady states from a welfare perspective and the effect of the pollution tax in the long run.



2a. $\rho < \rho_1$: Regime $\mathcal{R}_1$ | 2b. *CUSP* Point | 2c. $\rho > \rho_2$: Regime $\mathcal{R}_2$



2d. $\rho = \rho_1$: Bifurcation SN_1 | 2e. $\rho_1 < \rho < \rho_2$: Regime $\mathcal{R}_3$ | 2f. $\rho = \rho_2$: Bifurcation SN_2

Figure 2: Topological Regimes

5.1. Welfare analysis

Along the green steady state, economic growth and environmental quality are high, while they are low along the dark steady state. This does not necessarily mean that long-run welfare is higher along the green steady state: indeed, the welfare comparison between equilibria is not straightforward, as detailed below.

Using Eqs. (7) and (8), households' welfare U^i for each steady state i, with $i \in \{D, M, G\}$, writes as[14]

$$U^i = \int_0^{+\infty} e^{-\rho t} u(c_t^i, l_t^i)\, dt = \int_0^{+\infty} e^{-\rho t} \left\{ \log(c_t^i) - l_t^i \right\} dt.$$

At steady state i, consumption grows at the constant rate γ^i, while labor is constant (i.e. $l_t^i = l^i$). Hence, we obtain $c_t^i = K_0 c_k^i e^{\gamma^i t}$, where K_0 is the initial capital stock (normalized to one in our simulations), and c_k^i is the steady-state consumption-to-capital ratio. After some algebra, we find

$$U^i = \frac{1}{\rho^2} \left\{ \rho \left[\log(c_k^i) - l^i \right] + \gamma^i \right\}. \tag{20}$$

Formally, a steady state with a high consumption ratio, environmental quality, and economic growth (e.g. the green equilibrium) does not necessarily improve welfare compared to a steady state with a low consumption ratio, environmental quality, and economic growth (e.g. the dark equilibrium). Effectively, there is a tradeoff between economic growth (and therefore environmental quality) and leisure in household's utility function: less work will lead to more welfare but less growth. Similarly, firms can substitute pollution for labor, which has yet another ambiguous effect on household's welfare.

However, based on extensive simulation analysis, we can establish that the ranking of the three steady states in terms of welfare is similar to the ranking in terms of long-run economic growth, namely $U^D < U^M < U^G$ (see Table 2). This is because the effect of the consumption ratio and economic growth always exceeds the one that transits through labor supply in Eq. (20).

	$i = D$	$i = M$	$i = G$
γ^i	0.0018	0.0105	0.0259
c_k^i	0.0277	0.0407	0.0639
l^i	0.6395	0.7543	0.8435
$\mathbf{U^i}$	$\mathbf{-404.21}$	$\mathbf{-290.92}$	$\mathbf{-100.02}$

Table 2: Long-run welfare (baseline calibration)

These findings suggest that it makes sense to try to reach the green steady state through the use of pollution taxes. However, the multiplicity in our model raises some challenges for the formulation of the environmental policy, as we will see.

5.2. The effect of the pollution tax in the long run

So far, we studied the different regimes by varying ρ, since this parameter enabled us to obtain simple analytical results. However—naturally—the different bifurcations can

[14] We focus on welfare along the balanced-growth path, and ignore transitional dynamics; indeed, since the steady state can be indeterminate, aiming at assessing welfare along the transition path would not make sense.

emerge from varying any other parameter of the model. Using our benchmark calibration, we investigate the long-run effect of environmental policy (in comparative statics) by looking at the role of the pollution tax π_k for the existence of the different steady states. To this end, Figure 3 characterizes the three regimes with respect to the pollution tax π_k. Regime $\mathcal{R}_1$ emerges for $\pi_k < \pi_1 \simeq 0.5654$, regime $\mathcal{R}_2$ occurs for $\pi_k > \pi_2 \simeq 0.57$, and regime $\mathcal{R}_3$ appears for intermediate values $\pi_1 < \pi_k < \pi_2$.

Intuitively, if the pollution tax is high, so is environmental quality along the BGP, and only green-type steady states appear (Regime $\mathcal{R}_2$). Conversely, under low pollution taxes, the environmental quality is so deteriorated that only dark-type steady states emerge (Regime $\mathcal{R}_1$). In these regimes, an increase in the pollution tax decreases polluting emissions, which improves environmental quality, total factor productivity, and thus economic growth. Lastly, for intermediate levels of pollution taxes, both equilibria coexist, and an additional steady state (M) arises, in which the environment quality and economic growth negatively depend on pollution taxes.

From an economic standpoint, the existence of this steady state comes from accounting for an endogenous labor supply, which makes the output ratio to depend negatively on the consumption ratio. The counter-intuitive effect of pollution taxes on this equilibrium comes from households' tradeoff between work and leisure. When π_k increases, the environmental quality—and thus the output ratio—rise. However, facing higher taxes, the return to labor decreases, and the consumption ratio increases (since leisure and consumption are complements in households' utility), which negatively affects the output ratio. If the latter effect exceeds the former, economic growth decreases and so does environmental quality, because the resources for government abatement spending are reduced.

A last point deserves particular attention from an economic policy perspective. The occurrence of the cusp singularity can generate a hysteresis phenomenon, as described in Figure 3. To understand this phenomenon, assume that the economy is initially located in point A, namely on the segment of the steady-state curve associated with high environmental quality (i.e. a G-type steady state), characterized by the pollution tax π_2. If π_k decreases from π_2 until π_1, the steady state moves along the segment of G-type steady states until point LP_1. If π_k further decreases, the economy switches to regime $\mathcal{R}_1$: the steady state jumps to point B on the segment of D-type steady states, i.e. the economy suddenly falls from the green to the dark equilibrium. However, starting from point B, an increase of π_k does not return the economy back to a green steady state, but leaves it on the segment of D-type steady states; in particular, even if the pollution tax is restored to its initial value π_2, the economy will not return to the initial point A but will remain in point LP_2 .

Hence, for small changes of the pollution tax, the steady state can warp in a non-reversible way: a lax environmental policy (i.e. $\pi_k < \pi_1$) may condemn the economy to an irreversible steady state with low environmental quality, i.e. an environmental-poverty trap. Of course, such an analysis is only based on comparative statics of the steady states, and must be further investigated from a dynamic perspective (see the next section).

