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On a conjecture of Dyer on the join in the weak order of a Coxeter group

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Abstract. In his article of 2019, Dyer states some conjectures about the weak order of a Coxeter group. Among them, one affirms that the extended weak order is a lattice and gives an algebraic-geometric characterization of the join of two elements in this poset. The first assertion has been recently proven for affine types by Barkley and Speyer. In this paper, we prove the second for Coxeter groups of type A and I.

Keywords: Coxeter groups, weak order, join, Bruhat paths, biclosed sets.

1 Introduction

Coxeter groups are abstract groups having a simple presentation. They are very relevant in mathematics, for instance, dihedral groups and, more generally, the symmetry group of any regular polytope are Coxeter groups. They can be described in terms of reflections, in particular, finite Coxeter groups coincide with finite Euclidean reflection groups. Throughout the paper, we assume the reader familiar with basic properties of Coxeter groups and we refer to [3] and [6] for any undefined notation.

Let (W, S) be a Coxeter system, so W is a Coxeter group and S its set of generators. The set of *reflections* of W is $T = \{wsw^{-1} \mid w \in W, s \in S\}$ and the elements of S are called *simple* reflections. The *Bruhat graph* of W, $B(W)$, is the directed graph having W as vertex set and where there is an arrow from u to v, $u \xrightarrow{t} v$, if and only if there is $t \in T$ such that $v = tu$ and $\ell(u) < \ell(v)$, where ℓ denotes the length function.

One of the most important partial orders on W is the *(right) weak order* which can be defined by the prefix property: $u \leq_R v$ if and only if a reduced expression for u is the prefix of a reduced expression for v. It is well-known that $(W, \leq_R)$ is a meet-semilattice and so, when W is finite, it is a lattice. On the contrary, if W is infinite, $(W, \leq_R)$ is never a lattice.

In [4], Dyer introduces a generalization of this poset called *extended weak order*. Let Φ^+ be the set of positive roots of (W, S); then $A \subseteq \Phi^+$ is *closed* if for any $\alpha, \beta \in A$,

$$\{a\alpha + b\beta \mid a, b \in \mathbb{R}_{\geq 0}\} \cap \Phi^+ \subseteq A,$$

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is *coclosed* if its complement A^c is closed, and *biclosed* if it is both closed and coclosed. The set of biclosed subsets of Φ^+ is denoted by $\mathscr{B}(\Phi^+)$ and once it is ordered by inclusion, gives rise to the so called extended weak order $(\mathscr{B}(\Phi^+), \subseteq)$. It generalizes the weak order: when W is finite, the two posets are isomorphic, while if W is infinite, $(W, \leq_R)$ is isomorphic to a subposet of $(\mathscr{B}(\Phi^+), \subseteq)$.

A natural example of a biclosed subset of Φ^+ is the *inversion set* of an element $w \in W$ defined as: $\Phi_w = \Phi^+ \cap w(\Phi^-) = \{\alpha \in \Phi^+ \mid \exists \beta \in \Phi^-,\ \alpha = w(\beta)\}$. Indeed, let $\alpha, \beta \in \Phi_w$ and suppose there exist $a, b \in \mathbb{R}_{\geq 0}$ such that $a\alpha + b\beta \in \Phi^+$; then we know $w^{-1}(\alpha), w^{-1}(\beta) \in \Phi^-$. Therefore,

$$w^{-1}(a\alpha + b\beta) = aw^{-1}(\alpha) + bw^{-1}(\beta) \in \Phi^-,$$

namely, $a\alpha + b\beta \in \Phi_w$. Similarly, if $\alpha, \beta \in \Phi_w^c$, then $w^{-1}(\alpha), w^{-1}(\beta) \in \Phi^+$ and, under the same assumptions, we get $w^{-1}(a\alpha + b\beta) \in \Phi^+$, i.e. $a\alpha + b\beta \in \Phi_w^c$. Actually, Dyer proved that these are the only finite examples.

Lemma 1 ([4], §4.1)**.** *Let $A \subseteq \Phi^+$ and suppose A is finite. Then $A \in \mathscr{B}(\Phi^+)$ if and only if there exists $w \in W$ such that $A = \Phi_w$.*

In [4], Dyer conjectures that the extended weak order is a lattice for any Coxeter system and states a second conjecture characterizing the (conjectural) join of two biclosed sets in $(\mathscr{B}(\Phi^+), \subseteq)$. The first conjecture has been recently proven for affine types by Barkley and Speyer in [1] thanks to a combinatorial description of biclosed sets they gave in [2]; the second, to our knowledge, is still open even for finite Coxeter systems and its original statement is the following.

Let $\tau : \mathscr{P}(\Phi^+) \to \mathscr{P}(W)$ be the map from the power set of Φ^+ to the power set of W sending any $A \subseteq \Phi^+$ to the set

$$\tau(A) = \{w \in W \mid w = s_{\alpha_1} \cdots s_{\alpha_n},\ \ell(s_{\alpha_1}) < \ell(s_{\alpha_1} s_{\alpha_2}) < \cdots < \ell(s_{\alpha_1} \cdots s_{\alpha_n}),\ \alpha_1, \ldots, \alpha_n \in A\}.$$

Conjecture 2 ([4], Section 2.8)**.** *Let $A, B \in \mathscr{B}(\Phi^+)$; then the join of A and B in $(\mathscr{B}(\Phi^+), \subseteq)$ is the following set:*

$$\{\alpha \in \Phi^+ \mid s_\alpha \in \tau(A \cup B)\}.$$

In this paper, we prove Conjecture 2 in the case of Coxeter systems of type A and I. Since both types are finite, we use a re-formulation of Conjecture 2 which was presented to us by Hohlweg [5] and that we describe in what follows.

