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Published Version:
Conte, G., Perdon, A.M., Zattoni, E. (2021). A structural approach to unknown inputs observation for switching linear systems. AUTOMATICA, 129, 1-12 [10.1016/j.automatica.2021.109572].

Availability:
This version is available at: https://hdl.handle.net/11585/870179 since: 2022-02-26

Published:
DOI: http://doi.org/10.1016/j.automatica.2021.109572

A Structural Approach to Unknown Inputs Observation for Switching Linear Systems

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Abstract

The problem of devising an asymptotic observer for a given function of the state of a switching linear system in the presence of unknown inputs is considered. Solvability is studied both in the case of sufficiently large dwell time and in that of dwell time greater than a fixed threshold. A complete characterization of solvability in terms of necessary and sufficient conditions is given in both cases. It is shown that the necessary and sufficient conditions can be checked in practice in the first case and, under slightly more restrictive hypotheses, also in the second case by means of algorithmic procedures, which also provide a method to synthesize the observer sought for. The employed methodology makes use of geometric concepts to reveal the structural aspects of the problem and to derive its solutions. In particular, a key role is played by the novel notion of robust conditioned invariant subspace that is minimal with respect to the properties of containing a given subspace and of being externally stabilizable.

Key words: Switching linear systems; asymptotic observers; unknown-input observers; structural methods.

1 Introduction

A switching linear system Σ_σ consists of a finite family $\{\Sigma_i,\, i \in I\}$ of linear systems, called modes, and of a switching signal $\sigma(t)$ that defines the active mode at each time instant as well as the transition, at the so-called switching times, from one mode to another. Herein, switching signals are assumed to be piecewise constant time functions with values in a finite index set I that exhibit a finite number of discontinuity points in each finite interval. Linear switching systems can effectively model several complex phenomena that arise in the behavior of mechanical and electrical systems, as well as in other application areas. Examples are given by structural modifications due to abrupt changes in the mechanical configuration, to the action of switches, to failure or degradation of performances in subsystems or components.

In the framework of switching linear systems, a considerable research effort has been devoted to the investigation of stability and stabilization issues as well as to the study of classical control and estimation problems. This is documented by the wide recent literature, which counts, amongst others, a relevant number of surveys (Lin and Antsaklis, 2009; Zhu and Antsaklis, 2015), journal special issues (Antsaklis and Nerode, 1998; Morse et al., 1999; Di Bernardo et al., 2003; Cassandras and Giua, 2008; Giua et al., 2011; Heemels and De Schutter, 2013; Fränzle et al., 2018) and books (Van der Schaft and Schumacher, 2000; Savkin and Evans, 2002; Sun and Ge, 2005, 2011; Zhao et al., 2017).

In particular, the problem of observing the full state of a switching linear system and the problem of observing a linear function of the state in the presence of unknown inputs have been studied by several authors through different approaches. As to the design of full state observers in case the input is known, in (Alessandri and Coletta, 2001), sufficient conditions for the existence of state observers which guarantee quadratic stability of the estimation error dynamics are given in terms of LMIs, on the assumption that the switching signal is known. Instead, in (Pettersson, 2006), Lyapunov techniques are used to tackle the full state estimation problem both in the case of known switching signals and in that of unknown ones. As to the estimation of a linear function of the state in the presence of unknown inputs, system invertibility and a parametrization of the observer gain are at the basis of unknown input observer design for discrete-time switching systems in (Sundaram and Hadjicostis, 2006). High-order sliding mode techniques are exploited in (Bejarano and Pisano, 2011) and in (Bejarano et al., 2011), where sufficient constructive conditions for the solution of the unknown-input observation problem, either for arbitrary switching signals or for a restricted class of switching

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signals, are stated under specific assumptions. A time-varying state transformation is exploited to gather the partial information derived from the individual modes and obtain a reduced-order unknown-input observer in (Defoort et al., 2012). Further developments along these lines, made possible by the combination with sliding mode techniques, are presented in (Van Gorp et al., 2014). Coordinate transformations, together with Lyapunov and LMI techniques, enable the synthesis of unknown input observers in (Huang and Chen, 2014). Time-varying coordinate transformations and dwell-time techniques are employed in (Ma et al., 2017) with the same purpose.

In the papers cited above, the objective is to characterize the existence of a suitable asymptotic observer through sufficient conditions and to find viable methods for its construction. Instead, in the first part of this work, the solvability of the considered unknown input observation problems is fully characterized by necessary and sufficient conditions, which leverage on distinctive, structural and qualitative properties of suitably defined subspaces of the system state space. This is achieved by adopting a structural point of view, based on geometric notions and tools, that, together with the theoretical results, is able to provide a complete computational support for the synthesis of the desired unknown input observers.

In more details, two different formulations of the unknown input observation problem are investigated in this work. In the first formulation, the class of switching signals for which a solution is sought for is not specified, meaning that the estimation error is required to converge asymptotically to zero for sufficiently slow switching. In the second, stronger formulation, the class of switching signals is specified and the observer should guarantee asymptotic convergence to zero of the estimation error for all switching signals in that class.

The first main theoretical contribution of the paper, in Theorem 2, consists in stating necessary and sufficient solvability conditions for both problems by means of a structural condition and a qualitative one. This formulation allows a deep insight into the structure of the problems, but it does not provide a direct procedure to check the existence of solutions in concrete situations. In order to overcome this difficulty, the solvability conditions are formulated in an equivalent but more constructive way, respectively in Theorem 3 for the slow switching case and, under sliglthy more restrictive hypothesis, in Theorem 4 for the case in which the class of switching signals is specified. In this new formulations, the solvability conditions are checkable by means of viable, algorithmic procedures, which are based on the computation of specific subspaces of the system state space and on the analysis of their properties and which also allow the practical construction of solutions.

The key elements of our approach is the notion of robust conditioned invariance that applies to the subspaces of the state space of a switching system and that was described in (Conte et al., 2017a). In the case of slow switching, solvability of the unknown input observation problem is characterized in terms of the minimal conditioned invariant subspace in a suitable set that enjoys the property of external stabilizability. The existence of such object is proved by devising a viable algorithm for its construction, which can be used for checking solvability and, possibly, for constructing solutions. In the case in which the class of switching signals is specified, under a slightly restrictive structural assumption, solvability is characterized in terms of the maximal conditioned invariant subspace in a suitable set and by giving a practical characterization of its external stabilizability. Also in this case, we give a procedure to construct such object, which can be used for checking solvability and, possibly, for constructing solutions.

Some, earlier results on the design of unknown input observers for switching linear systems in the conceptual framework of the structural approach were presented in (Conte et al., 2017b). The class of observers considered in that paper as possible solutions of a general unknown input observation problem is strictly smaller than the class of observer considered here, causing the solvability conditions found there to be strictly more restrictive than those given here. For the same reason, the results as well as the theoretical issues and the computational tools presented in this paper turn out to be more complex than those discussed in (Conte et al., 2017b), although they are of the same nature, and new arguments are required to confirm their validity. Moreover, Theorem 2 gives a deep insight into the structure of the general unknown input observation problem that cannot be obtained from the results of the previous conference paper.

The paper is organized as follows. In Section 2, the class of switching linear systems dealt with is introduced and some basic facts about their stability are reviewed. In Section 3, the unknown input observation problem is formally stated, making distinction between the case where the class of switching signals for which the observation error asymptotically convergences to zero is not specified (Unknown Input Observation Problem or UIOP) and that in which asymptotic convergence is required to hold for all switching signals in an assigned class (Strong Unknown Input Observation Problem or SUIOP). In Section 4, the geometric notions of robust conditioned invariance and of external stabilizability for a subspace of the system state space are introduced and then used to state and to prove, in Theorem 2, the necessary and sufficient solvability conditions for both problems.The “if” part of the proof of the theorem provides a viable procedure to construct an observer that solves the problem, starting from the existence of a subspace that satisfies the conditions of the statement. In Section 5, the solvability conditions are formulated in a more constructive way for the UIOP, in Theorem 3, and for the SUIOP, in Theorem 4, with the advantage of stating conditions that can be practically checked. This is obtained by introducing, in Proposition 5, a number of geometric objects that enjoy special structural properties. The existence of such objects, which are the extremal elements in suitable lattices of robust conditioned invariant subspaces, is proved in Section 6 by providing algorithmic procedures to construct them. In Section 7, a characterization of the qualitative, crucial property of external stabilizabilty of a robust conditioned invariant subspace is introduced. A worked out numerical example aimed at illustrating how to apply the procedure, based on the previous theoretical results, for the synthesis of an unknown input observer is presented in

Section 8. Finally, Section 9 contains concluding remarks and directions for future work.

Notation: The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ are used to denote the sets of real numbers, non negative real numbers and non negative integer numbers, respectively. Real vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The subspace orthogonal to a given subspace $\mathcal{V}$ is denoted by $\mathcal{V}^\perp$. Linear maps between vector spaces and the associated matrices are denoted by the same slanted capital letters, like A. Therefore, the statements $A \in \mathbb{R}^{p\times q}$ and $A : \mathbb{R}^q \to \mathbb{R}^p$ are consistent. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The image of a subspace $\mathcal{V}$ under a map A is simply denoted by $A\mathcal{V}$. The transpose of A is denoted by $A^\top$. The symbols I_n, $0_{m\times n}$ and 0_n are respectively used for the identity matrix of dimension n, for the $m \times n$ zero matrix and for the n-dimensional zero vector (subscripts are omitted if the dimensions are clear from the context).

2 Switching Linear Systems

A continuous-time switching linear system Σ_σ is a dynamical system described by the equations

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) \;=\; A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t), \\ y(t) \;=\; C_{\sigma(t)}\, x(t), \end{cases} \tag{1}$$

where $t \in \mathbb{R}^+$ is the time, $x \in \mathcal{X} = \mathbb{R}^n$ the state, $u \in \mathcal{U} = \mathbb{R}^m$ the input, $y \in \mathcal{Y} = \mathbb{R}^p$ the output. Letting $I = \{1, \ldots, N\}$ denote a finite index set, the function $\sigma : \mathbb{R}^+ \to I$ is a piecewise-constant, left continuous function that represents the time-driven switching signal. For any value $i \in I$, A_i, B_i, C_i are real matrices of suitable dimensions. In particular, B_i is assumed to be full column-rank and C_i full row-rank for all $i \in I$. The time-invariant linear systems

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) \;=\; A_i\, x(t) + B_i\, u(t), \\ y(t) \;=\; C_i\, x(t), \end{cases} \quad \text{with } i \in I,$$

are called the *modes* of Σ_σ. The indexed family $\Sigma = \{\Sigma_i\}_{i\in I}$ is said the family of modes and the active mode at the time t is specified by the value of $\sigma(t) \in I$.

The discontinuity points of σ are referred to as the *switching times*. For any switching signal σ, the *dwell time*, henceforth denoted by τ_σ, is defined as the lower bound of the set of the lengths of the time intervals between two consecutive switching times. The symbol $\mathscr{S}_\alpha$, where $\alpha \in \mathbb{R}^+$, denotes the set of the switching signals such that $\tau_\sigma \geq \alpha$. With a slight abuse of notation, $\mathscr{S}_0$ denotes the set of the switching signals that only cause finitely many switches in any time interval of finite length, so to avoid Zeno phenomena.

Given $\sigma \in \mathscr{S}_0$, the discontinuity points of σ form a finite or countable, ordered set $T_\sigma = \{t_0 = 0, t_1, t_2, ...\}$. With this notation, given $\sigma \in \mathscr{S}_0$, an initial state $x(0) = x_0 \in \mathcal{X}$ and an input function $u(t)$ defined over the interval $[0, \bar{t})$, the solution $x(t)$ to the state equations of (1) can be recursively defined for any $t \in [0, \bar{t}]$ by

$$x(t) = e^{A_{i_k}(t-t_{k-1})}x(t_{k-1}) + \int_{t_{k-1}}^{t} e^{A_{i_k}(t-\tau)}B_{i_k}u(\tau)d\tau$$

where $t \in (t_{k-1}, t_k]$ and $\sigma(t) = i_k$ for $t \in (t_{k-1}, t_k]$. Since $x(t)$ depends on the choice of σ and on t, $x(0)$ and $u(t)$ over $[0, t)$, we will use the notation $x(t) = \phi_\sigma(t, x(0), u(\tau)|_{[0,t)})$.