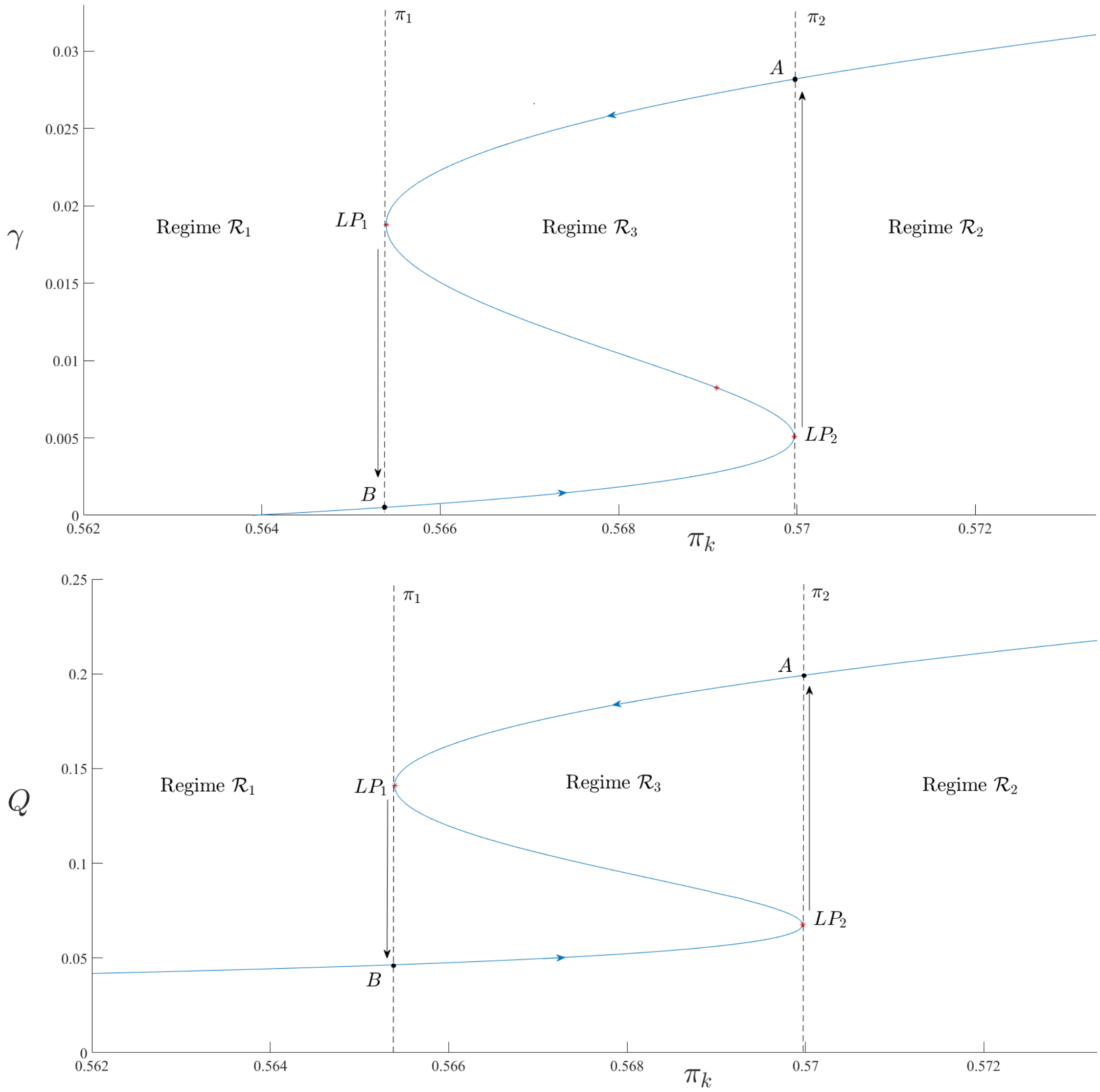


Figure 3: Environmental quality and the pollution-tax

Our findings contrast with respect to those of Itaya (2008). While in his model the environmental tax improves long-run economic growth only when the steady state is locally indeterminate, in our setup the environmental tax is *always* growth-improving along the well-determinate steady states D and G. This difference comes fundamentally from the way environment is introduced in the model.

Itaya (2008) considers pollution as by-product that does not affect total productivity but exerts a negative externality on households' utility. High expected pollution induces households to raise current labor supply (because pollution and leisure are substitute in utility), which in turn raises output and economic growth. This higher growth generates higher pollution in the future, hence the emergence of expectation-driven fluctuations (local indeterminacy). In addition, any increase in the environmental tax rises the relative price of pollution and the inducement to work, boosting economic growth. Therefore, the local indeterminacy of the steady state and the positive association between growth and the environmental tax go hand-in-hand.

On the contrary, in our model pollution does not affect households' utility but is an input in the production function. In contrast to Itaya (2008), first, a higher environmental tax reduces the return to labor, so that households reduce their labor supply, which has a negative effect on output and economic growth. Second, the environment exerts a positive externality on total factor productivity; therefore, a rise in environmental taxes reduces pollution emissions and improves long-run economic growth through this channel.[15] The second effect prevails when the steady state is well-determined, but the first effect may dominate when the economy is subject to multiple equilibria and indeterminacy. Therefore, in contrast with Itaya (2008), in our model indeterminacy is not the price to be paid for environmental policy to be pro-growth. Instead, indeterminacy is an undesirable property arising from inadequate environmental policies regarding the environmental tax.

From an economic policy perspective, this feature yields mixed results. On the one hand, if the economy is initially located in the neighborhood of point B, the environmental policy must be handled with care. Indeed, increasing the pollution tax can generate indeterminacy by switching the economy from regime $\mathcal{R}_1$ to regime $\mathcal{R}_3$, with the risk of generating large aggregate fluctuations, as we will see. On the other hand, a large increase in the pollution tax, from $\pi_k < \pi_1$ to $\pi_k > \pi_2$, would likely allow the economy to escape the environmental trap and move towards the green steady state in the long run. This pleads in favor of a "big push" in pollution taxes and calls for the adoption of tight taxation policies of the polluting input ($\pi_k > \pi_2$).

However, severe difficulties could arise in the implementation of such an environmental policy, because high pollution taxes are likely to generate social conflicts. First, since environment is a public good, environmental policies involve continuous struggles over implicit property rights, because property rights on natural resources, such as air or water, are often poorly defined. Since the taxes associated with these implicit property rights can be misunderstood by citizens, this makes it difficult to solve such social conflicts. Second, environmental policies can generate intergenerational conflicts arising between the generations alive at the time the society imposes the environmental tax, and the generations alive at the future times (see e.g. Karp and Rezai, 2014). All the more in the current times when, notably in high-tax countries, households are generally characterized by a high degree of "fiscal fatigue" (Ghosh et al., 2013), tight environmental policies could

[15] Besides, while in Itaya (2008) the steady-state is unique, there are three steady states in our model. As a result, the environmental tax exerts an unambiguously-positive effect on long-run economic growth in regimes $\mathcal{R}_1$ and $\mathcal{R}_2$, whereas this effect can be positive or negative in regime $\mathcal{R}_3$.

be difficult to implement, however at the cost of potentially plunging these countries towards an undesirable environmental trap.

6. Global dynamics

Having identified the different regimes, we now analyze their implications for the global dynamics of the economy, as well as their consequences for the paths of environmental quality and pollution. Depending on the initial value of the environmental quality (Q_0) and the initial consumption ratio (c_{k0}) reflecting households' expectations,[16] several scenarios can appear: the economy can be trapped in the environmental-poverty trap, experiment large oscillating trajectories, or converge towards the green steady state. In this section we detail the global dynamics in the more general regime $\mathcal{R}_3$.[17]

6.1. Local and global indeterminacy

According to our discussion in the previous section, the existence of regime $\mathcal{R}_3$ is linked to the cusp bifurcation. As we have seen, depending on parameters' values, one or three equilibria may occur in the neighborhood of the cusp point. Accordingly, the emergence of global indeterminacy comes from the existence of the steady state M.

Proposition 3. *For any initial environmental quality Q_0 close to Q^M, the model is globally indeterminate. In addition, if M is stable, there is also local indeterminacy.*

Proof. In regime $\mathcal{R}_3$, there are two saddle-path steady-states (D and G). Since (i) the dynamic system (18) is smooth on $(c_k, Q) \in (\rho, +\infty) \times (0, \bar{Q})$, and ($ii$) the dynamic involves one predetermined variable (the environmental quality) and one jumpable variable (the consumption ratio), if Q_0 is close to Q^M the initial consumption ratio c_{k0} can jump to several values that are consistent with the household's optimal behavior and the transversality condition. Specifically, there is one value of c_{k0} that puts the economy on the stable manifold that converges to G and another value that puts the economy on the stable manifold that converges to D. Additionally, if M is stable, there is also an infinite set of trajectories converging to M. ■

Figure 4 illustrates Proposition 3 in the absence of cyclical dynamics, i.e. for values of π_k quite distant from the Hopf bifurcation (as we will see, for values close to π_k^h, a

[16] In a competitive equilibrium, if the system admits a unique saddle steady state, the initial consumption ratio c_{k0} needs to "jump" to put the economy on the unique stable manifold towards the steady state for the transversality condition (12) to be verified. In contrast, in the presence of multiple reachable steady states, several values of c_{k0} are consistent with the transversality condition. Thus, multiple paths converging to different steady states are feasible, leading to a selection problem.

[17] In regimes $\mathcal{R}_1$ and $\mathcal{R}_2$ associated with a unique saddle-path steady state, the model is globally well-determined. Indeed, for any predetermined value of the environmental quality ($Q_0 \in (0, \bar{Q})$), the initial consumption ratio (c_{k0}) jumps to put the economy on the stable manifold that converges towards the unique steady state.