The weak order is characterized by inversion sets: for any $u, v \in W$, $u \leq_R v$ if and only if $\Phi_u \subseteq \Phi_v$. As mentioned earlier, if W is finite, by Lemma 1, the map $w \mapsto \Phi_w$ is an isomorphism between the posets $(W, \leq_R)$ and $(\mathscr{B}(\Phi^+), \subseteq)$; when W is infinite, the map $w \mapsto \Phi_w$ is still an injective morphism. Note that the extended weak order always has a maximal element which is Φ^+, while the weak order does not have one.

We use the classical bijection between reflections and positive roots, (see [6, Section 1.14]), to work with the *left-reflection set* $T_L(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$ instead of the inversion set Φ_w. In this setting, $u \leq_R v$ if and only if $T_L(u) \subseteq T_L(v)$; in particular, every element $w \in W$ is uniquely identified by $T_L(w)$ or equivalently by Φ_w.

Definition 3 ((u, v)-Bruhat path)**.** *Let $u, v \in W$ and consider the Bruhat graph $B(W)$. A* (u, v)-Bruhat path *is any (directed) path in $B(W)$ starting from the vertex e and whose edges have labels in the set $T_L(u) \cup T_L(v)$. We denote by $V_W(u, v)$ the set of vertices of all the (u, v)-Bruhat paths in $B(W)$.*

The conjecture can be re-formulated as follows:

Conjecture 4. *Let W be a finite Coxeter group and $u, v \in W$. Then*

$$T_L(u \vee_R v) = T \cap V_W(u, v).$$

Conjecture 4 states that the left-reflection set of the join $u \vee_R v$ is the set of reflections reached by all possible (u, v)-Bruhat paths.

We show that the two formulations are equivalent in the finite case.

Theorem 5. *Let W be a finite Coxeter group. Then Conjecture 2 and Conjecture 4 are equivalent.*

Proof. Since W is a finite Coxeter group, then by Lemma 1, we know that any element of $\mathscr{B}(\Phi^+)$ is of the form Φ_w for some $w \in W$. So, if we consider $\Phi_u, \Phi_v \in \mathscr{B}(\Phi^+)$; then Conjecture 2 says that

$$\Phi_u \vee \Phi_v = \{\alpha \in \Phi^+ \mid s_\alpha \in \tau(\Phi_u \cup \Phi_v)\} = \{\alpha_t \in \Phi^+ \mid t \in \tau(\Phi_u \cup \Phi_v)\}. \tag{1.1}$$

Since the posets $(W, \leq_R)$ and $(\mathscr{B}(\Phi^+), \subseteq)$ are isomorphic, then $\Phi_u \vee \Phi_v = \Phi_{u \vee_R v}$ and this set corresponds to the left-reflection set $T_L(\sigma \vee_R \tau)$ through the classical bijection of positive roots with reflections. We need to unpack equation (1.1) using the definition of the function τ: a reflection t is in $\tau(\Phi_u \cup \Phi_v)$ if there exist $\alpha_1, \ldots, \alpha_n \in \Phi_u \cup \Phi_v$ such that $t = s_{\alpha_1} \cdots s_{\alpha_n}$ and

$$\ell(s_{\alpha_1}) < \ell(s_{\alpha_1} s_{\alpha_2}) < \cdots < \ell(s_{\alpha_1} \cdots s_{\alpha_n}). \tag{1.2}$$

Hence, it is sufficient to show that this condition on a reflection $t \in T$ is equivalent to $t \in V_W(u, v)$. Since t is a reflection, $t = t^{-1} = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$ and equation (1.2) implies $\ell(s_{\alpha_1}) < \ell(s_{\alpha_2} s_{\alpha_1}) < \cdots < \ell(s_{\alpha_n} \cdots s_{\alpha_1})$. Finally, recalling the correspondence between Φ_w and $T_L(w)$, we get

$$\alpha_1, \ldots, \alpha_n \in \Phi_u \cup \Phi_v \iff s_{\alpha_1}, \ldots, s_{\alpha_n} \in T_L(u) \cup T_L(v),$$

therefore, $t \in V_W(u, v)$ as we have the path

$$e \xrightarrow{s_{\alpha_1}} s_{\alpha_1} \xrightarrow{s_{\alpha_2}} s_{\alpha_2} s_{\alpha_1} \xrightarrow{s_{\alpha_3}} \cdots\cdots \xrightarrow{s_{\alpha_n}} s_{\alpha_n} \cdots s_{\alpha_1} = t.$$

So, Conjecture 2 and Conjecture 4 are equivalent whenever W is a finite Coxeter group. □