Definition 1 *A switching linear system Σ_σ is said to be globally asymptotically stable over $\mathscr{S}_\alpha$ if its state goes to* 0 *as the time goes to infinity, for any initial condition and for any switching signal $\sigma \in \mathscr{S}_\alpha$.*

Stability of switching systems depends on the stability properties of the modes and also on the switching signal. Asymptotic stability of all the modes is not sufficient to imply global asymptotic stability for arbitrary switching (see, e.g., Lin and Antsaklis, 2009). However, global asymptotic stability is guaranteed if all the modes are asymptotically stable and the switching signal is suitably constrained, as is specified in the following theorem.

Theorem 1 *(see Morse, 1996, Lemma 2) Let all the modes Σ_i, with $i \in I$, of Σ_σ be asymptotically stable. Then, there exists $\alpha \in \mathbb{R}^+$ such that Σ_σ is globally asymptotically stable over $\mathscr{S}_\alpha$.*

The proof of Theorem 1 given in (Morse, 1996, Lemma 2) shows that if, for all $i \in I$, we have $||e^{A_i t}|| \leq e^{(a_i - \lambda_i t)}$ for some $a_i, \lambda_i \in \mathbb{R}$, with $\lambda_i > 0$, for all $t \geq 0$, then Σ_σ is globally asymptotically stable, in particular, over $\mathscr{S}_\alpha$ for $\alpha \geq \max_{i\in I} a_i/\lambda_i$. We speak, in that case, of *slow switching.*

Stability over $\mathscr{S}_\alpha$ for a given α is more difficult to be characterized. Sufficient conditions in terms of LMIs can be found in (Chesi et al., 2012) and in the references therein.

3 Unknown-Input Observer Problems

Given the switching linear system Σ_σ described by (1), where the input $u(t)$ is assumed to be unknown, and a family of matrices $\mathscr{H} = \{H_i,\, H_i \in \mathbb{R}^{q\times n}\}_{i\in I}$, the observation problem in the presence of the unknown input consists in finding, if possible, an asymptotic estimate, denoted by $w(t)$, of the state function $H_{\sigma(t)}\, x(t)$, by exploiting the knowledge of the output $y(t)$ and of the active mode $\Sigma_{\sigma(t)}$ of the switching linear system Σ_σ. Since the instantaneous knowledge of the active mode of Σ_σ is equivalent to the knowledge, at the time t, of the value taken by σ at the same time t for all $t \in \mathbb{R}^+$, it will henceforth be said that σ is assumed to be measurable.

The estimate $w(t)$ is regarded as the output of a switching linear system $\Sigma_{O\sigma}$, the candidate *observer*, described by

$$\Sigma_{O\sigma} \equiv \begin{cases} \dot{z}(t) \;=\; A_{O\sigma(t)}\, z(t) + B_{O\sigma(t)}\, y(t), \\ w(t) \;=\; C_{O\sigma(t)}\, z(t) + D_{O\sigma(t)}\, y(t). \end{cases} \tag{2}$$

The real vector spaces $\mathcal{Z}=\mathbb{R}^r$ and $\mathcal{W}=\mathbb{R}^q$ represent the state space and the output space of $\Sigma_{O\sigma}$, respectively. The fundamental properties of the estimate $w(t)$ are expressed by considering the so-called *estimation error*, which is defined by

$$e(t) = w(t) - H_{\sigma(t)}\, x(t). \qquad (3)$$

The main requirements on $\Sigma_{O\sigma}$ are that the time evolution of $e(t)$ is not influenced by the input $u(t)$ and that $e(t)$ goes asymptotically to 0 as t goes to $+\infty$ for all σ belonging to a set $\mathscr{S}_\alpha$ of interest. This means that $w(t)$ converges to $H_{\sigma(t)}x(t)$ as t goes to $+\infty$ independently of $u(t)$ for all σ belonging to $\mathscr{S}_\alpha$. The set $\mathscr{S}_\alpha$ can be defined by assigning explicitely α or by asking that the previous requirement is satisfied for α sufficiently large, i.e. for sufficiently slow switching.

By writing $e(t) = \phi_\sigma(t, x(0), z(0), u(\tau)|_{[0,t)})$, we state formally the above requirement about independence of the evolution of $e(t)$ from $u(t)$ as follows:

$\mathfrak{R}.1$ *For all $\bar\sigma \in \mathscr{S}_0$, all $x_0 \in \mathcal{X}$, all $z_0 \in \mathcal{Z}$ and all pairs of input signals $u_1(t)$ and $u_2(t)$, one has, for all $t \in \mathbb{R}^+$,*

$$\phi_{\bar\sigma}(t, x_0, z_0, u_1(\tau)|_{[0,t)}) = \phi_{\bar\sigma}(t, x_0, z_0, u_2(\tau)|_{[0,t)}).$$

and the requirement about asymptotic convergence to 0 of $e(t)$ as follows:

$\mathfrak{R}.2$ *For all $\bar\sigma \in \mathscr{S}_\alpha$ with α given or, respectively, with α sufficiently large, all $x_0 \in \mathcal{X}$, all $z_0 \in \mathcal{Z}$ and all input signals $u(t)$, one has*

$$lim_{t\to+\infty}\phi_{\bar\sigma}(t, x_0, z_0, u(t)|_{[0,t)}) = 0.$$

It is also quite natural to ask for global asymptotic stability over $\mathscr{S}_\alpha$ (either for a given α or for α sufficiently large) of the observer $\Sigma_{O\sigma}$, which is used in practice to obtain the desired estimate. In the rest of the paper we will consider only candidate observers which satisfy such condition.

A further natural requirement on $\Sigma_{O\sigma}$ is that for any state $x_0 \in \mathcal{X}$ there exists a state $z_0 \in \mathcal{Z}$ such that, initializing Σ_σ at $t = 0$ in $x(0) = x_0$ and $\Sigma_{O\sigma}$ in $z(0) = z_0$, one has $e(t) = 0$ for all $t \in \mathbb{R}^+$, for all the input signals $u(\cdot)$ and for all the switching signals $\sigma \in \mathscr{S}_0$. Actually, this means that, if $x(0)$ is known, the observer can be initialized in such a way that its output coincides with the function $H_{\sigma(t)}x(t)$ to be estimated. Without loss of generality, we can also assume that the state space $\Sigma_{O\sigma}$ has only as many elements as needed to fulfil the former condition. We state formally the above requirement as follows.

$\mathfrak{R}.3$ *There exists a linear map $P: \mathcal{X} \to \mathcal{Z}$, with $\operatorname{Im} P = \mathcal{Z}$, such that for all $\bar\sigma \in \mathscr{S}_0$, all $x_0 \in \mathcal{X}$ and all input signals $u(t)$, one has, for all $t \in \mathbb{R}^+$,*

$$e(t) = \phi_{\bar\sigma}(t, x_0, Px_0, u(\tau)|_{[0,t)}) = 0.$$

Any map $P: \mathcal{X} \to \mathcal{Z}$ such that $\mathfrak{R}.3$ holds is said to be an *exact initialization map*. Moreover, $\Sigma_{O\sigma}$ is said to be *exactly initialized* if, for any given initial condition $x(0) = x_0$ of Σ_σ, the initial condition of $\Sigma_{O\sigma}$ is set to $z(0) = P\,x_0$.

Note that $\mathfrak{R}.1$ and $\mathfrak{R}.3$ are structural requirements, since their formulation does not involve any constraint on the dwell time. Instead, $\mathfrak{R}.2$ is a qualitative requirement that makes sense either for a given α or for α sufficiently large and, in both cases, the set of the dwell times of the considered switching signals has a lower bound. Accordingly, the unknown-input observation problem can be given two different statements, depending on whether α is a-priori assigned or not, as follows.

Problem 1 (Unknown-Input Observation Problem) *Let the switching linear system Σ_σ and the family of linear maps $\mathscr{H} = \{H_i,\ H_i: \mathcal{X} \to \mathbb{R}^q\}_{i\in I}$ be given. Let the switching signal σ be measurable. The Unknown-Input Observer Problem (UIOP) consists in finding an observer $\Sigma_{O\sigma}$ of the form (2) such that the qualitative requirement $\mathfrak{R}.2$ is satisfied for some $\alpha \in \mathbb{R}^+$, together with the structural requirements $\mathfrak{R}.1$ and $\mathfrak{R}.3$.*

Problem 2 (Strong Unknown-Input Observation Problem) *Let the switching linear system Σ_σ, the family of linear maps $\mathscr{H} = \{H_i,\ H_i: \mathcal{X} \to \mathbb{R}^q\}_{i\in I}$ and a positive real constant α be given. Let the switching signal σ be measurable. The Strong Unknown-Input Observer Problem (SUIOP) consists in finding an observer $\Sigma_{O\sigma}$ of the form (2) such that the qualitative requirement $\mathfrak{R}.2$ is satisfied for the given α, together with the structural requirements $\mathfrak{R}.1$ and $\mathfrak{R}.3$.*

Remark 1 *In (Conte et al., 2017b), the authors considered a more restrictive formulation of the unknown input observation problem for switching systems by limiting the observer to have strictly proper modes, instead of proper modes as in (2). That choice allowed a simpler development of the theory, but, in the light of the present paper, it appears to be unnecessarily restrictive, since it prevents to exploit fully in the construction of an observer the information contained in $y(t)$. As a result, the solvability conditions found in (Conte et al., 2017b) are more restrictive than those given here in Theorem 2.*

4 Solution to the unknown input observation problems

In this section, necessary and sufficient solvability conditions for the unknown-input observation problems stated above are presented in structural terms. In this regard, it is worth mentioning that the structural notions and some of the geometric objects illustrated here have been first considered in the framework of parameter depending systems in Basile and Marro (1987); Conte et al. (1991). More recently, they were considered in relation to observation problems in (Conte et al., 2017b,a).

4.1 Structural notions

The structural notion that is most relevant to the investigation of the unknown-input observation problems considered herein is that of conditioned invariance. A subspace $\mathcal{S}_R \subseteq \mathcal{X}$ is said to be a *(robust) conditioned invariant* *subspace for* Σ_σ if $\mathcal{S}_R$ is a conditioned invariant subspace for all the modes of Σ_σ: i.e., if

$$A_i(\mathcal{S}_R \cap \operatorname{Ker} C_i) \subseteq \mathcal{S}_R \quad \text{for all } i \in I. \tag{4}$$

In the papers (Basile and Marro, 1987; Conte et al., 1991), where the concepts of invariance (i.e., invariance with respect to the system dynamics and controlled invariance, since the problem tackled therein was a decoupling problem) were first defined for families of linear time-invariant systems, these notions were distinguished from the classic ones by the adjective *robust*. In this work, the switching linear systems for which the considered invariance properties hold are always specified. Thus, the adjective *robust* becomes redundant and, when speaking of conditioned invariant subspaces, it is dropped, although we keep the subscript $_R$.

Given a conditioned invariant subspace $\mathcal{S}_R$ for Σ_σ, an indexed family of linear maps $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ such that

$$(A_i + G_i\, C_i)\, \mathcal{S}_R \subseteq \mathcal{S}_R \quad \text{for all } i \in I \tag{5}$$

is said to be a *friend* of $\mathcal{S}_R$.

Proposition 1 *A subspace $\mathcal{S}_R \subseteq \mathcal{X}$ is a conditioned invariant subspace for Σ_σ if and only if there exists an indexed family of linear maps $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ such that (5) holds.*

Proof. Note that (4) is equivalent to $A_i^\top \mathcal{S}_R^\perp \subseteq \mathcal{S}_R^\perp + \operatorname{Im} C_i^\top$ for all $i \in I$. Then, given a matrix $P^\top$ whose columns form a basis of $\mathcal{S}_R^\perp$, the previous inclusion is equivalent to the existence of an indexed family of pairs of real matrices $\{(L_i, M_i)\}_{i \in I}$ such that

$$A_i^\top P^\top = P^\top L_i^\top + C_i^\top M_i^\top \quad \text{for all } i \in I. \tag{6}$$

By transposing (6) and taking, for any $i \in I$, a matrix G_i such that $P\, G_i = -\, M_i$, one gets

$$P\,(A_i + G_i\, C_i) = L_i\, P \quad \text{for all } i \in I. \tag{7}$$

Since $P^\top$ is a basis matrix of $\mathcal{S}_R^\perp$, (7) implies $P\,(A_i + G_i\, C_i)\, \mathcal{S}_R = L_i\, P\, \mathcal{S}_R = \{0\}$ for all $i \in I$, which, in turn, implies (5), since P is full row rank. Conversely, given a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$, let $M_i = -\, P\, G_i$ and construct L_i, with $i \in I$, by solving the set of linear equations (6). Since, as seen above, (6) is equivalent to (5), the thesis follows. □

It is important to remark that the proof of the equivalence between conditioned invariance and existence of a friend in the framework of switching linear systems provides an algorithmic procedure for deriving a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$ from the indexed family $\{(L_i, M_i)\}_{i \in I}$ and viceversa.