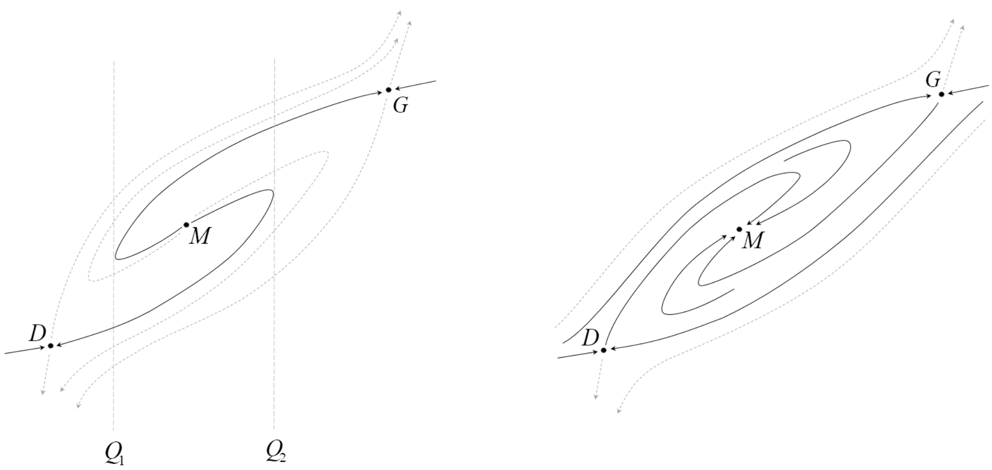
limit-cycle can appear). We will consider two cases, namely when M is unstable (case 1) and when M is stable (case 2)

Case 1. If M is unstable ($\pi_k \ll \pi_k^h$, see Figure 4a), there are two environmental quality thresholds Q_1 and Q_2, with $Q_1 > Q^D$ and $Q_2 < Q^G$,[18] such as, if the initial environmental value is $Q_0 < Q_1$ (respectively $Q_0 > Q_2$), the initial consumption ratio c_{k0} jumps to put the economy on the saddle path that converges towards the dark (respectively green) equilibrium. Consequently, the initial environmental quality (Q_0) allows selecting the long-run equilibrium. If the ecosystem is initially poorly-endowed in natural capital ($Q_0 < Q_1$), the economy will be trapped in a region with low environmental quality and low growth. As the dark environmental equilibrium is saddle-path stable, the environment is deteriorating in an irreversible way: the economy is dragged towards the environmental-poverty trap. On the contrary, in the case of a high initial natural capital endowment ($Q_0 > Q_2$), the economy will converge towards the green long-run equilibrium.

However, if the initial environmental quality is such as $Q_0 \in (Q_1, Q_2)$, the green and the dark equilibria can be reached depending on the initial jump of the consumption ratio, i.e. there is global indeterminacy. In this case, the steady state is subject to "animal spirits" in the form of self-fulfilling prophecies. Such a global indeterminacy is intuitive from an economic standpoint. Suppose that households initially expect high future environmental quality. Due to the externality on the productivity ($A(Q)$), output and the expected net return of capital will be high. Then, at the initial time households increase their savings, such that the initial consumption ratio (c_{k0}) will be low and the initial hours worked will be high. This means that output will also be high, which generates large abatement public spending initially ($G_0 = gY_0$) that ensure a good environmental quality in the future. Conversely, following the same mechanism a low expected natural capital is self-fulfilling and may lead to the dark steady-state D. In other words, forward-looking households can validate in equilibrium any expectation on the environmental quality that can be reached in the future. Consequently, the short-run and long-run behavior of the economy depends both on history (i.e. the initial state of the environment Q_0) and on expectations (i.e. the initial jump of the consumption ratio c_{k0}).

Case 2. If M is stable ($\pi_k \gg \pi_k^h$, see Figure 4b), global indeterminacy emerges for any initial state of the environment $Q_0 > 0$, but history still plays a role in the possibility of having three reachable steady states if $Q_0 \in (Q^D, Q^G)$, or only two in the opposite case.

[18] The critical value Q_1 (Q_2) is the minimum (maximum) value of environmental quality along the stable manifold that converges towards the saddle-path steady-state G (D).



4a. Bi-stability ($\pi_k \ll \pi_k^h$) 4b. Three long-run solutions ($\pi_k \gg \pi_k^h$)
Figure 4: Dynamics in the (Q, c_k)-plane for π_k far from π_k^h

6.2. Limit-cycles and the homoclinic orbit

Global indeterminacy can also arise from fluctuations. This is the case in the neighborhood of the Hopf bifurcation that occurs at $\pi_k = \pi_k^h$ as established by Corollary 1. Thus, the stability of the steady state M changes between the two sides of the Hopf bifurcation: if $\pi_k < \pi_k^h$, M is unstable (Figures 5a and 5b), while M is stable if $\pi_k > \pi_k^h$ (Figure 5c).

This Hopf bifurcation can be supercritical, generating a stable limit cycle; or subcritical, generating an unstable closed orbit, depending on the value of the parameters. More precisely, *Appendix D* shows that in regime $\mathcal{R}_3$ a supercritical Hopf bifurcation does exist. We focus on this case, which is the most interesting one, since stable limit-cycles emerge for π_k slightly lower than π_k^h (Figure 5b). If π_k decreases, the limit-cycle that surrounds point M enlarges and—at the limit—it merges with the stable and unstable manifolds of D through a saddle-loop bifurcation, generating a homoclinic orbit, i.e. a large cycle with infinite amplitude (Figure 5a). The existence of this homoclinic orbit follows from the occurrence of a Bogdanov-Takens (BT) bifurcation. The following proposition shows that a BT bifurcation arises when steady states M and D collide.

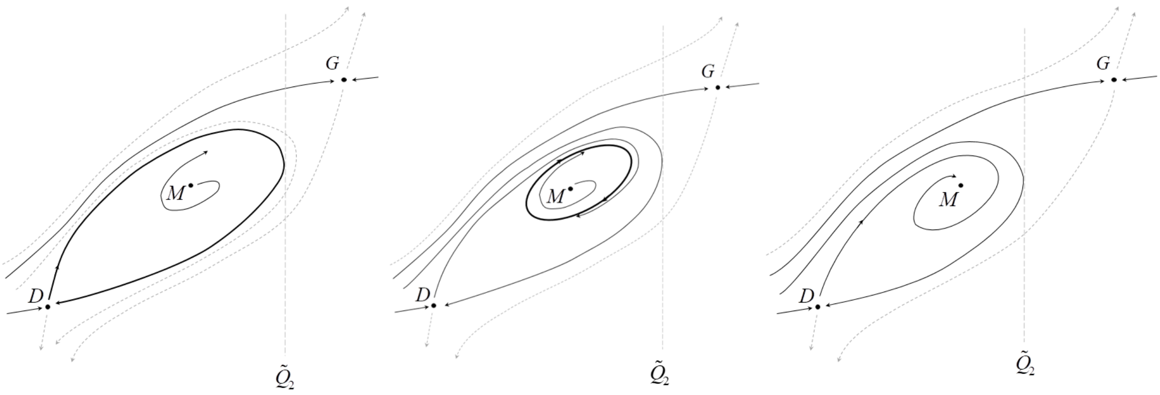
Proposition 4. *There is a BT bifurcation in the neighborhood of steady state M that appears for the pair of parameter (ρ^{bt}, π_k^{bt}). For nearby parameter values, the economy can experiment a homoclinic orbit.*

Proof. See *Appendix D*. ■

From an economic perspective, establishing formally the presence of a BT bifurcation has crucial implications for the dynamics of the economy. For example, Benhabib et al.

(2001) and Sniekers (2018) reveal the presence of inflation and unemployment fluctuations, respectively. In our model, the BT bifurcation allows uncovering fluctuations in pollution, environmental quality, and economic growth, as follows.

For values of π_k close to π_k^h, the initial stock of the environment exerts—once again—a threshold effect. If $Q_0 > \tilde{Q}_2$,[19] the economy is well-determinate and converges towards the green equilibrium. In contrast, if $Q_0 < \tilde{Q}_2$ there is global indeterminacy: according to the initial jump of c_{k0}, the economy can converge towards (i) the limit-cycle, (ii) point D, or (iii) point G. If $\pi_k > \pi_k^h$, M is stable. Conversely, if $\pi_k < \pi_k^h$, M is unstable and cannot be reached. For values of π_k slightly less than π_k^h, a stable limit-cycle births, and it enlarges as π_k decreases. If π_k decreases further, the homoclinic orbit appears: the path of economic growth and environmental quality are subject to long-lasting fluctuations, and the economy ultimately converges towards the environmental-poverty trap. Note that the homoclinic orbit is the largest possible limit-cycle: if π_k decreases again, the limit-cycle vanishes and the dynamics are similar to those depicted in Figure 4a.