2 Dihedral groups

We start verifying Conjecture 4 for the case of finite dihedral groups, $I_2(m)$. They represent the most simple examples for this conjecture since they are generated by only two reflections $S = \{s, r\}$ with the relation $(sr)^m = e$. For instance, consider $I_2(4)$, the symmetry group of the square, here we have $s \vee_R r = srsr$ and $T_L(s) \cup T_L(r) = \{s, r\}$; therefore any reflection in $T = \{s, r, srs, rsr\}$ is a vertex of a (s, r)-Bruhat path. Indeed, we have the following two paths:

$$e \xrightarrow{s} s \xrightarrow{r} rs \xrightarrow{s} srs, \qquad e \xrightarrow{r} r \xrightarrow{s} sr \xrightarrow{r} rsr.$$

Hence, $T \cap V_{I_2(4)}(s, r) = T = T_L(srsr)$. Now, consider $s \vee_R srs = srs$; we know $T_L(s) \subseteq T_L(srs) = \{s, srs, rsr\}$, so any (s, srs)-Bruhat path has labels in $T_L(srs)$. This implies

$$\{s, srs, rsr\} \subseteq T \cap V_{I_2(4)}(s, srs).$$

Furthermore, since a simple reflection is in a (u, v)-Bruhat path if and only if it is an element of $T_L(u) \cup T_L(v)$, we get $r \notin V_{I_2(4)}(s, srs)$; yielding $T_L(s \vee_R srs) = T \cap V_{I_2(4)}(s, srs)$.

Theorem 6. *Let $u, v \in I_2(m)$; then*

$$T_L(u \vee_R v) = T \cap V_{I_2(m)}(u, v).$$

Proof. First, suppose that $u \not\leq_R v$ and $v \not\leq_R u$. This implies that $u \vee_R v = w_0$, the maximal element. Since $T_L(u \vee_R v) = T$, we need to check that any reflection is a vertex of a (u, v)-Bruhat path. This is not hard as it suffices to notice that the reduced expressions of u and v begin with two different letters. Thus, both r and s are in $T_L(u) \cup T_L(v)$. This yields that any element of the group is a vertex of a (u, v)-Bruhat path since s and r generate the group, so $T = T \cap V_{I_2(m)}(u, v)$.

We are left to prove the other case: $u \leq_R v$ or $v \leq_R u$. We can suppose that both $u \neq w_0$ and $v \neq w_0$, otherwise we would get $\{s, r\} \subseteq T_L(u) \cup T_L(v)$ and we could conclude as above. Assume, without loss of generality, that $u \leq_R v$ and the reduced expression of u starts with s. Then the reduced expression of v is either $(sr)^h$ or $(sr)^h s$ for some $h \in \mathbb{Z}_{\geq 0}$ and, in both cases,

$$T_L(v) = \{s, srs, \ldots, (sr)^d s\}, \tag{2.1}$$

for some $d \in \mathbb{Z}_{\geq 0}$, where some expressions may not be reduced. Since $u \vee_R v = v$, we have to check that $T_L(v) = T \cap V_{I_2(m)}(u, v)$. Furthermore, $u \leq_R v$ is equivalent to $T_L(u) \subseteq T_L(v)$, so, in this case, (u, v)-Bruhat paths have labels in $T_L(v)$. Naturally, $T_L(v) \subseteq V_{I_2(m)}(u, v)$, so it suffices to check that $(T \setminus T_L(v)) \cap V_{I_2(m)}(u, v) = \emptyset$. We split the proof in two cases.

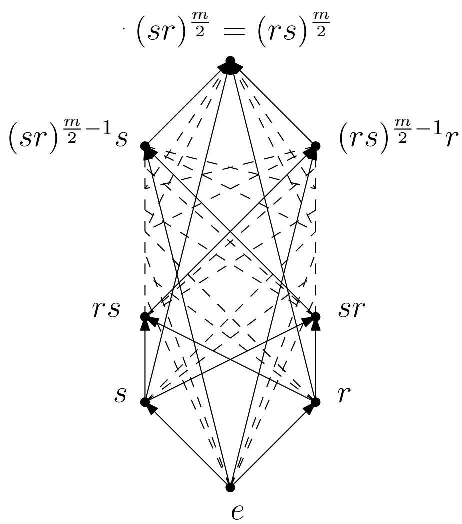

Figure 1: $B(I_2(m))$, if m is even.

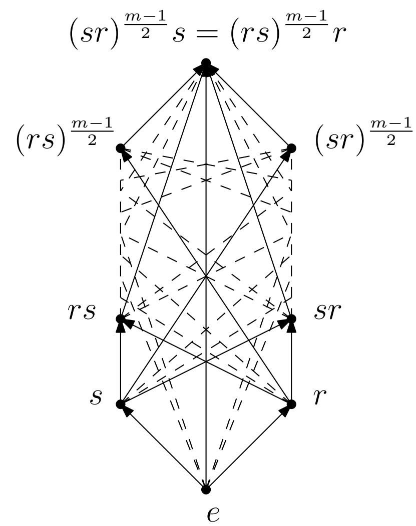

Figure 2: $B(I_2(m))$, if m is odd.

Case 1: all the expressions in the set in (2.1) are reduced, equivalently, $d < \frac{m-1}{2}$. We need to check which vertices we can get in a generic (u, v)-Bruhat path. We can take a look at the Bruhat graphs in Figures 1 and 2.