Any friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of a conditioned invariant subspace $\mathcal{S}_R$ for Σ_σ can be seen as a family of output injections which, applied to the corresponding modes of Σ_σ, define a new switching linear system

$$\Sigma_\sigma^{\mathscr{G}} \equiv \begin{cases} \dot{x}(t) \;=\; (A_{\sigma(t)} + G_{\sigma(t)}\, C_{\sigma(t)})\, x(t) + B_{\sigma(t)}\, u(t), \\ y(t) \;=\; C_{\sigma(t)}\, x(t). \end{cases}$$

By (5), the subspace $\mathcal{S}_R$ is invariant for the switching linear system $\Sigma_\sigma^{\mathscr{G}}$. Thus, $\Sigma_\sigma^{\mathscr{G}}$ induces a switching linear dynamics on the subspace $\mathcal{S}_R$ and a switching linear dynamics on the quotient space $\mathcal{X}/\mathcal{S}_R$. The former will be denoted by $\Sigma_\sigma^{\mathscr{G}}|_{\mathcal{S}_R}$ and the latter will be denoted by $\Sigma_\sigma^{\mathscr{G}}|_{\mathcal{X}/\mathcal{S}_R}$.

Proposition 2 *Let $\mathcal{S}_R \subseteq \mathcal{X}$ be a conditioned invariant subspace for Σ_σ and let $P^\top$ be a matrix whose columns are a basis of $\mathcal{S}_R^\perp$. Then, given a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$, the switching linear dynamics induced by $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ is described, up to a change of basis in $\mathcal{X}/\mathcal{S}_R$, by the indexed family of matrices $\{L_i\}_{i \in I}$, where $\{(L_i, M_i)\}_{i \in I}$ is a family of pairs which satisfy (6) with $M_i = -PG_i$ for all $i \in I$.*

Proof. The modes of $\Sigma_\sigma^{\mathscr{G}}$ are described by

$$\Sigma_i^{\mathscr{G}} \equiv \begin{cases} \dot{x}(t) \;=\; (A_i + G_i\, C_i)\, x(t) + B_i\, u(t), \\ y(t) \;=\; C_i\, x(t), \end{cases} \quad \text{with } i \in I.$$

By applying the change of basis $x = T\,\xi = [P^\top S]\xi$, where $P^\top$ is a matrix whose columns are a basis of $\mathcal{S}_R^\perp$ and S is a matrix whose columns are a basis of $\mathcal{S}_R$, one gets

$$\Sigma_i^{\mathscr{G}} \equiv \begin{cases} \dot{\xi}(t) \;=\; A_i'\, \xi(t) + B_i'\, u(t), \\ y(t) \;=\; C_i'\, \xi(t), \end{cases} \quad \text{with } i \in I,$$

where, in particular,

$$A_i' = T^{-1}(A_i + G_i\, C_i)\, T = \begin{bmatrix} A_{11i}' & 0 \\ A_{21i}' & A_{22i}' \end{bmatrix} \quad \text{for all } i \in I.$$

The lower block-triangular form of A_i' shows that, for all $i \in I$, the matrix A_{11i}' describes the dynamics induced by $\Sigma_i^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$, while A_{22i}' describes that induced on $\mathcal{S}_R$. Let $\Pi : \mathcal{X} \to \mathcal{X}/\mathcal{S}_R$ denote the canonical projection. It is easy to see that $A_{11i}'\,\Pi = \Pi\, T^{-1}(A_i + C_i\, G_i)\, T$ and, since $\Pi\, T^{-1} = P$, also that $A_{11i}'\, P = P\,(A_i + C_i\, G_i)$. Hence, as P is a full row rank matrix, the comparison of the latter equation with (7) proves that $A_{11i}' = L_i$, where $\{(L_i, M_i)\}_{i \in I}$, with $M_i = -\, P\, G_i$ for all $i \in I$, is a family of pairs which satisfy (6). □

The notions of *external stabilizability* and *external α-stabilizability* of a conditioned invariant subspace $\mathcal{S}_R$ can now be introduced as follows.

Definition 2 A conditioned invariant subspace $\mathcal{S}_R$ for Σ_σ is said to be

- *externally stabilizable* if there exists a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$ such that the dynamics induced by $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \in \mathbb{R}^+$;
- *externally α-stabilizable* if, given $\alpha \geq 0$, there exists a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$ such that the dynamics induced by $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ is globally asymptotically stable over $\mathscr{S}_\alpha$.

4.2 Solvability conditions

Using the notions introduced above, we can now state the necessary and sufficient conditions for the solvability of the UIOP and the SUIOP.

Theorem 2 *Let the switching linear system Σ_σ, the family of linear maps $\mathscr{H}=\{H_i,\, H_i:\mathcal{X}\to\mathbb{R}^q\}_{i\in I}$ and the positive real constant α be given. Then, the UIOP defined by Σ_σ and $\mathscr{H}$ and, respectively, the SUIOP defined by Σ_σ, $\mathscr{H}$ and α are solvable if and only if there exists a conditioned invariant subspace $\mathcal{S}_R$ with $\operatorname{Im} B_i \subseteq \mathcal{S}_R$ for all $i\in I$ such that*

- **S**) $S_R \cap \operatorname{Ker} C_i \subseteq \operatorname{Ker} H_i$ *for all $i\in I$ (structural condition);*
- **Q**) S_R *is externally stabilizable or, respectively, externally α-stabilizable (qualitative condition).*

In order to prove Theorem 2, it is convenient to introduce, in relation to a candidate observer $\Sigma_{O\sigma}$ of the form (2), the auxiliary variable

$$e_{aux}(t)=z(t)-P\,x(t), \tag{8}$$

whose value at $t=0$ is the initialization error $z(0)-P\,x(0)$. Moreover, it is useful to state the following two propositions.

Proposition 3 *Given a switching linear system Σ_σ of the form (1), a family of linear maps $\mathscr{H}=\{H_i, H_i:\mathcal{X}\to\mathbb{R}^q\}$ and a candidate observer $\Sigma_{O\sigma}$ of the form (2), let $P:\mathcal{X}\to\mathcal{Z}$ be a linear map such that $\mathfrak{R}.3$ holds. Then, P satisfies*

$$H_i = C_{Oi}P + D_{Oi}C_i \quad \textit{for all } i\in I. \tag{9}$$

Proof. $\mathfrak{R}.3$ implies, in particular,

$$\begin{aligned} e(0) &= \phi_\sigma(0,x_0,Px_0,u(0)) \\ &= C_{O\sigma(0)}z(0)+D_{O\sigma(0)}y(0)-H_{\sigma(0)}x_0 \\ &= C_{O\sigma(0)}Px_0+D_{O\sigma(0)}C_{\sigma(0)}x_0-H_{\sigma(0)}x_0 \\ &= (C_{O\sigma(0)}P+D_{O\sigma(0)}C_{\sigma(0)}-H_{\sigma(0)})x_0=0 \end{aligned}$$

for all $x_0\in\mathcal{X}$ and for all $\sigma\in\mathscr{S}_0$. Taking σ in such a way that $\sigma(0)=i$, the thesis follows. □

Then, taking into account (1), (2), (3), (8) and (9), the estimation error $e(t)$ can be modeled as the output of the switching linear system $\Sigma_{E\sigma}$ described by

$$\Sigma_{E\sigma}\equiv\begin{cases} \dot e_{aux}(t) = A_{O\sigma(t)}e_{aux}(t)-P\,B_{\sigma(t)}\,u(t)+ \\ \quad (B_{O\sigma(t)}\,C_{\sigma(t)}-P\,A_{\sigma(t)}+A_{O\sigma(t)}\,P)\,x(t)\,. \\ e(t) = C_{O\sigma(t)}\,e_{aux}(t) \end{cases} \tag{10}$$

Proposition 4 *Given a switching linear system Σ_σ of the form (1), a family of linear maps $\mathscr{H}=\{H_i, H_i:\mathcal{X}\to\mathbb{R}^q\}$ and a candidate observer $\Sigma_{O\sigma}$ of the form (2), let $P:\mathcal{X}\to\mathcal{Z}$ be a linear map such that requirement $\mathfrak{R}.3$ is fulfilled. Then the requirement $\mathfrak{R}.1$ is fulfilled if and only if $\Sigma_{E\sigma}$ is an autonomous switching linear system: i.e., its state evolution is not affected by any exogenous inputs.*

Proof. Since we assume that $\Sigma_{O\sigma}$ has no unobservable states, also $\Sigma_{E\sigma}$, having the same dynamics and the same output map of $\Sigma_{O\sigma}$, has no unobservable states. Then, it is easy to see that $u(t)$ does not influence the time evolution of $e(t)$ (i.e. $\mathfrak{R}.1$ is fulfilled) if and only if $u(t)$ does not influence the time evolution of $e_{aux}(t)$ either directly, i.e through the input channel defined by $PB_{\sigma(t)}u(t)$, or indirectly, i.e. through the forced component of $x(t)$ and the input channel defined by $(B_{O\sigma(t)}\,C_{\sigma(t)}-P\,A_{\sigma(t)}+A_{O\sigma(t)}\,P)$. In other terms, if and only if

$$P\,B_i=0 \quad \text{for all } i\in I \tag{11}$$

$$B_{Oi}\,C_i-P\,A_i+A_{Oi}\,P=0 \quad \text{for all } i\in I \tag{12}$$

or, equivalently, if and only if $\Sigma_{E\sigma}$ is an autonomous system. □

4.3 Proof of Theorem 2

If. Let P be a $q\times n$ matrix with full row rank such that $\operatorname{Ker} P=\mathcal{S}_R$, so that, denoting by $\mathcal{Z}=\mathbb{R}^{n-q}$ the quotient space $\mathcal{X}/\mathcal{S}_R$, $P:\mathcal{X}\to\mathcal{Z}$ is the projection of $\mathcal{X}$ onto it. Since $\mathcal{S}_R\cap\operatorname{Ker} C_i\subseteq\operatorname{Ker} H_i$ for all $i\in I$, there exist matrices C_{Oi} and D_{Oi}, of suitable dimensions such that

$$C_{Oi}P \;+\; D_{Oi}\,C_i = H_i, \quad \text{for all } i\in I. \tag{13}$$

Moreover, as seen in Proposition 2, for any friend $\mathscr{G}=\{G_i,\,G_i:\mathcal{Y}\to\mathcal{X}\}_{i\in I}$, the dynamics induced on $\mathcal{Z}$ by that of $\Sigma_\sigma^{\mathscr{G}}$ is described by a family of matrices $\mathscr{L}=\{L_i, L_i\in\mathbb{R}^{q\times q}\}_{i\in I}$ such that $\{(L_i,-PG_i)\}_{i\in I}$ is a family of pairs which satisfy (6) and

$$L_iP=P(A_i+G_iC_i), \quad \text{for all } i\in I. \tag{14}$$

Choose $\mathscr{G}=\{G_i,\,G_i:\mathcal{Y}\to\mathcal{X}\}_{i\in I}$ in such a way that the dynamics induced on $\mathcal{Z}$ by that of $\Sigma_\sigma^{\mathscr{G}}$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for all α sufficiently large or, respectively, for the given α and consider the observer system

$$\Sigma_{O\sigma}\equiv\begin{cases} \dot z(t) = L_{\sigma(t)}z(t)-PG_{\sigma(t)}y(t) \\ w(t) = C_{O\sigma(t)}z(t)+D_{O\sigma(t)}y(t) \end{cases} \tag{15}$$

with $z\in\mathcal{Z}$, which is of the form (2) for $A_{O\sigma(t)}=L_{\sigma(t)}$ and $B_{O\sigma(t)}=-PG_{\sigma(t)}$. The estimation error $e(t)=w(t)-H_{\sigma(t)}x(t)$ can be expressed as the output of a suitable switching linear system by introducing the auxiliary variable $e_{aux}(t)=z(t)-Px(t)$. In fact, by (13), we have $e(t)=w(t)-H_{\sigma(t)}x(t)=C_{O\sigma(t)}z(t)+D_{O\sigma(t)}C_{\sigma(t)}x(t)-H_{\sigma(t)}x(t)=C_{O\sigma(t)}z(t)-C_{O\sigma(t)}Px(t)=C_{O\sigma(t)}e_{aux}(t)$ and the evolution of $e_{aux}(t)$ is described by $\dot e_{aux}(t)=L_{\sigma(t)}z(t)-PG_{\sigma(t)}C_{\sigma(t)}x(t)-P(A_{\sigma(t)}x(t)+B_{\sigma(t)}u(t))$.