5a. The homoclinic orbit 5b. $\pi_k < \pi_k^h$ (limit-cycle) 5c. $\pi_k > \pi_k^h$

Figure 5: Dynamics in the (Q, c_k)-plane for π_k close to π_k^h

[19]The level $\tilde{Q}_2$ defines the maximum of environmental quality levels along the stable manifold that converges towards D.

On the policy side, these global dynamics highlight the difficulties of defining an appropriate environmental policy: in regime $\mathcal{R}_3$ (i.e. $\pi_1 < \pi_k < \pi_2$) low pollution taxes are likely to generate large fluctuations in the long run, while high pollution taxes are likely to produce local indeterminacy since the steady state M becomes stable. Outside regime $\mathcal{R}_3$, a lax environmental fiscal policy ($\pi_k < \pi_1$) locks the economy in an environmental-poverty trap (regime $\mathcal{R}_2$). Consequently, the only policy to secure high economic growth and good environmental quality in the long run without aggregate fluctuation is a "big push" in pollution taxes ($\pi_k > \pi_2$); however, as previously acknowledged, such a policy would be difficult to implement in practice. Lastly, the presence of the homoclinic orbit has equally deep policy implications: when parameters (ρ, π_k) are close to (ρ^{bt}, π_k^{bt}), the economy can experiment large fluctuations in growth and polluting emissions, or can slowly converge towards the environmental-poverty trap along this homoclinic orbit.

6.3. A numerical illustration

Two features deserve attention. First, the oscillating trajectories that our model produces arise in the absence of any stochastic shock: the interaction between environment and economic growth is sufficient to generate endogenous fluctuations without the need of an exogenous "impulse". Second, global indeterminacy implies that, for the same set of parameters' values, various trajectories can emerge depending on households' expectations and the initial level of natural capital. This subsection presents a numerical illustration of these two features.

In our benchmark calibration, the Hopf bifurcation arises at $\pi_k \simeq 0.5691001$. In the neighborhood of point M, different dynamics emerge. If π_k is close but lower than the value defining the Hopf bifurcation, there are stable limit-cycles. This implies that a small change of a parameter would not eliminate the cyclical dynamics of environmental quality and growth; indeed, we can show that the limit-cycle enlarges as π_k decreases (see Figure 6).

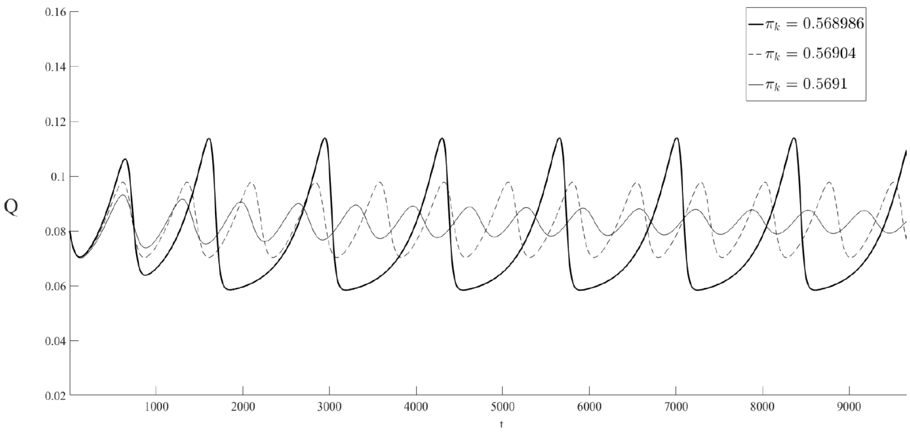
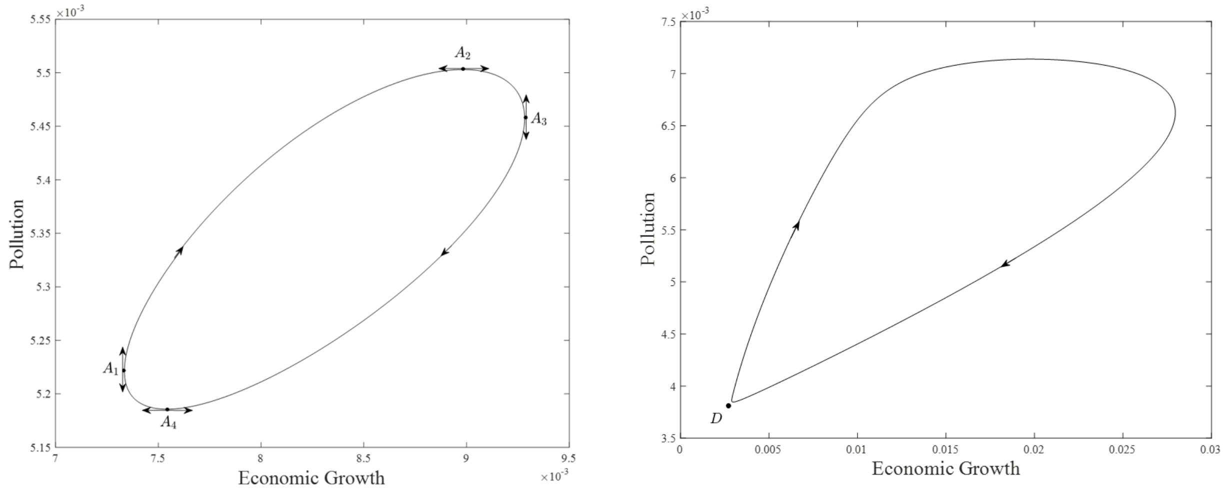


Figure 6: Stable limit-cycles for π_k close to its value defining the Hopf bifurcation

From an economic standpoint, the cyclical dynamics are produced by the interaction between the effect of growth on abatement public spending and the effect of pollution on the environmental quality. Figure 7a presents the asymptotic behavior of economic growth and pollution along a typical limit-cycle. Starting for example from point A_1, growth and pollution initially increase (*phase 1*). As growth rises, new fiscal resources are available for abatement spending, which improves environmental quality despite the increase in pollution. At point A_2, these fiscal resources are even sufficient to reduce pollution, so that pollution and growth move in opposite directions (*phase 2*). However, pollution remains high, and at point A_3 the quality of the environment begins to decline, so that total factor productivity is reduced: the economy displays a path such as both growth and pollution decline (*phase 3*). Indeed, since the decrease in pollution is not sufficient to offset the reduction in fiscal resources for abatement, the environmental quality deteriorates. Moreover, at point A_4, following the reduction in pollution, environmental quality begins to recover, but pollution increases following the sharp drop in abatement expenditures (*phase 4*). Finally, the economy returns to the initial point A_1, where the quality of the environment has improved sufficiently to generate a further increase in growth.



7a. $\pi_k = 0.5691$ 7b. $\pi_k = 0.5689851$

Figure 7: The pollution-growth limit-cycle

The core of these cyclical dynamics lies in the substitution effect between polluting and non-polluting inputs in the production technology. When the environmental tax is high, firms use less pollutants, so that the oscillating profile of pollution and growth is of lesser amplitude. This is the reason why the cycle widens when π_k decreases.

Interestingly, as the amplitude of the cycle becomes larger, it presents a strong asymmetry characterized by sudden degradations of the quality of the environment and economic growth, followed by a progressive recovery (as in Figure 6 for $\pi_k = 0.568986$). Our model is therefore able to produce violent episodes of environmental economic crises. The economic explanation of these crises comes from the existence of the homoclinic trajectory in the vicinity of the dark equilibrium. At $\pi_k \approx 0.568985$, the limit cycle expands so much that it coincides with the stable and the unstable manifolds of the saddle point D: at this point, there is a saddle-loop bifurcation. As proven in Proposition 4, a Bogdanov-Takens bifurcation emerges, which ensures the presence of a homoclinic orbit. For trajectories close to the homoclinic orbit, the cyclical behaviour of the economy is very asymmetric, because the dark equilibrium attracts economic growth and environmental quality trajectories, creating violent episodes of crises, and then repels them away, resulting in long periods of growth and environmental recovery. Figure 7b depicts the profile of pollution and growth among a quasi-homoclinic orbit for $\pi_k = 0.5689851$.