Let $(sr)^i s$ and $(sr)^j s$ be respectively the first and the second labels of a Bruhat path. Then the third vertex of the path is

$$(sr)^j s (sr)^i s = (sr)^j (rs)^i = \begin{cases} (sr)^{j-i}, & \textit{if } j > i, \\ (rs)^{i-j}, & \textit{otherwise}, \end{cases} \tag{2.2}$$

where the last equality distinguishes between two possible reduced expressions. Moreover, by definition of Bruhat path, we also need to require that the length increases from a vertex to its subsequent, hence, in equation (2.2) we ask that

$$\ell((sr)^i s) < \ell((sr)^j s (sr)^i s) \iff \begin{cases} 2i+1 < 2(j-i), & \textit{if } j > i, \\ 2i+1 < 2(i-j), & \textit{otherwise}. \end{cases}$$

The second inequality is necessarily false, so, the reduced expression of the third vertex can only be of the form $(sr)^l$, with $l < d$. Let $(sr)^k s$ be the third label of the path; then the next vertex is

$$(sr)^k s (sr)^l = (sr)^k (rs)^{l-1} r = \begin{cases} (sr)^{k-l} s, & \textit{if } k > l-1, \\ (rs)^{l-k-1} r, & \textit{otherwise}, \end{cases}$$

where, of course, we have to require that the length increases, i.e.

$$\ell((sr)^l) < \ell((sr)^k s (sr)^l) \iff \begin{cases} 2l < 2(k-l)+1, & \textit{if } k > l-1, \\ 2l < 2(l-k-1)+1, & \textit{otherwise}. \end{cases}$$

Again, the second inequality is false, so we can only get a vertex whose reduced expression is $(sr)^b s$, with $b < d$. At this point, it is easy to note that any reflection whose reduced expression is of the form $(rs)^i r$, cannot be a vertex of the path. Furthermore, any other reflection in $T \setminus T_L(v)$ cannot be a vertex of a (u, v)-Bruhat path. Indeed, let t be such a reflection; then its reduced expression is $(sr)^j s$, with $d < j \leq \frac{m-1}{2}$. Now, if $t \in V_{I_2(m)}(u, v)$, from previous calculations, we know that it must be the product of $(sr)^i s$ and $(sr)^l$ which are respectively the labels of the edge which ends in t and the vertex preceding t. Moreover, as we already observed, i and l must be smaller than d and $i > l - 1$. So, $(sr)^i s(sr)^l = (sr)^{i-l} s = (sr)^j s$, and $i - l \leq d - l < j - l < j$. Which is a contradiction, yielding $T \cap V_{I_2(m)}(u, v) = T_L(v)$.

Case 2: the set in (2.1) contains expressions that are not reduced.
Now, if m is even, we rewrite the elements of $T_L(v)$ only using reduced expressions as

$$T_L(v) = \{s, srs, \ldots, (sr)^{\frac{m}{2}-1} s, (rs)^{\frac{m}{2}-1} r, (rs)^{\frac{m}{2}-2} r \ldots, (rs)^{m-d-1} r\},$$

whereas if m is odd we write it as

$$T_L(v) = \{s, srs, \ldots, (sr)^{\frac{m-1}{2}} s = (rs)^{\frac{m-1}{2}} r, (rs)^{\frac{m-3}{2}} r \ldots, (rs)^{m-d-2} r\}.$$

We discuss both cases at the same time. Note that any reflection whose reduced expression starts with s is in $T_L(v)$. We consider $k := \min\{n \in \mathbb{Z}_{\geq 0} \mid (rs)^n r \in T_L(v)\}$ and prove that if $0 \leq j < k$, then $(rs)^j r \notin V_{I_2(m)}(u, v)$.
Suppose $(rs)^j r \in V_{I_2(m)}(u, v)$, with $0 \leq j < k$; then the first label of the path must be $(sr)^i s$ for some $i < j$, since we need to start with a reflection that is shorter in length. At this point, we have two possibilities for the second label of the path. If it is $(rs)^l r$, for some $k \leq l \leq d$, then the third vertex of the path is

$$(rs)^l r(sr)^i s = (rs)^l (rs)^{i+1} = \begin{cases} (rs)^{l+i+1}, & \text{if } l + i + 1 \leq \frac{m}{2}, \\ (sr)^{m-l-i-1}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that, since $l \geq k$, the first case of equation (2.3) gives an element which is already longer than $(rs)^j r$, so we do not consider it.
If the second label is $(sr)^l s$, for some $i < l \leq d$, then, as we computed earlier, the reduced expression of the third vertex is $(sr)^b$. So, whatever is the reduced expression of the second label, in order for $(rs)^j r$ to be a vertex of the path, the third vertex is $(sr)^b$ for some b. Starting from this vertex, if we label the third edge by $(sr)^l s$, with $l < \frac{m-1}{2}$, then, as in case 1, we get a fourth vertex whose reduced expression is $(sr)^c s$, for some c. Hence, we would be back to a vertex with the same form of the second one. For this reason, we only have to discuss the other possibility. If the third label is $(rs)^l r$, with $k \leq l \leq d$, then the fourth vertex of the path is

$$(rs)^l r(sr)^b = (rs)^{l+b} r = \begin{cases} (rs)^{l+b} r, & \text{if } l + b \leq \frac{m-1}{2}, \\ (sr)^{m-l-b-1} s, & \text{otherwise.} \end{cases} \tag{2.4}$$