Since $\operatorname{Im} B_i \subseteq \mathcal{S}_R = \operatorname{Ker} P$, we have $PB_i = 0$ and, therefore, using (14) we obtain the auxiliary error system

$$\Sigma_{E\sigma} \equiv \begin{cases} \dot{e}_{aux}(t) = L_{\sigma(t)} e_{aux}(t) \\ e(t) = C_{O\sigma(t)} e_{aux}(t) \end{cases} \quad (16)$$

Equation (16) shows that $\mathfrak{R}.1$ is satisfied since $\Sigma_{E\sigma}$ is an autonomous system and that $\mathfrak{R}.2$ is satisfied since $\Sigma_{E\sigma}$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for α sufficiently big or, respectively, for the given α. Moreover, choosing $z(0) = Px(0)$, we have $e(t) = 0$ for all $t \in \mathbb{R}^+$, for all $\sigma \in \mathscr{S}_0$ and for all input signals $u(t)$ and, therefore, also $\mathfrak{R}.3$ is satisfied.

Only if. Let $\Sigma_{O\sigma}$ be an observer of the form (2) such that requirements $\mathfrak{R}.1$, $\mathfrak{R}.3$ are satisfied and requirement $\mathfrak{R}.2$ is satisfied for α sufficiently big or, respectively, for the given α. Consider the auxiliary variable $e_{aux}(t) = z(t) - Px(t)$, where the matrix P that verifies (9) is determined by $\mathfrak{R}.3$, and the auxiliary error system $\Sigma_{E\sigma}$ is given by (10). By $\mathfrak{R}.1$ and (10) we have

$$\begin{aligned} &PB_i = 0 \\ &PA_i - B_{Oi}C_i - A_{Oi}P = 0 \quad \text{for all } i \in I. \end{aligned} \quad (17)$$

The second equality in (17) implies

$$\begin{aligned} &PA_i(\operatorname{Ker} P \cap \operatorname{Ker} C_i) = \\ &(B_{Oi}C_i + A_{Oi}P)(\operatorname{Ker} P \cap \operatorname{Ker} C_i) = \{0\} \\ &\text{for all } i \in I, \end{aligned} \quad (18)$$

which is equivalent to $A_i(\operatorname{Ker} P \cap \operatorname{Ker} C_i) \subseteq \operatorname{Ker} P$ for all $i \in I$. Therefore, $\mathcal{S}_R = \operatorname{Ker} P$ is a conditioned invariant subspace for Σ_σ which contains $\operatorname{Im} B_i$ for all $i \in I$. Since $\operatorname{Im} P = \mathcal{Z}$, we can write $B_{Oi} = -PG_i$ for suitable matrices G_i for all $i \in I$. So, the second equality in (17) gives $P(A_i + G_iC_i) = A_{Oi}P$ for all $i \in I$ which says that $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ is a friend of $\mathcal{S}_R$ and that the dynamics induced on $\mathcal{X}/\mathcal{S}_R$ by that of $\Sigma_\sigma^{\mathscr{G}}$ is described by the family of matrices $\{A_{Oi}\}_{i \in I}$. Since the dynamics of $\Sigma_{O\sigma}$ is described by the same family of matrices and it is globally asymptotically stable over $\mathscr{S}_\alpha$ for α sufficiently big or, respectively, for the given α, $\mathcal{S}_R$ is proved to be externally stabilizable or, respectively, α-externally stabilizable. Finally, since $H_i = C_{Oi}P + D_{Oi}C_i$ for all $i \in I$ by Proposition 3, we have $\mathcal{S}_R \cap \operatorname{Ker} C_i \subseteq \operatorname{Ker} H_i$ for all $i \in I$. □

Remark 2 *It is important to note that the "If" part of the proof of Theorem 2 shows how to construct, under the structural condition* **S**), *an observer $\Sigma_{O\sigma}$ of the form (2), which solves the UIOP or, respectively, the SUIOP if the qualitative condition* **Q**), *in the appropriate formulation, is satisfied. If this is not the case, $\Sigma_{O\sigma}$ is not globally asymptotically stable, but if it is correctly initialized at $z(0) = Px(0)$ it generates an output $w(t)$ that equals $H_{\sigma(t)}x(t)$, i.e. we have $w(t) = H_{\sigma(t)}x(t)$ for all $t \in \mathbb{R}^+$. This means that under the structural condition* **S**) *the value of $H_{\sigma(t)}x(t)$ can be computed also in presence of unknown inputs, provided that the initial condition $x(0)$ of Σ_σ is known.*

Remark 3 *Note that Theorem 2 gives a complete characterization of solvability of the considered observation problems without assuming any restrictive hypothesis, like e.g. that of being bounded, on the unknown input $u(t)$. This is one of the advantages offered by the point of view based on structural notions, whose geometric nature does not involve qualitative aspects of the signals at issue.*

Remark 4 *In order to gain more insight into the meaning of Theorem 2, it is useful to anticipate, from the following section, that, given a subspace $\mathcal{W} \subseteq \mathcal{X}$, the set of all conditioned invariant subspaces for Σ_σ containing $\mathcal{W}$ can be shown, by standard arguments of linear algebra, to be a lower semilattice with respect to the inclusion and the intersection of subspaces. This implies that there exists a minimal element of such set, which is denoted by $\mathcal{S}_R^*(\mathcal{W})$ or simply by $\mathcal{S}_R^*$ if $\mathcal{W} = \sum_{i \in I} \operatorname{Im} B_i$. Then, thank to minimality of $\mathcal{S}_R^*$, the existence of a conditioned invariant subspace S_R that contains $\operatorname{Im} B_i$ for all $i \in I$ and satisfies the structural condition* **S**) *of Theorem 2 can be globally expressed in term of S_R^* by the equivalent condition*

S′) $S_R^* \cap \operatorname{Ker} C_i \subseteq \operatorname{Ker} H_i$ *for all $i \in I$.*

Minimality of $\mathcal{S}_R^*$ qualifies **S′**) *as the structural obstruction to the solvability of the UIOP and SUIOP, as well as to the existence of an observer that allows the computation of $H_{\sigma(t)}x(t)$ if $x(0)$ is known.*

Remark 5 *The problem of estimating asymptotically the full state x of Σ_σ for all $\sigma \in \mathscr{S}_\alpha$, either for an α sufficiently big or for a given α, can be viewed as a special case of the UIOP or of the SUIOP in which $H_i = I_n$ for all $i \in I$. Then, it follows from Theorem 2 that the full state asymptotic estimation problem is solvable for α sufficiently big or, respectively, for a given α if and only if there exists an externally stabilizable or, respectively, an externally α-stabilizable conditioned invariant subspace $\mathcal{S}_R$ such that $\operatorname{Im} B_i \subseteq \mathcal{S}_R$ and $\mathcal{S}_R \cap \operatorname{Ker} C_i = \{0\}$ for all $i \in I$. In case $x(0)$ is known, no qualitative condition is required. Hence, the computation of $x(t)$ for $t \in \mathbb{R}^+$ is possible if and only if the condition $\mathcal{S}_R^* \cap \operatorname{Ker} C_i = \{0\}$ is satisfied for all $i \in I$. This appears to agree with intuition, since the subspace $\mathcal{S}_R^* \cap \operatorname{Ker} C_i$ describes the component of the state vector x that is influenced by the unknown input and that is filtered out by the output map when the mode Σ_i is the active one. Quite obviously, such component cannot be estimated or computed, even if $x(0)$ is known, during any time interval in which Σ_i is the active mode. Since there is no canonical way to decompose $\mathcal{X}$ as a direct sum of the form $\mathcal{X} = (\mathcal{S}_R^* \cap \operatorname{Ker} C_i) \oplus \mathcal{W}_i$, but infinitely many choices of the summand $\mathcal{W}_i$ are possible, the best way to represent the maximal content of information about $x(t)$ that can be estimated consists in considering the image of x in the quotient space $\mathcal{X}/\mathcal{S}_R^*$. The function $\mathscr{H}$ in the statement of the UIOP and of the SUIOP describes, in each specific situation, part of this information. Note that in case $H_i = I_n$ and, in addition, the unknown-input distribution matrix is zero, i.e. the system is actually not subject to unknown* *inputs, the unknown input observation problem reduces to that of finding a reduced-order observer for the full state. In such situation, we have $\mathcal{S}_R^* = \{0\}$ and the structural condition we have discussed in the previous remark is always verified. Asymptotic estimation of the full state, then, depends only on the existence of a conditioned invariant subspace $\mathcal{S}_R$ that is externally stabilizable or, respectively, externally α-stabilizable, such that $\mathcal{S}_R \cap \operatorname{Ker} C_i = \{0\}$.*

The solvability conditions given in Theorem 2 are not constructive since the theorem gives no indication on how to ascertain the existence of a conditioned invariant subspace $\mathcal{S}_R$ with the required properties. To overcome this limitation, constructive necessary and sufficient conditions that can be checked by algorithmic procedures are derived in the next section.

5 Constructive Solvability Conditions

In this section, we characterize the solvability of the considered unknown input observation problems by means of constructive, necessary and sufficient conditions that can be practically checked. In order to formulate such conditions, it is necessary to introduce some geometric objects and to state their properties. This is done in the following Proposition 5, whose proof is given in Section 6 by providing finite construction procedures, under suitable hypotheses, for all the considered geometric objects. Together with the construction of $\Sigma_{O\sigma}$ given in the proof of Theorem 2, such procedures provide a viable way to check the existence of solutions to the UIOP or to the SUIOP and, in case, to construct them.

Proposition 5 *Given a switching system Σ_σ of the form (1), the following facts hold true*

1. *The set of all conditioned invariant subspaces for Σ_σ containing $\operatorname{Im} B_i$ for all $i \in I$ has a minimal element, which is denoted by $\mathcal{S}_R^*$.*
2. *The set of all externally stabilizable conditioned invariant subspaces for Σ_σ containing $\operatorname{Im} B_i$ for all $i \in I$ has a minimal element, which is denoted by $\mathcal{S}_{Rg}^*$ and is called the minimal good conditioned invariant subspace.*
3. *Let $\{\mathcal{K}_i\}_{i\in I}$ be an indexed family of subspaces of $\mathcal{X}$ and assume that*

$$\mathcal{S}_R^* + \operatorname{Ker} C_i = \mathcal{X} \quad \text{for all } i \in I. \tag{19}$$

Then, the set of all conditioned invariant subspaces $\mathcal{S}_R$ for Σ_σ satisfying the conditions

$$\operatorname{Im} B_i \subseteq \mathcal{S}_R \quad \text{for all } i \in I \tag{20}$$
$$\mathcal{S}_R \cap \operatorname{Ker} C_i \subseteq \mathcal{K}_i \quad \text{for all } i \in I \tag{21}$$

is either empty or it has a maximal element, which is denoted by $\mathcal{S}_{Rm}(\{\mathcal{K}_i\}_{i\in I})$.

Using $\mathcal{S}_R^*$, $\mathcal{S}_{Rg}^*$ and $\mathcal{S}_{Rm}(\{\mathcal{K}_i\}_{i\in I})$, it is possible to state the solvability conditions of the UIOP and of the SUIOP given in Theorem 2 in a constructive way as follows.