In the vicinity of the homoclinic orbit, our model is able to generate (very) long-lasting fluctuations of economic growth and the environment, consistent with the observed long-run fluctuations in Earth's climate and in greenhouse gases (see e.g. Snyder, 2016). Moreover, thanks to multiplicity, our model shows that an economy may experience very different long-run scenarios, for the same set of structural parameters and initial conditions for predetermined variables (and, in particular, the initial stock of environmental quality). An illustration is provided in Figure 8, which depicts an orbit close to the homoclinic orbit: the economy experiences strong fluctuations that move it close to the environmental trap (Q^D) for a very long time. However, due to multiplicity, for the same initial natural capital stock Q_0, the economy can also join the unique stable manifold that converges towards the green steady state (Q^G) through an appropriate jump of the initial consumption ratio c_{k0}.

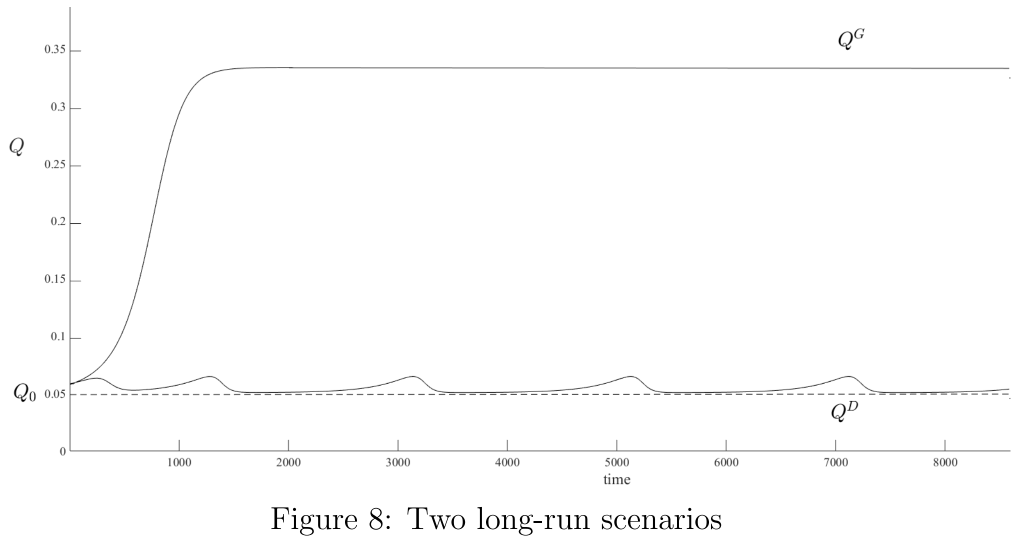


Figure 8: Two long-run scenarios

Consequently, even if our model does not include stochastic shocks, it does not lead to historical determinism. Initial conditions alone do not predict the future trajectory of pollution and growth, which instead equally depends upon households' expectations about the future state of natural resources. Thus, various scenarios are possible, and in particular societies are not necessarily condemned to face a growth-based development path that is at odds with environmental preservation. In particular, these features give rise to new perspectives on the Environmental Kuznets Curve.

6.4. Implications for the Environmental Kuznets Curve

Along a quasi-homoclinic orbit (Figure 7b), phases 2, and especially 4, of the cycle are much shorter than phases 1 and 3, so that growth and pollution are generally positively associated, both when they grow and when they decline. In this way, the well-known Environmental Kuznets Curve (EKC) suggested by many studies (see the seminal contribution of Grossman and Krueger, 1995) can be viewed as resulting from the cyclical behaviour of the variables. For example, in Figure 7, pollution rises during the phase 1 of the cycle (between points A1-A2), but decreases during phases 2 and 3 (between points A2-A3). Since, beyond its oscillations, economic growth is always positive along the cycle, this creates a hump-shaped relation between pollution and the (logarithm of) GDP (see Figure 9).

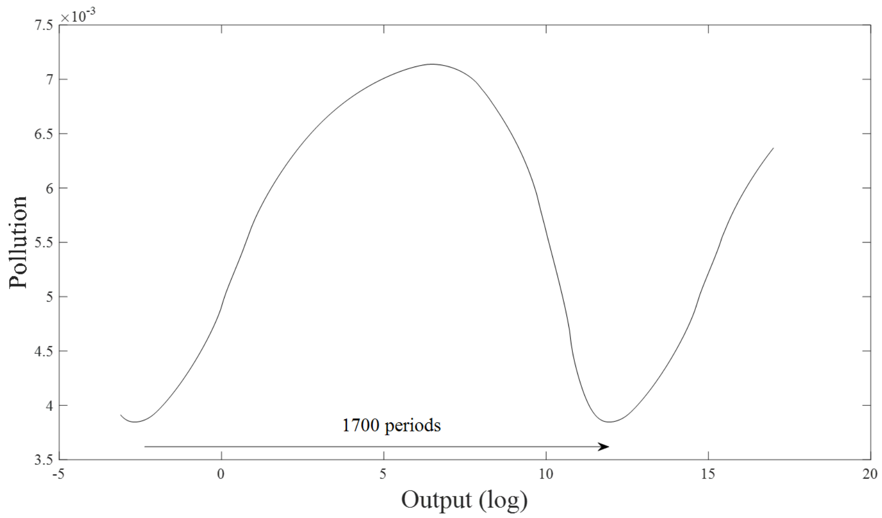


Figure 9: The EKC as a cyclical phenomenon ($\pi_k = 0.5689851$)

In the theoretical literature, the EKC is often derived from static models (see e.g. Andreoni and Levinson, 2001), or from out-of-equilibrium (i.e. non-converging) paths in dynamics models (see e.g. Dinda, 2005). The originality of our approach is to consider the EKC as an equilibrium relationship that appears along the cyclical dynamics. More precisely, in our model the EKC is viewed as (part of) a stationary orbit around the steady state, and not as a relation that emerges during the adjustment path toward the steady state. At the limit, i.e. when the limit-cycle approaches the homoclinic orbit, the EKC becomes a very long-lasting feature, which can replicate the profile of pollution and income during the lifelong process of economic development. For example, in Figure 9, the duration of the cycle is so large that it results in a phase of rising and falling pollution (i.e. an EKC curve) that lasts for more than 1700 periods; if we interpret the periods in months, the EKC would occur during a period of the order of 140 years, consistent with economic and environmental transition since the industrial revolution (see e.g. Panayotou, 1993).

These results convey a fairly pessimistic view over the inverted U-shaped relationship between the environment and economic development. First, contrasting with the standard message derived from an EKC, a negative correlation between economic growth and polluting emissions does not suggest, in our setup, that the economy has reached a critical income level such that the goals of economic growth and environmental protection will always go hand in hand, but rather that the economy is in a (possibly very long-lasting) transitional phase where it may well converge towards a poverty-environment trap (as in Figure 7b). Second, that the economy is in the declining part of the EKC (i.e. with reducing emissions) is not necessarily beneficial. In our model, pollution emissions come from the production process. Hence, a strong economic activity generates both large public resources for abatement and high pollution flows. Consequently, a decrease in emissions in the long run may signal that the economy will be trapped in the dark equilibrium with low abatement capacities. In such a case, in contrast with Dasgupta et al. (2002), the potential irreversibility of environmental damages does not necessarily play against the existence of the EKC.

7. Concluding remarks

This paper has shown that accounting for the dual interaction between households' optimal saving behavior and the law of motion of the environment gives rise to complex dynamics in the relationship between economic growth and environmental quality. Our analysis provides three novel results.

First, the growth-environment interaction yields multiple equilibria (i.e. a "dark", an intermediate, and a "green" equilibrium) that trigger local and global indeterminacy. From an economic standpoint, it follows that the dynamic of the economy is subject to multiple self-fulfilling paths in the short- and long-run. Households' expectations and the initial state of the environment determine whether in the long-run the economy experiences an appealing equilibrium with high growth and environmental quality, an undesirable environmental-poverty trap, or long-lasting endogenous fluctuations.