The first case of equation (2.4) is a reduced expression of the form we are looking for, but its length is necessarily more than $2k+1$. This means that we cannot get $(rs)^j r$, with $j < k$ as a vertex of a (u,v)-Bruhat path. This concludes the proof, as $T_L(v) = T \cap V_{I_2(m)}(u,v)$ and $T_L(u \vee_R v) = T_L(v)$. □

3 Symmetric groups

We consider the combinatorial description of the Coxeter system (W,S) of type A_{n-1}, where W is the symmetric group S_n and $S = \{(i\, i+1) \mid i \in [n-1]\}$ is the set of simple transpositions; moreover, the set of reflections is given by the transpositions: $T = \{(i\, j) \mid 1 \leq i < j \leq n\}$. We denote a permutation with its *one-line* notation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and we call *inversion* of σ a pair $(i,j) \in [n] \times [n]$, such that $i < j$ and $\sigma(i) > \sigma(j)$. The set of inversions of σ is denoted by $\mathrm{Inv}(\sigma)$ and its cardinality by $\mathrm{inv}(\sigma)$, which is known to be equal to $\ell(\sigma)$, the Coxeter length of σ. Observe that if $i,j \in [n]$ are such that $i < j$ and $\sigma \in S_n$, the following are equivalent:

$$(i)\ (a\ b) \in T_L(\sigma); \qquad (ii)\ (\sigma^{-1}(b), \sigma^{-1}(a)) \in \mathrm{Inv}(\sigma); \qquad (iii)\ (a,b) \in \mathrm{Inv}(\sigma^{-1}).$$

Therefore, for $\sigma, \tau \in S_n$, we have $\sigma \leq_R \tau$ if and only if $\mathrm{Inv}(\sigma^{-1}) \subseteq \mathrm{Inv}(\tau^{-1})$. Before proving the conjecture for symmetric groups, we check its statement in the following example.

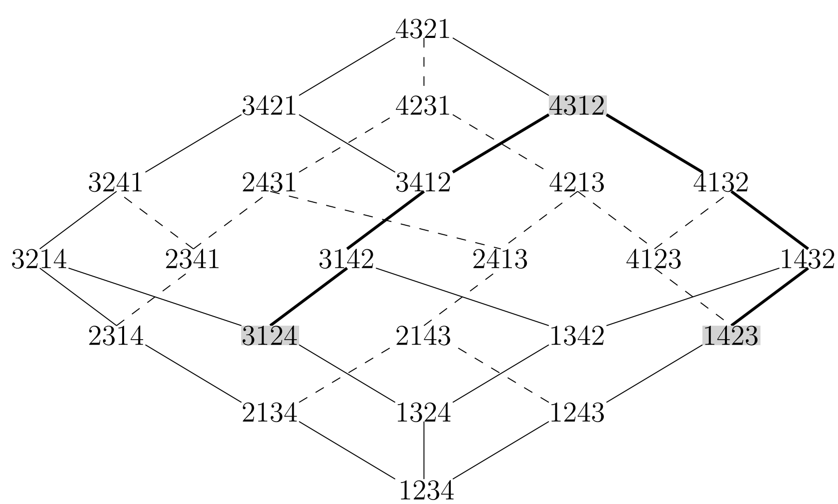

Figure 3: Hasse diagram of $(S_4, \leq_R)$ from example 7.

Example 7. *Consider* $\sigma = 3124$ *and* $\tau = 1423$ *in* S_4*. We have* $T_L(\sigma) = \{(1\,3),(2\,3)\}$, $T_L(\tau) = \{(2\,4),(3\,4)\}$ *and from Figure 3 we see that* $\sigma \vee_R \tau = 4312$*, so* $T_L(\sigma \vee_R \tau) = \{(1\,3),(1\,4),(2\,3),(2\,4),(3\,4)\}$*. Let us verify that the conjecture holds for this example, i.e.*