Theorem 3 *Let the switching linear system Σ_σ and the family of linear maps $\mathscr{H} = \{H_i,\, H_i : \mathcal{X} \to \mathbb{R}^q\}_{i\in I}$ be given. Then, the corresponding UIOP is solvable if and only if*

$$\mathcal{S}_{Rg}^* \cap \operatorname{Ker} C_i \subseteq \operatorname{Ker} H_i \quad \text{for all } i \in I. \tag{22}$$

Proof. *If.* $\mathcal{S}_{Rg}^*$ is externally stabilizable and, therefore, by (22), it satisfies the conditions **S**) and **Q**) of Theorem 2. Hence, the UIOP is solvable.
Only if. If the UIOP is solvable, there exists by Theorem 2 a conditioned invariant subspace $\mathcal{S}_R$ that satisfies the conditions **S**) and **Q**). In particular, by **Q**), it is externally stabilizable and, therefore, it is contained in $\mathcal{S}_{Rg}^*$. Then, it follows from **S**) that condition (22) is satisfied. □

Theorem 4 *Let the switching linear system Σ_σ, the family of linear maps $\mathscr{H} = \{H_i,\, H_i : \mathcal{X} \to \mathbb{R}^q\}_{i\in I}$ and the positive real constant α be given. Let condition (19) hold. Then, the corresponding SUIOP is solvable if and only if the folllowing conditions are satisfied*

- **S′**) *$\mathcal{S}_R^* \cap \operatorname{Ker} C_i \subseteq \operatorname{Ker} H_i$ for all $i \in I$ (structural condition);*
- **Q′**) *$\mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ is externally α-stabilizable (qualitative condition).*

Proof. *If.* Note that, by **S′**), the set of all conditioned invariant subspaces that satisfy (20) and (21) with $\mathcal{K}_i = \operatorname{Ker} H_i$ for $i \in I$ is not empty. Hence, by (19), its maximal $\mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ exists and the external α-stabilizability condition is well-posed. Moreover, $\mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ satisfies the conditions **S**) and **Q**) of Theorem 2 and the construction of $\Sigma_{O\sigma}$ seen in its proof provides an observer that is globally asymptotically stable over $\mathscr{S}_\alpha$. Hence, the SUIOP is solvable.
Only if. If the SUIOP is solvable, there exists, by Theorem 2, a conditioned invariant subspace $\mathcal{S}_R$ that satisfies the conditions **S**) and **Q**) for the given α. In particular, by **Q**), there exists a friend $\mathscr{G} = \{G_i; G_i : \mathcal{X} \to \mathcal{Y}\}$ of $\mathcal{S}_R$ such that the dynamics induced on $\mathcal{X}/\mathcal{S}_R$ by that of $\Sigma_\sigma^{\mathscr{G}}$ coincides with the dynamics of the observer and, hence, it is globally stable over $\mathscr{S}_\alpha$. Moreover, by **S**), the set of all conditioned invariant subspaces that satisfy (20) and (21) with $\mathcal{K}_i = \operatorname{Ker} H_i$ for $i \in I$ is not empty and $\mathcal{S}_R \subseteq \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$. Together with (19), this implies that $\mathscr{G}$ is also a friend of $\mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$. We show this by proving that $(A_i + G_iC_i)s \in \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ for all $s \in \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ and all $i \in I$. In fact, since $\mathcal{S}_R^* \subseteq \mathcal{S}_R$, for all $i \in I$ we have $\mathcal{S}_R + \operatorname{Ker} C_i = \mathcal{X}$ and any $s \in \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ can be written as $s = s_i + k_i$, with $s_i \in \mathcal{S}_R$ and $k_i \in \operatorname{Ker} C_i$. Since both s_i and s belong to $\mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$, it follows that $k_i = s - s_i$ actually belongs to $\operatorname{Ker} C_i \cap \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$. Now, $(A_i + G_i C_i)\, s_i$ belongs to $\mathcal{S}_R \subseteq \mathcal{S}_{Rm}(\{\operatorname{Ker} H_i\}_{i\in I})$ for all $i \in I$, because $s_i \in \mathcal{S}_R$ and $\mathscr{G}$ is a friend of

$\mathcal{S}_R$, and $A_i\,k_i$ belongs to $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$, because $k_i \in \mathrm{Ker}\,C_i \cap \mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ and $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ is a conditioned invariant subspace for Σ_σ. Then, $(A_i+G_i\,C_i)\,s=(A_i+G_i\,C_i)\,(s_i+k_i)=(A_i+G_i\,C_i)\,s_i + A_i\,k_i$ belongs to $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ for all $i\in I$ and $\mathscr{G}$ is proved to be a friend of $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$. The quotient $\mathcal{X}/\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ can therefore be viewed as $(\mathcal{X}/\mathcal{S}_R)\,/\,(\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})/\mathcal{S}_R)$ and the dynamics induced on $\mathcal{X}/\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ by that of $\Sigma_\sigma^{\mathscr{G}}$ can be viewed as the dynamics induced on the quotient of $\mathcal{X}/\mathcal{S}_R$ modulo $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})/\mathcal{S}_R$ by that induced on $\mathcal{X}/\mathcal{S}_R$. It follows that the dynamics induced on $\mathcal{X}/\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for the given α and so $\mathcal{S}_{Rm}(\{\mathrm{Ker}\,H_i\}_{i\in I})$ is externally α-stabilizable. □

Remark 6 *As stated at the beginning of this section, the conditioned invariant subspaces $\mathcal{S}_R^*$, $\mathcal{S}_{Rg}^*$ and, under more restrictive hypotheses, $\mathcal{S}_{Rm}^*(\{\mathrm{Ker}\,H_i\}_{i\in I})$ can be computed by means of algorithmic procedures that will be given in the next section, when proving Proposition 5. This implies, in particular, that the necessary and sufficient conditions stated in Theorem 3 can be practically checked. Moreover, after constructing $\mathcal{S}_{Rm}^*(\{\mathrm{Ker}\,H_i\}_{i\in I})$, its external stabilizability properties can be analysed using the results of Section 7 and the methods described in Chesi et al. (2012) and in Xiang (2016). It follows that the necessary and sufficient condition stated in Theorem 4 can also be practically checked.*

Remark 7 *Note that, while in Theorem 2 the solvability condition is split into the structural requirement* **S**) *and the qualitative requirement* **Q**)*, in Theorem 3 both requirements, the structural one and the qualitative one, are jointly expressed by condition (22). This is due to the fact that the conditioned invariant subspace $\mathcal{S}_{Rg}^*$, being externally stabilizable by definition, satisfies the qualitative requirement.*

6 Construction of the Key Subspaces for Σ_σ

In this section, we prove Proposition 5 by showing that the geometric subspaces introduced are well defined and by providing the algorithmic procedures to construct them. Since those subspaces are the key objects for characterizing the solvability of the UIOP and of the SUIOP in Theorem 3 and in Theorem 4, the results of this section give constructive, viable procedures to check the existence of solutions in any specific case and, together with the construction of $\Sigma_{O\sigma}$ given in Theorem 2, to construct them.

6.1 The Minimal Conditioned Invariant Subspace

The aim of this subsection is to prove Proposition 5-1 by showing the existence of the minimal element in the set of all conditioned invariant subspaces that contain a given subspace and to provide an algorithmic procedure for its construction. Let us start by remarking that, given a subspace $\mathcal{W}$ of $\mathcal{X}$, the set of all conditioned invariant subspaces for Σ_σ containing $\mathcal{W}$ can be shown by standard linear algebra arguments to be a lower semilattice with respect to the inclusion and the intersection of subspaces. Hence, it has a minimal element, which is denoted by $\mathcal{S}_R^*(\mathcal{W})$, and, taking $\mathcal{W}=\sum_{i\in I}\mathrm{Im}B_i$, Proposition 5-1 is proved. The following proposition provides an algorithmic procedure to compute it.

Proposition 6 (Construction of $\mathcal{S}_R^*(\mathcal{W})$) Given a subspace $\mathcal{W}\subseteq\mathcal{X}$, the sequence of subspaces $\mathcal{S}_{Rk}$, with $k\in\mathbb{Z}^+$, generated by

$$\begin{cases} \mathcal{S}_{R0}=\mathcal{W}, \\ \mathcal{S}_{R(k+1)}=\mathcal{S}_{Rk}+\sum_{i\in I}A_i\,(\mathcal{S}_{Rk}\cap\mathrm{Ker}\,C_i) \end{cases} \tag{23}$$

converges to $\mathcal{S}_R^*(\mathcal{W})$ in n steps at most.

Proof. The sequence of subspaces $\mathcal{S}_{Rk}$, with $k\in\mathbb{Z}^+$, generated by (23), is nondecreasing and, as the dimension of $\mathcal{X}$ is finite and equal to n, there exists $j<n$ such that $\mathcal{S}_{R(j+1)}=\mathcal{S}_{Rj}+\sum_{i\in I}A_i\,(\mathcal{S}_{Rj}\cap\mathrm{Ker}\,C_i)=\mathcal{S}_{Rj}$. The subspace $\mathcal{S}_{Rj}$, which represents the last term of the sequence, is a conditioned invariant subspace for Σ_σ since, as is shown by the latter equation, $A_i\,(\mathcal{S}_{Rj}\cap\mathrm{Ker}\,C_i)\subseteq\mathcal{S}_{Rj}$ for all $i\in I$. Moreover, $\mathcal{S}_{Rj}\supseteq\mathcal{W}$ since $\mathcal{S}_{R(k+1)}\supseteq\mathcal{S}_{Rk}$ for all $k\in\mathbb{Z}^+$ and $\mathcal{S}_{R0}=\mathcal{W}$. To prove minimality of $\mathcal{S}_{Rj}$, let $\mathcal{S}_R$ be a conditioned invariant subspace for Σ_σ containing $\mathcal{W}$. It will be shown by induction that $\mathcal{S}_R\supseteq\mathcal{S}_{Rk}$ for all $k\in\mathbb{Z}^+$, which implies $\mathcal{S}_R\supseteq\mathcal{S}_{Rj}$, in particular. First, note that $\mathcal{S}_R\supseteq\mathcal{S}_{R0}$ since $\mathcal{S}_{R0}=\mathcal{W}$. Secondly, note that $\mathcal{S}_{Rk}\subseteq\mathcal{S}_R$ for some $k\in\mathbb{Z}^+$ implies $A_i\,(\mathcal{S}_{Rk}\cap\mathrm{Ker}\,C_i)\subseteq A_i\,(\mathcal{S}_R\cap\mathrm{Ker}\,C_i)\subseteq\mathcal{S}_R$ for all $i\in I$. Hence, $\mathcal{S}_{R(k+1)}\subseteq\mathcal{S}_R$ since both terms at the right-hand side of the second of (23) are contained in $\mathcal{S}_R$. □

Remark 8 *The characterization of $\mathcal{S}_R^*(\mathcal{W})$ as the limit of the sequence defined by (23) and the fact that convergence is obtained in a finite number of steps enable direct computation of a basis for $\mathcal{S}_R^*(\mathcal{W})$. Then, a friend $\mathscr{G}=\{G_i,\,G_i:\mathcal{Y}\to\mathcal{X}\}_{i\in I}$ of $\mathcal{S}_R^*(\mathcal{W})$ can be obtained as illustrated in Section 4. To make a comparison with the linear time-invariant case (which applies to each mode Σ_i of Σ_σ), let $\mathcal{S}_i^*(\mathcal{W})$, with $i\in I$, denote the minimal conditioned invariant subspace for the mode Σ_i containing the given subspace $\mathcal{W}$ (see Basile and Marro, 1992, Section 4.1.1). From (23) it follows that $\mathcal{S}_R^*(\mathcal{W})$ contains $\mathcal{S}_i^*(\mathcal{W})$ for all $i\in I$ and that $\mathcal{S}_R^*(\mathcal{W})$ may be larger than $\sum_{i\in I}\mathcal{S}_i^*(\mathcal{W})$.*

Notation We recall that $\mathcal{S}_R^*(\sum_{i\in I}\mathrm{Im}B_i)$ is simply denoted by $\mathcal{S}_R^*$.

6.2 The Minimal Good Conditioned Invariant Subspace

The aim of this subsection is to prove Proposition 5-2 by showing the existence of the minimal element in the set of all externally stabilizable, conditioned invariant subspaces that contain a given subspace and to provide an algorithmic procedure for its construction. We start by recalling that, for each mode Σ_i of the switching linear system Σ_σ, it is possible to introduce the minimal externally stabilizable conditioned invariant subspace for Σ_i containing the subspace $\mathcal{W}$, which is denoted by $\mathcal{S}_{Rgi}^*(\mathcal{W})$. Such subspace can be defined dualizing the definition of maximal internally stabilizable controlled invariant subspace contained in a given subspace originally proposed in (Wonham, 1985, Section 5.6). The same reference provides also an algorithm that, in dual formulation, produces $\mathcal{S}^*_{Rgi}(\mathcal{W})$. The next proposition provides an algorithm to construct $\mathcal{S}^*_{Rg}(\mathcal{W})$ that employs the subspaces $\mathcal{S}^*_{Rgi}(\mathcal{W})$ and, at the same time, it gives a constructive proof of Proposition 5-2 by taking $\mathcal{W}=\sum_{i\in I}\operatorname{Im}B_i$.