By illustrating the richness of growth-environment relationships, our analysis contributes to the understanding of the observed cross-country heterogeneities in the profile of emissions highlighted by empirical studies (see e.g. López-Menéndez et al., 2014). Indeed, although the empirical literature is inconclusive about the economic forces leading to different growth-environment relationships, some influential papers (see e.g. Chimeli and Braden, 2005) suggest that such cross-country heterogeneities are primarily explained by differences in total factor productivity (TFP). This result provides empirical support for the realism of our findings, because our theoretical analysis shows precisely that the endogenous TFP is one of the two ingredients leading to various growth-environment paths. In this vein, the popular Environmental Kuznets Curve emerges as a special case in our model, in the form of an (equilibrium) long-lasting feature. In light of this finding, the policy message that is usually defended by studies devoted to the EKC may have to be restated: a negative growth-emissions correlation may cover a poverty-environment trap in the long-run.

Second, the growth-environment interplay reveals a novel perspective on the possible role of an environmental tax. Closely related to the presence of a hysteresis phenomenon, we reveal the presence of an irreversible environmental-poverty trap. Following the seminal paper of Azariadis and Drazen (1990), numerous studies attempt to explain the emergence of such traps (see e.g. Azariadis and Stachurski, 2005; Antoci et al., 2011), mostly related with exogenous threshold externalities affecting the abatement knowledge technology (see e.g. Xepapadeas, 1997; Prieur, 2009). Contrasting with such exogenous technological breaks, the poverty trap that we reveal is the consequence of complex growth-environment interactions that open the door for a crucial role for environmental taxes: our analysis shows that a too low environmental tax condemns the economy to the environmental-poverty trap.

Moreover, increasing the environmental tax to average levels is associated with aggregate fluctuations in the short run and possibly-large growth and pollution oscillations in the long run. On the contrary, more encouraging perspectives are observed for large levels of environmental taxes, which may secure both high growth and environmental quality. Nevertheless, such an environmental taxes "big push" would be difficult to implement in practice, particularly in a context of poorly-defined property rights on global environmental goods or reduced environmental intergenerational altruism arising from a too high fiscal pressure on the current generation.

Third, accounting for the two-sided growth-environment dynamics allows reassessing the role of indeterminacy for environmental policies. While the important work of Itaya (2008) reveals that environmental taxes generate a tradeoff between higher growth and a determinate equilibrium, our finding is that—provided that the equilibrium is determinate and unique—environmental taxes unambiguously foster economic growth. Therefore, while growth-enhancing environmental policies are found to be equally consistent with a determinate equilibrium, our analysis shows that indeterminacy may be the result of environmental policies that are not sufficiently ambitious in terms of environmental taxes. In this case, such policies should be accompanied by an appropriate communication over the environmental (and macroeconomic) future goals, designed for channeling households' expectations in order to attain the desirable level of environmental quality.

We have little doubt that the analysis of nonlinear dynamics in environmental growth models deserves future research. Related to our analysis, one could relax the balanced-budget hypothesis by authorizing the debt-financing of abatement expenditure as in e.g. Boly et al. (2022), with the aim of further investigating theoretically the growth-environment relationship in higher-dimension models (see e.g. Bosi and Desmarchelier, 2019), and assessing empirically (using for example the methodology of Barnett and Chen, 2015) the presence of various dynamic paths in CO_2 and other pollutants.

Moving away from our present analysis, future research could deal with the issue of adaptation to climate change. Our model could be extended by integrating pollution in households' utility through e.g. a damage function in the spirit of Bréchet et al. (2013) and Le Kama and Pommeret (2017). In this extended setup, one could investigate the role of various types of public expenditure, and particularly public spending for abatement (as in our model) and public investment for environmental adaptation, which may exert potentially-conflicting consequences for the dynamics of the environment.

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Appendix A. Solution of Households' programme.

The household maximizes (7) subject to (9), with k_0 given. Using (8), the current Hamiltonian writes

$$H = \log(c_t) - l_t + \lambda_t \left[r_t k_t + (1 - \tau_t)\, w_t l_t - c_t \right],$$

where λ_t is the co-state variable associated with k_t.

The first-order conditions are

$$/c_t \qquad 1/c_t = \lambda_t, \tag{A.1}$$
$$/l_t \qquad 1 = (1 - \tau_t) w_t, \tag{A.2}$$
$$/k_t \qquad \dot{\lambda}_t / \lambda_t = \rho - r_t, \tag{A.3}$$

and the transversality condition is

$$\lim_{t \to +\infty} \{ \exp(-\rho t)\; k_t / c_t \} = 0.$$

By differentiating (A.1), and after some simple manipulations, we obtain Eqs. (10) and (11) of the main text.

Appendix B.

We restrict the analysis to strictly-positive solutions, namely $Q \in (0, \bar{Q})$ and $c_k \in (\rho, +\infty)$. The first relation is the $\dot{c}_k = 0$ locus, which comes from Eq. (18a)

$$Q = \Psi(c_k) := \kappa_0 (c_k - \rho)^{\varepsilon/\delta} c_k^{\beta/\delta},$$

where $\kappa_0 = [\lambda A^{1/\varepsilon} (1 - \alpha - g)]^{-\varepsilon/\delta} > 0$. This relation describes an increasing continuous curve in the (c_k, Q)-plane, with $c_k \in (\rho, +\infty)$.

The second relation is the $\dot{Q} = 0$ locus, which comes from Eq. (18b)

$$c_k = \Phi(Q) := \frac{\kappa_1 Q^{\delta/\beta}}{E(Q)^{\varepsilon/\beta}},$$

where $\kappa_1 = \left[\left(\frac{1-\alpha-\beta}{\pi_k} \right) \lambda A^{1/\varepsilon} \right]^{\varepsilon/\beta}$. Considering $E(Q) = vQ(\bar{Q} - Q)$, the shape of the mapping $\Phi(Q)$ depends on the behavior of the ratio $Q^{\delta-\varepsilon}(\bar{Q} - Q)^{-\varepsilon}$.

There are three cases.

i. $\varepsilon > \delta$. The mapping $\Phi(\cdot)$ describes a U-shaped curve in the (c_k, Q)-plane, with two vertical asymptotes at 0 and $\bar{Q}$, i.e. $\Phi(0) = \Phi(\bar{Q}) = +\infty$.

ii. $\varepsilon = \delta$. The mapping $\Phi(\cdot)$ describes an increasing curve in the (c_k, Q)-plane, with $\Phi(0) = \kappa_1 / (v\bar{Q})^{\varepsilon/\beta} > 0$ and $\Phi(\bar{Q}) = +\infty$.

iii. $\varepsilon < \delta$. The mapping $\Phi(\cdot)$ describes yet again an increasing curve in the (c_k, Q)-plane, with $\Phi(0) = 0$ and $\Phi(\bar{Q}) = +\infty$.

We thereafter focus on the case *iii*, since the condition $\varepsilon < \delta$ leads to the most general configuration, i.e. in which three long-run steady states can appear.

Lemma 1. If $\varepsilon < \delta < \varepsilon + \beta$, the curve driven by the mapping $\Phi(Q)$ is first concave and then convex in the (c_k, Q)-plane (see Figure 1 in the main text).

Proof. Using $\tilde{\delta} := \delta/\beta$ and $\tilde{\varepsilon} := \varepsilon/\beta$, we compute: $\Phi''(Q) = \kappa_1 v^{-\tilde{\varepsilon}} Q^{\tilde{\delta}-\tilde{\varepsilon}-2}(\bar{Q} - Q)^{-\tilde{\varepsilon}-2} h(Q)$, with

$$h(Q) = (\tilde{\delta} - \tilde{\varepsilon})(\tilde{\delta} - \tilde{\varepsilon} - 1)(\bar{Q} - Q)^2 + 2\tilde{\varepsilon}(\tilde{\delta} - \tilde{\varepsilon})Q(\bar{Q} - Q) + \tilde{\varepsilon}(\tilde{\varepsilon} + 1)Q^2.$$