$$T_L(\sigma \vee_R \tau) = T \cap V_{S_4}(\sigma, \tau). \tag{3.1}$$

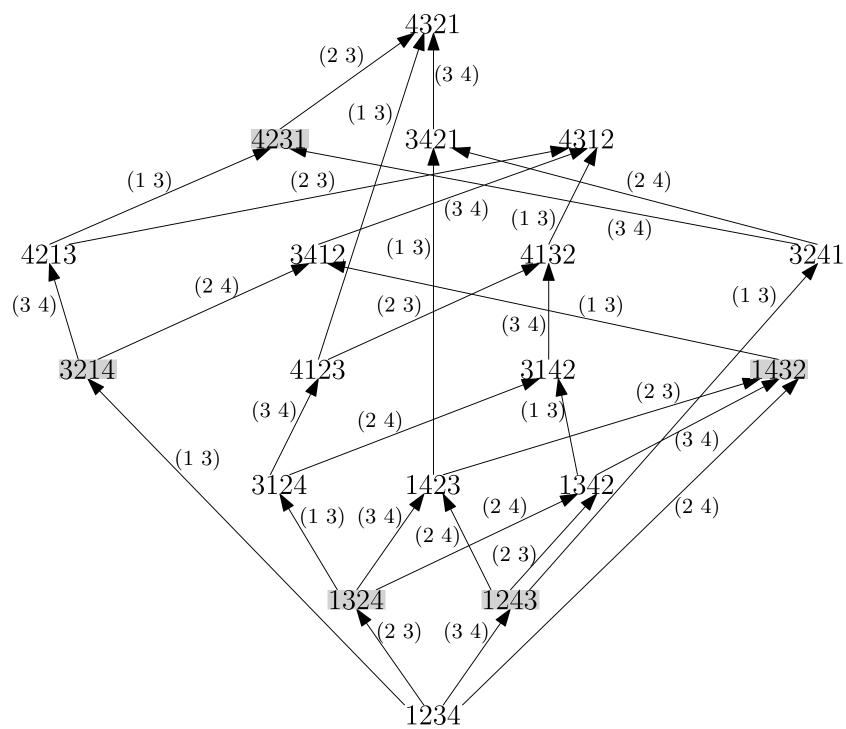

Figure 4: A subgraph of $B(S_4)$ representing every possible (σ, τ)-Bruhat path where reflections are highlighted.

By definition of (σ,τ)-Bruhat path, we have $T_L(\sigma) \cup T_L(\tau) \subseteq T \cap V_{S_4}(\sigma,\tau)$, as $T_L(\sigma) \cup T_L(\tau)$ is the set of labels of (σ,τ)-Bruhat paths; furthermore we have the path

$$1234 \xrightarrow{(1\ 3)} 3214 \xrightarrow{(3\ 4)} 4213 \xrightarrow{(1\ 3)} 4231 = (1\ 4)\,,$$

thus, $(1\ 4) \in V_{S_4}(\sigma,\tau)$ and $T_L(\sigma \vee_R \tau) \subseteq T \cap V_{S_4}(\sigma,\tau)$. Now, since $T \setminus T_L(\sigma \vee_R \tau) = \{(1\ 2)\}$, we can look at all possible (σ,τ)-Bruhat paths represented in Figure 4 and check that $(1\ 2) \notin V_{S_4}(\sigma,\tau)$, so equation (3.1) holds. We can also argue that $(1\ 2)$ is a simple reflection, therefore it is a vertex of a (σ,τ)-Bruhat path if and only if it is in $T_L(\sigma) \cup T_L(\tau)$.

In order to prove Conjecture 4 we use a known result about left-reflection sets of the join of two permutations. First, we say a set of transpositions $J \subseteq T$ is *transitively closed* if for any $(i\ j), (j\ k) \in J$ with $i < j < k$, then $(i\ k) \in J$. Recall that $(i\ j) \in T_L(\sigma)$ if and only if $i < j$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$, so $T_L(\sigma)$ is transitively closed. The following theorem is well-known and its proof can be found in [7, Theorem 1(b)].

Theorem 8. *Let $\sigma, \tau \in S_n$ and denote by J^{tc} the transitive closure of a subset $J \subseteq T$; then*

$$T_L(\sigma \vee_R \tau) = (T_L(\sigma) \cup T_L(\tau))^{tc}.$$

We prove the conjecture by double inclusion. We start by showing that any left-reflection of the join $\sigma \vee_R \tau$ is a vertex of a (σ,τ)-Bruhat path. A Bruhat path is called *palindromic* if its sequence of labels is palindromic.

Theorem 9. *Let $\sigma,\tau \in S_n$ and consider $t \in T_L(\sigma \vee_R \tau)$; then there exists a palindromic (σ,τ)-Bruhat path from e to t.*

Proof. Let $t = (a\ b)$, with $1 \le a < b \le n$ and suppose $t \in T_L(\sigma) \cup T_L(\tau)$; then $e \xrightarrow{t} t$ is a palindromic (σ,τ)-Bruhat path. On the other hand, if $t \notin T_L(\sigma) \cup T_L(\tau)$, by Theorem 8, there is a chain $a = i_0 < i_1 < i_2 < \cdots < i_{l-1} < i_l = b$, with $l \ge 2$, such that $(i_r\ i_{r+1}) \in T_L(\sigma) \cup T_L(\tau)$, for any $r \in \{0,1,\ldots,l\}$. Furthermore, t is obtained by the product of transpositions as follows:

$$t = (a\ b) = (a\ i_1)\,(i_1\ i_2)\cdots(i_{l-2}\ i_{l-1})\,(i_{l-1}\ b)\,(i_{l-2}\ i_{l-1})\cdots(a\ i_1)\,.$$

Now, we define the following palindromic path and show that it is a (σ,τ)-Bruhat path. We do this in two steps.

$$e \xrightarrow{(a\ i_1)} (a\ i_1) \xrightarrow{(i_1\ i_2)} (i_1\ i_2)\,(a\ i_1) \xrightarrow{(i_2\ i_3)} \cdots \xrightarrow{(i_1\ i_2)} (a\ i_1)\,(a\ b) \xrightarrow{(a\ i_1)} (a\ b)\,. \tag{3.2}$$

Step 1: we check that for any $k \in [l-1]$,

$$\mathrm{inv}((i_k\ i_{k+1})\cdots(i_1\ i_2)\,(a\ i_1)) > \mathrm{inv}((i_{k-1}\ i_k)\cdots(i_1\ i_2)\,(a\ i_1)). \tag{3.3}$$