Proposition 7 (Construction of $\mathcal{S}^*_{Rg}(\mathcal{W})$) Given a subspace $\mathcal{W}\subseteq\mathcal{X}$, the sequence of subspaces $\mathcal{S}_{Rk}$, with $k\in\mathbb{Z}^+$, generated by

$$\begin{cases}\mathcal{W}_0=\mathcal{W},\\ \mathcal{K}_0=\sum_{i\in I}\mathcal{S}^*_{Rgi}(\mathcal{W}_0),\\ \mathcal{S}_{R0}=\mathcal{S}^*_R(\mathcal{K}_0),\end{cases} \tag{24a}$$

$$\begin{cases}\mathcal{W}_{k+1}=\mathcal{S}_{Rk},\\ \mathcal{K}_{k+1}=\sum_{i\in I}\mathcal{S}^*_{Rgi}(\mathcal{W}_{k+1}),\\ \mathcal{S}_{R(k+1)}=\mathcal{S}^*_R(\mathcal{K}_{k+1}),\end{cases} \tag{24b}$$

converges to $\mathcal{S}^*_{Rg}(\mathcal{W})$ in at most n steps.

Proof. The sequence of subspaces $\mathcal{S}_{Rk}$, with $k\in\mathbb{Z}^+$, generated by (24a)–(24b) is nondecreasing and, since $\mathcal{X}$ has dimension equal to n, it becomes stationary for some $j<n$. The last term of the sequence, i.e. the subspace $\mathcal{S}_{Rj}$, is a conditioned invariant subspace for Σ_σ by the last equation of (24b) (or the last equation of (24a) if $j=0$) and the definition of $\mathcal{S}^*_R(\cdot)$. Moreover, $\mathcal{S}_{Rj}\supseteq\mathcal{W}$ since $\mathcal{S}_{R(k+1)}\supseteq\mathcal{S}_{Rk}$ for all $k\in\mathbb{Z}^+$ and $\mathcal{S}_{R0}\supseteq\mathcal{W}$ by (24a). Furthermore, $\mathcal{S}_{Rj}$ is externally stabilizable (as a conditioned invariant subspace for Σ_σ) by Theorem 1. To prove minimality of $\mathcal{S}_{Rj}$, let $\mathcal{S}_R$ be an externally stabilizable conditioned invariant subspace for Σ_σ containing $\mathcal{W}$. It will be shown by induction that $\mathcal{S}_R\supseteq\mathcal{S}_{Rk}$ for all $k\in\mathbb{Z}^+$, which implies, in particular, $\mathcal{S}_R\supseteq\mathcal{S}_{Rj}$. First, note that $\mathcal{S}_R\supseteq\mathcal{W}_0$, since $\mathcal{W}_0=\mathcal{W}$. Then, $\mathcal{S}_R\supseteq\mathcal{S}^*_{Rgi}(\mathcal{W}_0)$ for all $i\in I$, by minimality of $\mathcal{S}^*_{Rgi}(\mathcal{W}_0)$ with respect to Σ_i, and, by the second equation of (24a), $\mathcal{S}_R\supseteq\mathcal{K}_0$. Consequently, $\mathcal{S}_R\supseteq\mathcal{S}_{R0}$, by minimality of $\mathcal{S}_{R0}$ with respect to Σ_σ. Secondly, note that the assumption $\mathcal{S}_{Rk}\subseteq\mathcal{S}_R$ for some $k\in\mathbb{Z}^+$ implies $\mathcal{S}_{R(k+1)}\subseteq\mathcal{S}_R$. In fact, $\mathcal{S}_R\supseteq\mathcal{S}_{Rk}$ implies $\mathcal{S}_R\supseteq\mathcal{W}_{k+1}$ since $\mathcal{W}_{k+1}=\mathcal{S}_{Rk}$. Then, as before, $\mathcal{S}_R\supseteq\mathcal{S}^*_{Rgi}(\mathcal{W}_{k+1})$ for all $i\in I$, by minimality of $\mathcal{S}^*_{Rgi}(\mathcal{W}_{k+1})$ with respect to Σ_i, and, by the second equation of (24b), $\mathcal{S}_R\supseteq\mathcal{K}_{k+1}$. Thus, $\mathcal{S}_R\supseteq\mathcal{S}_{R(k+1)}$, by minimality of $\mathcal{S}_{R(k+1)}$ with respect to Σ_σ. □

The minimal externally stabilizable conditioned invariant subspace for Σ_σ containing a given subspace described herein is the dual counterpart of the maximal internally stabilizable controlled invariant subspace for Σ_σ contained in a given subspace introduced in (Zattoni et al., 2016, Section III). Duality is reflected in the respective computational algorithms.

Notation We recall that $\mathcal{S}^*_{Rg}(\sum_{i\in I}\operatorname{Im}B_i)$ is simply denoted by $\mathcal{S}^*_{Rg}$.

6.3 Maximal Conditioned Invariant Subspaces

The aim of this subsection is to prove Proposition 5-3 by showing the existence, under condition (19), of the maximal element in the set of all conditioned invariant subspaces that statisfy the conditions (20) and (21) and to provide, under suitable more restrictive hypotheses, an algorithmic procedure for its construction. We start by stating the following two preliminary results.

Proposition 8 *Let (19) hold. Let $\mathcal{S}_R$ be a conditioned invariant subspace for Σ_σ containing $\operatorname{Im}B_i$ for all $i\in I$ and let $\mathscr{G}=\{G_i,\,G_i:\mathcal{Y}\to\mathcal{X}\}_{i\in I}$ be a friend of $\mathcal{S}_R$. Then, $\mathscr{G}$ is a friend of any conditioned invariant subspace $\mathcal{S}'_R$ for Σ_σ such that $\mathcal{S}_R\subseteq\mathcal{S}'_R$.*

Proof. The same arguments used in the proof of Theorem 4 to show that any friend $\mathscr{G}$ of a conditioned invariant subspace $\mathcal{S}_R$ contained in $\mathcal{S}_{Rm}(\{\operatorname{Ker}H_i\}_{i\in I})$ is also a friend of $\mathcal{S}_{Rm}(\{\operatorname{Ker}H_i\}_{i\in I})$ apply for any conditioned invariant subspace $\mathcal{S}'_R$ in which $\mathcal{S}_R$ is contained. □

Proposition 9 *Let (19) hold. Then, the sum of any two conditioned invariant subspaces for Σ_σ containing $\operatorname{Im}B_i$ for all $i\in I$ is a conditioned invariant subspace for Σ_σ containing $\operatorname{Im}B_i$ for all $i\in I$.*

Proof. Let $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ be two conditioned invariant subspaces for Σ_σ containing $\operatorname{Im}B_i$ for all $i\in I$. Hence, the subspace $\operatorname{Im}B_i$, being contained in both $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$, is contained in $\mathcal{S}_{R1}+\mathcal{S}_{R2}$ for all $i\in I$. Moreover, let $\mathscr{G}=\{G_i,\,G_i:\mathcal{Y}\to\mathcal{X}\}_{i\in I}$ be a friend of $\mathcal{S}^*_R$. Since both $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ contain $\mathcal{S}^*_R$ by minimality of the latter, $\mathscr{G}$ is a friend of both $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ by Proposition 8. Therefore, we have $(A_i+G_i\,C_i)(\mathcal{S}_{R1}+\mathcal{S}_{R2}) = (A_i+G_i\,C_i)\mathcal{S}_{R1}+(A_i+G_i\,C_i)\mathcal{S}_{R2}\subseteq\mathcal{S}_{R1}+\mathcal{S}_{R2}$ for all $i\in I$. □

We can now prove Proposition 5-3.

Proof of Proposition 5-3. Since $\mathcal{X}$ has finite dimension, to prove the statement it is sufficient to show that, if $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ are two conditioned invariant subspaces for Σ_σ that satisfy (20) and (21), then the subspace $\mathcal{S}_{R1}+\mathcal{S}_{R2}$ is a conditioned invariant subspace for Σ_σ satisfying the same conditions. Since (19) holds by assumption, $\mathcal{S}_R=\mathcal{S}_{R1}+\mathcal{S}_{R2}$ is a conditioned invariant subspace for Σ_σ by Proposition 9 and it satisfies (20) because $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ do it. Now, take any $i\in I$ and a vector $x\in\mathcal{K}_i^\perp$ and note that, since $\mathcal{S}_{R1}^\perp+(\operatorname{Ker}C_i)^\perp\supseteq\mathcal{K}_i^\perp$ and $\mathcal{S}_{R2}^\perp+(\operatorname{Ker}C_i)^\perp\supseteq\mathcal{K}_i^\perp$ because of (20), x can be written as $x=s_{1i}+c_{1i}=s_{2i}+c_{2i}$ with $s_{1i}\in\mathcal{S}_{R1}^\perp$, $s_{2i}\in\mathcal{S}_{R2}^\perp$, and $c_{1i},c_{2i}\in(\operatorname{Ker}C_i)^\perp$ for the chosen index i. Since $\mathcal{S}_{R1}^\perp\subseteq(\mathcal{S}^*_R)^\perp$ and $\mathcal{S}_{R2}^\perp\subseteq(\mathcal{S}^*_R)^\perp$ by minimality of $\mathcal{S}^*_R$, it follows that s_{1i} and s_{2i} belong to $(\mathcal{S}^*_R)^\perp$ for all $i\in I$. As a consequence, we have that $s_{1i}-s_{2i}=c_{2i}-c_{1i}$ belongs to $(\mathcal{S}^*_R)^\perp\cap(\operatorname{Ker}C_i)^\perp=\{0\}$, where the last equality is due to (19). Hence, $s_{1i}=s_{2i}$ belongs to $(\mathcal{S}_{R1}^\perp\cap\mathcal{S}_{R2}^\perp)$ and, therefore, x belongs to $(\mathcal{S}_{R1}^\perp\cap\mathcal{S}_{R2}^\perp)+(\operatorname{Ker}C_i)^\perp$. This implies $(\mathcal{S}_{R1}^\perp\cap\mathcal{S}_{R2}^\perp)+(\operatorname{Ker}C_i)^\perp\supseteq\mathcal{K}_i^\perp$ and, taking the orthogonal spaces, $\mathcal{S}_{R1}+\mathcal{S}_{R2}\cap(\operatorname{Ker}C_i)\subseteq\mathcal{K}_i$ for the chosen $i\in I$. Repeating the same argument for all $i\in I$, shows that the subspace $\mathcal{S}_{R1} + \mathcal{S}_{R2}$ satisfies (21). Hence, the set of all conditioned invariant subspaces $\mathcal{S}_r$ for Σ_σ satisfying (20) and (21) is a lattice with respect to inclusion, sum and intersection of subspaces and, if it is not empty, it has a maximal element $S_{Rm}(\{\mathcal{K}_i\}_{i\in I})$. □

Concerning condition (19), it is worth noting that it is akin to, but weaker than, right invertibility of all modes. In fact, using the notation employed in Remark 8, the latter would amount to $\mathcal{S}^*_{Ri}(\operatorname{Im} B_i) + \operatorname{Ker} C_i = \mathcal{X}$ for all $i \in I$ and $\mathcal{S}^*_R \supseteq \mathcal{S}^*_{Ri}(\sum_{i\in I} \operatorname{Im} B_i) \supseteq \mathcal{S}^*_{Ri}(\operatorname{Im} B_i)$ for all $i \in I$. Now, to provide a procedure to construct $S_{Rm}(\{\mathcal{K}_i\}_{i\in I})$, we have to strengthen the condition (19) by assuming that $\mathcal{X}$ is equal to the sum between $\mathcal{S}^*_R$ and a subspace smaller than the intersection of all the $\operatorname{Ker} C_i$. So, let us consider the sequence of subspaces $\mathcal{V}_k \subseteq \mathcal{X}$ with $k \in \mathbb{Z}^+$, generated by

$$\begin{cases} \mathcal{V}_0 = \bigcap_{i\in I} (\mathcal{K}_i \cap \operatorname{Ker} C_i), \\ \mathcal{V}_{k+1} = \mathcal{V}_k \cap \left(\bigcap_{i\in I} A_i^{-1}(V_k + \operatorname{Im} B_i)\right). \end{cases} \tag{25}$$