Thus, $h \in C^\infty([0, \bar{Q}])$, $h(0) = (\tilde{\delta} - \tilde{\varepsilon})(\tilde{\delta} - \tilde{\varepsilon} - 1)\bar{Q}^2 < 0$, and $h(\bar{Q}) = \tilde{\varepsilon}(\tilde{\varepsilon} + 1)\bar{Q}^2 > 0$, as $\delta - \varepsilon < \beta$. In addition, we have

$$h'(Q) = -2(\tilde{\delta} - \tilde{\varepsilon})(\tilde{\delta} - \tilde{\varepsilon} - 1)(\bar{Q} - Q) + 2\tilde{\varepsilon}(\tilde{\delta} - \tilde{\varepsilon})(\bar{Q} - 2Q) + 2\tilde{\varepsilon}(\tilde{\varepsilon} + 1)Q = \lambda_2\bar{Q} - \lambda_1 Q,$$

where $\lambda_2 := -2(\tilde{\delta} - \tilde{\varepsilon})(\tilde{\delta} - \tilde{\varepsilon} - 1) + 2\tilde{\varepsilon}(\tilde{\delta} - \tilde{\varepsilon}) > 0$, and $\lambda_1 := -2(\tilde{\delta} - \tilde{\varepsilon})(\tilde{\delta} - \tilde{\varepsilon} - 1) + 4\tilde{\varepsilon}(\tilde{\delta} - \tilde{\varepsilon}) - 2\tilde{\varepsilon}(\tilde{\varepsilon} + 1)$. There are two cases.

i. If $\lambda_1 \leq 0$, then $h'(Q) > 0$ for all $Q \in (0, \bar{Q})$.

ii. If $\lambda_1 > 0$, then $h'(Q) > 0 \Leftrightarrow Q < \bar{Q}(\lambda_2/\lambda_1)$. Yet, we have $\lambda_2 > \lambda_1$ such that $h'(Q) > 0$ for any $Q \in [0, \bar{Q}]$. Indeed, $\lambda_2 - \lambda_1 = 2\tilde{\varepsilon}[-\tilde{\delta} + 2\tilde{\varepsilon} + 1] > 0$.

Consequently, according to the Intermediate Value Theorem, there is a unique critical value $\check{Q} \in (0, \bar{Q})$ such that $\Phi''(Q) < 0$ for $Q \in (0, \check{Q})$, and $\Phi''(Q) > 0$ for $Q \in (\check{Q}, \bar{Q})$. ■

Finally, as $\Psi(\rho) = 0$, $\Phi(0) = 0$, and $\Phi(\bar{Q}) = +\infty$, the loci $\dot{c}_k = 0$ and $\dot{Q} = 0$ can cross once or thrice, depending on the value of ρ. According to the Intermediate Value Theorem, there are two critical values ρ_1 and ρ_2, where $0 < \rho_1 < \rho_2$, such that:

- if $\rho < \rho_1$: there is only one crossing-point, defining the dark steady state (point D in Figure 1);
- if $\rho_1 < \rho < \rho_2$: there are three crossing-points, namely the green steady state (point G), the intermediate steady state (point M), and the dark steady state (point D);
- if $\rho > \rho_2$: only the green steady state exists.

Appendix C. Local stability

The Jacobian matrix $\mathbf{J}^i$ at the steady-state i, $i \in \{D, M, G\}$ is

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CQ^i \\ QC^i & QQ^i \end{pmatrix},$$

where, using (18),

$$CC^i = (\alpha + g - 1)c_k^i y_{kc}^i + c_k^i > 0, \tag{C.1}$$
$$CQ^i = (\alpha + g - 1)c_k^i y_{kQ}^i < 0, \tag{C.2}$$
$$QQ^i = E'(Q^i) - y_{kQ}^i(1 - \alpha - \beta)/\pi_k < 0, \text{if } \varepsilon < \delta, \tag{C.3}$$
$$QC^i = -y_{kc}^i(1 - \alpha - \beta)/\pi_k > 0, \tag{C.4}$$

with $y_{kc}^i := \frac{\partial y_k^i}{\partial c_k^i} = -\left(\frac{\beta}{\varepsilon}\right)\frac{y_k^i}{c_k^i} < 0$, and $y_{kQ}^i := \frac{\partial y_k^i}{\partial Q^i} = \left(\frac{\delta}{\varepsilon}\right)\frac{y_k^i}{Q^i} > 0$.

We show first that $QQ^i < 0$. Using Eqs. (18b) and (C.3), we have $QQ^i = E'(Q) - (\delta/\varepsilon)E(Q)/Q = v[(\varepsilon - \delta)(\bar{Q} - Q) - \varepsilon Q]/\varepsilon < 0$, since $\delta > \varepsilon$.

In our two-dimensional system, we can study the local stability of steady states by inspecting the slope of $\dot{c}_k = 0$ (the slope of $\Psi(c_k)$ in Figure 1, denoted by s_c^i) and $\dot{Q} = 0$ (the slope of $\Phi(Q)$ in Figure 1, denoted by s_Q^i) in the neighbourhood of each BGP i, in the (c_k,Q)-plane.

First, using the Implicit Function Theorem we compute from Eqs. (C.1)-(C.4)
$s_c^i = -CQ^i/CC^i > 0$ and $s_Q^i = -QQ^i/QC^i > 0$.

Second, the trace and the determinant of the jacobian matrix are $\text{Tr}(\mathbf{J}^i) = CC^i + QQ^i$ and $\det(\mathbf{J}^i) = CC^iQQ^i - CQ^iQC^i = CC^iQC^i(s_c^i - s_Q^i)$. Since $CC^i > 0$ and $QC^i < 0$, we have $\det(\mathbf{J}^i) < 0$ if $s_c^i < s_Q^i$, as for the points D and G of Figure 1. At point M, $s_c^i > s_Q^i$ such that $\det(\mathbf{J}^i) > 0$, and a Hopf bifurcation emerges when $CC^i = -QQ^i$, such that $\text{Tr}(\mathbf{J}^i) = 0$.

It follows that points D and G, if they exist, are saddle-path stable because $\mathbf{J}^D$ and $\mathbf{J}^G$ contains two opposite-sign eigenvalues. If M exists, M is either stable ($\mathbf{J}^M$ contains two negative eigenvalues) or instable ($\mathbf{J}^M$ contains two positive eigenvalues). A Hopf bifurcation occurs when $CC^M + QQ^M = 0$; namely, from Eqs. (C.1) and (C.3), at $\pi_k = \pi_k^h$, where

$$\pi_k^h = \left(\frac{\delta}{\varepsilon}\right)\frac{y_k^M(1 - \alpha - \beta)}{Q^M[E'(Q^M) + CC^M]} > 0.$$

Appendix D. Location of regimes and bifurcations

In the (ρ, π_k)-plane, the two saddle-node bifurcations are depicted by the curves $\mathcal{SN}_1$ and $\mathcal{SN}_2$ that represent the limit-points between regimes $\mathcal{R}_1$ and $\mathcal{R}_3$, and $\mathcal{R}_3$ and $\mathcal{R}_2$, respectively (see Figure A1). The cusp point (labelled CP) occurs at the intersection of these two bifurcation curves, such that for higher levels of the discount rate or lower values of the pollution tax regime $\mathcal{R}_3$ vanishes. Regime $\mathcal{R}_1$ arises below the lower branch that joins the cusp point, namely if $\rho < \rho^c$ and π_k is low enough. Regime $\mathcal{R}_2$ appears above the upper branch that joins the cusp point, namely if $\pi_k > \pi_k^c$ and ρ is high enough. If $\pi_k < \pi_k^c$ and $\rho > \rho^c$, regimes $\mathcal{R}_1$ or $\mathcal{R}_2$ can emerge depending on the size of these parameters. In all of these configurations, the long-run steady state is unique (and well determinate, as we have seen). In contrast, for values of π_k and ρ located inside the two branches that join the cusp point, there is multiplicity because three steady states emerge (Regime $\mathcal{R}_3$). In Figure A1, the curve $\mathcal{H}$ depicts the locus of Hopf bifurcations.