It can be shown by standard arguments of linear algebra that such sequence converges in a finite number of steps (at most in $\dim(\bigcap_{i\in I}(\mathcal{K}_i \cap \operatorname{Ker} C_i) + 1)$ steps). Its limit, that we denote by $\mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$, is known to be the maximal subspace in the set of all subspaces $\mathcal{V} \subseteq \bigcap_{i\in I} (\mathcal{K}_i \cap \operatorname{Ker} C_i) \subseteq \mathcal{X}$ that are robust controlled invariant with respect to the switching dynamics of Σ_σ (or controlled invariant for Σ_σ), namely such that $A_i\mathcal{V} \subseteq \mathcal{V} + \operatorname{Im} B_i$ for all $i \in I$. Controlled invariance with respect to a switching dynamics was first considered in (Otsuka, 2010) in relation to the disturbance decoupling problem and it was further investigated and used to deal with other control problems in (Conte and Perdon, 2011), (Conte et al., 2014)). If $\mathcal{K}_i = \operatorname{Ker} C_i$ for all $i \in I$ (and hence in (25) $\mathcal{V}_0 = \bigcap_{i\in I} \operatorname{Ker} C_i$), we denote $\mathcal{V}^*_R(\{\operatorname{Ker} C_i\}_{i\in I})$ simply by $\mathcal{V}^*_R$. Note that, by construction, we have $\mathcal{V}^*_R \subseteq \bigcap_{i\in I} \operatorname{Ker} C_i$ and $\mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})) \subseteq \bigcap_{i\in I}(\mathcal{K}_i \cap \operatorname{Ker} C_i) \subseteq \bigcap_{i\in I} \operatorname{Ker} C_i$ and therefore, by maximality of $\mathcal{V}^*_R$, also $\mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \subseteq \mathcal{V}^*_R$.
We assume now that the following condition is satisfied

$$\mathcal{S}^*_R + \mathcal{V}^*_R = \mathcal{X}. \tag{26}$$

Remark 9 *Since $\mathcal{V}^*_R \subseteq \bigcap_{i\in I} \operatorname{Ker} C_i$, the condition (26) is stronger than the condition (19). Actually, letting $\mathcal{V}^*_i$, with $i \in I$, denote the maximal controlled invariant subspace for the mode Σ_i contained in $\operatorname{Ker} C_i$ (see (Basile and Marro, 1992)), it may be interesting to recall that $\mathcal{S}^*_i + \operatorname{Ker} C_i = \mathcal{X}$ is equivalent to $\mathcal{S}^*_i + \mathcal{V}^*_i = \mathcal{X}$. This means that in the classical non switching situation (19) and (26) are equivalent.*

The next proposition characterizes $S_{Rm}(\{\mathcal{K}_i\}_{i\in I})$ in such a way that an algorithmic procedure for its construction can be given.

Proposition 10 *Given a switching system Σ_σ of the form (1) and an indexed family $\{\mathcal{K}_i\}_{i\in I}$ of subspaces of $\mathcal{X}$, assume that $\mathcal{S}^*_R \cap \operatorname{Ker} C_i \subseteq \mathcal{K}_i$ for all $i \in I$ and that condition (26) holds. Then, the maximal conditioned invariant subspace for Σ_σ that satisfies (20) and (21) is equal to the subspace $\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$, that is $S_{Rm}(\{\mathcal{K}_i\}_{i\in I}) = \mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$*

Proof. We show, first of all, that $\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$ verifies the conditions (20) and (21) and that it is a conditioned invariant subspace for Σ_σ. In fact, we have $\operatorname{Im} B_i \subseteq \mathcal{S}^*_R \subseteq \mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$ and $(\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \cap \operatorname{Ker} C_i = (\mathcal{S}^*_R \cap \operatorname{Ker} C_i) + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I} \subseteq \mathcal{K}_i$ for all $i \in I$. Moreover, $A_i((\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \cap \operatorname{Ker} C_i) = A_i((\mathcal{S}^*_R \cap \operatorname{Ker} C_i) + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \subseteq \mathcal{S}^*_R + (\mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) + \operatorname{Im} B_i) = \mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$ for all $i \in I$.
Now, to show maximality , let $\mathcal{S}$ be a conditioned invariant subspace for Σ_σ that satisfies (20) and (21). By minimality of $\mathcal{S}^*_R$, we have $\mathcal{S}^*_R \subseteq \mathcal{S}$ and the subspace $\mathcal{S}_1 = \mathcal{S} + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) = \mathcal{S} + (\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}))$, being a sum of conditioned invariant subspaces containing $\operatorname{Im} B_i$, is conditioned invariant by Proposition 9 and it contains $\operatorname{Im} B_i$ for all $i \in I$. Moreover, we have $\mathcal{S}_1 \cap \operatorname{Ker} C_i = (\mathcal{S} + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})) \cap \operatorname{Ker} C_i = (\mathcal{S} \cap \operatorname{Ker} C_i) + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \subseteq \mathcal{K}_i$ for all $i \in I$. Then, taking the maximal controlled invariant subspace for Σ_σ contained in $\bigcap_{i\in I} \operatorname{Ker} C_i$, namely $\mathcal{V}^*_R$, we consider the subspace $\mathcal{V} = \mathcal{S}_1 \cap \mathcal{V}^*_R = (\mathcal{S} + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})) \cap \mathcal{V}^*_R = (\mathcal{S} \cap \mathcal{V}^*_R) + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$. By $\mathcal{V} = (\mathcal{S} + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})) \cap \mathcal{V}^*_R \subseteq \mathcal{V}^*_R \subseteq \operatorname{Ker} C_i$ and $\mathcal{V} = (\mathcal{S} \cap \mathcal{V}^*_R) + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \subseteq (\mathcal{S} \cap \operatorname{Ker} C_i) + \mathcal{K}_i \subseteq \mathcal{K}_i$ for all $i \in I$, we have $\mathcal{V} \subseteq \bigcap_{i\in I}(\mathcal{K}_i \cap \operatorname{Ker} C_I)$. Moreover, $\mathcal{V}$ is controlled invariant for Σ_σ, since $A_i\mathcal{V} = A_i(\mathcal{S}_1 \cap \mathcal{V}^*_R) \subseteq A_i((\mathcal{S}_1 \cap \operatorname{Ker} C_i) \cap \mathcal{V}^*_R) \subseteq \mathcal{S}_1 \cap (\mathcal{V}^*_R + \operatorname{Im} B_i) \subseteq \mathcal{S}_1$. Therefore, we have $\mathcal{V} \subseteq \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$ by maximality of the latter subspace. On the other hand, note that any $s \in \mathcal{S}_1 \subseteq \mathcal{X}$ can be written as $s = s^* + c$ with $s^* \in \mathcal{S}^*_R$ and $c \in \bigcap_{i\in I} \operatorname{Ker} C_i$ by (26). Since $\mathcal{S}^*_R \subseteq \mathcal{S}_1$, this implies that c actually belongs to $\mathcal{S}_1 \cap \mathcal{V}^*_R$, and hence we have $\mathcal{S}_1 \subseteq \mathcal{S}^*_R + (\mathcal{S}_1 \cap \mathcal{V}^*_R)$. Finally, we get $\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I}) \supseteq \mathcal{S}^*_R + \mathcal{V} \supseteq \mathcal{S}^*_R + (\mathcal{S}_1 \cap \mathcal{V}^*_R) \supseteq \mathcal{S}_1 \supseteq \mathcal{S}$, which shows maximality of $\mathcal{S}^*_R + \mathcal{V}^*_R(\{\mathcal{K}_i\}_{i\in I})$. □

7 A Characterization of External Stabilizability

In order to handle the qualitative condition **Q')** of Theorem 4, in this section we investigate and characterize the properties of external stabilizability and external α-stabilizability. In particular, we provide algebraic necessary and sufficient conditions for external stabilizability of a conditioned invariant subspace and a result that simplifies the analysis of external α-stabilizability under specific assumptions.
The result of the proposition given below follows easily from Theorem 1.

Proposition 11 *A conditioned invariant subspace $\mathcal{S}_R \subseteq \mathcal{X}$ for Σ_σ is externally stabilizable if and only if there exists a friend $\mathscr{G} = \{G_i, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ such that the dynamics induced on $\mathcal{X}/\mathcal{S}_R$ by that of $\Sigma_i^{\mathscr{G}}$ is asymptotically stable for all $i \in I$.*

In Section 4, the dynamics induced on $\mathcal{X}/\mathcal{S}_R$ by that of $\Sigma_i^{\mathscr{G}}$ was shown to be defined by the matrix L_i if the indexed family of pairs of matrices $\{(L_i, M_i)\}_{i\in I}$, where $PG_i = -M_i$ and the columns of $P^\top$ are a basis of $\mathcal{S}_R^\perp$, satisfies (6). It is therefore worthwhile to consider the parametrization of the set of such families, and hence of the set of friends, given by the following proposition (see (Perdon et al., 2016, Lemma 1) for a dual results about controlled invariant subspaces).

Proposition 12 *Let $\mathcal{S}_R$ be a conditioned invariant subspace for Σ_σ of dimension $n-q$ and let $P^\top$ be a $n \times q$ matrix whose columns are a basis of $\mathcal{S}_R^\perp$. Let $\{L_i, M_i\}_{i\in I}$ be an indexed family of matrices that, together with P, verify (6). Then, letting $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}$ be, for all $i \in I$, a $(q+p)\times r_i$ matrix whose columns are a basis of* $\mathrm{Ker}\begin{bmatrix} P^\top & C_i^\top \end{bmatrix}$*, it follows that*

- *any other matrix $\bar{P}^\top$ whose columns are a basis of $\mathcal{S}_R^\perp$ is of the form $\bar{P}^\top = P^\top T^\top$ for some nonsingular $q\times q$ matrix T;*
- *for any other indexed family $\{\bar{L}_i, \bar{M}_i\}_{i\in I}$ such that $A_i^\top \bar{P}^\top = \bar{P}^\top \bar{L}_i^\top + C_i^\top \bar{M}_i^\top$, the matrices $\bar{L}_i$ and $\bar{M}_i$ are of the form $\bar{L}_i = TL_iT^{-1} - TQ_iN_{1i}^\top T^{-1}$ and $\bar{M}_i = TM_i - TQ_iN_{2i}^\top$ for some $q\times r_i$ matrix Q_i.*

Proof. The first statement is obvious. To prove the second statement, note that $A_i^\top \bar{P}^\top = \bar{P}\bar{L}_i^\top + C_i^\top \bar{M}_i^\top$ is equivalent to $A_i^\top P = PT^\top \bar{L}_i^\top (T^{-1})^\top + C_i^\top \bar{M}_i^\top (T^{-1})^\top$ and in turn, thank to (6), this is equivalent to $PT^\top \bar{L}_i^\top (T^{-1})^\top + C_i^\top \bar{M}_i^\top (T^{-1})^\top - PL_i^\top + C_i^\top M_i^\top = 0$. The last equality says that

$$[P\ C_i^\top]\begin{bmatrix} T^\top \bar{L}_i^\top (T^{-1})^\top - L_i^\top \\ \bar{M}_i^\top (T^{-1})^\top - M_i^\top \end{bmatrix} = 0 \qquad (27)$$

for all $i \in I$. Since N_{1i} and N_{2i} are two matrices such that the columns of $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}$ are a basis of Ker $[P^\top\ C_i^\top]$, the conclusion follows by transposing the equalities $T^\top \bar{L}_i^\top (T^{-1})^\top = L_i^\top + N_{1i}Q_i^\top$ and $\bar{M}_i^\top (T^{-1})^\top = M_i^\top + N_{2i}Q_i^\top$ which, for some Q_i, are implied by (27). □

As a consequence, we have the following results.