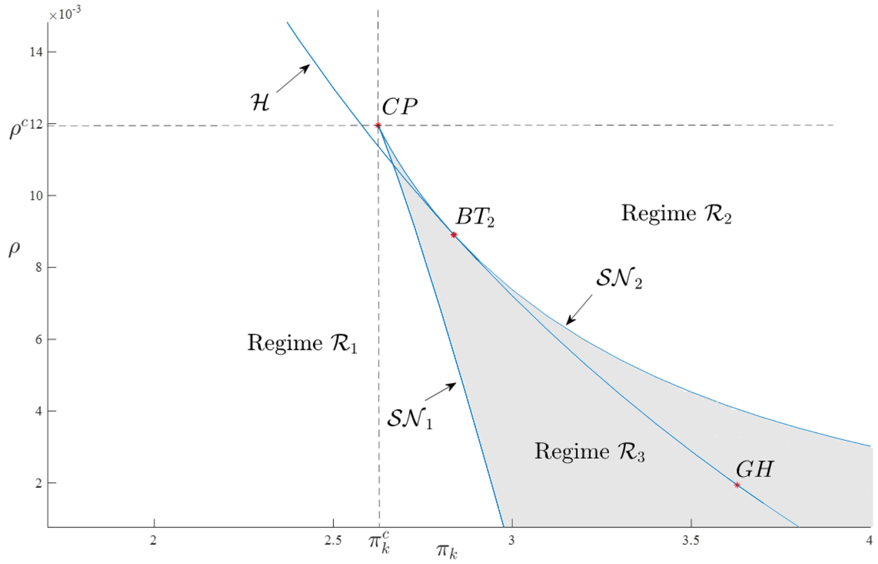

Figure A1: Bifurcation points as a function of the parameters (π_k, ρ)

In addition to the cusp bifurcation, our numerical analysis highlights two other kinds of codim-2 bifurcations.

The first is a Bogdanov-Takens (BT) bifurcation. In a two (or more) parameter system, such a bifurcation occurs when a Hopf bifurcation and a saddle-node bifurcation coincide in a single point of the parameter space. In Figure A1, it appears at the tangency point of a saddle-node curve $\mathcal{SN}$ and a Hopf-curve $\mathcal{H}$. The main interest of the BT bifurcation is the presence of a homoclinic orbit (i.e. a path that connects a steady state with itself), thus creating very long-lasting fluctuations. In our model, as there are two possible saddle-node bifurcations, a BT bifurcation can appear either (i) when steady-states D and M collide, or (ii) when steady-states G and M collide. In our benchmark calibration, only the second case arises for positive values of economic growth and environmental quality. Hence, the BT bifurcation is located at the crossing-point of the $\mathcal{SN}_2$ and $\mathcal{H}$ curves in Figure A1.

The second is a Generalized (Bautin) Hopf (GH) bifurcation. As emphasized in corollary 1, a (codim-1) Hopf bifurcation emerges in the neighborhood of the intermediate steady state M. This bifurcation is supercritical, generating a stable limit cycle, if the first Lyapunov coefficient is negative; or subcritical, generating an unstable closed orbit, if the coefficient is positive. The GH bifurcation appears at the limit case, namely when the first Lyapunov coefficient is zero. The existence of a GH bifurcation ensures the presence of stable limit cycles for nearby parameter values.

Table A1 computes the different codim 2 bifurcations and shows that they occur for low values of economic growth, namely between 0.1% and 1.2%.

	π_k	ρ	γ	Q
BT	0.55965	0.0107	0.0074	0.084
GH	0.61471	0.00665	0.0122	0.0795
CP	0.55582	0.01107	0.0108	0.104

Table 2: Codim 2 bifurcations

Appendix E. The Bogdanov-Takens bifurcation and homoclinic orbits

We prove the occurrence of a Bogdanov-Takens (BT) bifurcation and homoclinic orbits in the neighborhood of steady state M using a two-step proof. In the first step, based on our baseline calibration (see Table 1), we will show that there is a critical pair of parameters (ρ, π_k) that characterizes the BT singularity. In the second step, we demonstrate the existence of a homoclinic orbit around point M, using the argument that points D and M collide at the BT bifurcation.

Step 1: Preliminary. From Eqs. (C.1)-(C.4), the determinant and the trace of the jacobian matrix in the neighborhood of steady state M are

$$\det(\mathbf{J}^M) = c_k^M \left[(\alpha + g - 1)y_{kc}^M + 1\right] \left[E'(Q^M) - \frac{y_{kQ}^M(1-\alpha-\beta)}{\pi_k}\right] + \frac{c_k^M(\alpha+g-1)(1-\alpha-\beta)y_{kQ}^M y_{kc}^M}{\pi_k}, \quad \text{(E.1)}$$

$$\text{trace}(\mathbf{J}^M) = (\alpha + g - 1)c_k^M y_{kc}^M + c_k^M + E'(Q^M) - \frac{y_{kQ}^M(1-\alpha-\beta)}{\pi_k}. \quad \text{(E.2)}$$

Thanks to numerical simulations based on the baseline calibration, we derive a pair of parameters $(\rho^{bt}, \pi_k^{bt}) = (0.01, 0.657)$ such that $\det(\mathbf{J}^M) = \text{trace}(\mathbf{J}^M) = 0$; hence, the Jacobian matrix $\mathbf{J}^M$ has a double zero eigenvalue.

Step 2: Homoclinic orbit.
We prove the occurrence of the BT bifurcation by applying a theorem that allows us to transform our system into a simpler, topologically-equivalent planar system of differential equations with well-known bifurcation diagram. We conclude using a Lemma that ensures the occurrence of homoclinic orbits.

Theorem (Kuznetsov, 1998, Theorem 8.4, page 321) Suppose that a planar system

$$\dot{x} = f(x, \Lambda),\ x \in \mathbb{R}^2,\ \Lambda \in \mathbb{R}^2,$$

with smooth f, has at $\Lambda = 0$ the equilibrium $x = 0$ with a double zero eigenvalue

$$\lambda_{1,2} = 0.$$

Assume the following generic conditions are satisfied
(BT.0) the Jacobian matrix $A(0) = f_x(0, 0) \neq 0$;
(BT.1) $a_{20}(0) + b_{11}(0) \neq 0$;
(BT.2) $b_{20}(0) \neq 0$;
(BT.3) the map

$$(x, \Lambda) \mapsto \left(f(x, \Lambda), \operatorname{tr}\left(\frac{\partial f(x, \Lambda)}{\partial x}\right), \det\left(\frac{\partial f(x, \Lambda)}{\partial x}\right)\right)$$

is regular at point $(x, \Lambda) = (0, 0)$.

Then, there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparameterization, and smooth invertible parameter changes, which together reduce the system to

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 + O(||\eta||^3), \end{cases}$$

where $s := \operatorname{sgn}[b_{20}(a_{20}(0) + b_{11}(0))] = \pm 1$. ■

Let $\Lambda := (\rho - \rho^{bt}, \pi_k - \pi_k^{bt})$ and $x := (c_k - c_k^M, Q - Q^M)$. Clearly, at $\Lambda = 0$, the equilibrium $x = 0$ has a double zero eigenvalue. We need to ensure conditions (BT.0)-(BT.3).

Condition (BT.0). Using Eq. (C.1), at point M we have

$$CC^M = -\frac{\beta(\alpha + g - 1)}{\varepsilon} y_k^M + c_k^M.$$

From our baseline calibration, at $(\rho, \pi_k) = (\rho^{bt}, \pi_k^{bt})$ we compute $CC^M \simeq 0.345$. Consequently, the jacobian matrix $\mathbf{J}^M$ evaluated at $(\rho, \pi_k) = (\rho^{bt}, \pi_k^{bt})$ is non-zero.

Conditions (BT.1) and (BT.2). Numerically, we compute the generic BT parameters, and show that $a_{20}(0) + b_{11}(0) \neq 0$ and $b_{20}(0) \neq 0$ for a large constellation of parameters. Using our baseline calibration, we find $a_{20} \simeq -6.32e^{-4}$, $b_{20} \simeq 0.8$ and $b_{11} \simeq 3.32$.

Conditions (BT.3). Let $\phi : (x, \Lambda) \mapsto \big(f(x, \Lambda), \operatorname{Tr}(\mathbf{J}^M), \det(\mathbf{J}^M)\big)$. Numerically, we ensure that $\det(\phi(0, 0)) \neq 0$ for a large constellation of parameters.

Finally, according to the above-mentioned theorem, our system is topologically-equivalent to the following two-differential equation system in the neighborhood of equilibrium M

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2, \end{cases} \tag{E.3}$$

where β_1 and β_2 are combinations of parameters. The coefficient on $\eta_1\eta_2$ is -1, since the periodic orbit around point M is stable (the first Lyapunov coefficient is negative in our baseline calibration). Thus, the bifurcation diagram is usually depicted in the (β_1, β_2)-plane (Kuznetsov, 1998, section 8.4.2), where the origin corresponds to the BT bifurcation.