Proposition 13 *Let $\mathcal{S}_R \subseteq \mathcal{X}$ be a conditioned invariant subspace for Σ_σ and let $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y}\to\mathcal{X}\}_{i\in I}$ be one of its friend. Let $P^\top$ be a matrix whose columns are a basis of $\mathcal{S}_R^\perp$. Let $\{L_i, M_i\}_{i\in I}$ be an indexed family of matrices that, together with P, satisfies (6). Then, letting $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}$ be a $(q+p)\times r_i$ matrix whose columns are a basis of* Ker $\begin{bmatrix} P^\top & C_i^\top \end{bmatrix}$ *for $i \in I$, we have that $\mathcal{S}_R$ is externally stabilizable if and only if the pairs $(L_i, N_{1i}^\top)$ are detectable for all $i\in I$.*

Proof. It follows from the characterization of the dynamics induced on $\mathcal{X}/\mathcal{S}_R$ by that of $\Sigma_\sigma^{\mathscr{G}}$ given in Section 4, from Proposition 12 and Theorem 1. □

The above proposition gives a complete algebraic characterizion of external stabilizability for conditioned invariant subspaces. In addition, assuming that condition (19) holds for all $i \in I$, we have the following results.

Proposition 14 *Given a switching linear system Σ_σ of the form (1), assume that the condition (19) holds for all $i\in I$ and let $\mathcal{S}_R$ be a conditioned invariant subspace for Σ_σ such that $\mathrm{Im}\, B_i \subseteq \mathcal{S}_R$ for all $i \in I$. Then, the linear switching dynamics induced by that $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ does not depend on the choice of the friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y}\to\mathcal{X}\}_{i\in I}$ of $\mathcal{S}_R$.*

Proof. By minimality of $\mathcal{S}_R^*$, condition (19) implies $\mathcal{S}_R + \mathrm{Ker}\, C_i = \mathcal{X}$ for all $i \in I$. Therefore, letting $P^\top$ be a matrix whose columns are a basis of $\mathcal{S}_R^\perp$, we have $\{0\} = (\mathcal{S}_R + \mathrm{Ker}\, C_i)^\perp = \mathcal{S}_R^\top + \mathrm{Im}\, C_i^\top = \mathrm{Im}P^\top + \mathrm{Im}C_i^\top$ for all $i \in I$. This implies that, for all $i \in I$, any matrix $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}$ whose columns are a basis of $\mathrm{Ker}[P^\top\ C_i^\top]$ is of the form $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix} = \begin{bmatrix} 0 \\ N_{2i} \end{bmatrix}$, where N_{2i} is a matrix whose columns form a basis of $\mathrm{Ker}\, C_i$. By Proposition 12, any indexed family of pairs of matrices that satisfies (6) is of the form $\{(TL_iT^{-1}, TM_i - TQ_iN_{2i})\}_{i\in I}$, where $\{(L_i, M_i)\}_{i\in I}$ is an indexed family that satisfies (6), T is a change of basis in $\mathcal{X}/\mathcal{S}_R$ and the matrices Q_i are arbitrary matrices of suitable dimensions. Any friend of $\mathcal{S}_R$ is therefore of the form $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y}\to\mathcal{X}\}_{i\in I}$ with $PG_i = -TM_i + TQ_iN_{2i}$, and the linear switching dynamics induced by that of $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$, being characterized by the indexed family of matrices $\{L_i\}_{i\in I}$, does not depend on the choice of the friend $\mathscr{G}$. □

The important consequence of Proposition 14 is that, under its hypotheses, the dynamics induced by that of $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ depends only on $\mathcal{S}_R$. Therefore, external stabilizability of $\mathcal{S}_R$ over $\mathscr{S}_\alpha$ for a given α can be checked by using any of its friends $\mathscr{G}$. In practice, this can be done by employing, e.g., the LMI condition described in Chesi et al. (2012), or the necessary and sufficient conditions for stability over $\mathscr{S}_\alpha$ of Xiang (2016), with respect to the linear switching dynamics induced by that of $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ for an arbitrary friend $\mathscr{G}$.

8 An Illustrative Example on Unknown Input Observer Synthesis

Let us consider the UIOP described by the system Σ_σ of the form (1), where $I = \{1, 2\}$, with state space $\mathcal{X} = \mathbb{R}^4$ and

$$A_1 = \begin{bmatrix} 2.6 & 1.3 & -1.9 & 1.2 \\ -0.8 & -0.4 & 0.2 & -1.6 \\ -0.8 & -1.4 & 1.2 & -1.6 \\ -0.2 & -0.85 & -0.45 & 1.6 \end{bmatrix};$$

$$B_1 = [2\ \ -1\ 1\ 0]^\top;\ C_1 = \begin{bmatrix} 0.1 & 0.05 & -0.15 & 0.2 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.8 & -1.4 & 6.2 & -9.6 \\ 1.2 & 3.1 & -2.3 & 2.4 \\ 1.2 & 0.1 & 0.7 & 2.4 \\ -1 & -1 & 1 & 2 \end{bmatrix};$$

$$B_2 = [2\ 0\ 0\ \ -1]^\top; \quad C_2 = \begin{bmatrix} 0.1 & 0.3 & 0.1 & 0.2 \\ -1 & 0 & 0 & -2 \end{bmatrix}$$

and by the linear maps defined by

$$H_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \ H_2 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Computing $\mathcal{S}^*_{Rg}$ by the procedure described in Section 6, we have $\mathcal{S}^*_{Rg} = \text{span}\{[2\ \ -1\ 1\ 0]^\top; [2\ 0\ 0\ \ -1]^\top\}$ and it is easy to see that $\mathcal{S}^*_{Rg} \cap \text{Ker}\, C_i \subseteq \text{Ker}\ H_i$ for $i = 1, 2$. Therefore, the sufficient condition of Theorem 3 is satisfied and the UIOP is solvable. In order to construct a solution, we search for a friend $\mathscr{G}$ of $\mathcal{S}^*_{Rg}$ that makes the dynamics induced on $\mathcal{X}/\mathcal{S}^*_{Rg}$ globally asymptotically stable. To this aim, let us remark that a basis of $\left(S^*_{Rg}\right)^\perp$ is given, e.g., by the columns of

$$P^\top = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & -3 \\ 2 & 4 \end{bmatrix}$$

and that we have the equality $A_i^\top P^\top = P^\top L_i^\top + C_i^\top M_i^\top$ for $i = 1, 2$ (that is (6)) with, for instance,

$$L_1 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}; \ L_2 = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$M_1 = 0_{2\times 2}; \ M_2 = 0_{2\times 2}.$$

A basis of Ker $\begin{bmatrix} P^\top & C_i^\top \end{bmatrix}$, for $i = 1, 2$, is given by the vector $\begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}$ where, e.g.,

$$N_{11} = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}; \ N_{21} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$N_{12} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}; \ N_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The pair $(L_i, N_{1i}^\top)$ is detectable for $i = 1, 2$ and choosing, e.g., $Q_1^\top = [0\ \ -100]$; $Q_2^\top = [25\ \ 0]$ we have $A_i^\top P^\top = P^\top(L_i^\top + N_{1i}Q_i^\top) + C_i^\top(M_i^\top + N_{2i}Q_i^\top) = P^\top \bar{L}_i^\top + C_i^\top \bar{M}_i^\top$ with

$$\bar{L}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}; \ \bar{L}_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\bar{M}_1 = \begin{bmatrix} 0 & 0 \\ 100 & 0 \end{bmatrix}; \ \bar{M}_2 = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\bar{L}_1$ and $\bar{L}_2$ are Hurwitz. Then, choosing

$$G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ -10 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -10 & 0 \\ -7.5 & 0 \end{bmatrix},$$

so that $PG_i = -\bar{M}_i$ for $i = 1, 2$, we get the desired friend $\mathscr{G} = \{G_1, G_2\}$. Letting

$$C_{O1} = \begin{bmatrix} 0.8 & -0.4 \\ 0.4 & -0.4 \\ -0.1 & 0.3 \end{bmatrix}; \ D_{O1} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \\ 0 & -0.5 \end{bmatrix},$$

$$C_{O2} = \begin{bmatrix} 0 & 0 \\ 0.3 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \ D_{O2} = \begin{bmatrix} 0 & -1 \\ 0 & 0.5 \\ 0 & -0.5 \end{bmatrix}$$

we also have $H_i = C_{Oi}P + D_{Oi}C_i$ for $i = 1, 2$. Then, the observer $\Sigma_{O\sigma}$ of the form (2) defined by

$$\Sigma_{O\sigma} \equiv \begin{cases} \dot{z}(t) = \bar{L}_\sigma z(t) + P^\top G_{\sigma(t)} y(t) \\ w(t) = C_{O\sigma(t)} z(t) + D_{O\sigma(t)} y(t) \end{cases} \tag{28}$$

solves the problem. In particular, as seen in (16), the evolution of the estimation error $e(t)$ is given by

$$\Sigma_{E\sigma} \equiv \begin{cases} \dot{e}_{aux}(t) = \bar{L}_{\sigma(t)} e_{aux}(t) \\ e(t) = C_{O\sigma(t)} e_{aux}(t) \end{cases} \tag{29}$$

and, by inspecting $\bar{L}_i$, we see that $e_{aux}(t)$ goes asymptotically to 0 for all $\sigma \in \mathscr{S}_0$. Hence, so does $e(t)$. The behaviour of the three components of $e(t)$ is shown in Fig. 1 in the case in which, e.g., Σ_σ is initialized at $x(0) = [1\ \ -1\ 0.5\ 1.5]^\top$ and $\Sigma_{O\sigma}$ is initialized at $z(0) = [0\ 0]^\top$ (hence $e_{aux}(0) = [-1.5\ \ -5.5]^\top$) and the switching signal σ is given by

$$\sigma(t) = \begin{cases} 1 & \text{for } 0 \le t < 0.5 \\ 2 & \text{for } 0.5 \le t < 1.5 \\ 1 & \text{for } 1.5 \le t < 3 \\ 2 & \text{for } 3 \le t < 4 \\ 1 & \text{for } 4 \le t \end{cases}.$$

Note that $e(t)$ is discontinuous at the switching times due to the abrupt transition from C_{O1} to C_{O2} or viceversa.

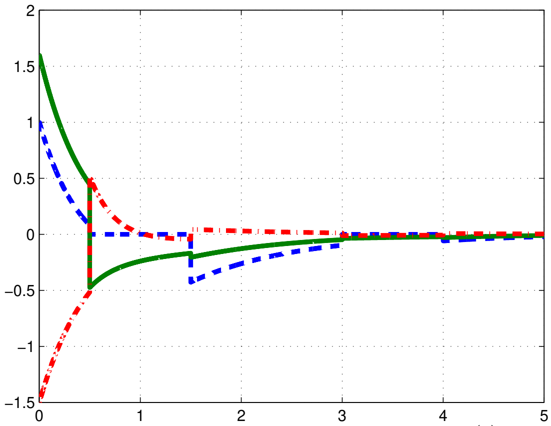

Fig. 1. Behavior of the three components of $e(t)$ – amplitude vs. time (s)

9 Conclusions and future work

Solvability of the unknown input observation problem has been dealt with in the linear switching framework by means of structural and geometric methods. This has made possible to obtain a complete characterization of such property, together with algorithmic procedures to synthesize solutions, both in the case in which the dwell time is fixed and in the case in which a sufficiently large dweel time is acceptable. The basic notion of good conditioned invariant subspace was shown to play a key role in obtaining such results. Future work along the same lines will aim at characterizing solvability with stronger requirements (like convergence to 0 in quadratic sense) on the estimation error.

The structural geometric approach developed here for switching systems also applies to (non-switching) systems which exhibit jumps in the state, also called impulsive systems (see, e.g., Conte et al., 2017c, 2019b, 2020b; Rios et al., 2020). However, in the case of systems with state jumps, the notions of invariance and conditioned invariance, respectively, are different from those given for switching systems, due to the fact that the system structure is different. Likewise, stability must be dealt with differently. Nevertheless, it is possible to combine the structural approach developed separately for switching systems and for impulsive systems to handle the case in which both these behaviours are present, as done in (see, e.g., Conte et al., 2019a; Zattoni et al., 2019) in relation to disturbance decoupling problems. This will be done in relation to observation problems in future work.

The structural approach developed here in investigating unknown input observation problems can be profitably employed to deal with the problem of fault detection and isolation and with the construction of residual generators in the framework of switching systems. This is the object of (Conte et al., 2020a).

Acknowledgements

The authors gratefully acknowledge the anonymous reviewers for their constructive comments.

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