Recall that the number of inversions of a permutation is equal to its length. We use the following notation:

$$(i_k\ i_{k+1})\cdots(i_1\ i_2)\,(a\ i_1) = \begin{bmatrix} a & i_1 & i_2 & \cdots & i_k & i_{k+1} \\ i_{k+1} & a & i_1 & \cdots & i_{k-1} & i_k \end{bmatrix}, \tag{3.4}$$

in which we denote only the indexes that are involved in our computation. Likewise, we have

$$(i_{k-1}\ i_k)\cdots(i_1\ i_2)\,(a\ i_1) = \begin{bmatrix} a & i_1 & i_2 & \cdots & i_k & i_{k+1} \\ i_k & a & i_1 & \cdots & i_{k-1} & i_{k+1} \end{bmatrix}. \tag{3.5}$$

From (3.4) and (3.5), we see that

$$\begin{aligned} &\mathrm{inv}((i_k\ i_{k+1})\cdots(i_1\ i_2)\,(a\ i_1)) \\ &= \mathrm{inv}((i_{k-1}\ i_k)\cdots(i_1\ i_2)\,(a\ i_1)) + (i_{k+1}-i_k) + (i_{k+1}-i_k-1) \end{aligned} \tag{3.6}$$

as in $(i_k\ i_{k+1})\cdots(i_1\ i_2)\,(a\ i_1)$ we have $(i_{k+1}-i_k)$ more inversions in which the first index is a and $(i_{k+1}-i_k-1)$ more inversions in which the second index is i_{k+1} and the first is not a. Thus, equation (3.6) yields

$$\mathrm{inv}((i_k\ i_{k+1})\cdots(i_1\ i_2)\,(a\ i_1)) = \mathrm{inv}((i_{k-1}\ i_k)\cdots(i_1\ i_2)\,(a\ i_1)) + 2(i_{k+1}-i_k) - 1$$

and, since $2(i_{k+1}-i_k)-1>0$, for any $k\in[l-1]$, the inequality in (3.3) is satisfied.

Step 2: we call $\sigma_k := (i_{l-k}\ i_{l-k+1})\cdots(i_{l-2}\ i_{l-1})\,(i_{l-1}\ b)\cdots(i_1\ i_2)\,(a\ i_1)$ and check that $\operatorname{inv}(\sigma_{k-1})<\operatorname{inv}(\sigma_k)$, for any $k\in\{2,3,\dots,l\}$. We use the notation introduced earlier getting

$$\sigma_{k-1}=\begin{bmatrix} a & i_1 & i_2 & \cdots & i_{l-k+1} & b \\ b & a & i_1 & \cdots & i_{l-k} & i_{l-k+1}\end{bmatrix}$$

and

$$\sigma_k=\begin{bmatrix} a & i_1 & i_2 & \cdots & i_{l-k} & i_{l-k+1} & b \\ b & a & i_1 & \cdots & i_{l-k-1} & i_{l-k+1} & i_{l-k}\end{bmatrix}.$$

We note that σ_k has $(i_{l-k+1}-i_{l-k}-1)$ less inversions in which the last index is i_{l-k+1} and $(i_{l-k+1}-i_{l-k})$ more inversions in which the second index is b. Thus,

$$\operatorname{inv}(\sigma_k)=\operatorname{inv}(\sigma_{k-1})-(i_{l-k+1}-i_{l-k}-1)+(i_{l-k+1}-i_{l-k})=\operatorname{inv}(\sigma_{k-1})+1>\operatorname{inv}(\sigma_{k-1}),$$

for any $k\in[l]$. So, we proved that the path in (3.2) is indeed a palindromic (σ,τ)-Bruhat path. □

Theorem 9 implies $T_L(\sigma\vee_R\tau)\subseteq T\cap V_{S_n}(\sigma,\tau)$, so we are left to prove the converse inclusion. In order to do that, we first prove a property of Bruhat paths from e to a reflection $(a\ b)\in T$.

Lemma 10. *Let $(a\ b)\in S_n$ with $a<b$; then all the edges of a Bruhat path from e to $(a\ b)$ are labeled by elements of $T_{ab}:=\{(i\ j)\mid a\le i<j\le b\}$.*

Proof. Let $(i_1\ j_1),(i_2\ j_2),\dots,(i_h\ j_h)\in T$ be the labels of the edges of a Bruhat path from e to $(a\ b)$ listed from last to first, i.e. $(a\ b)=(i_1\ j_1)\cdots(i_{h-1}\ j_{h-1})\,(i_h\ j_h)$.
We prove that $(i_r\ j_r)\in T_{ab}$, for any $r\in[h]$, by induction on r. Since $(a\ b)$ is the last vertex of the path, then the last edge is labeled by an element of $T_L((a\ b))\subseteq T_{ab}$, thus, $(i_1\ j_1)\in T_{ab}$ verifying the base case of induction. Suppose $r>1$ and assume, by inductive hypothesis, that $(i_l\ j_l)\in T_{ab}$, for any $l\in[r-1]$. Since the length in a Bruhat path must increase at every step, we know that

$$(i_r\ j_r)\in T_L((i_{r-1}\ j_{r-1})\cdots(i_1\ j_1)\,(a\ b)). \tag{3.7}$$

Call $\beta:=(i_{r-1}\ j_{r-1})\cdots(i_1\ j_1)\,(a\ b)$ and suppose $(i_r\ j_r)\notin T_{ab}$ and note that, since we assume $i_r<j_r$, equation (3.7) is equivalent to

$$\beta^{-1}(i_r)>\beta^{-1}(j_r). \tag{3.8}$$

Recall that by inductive hypothesis, $i_k,j_k\in\{a,a+1,\dots,b\}$, for any $k\in[r-1]$ and distinguish 3 cases:

1. if $\{i_r, j_r\} \cap \{a, a+1, \dots, b\} = \emptyset$, then equation (3.8) implies
$$i_r = (a\ b)\,(i_1\ j_2)\cdots(i_{r-1}\ j_{r-1})\,(i_r) > (a\ b)\,(i_1\ j_2)\cdots(i_{r-1}\ j_{r-1})\,(j_r) = j_r,$$
but $i_r < j_r$ from which we get a contradiction;

2. if $i_r < a$ while $j_r \in \{a, a+1, \dots, b\}$, then by equation (3.8), we get $i_r > \beta^{-1}(j_r)$; but $\beta^{-1}(j_r) \in \{a, a+1, \dots, b\}$ so, we have a contradiction since $a > i_r$;

3. if $i_r \in \{a, a+1, \dots, b\}$, $j_r > b$, equation (3.8) yields $\beta^{-1}(i_r) > j_r$ and, since $\beta^{-1}(i_r) \in \{a, a+1, \dots, b\}$, again we end up with a contradiction.

Therefore, necessarily $(i_r\ j_r) \in T_{ab}$, so, by induction, every edge of a Bruhat path from e to $(a\ b)$ is labeled by a reflection in T_{ab}. □

Before giving the proof of the last part of the conjecture we observe that if $\sigma \in S_n$ is product of elements of T_{ab}, then $\sigma(k) \in \{a, a+1, \dots, b\}$, for any $k \in \{a, a+1, \dots, b\}$ and $\sigma(i) = i$, for any $i \in [n] \setminus \{a, a+1, \dots, b\}$.

Theorem 11. *Let $\sigma, \tau \in S_n$; then*
$$T_L(\sigma \vee_R \tau) = T \cap V_{S_n}(\sigma, \tau).$$

Proof. By Theorem 9, we only need to prove that any $(a\ b) \in V_{S_n}(\sigma, \tau)$ is in $T_L(\sigma \vee_R \tau)$. Theorem 8 guarantees that we can show this by proving that $(a\ b)$ is in the transitive closure of $T_L(\sigma) \cup T_L(\tau)$; i.e. there exists a chain $a = i_1 < i_2 < \cdots < i_h = b$ such that $(i_j\ i_{j+1}) \in T_L(\sigma) \cup T_L(\tau)$, for any $j \in [h-1]$.

Let us consider a (σ, τ)-Bruhat path from e to $(a\ b)$:
$$e \xrightarrow{(j_0\ j_1)} (j_0\ j_1) \xrightarrow{(j_2\ j_3)} (j_2\ j_3)\,(j_0\ j_1) \xrightarrow{(j_4\ j_5)} \cdots\cdots \xrightarrow{(j_{k-1}\ j_k)} (a\ b)\,,$$
which implies
$$(a\ b) = (j_{k-1}\ j_k)\,(j_{k-3}\ j_{k-2})\cdots(j_0\ j_1)\,. \tag{3.9}$$
At this point, the crucial idea is to note that, since $(a\ b)$ maps a to b, among the labels of this path there must be transpositions of the following form: $(a\ c_1)\,, (c_1\ c_2)\,, \dots, (c_s\ b)\,,$ where we do not assume that for any $j \in [s-1]$, we have $c_j < c_{j+1}$. Indeed, in the product of equation (3.9) these transpositions map a to b. Now, we can observe that, by Lemma 10, $c_1, c_2, \dots, c_s \in \{a, a+1, \dots, b\}$ and, to prove that $(a\ b) \in (T_L(\sigma) \cup T_L(\tau))^{tc}$, it is sufficient to show that $a < c_1 < c_2 < \cdots < c_s < b$.

We already know that $a < c_1$ and $c_s < b$, hence it suffices to prove that $c_r < c_{r+1}$, for any $r \in [s-1]$. Let β be the first vertex of the path from which starts an edge labeled by $(c_r\ c_{r+1})$ and such that $\beta(a) = c_r$. Now, if we suppose $c_r > c_{r+1}$, then, since $\mathrm{inv}((c_r\ c_{r+1})\,\beta) > \mathrm{inv}(\beta)$, we get
$$\beta^{-1}(c_r) > \beta^{-1}(c_{r+1}). \tag{3.10}$$

Observe that, by what we noticed before, (3.10) implies $a = \beta^{-1}(c_r) > \beta^{-1}(c_{r+1})$, but since $c_{r+1} \geq a$, then we have $\beta^{-1}(c_{r+1}) \geq a$ as by Lemma 10, β is product of transpositions in T_{ab}. This is absurd, so, we obtain $c_r < c_{r+1}$ for any $r \in [s-1]$ and, finally,

$$(a\ b) \in (T_L(\sigma) \cup T_L(\tau))^{tc} = T_L(\sigma \vee_R \tau)$$

which concludes the proof. □

4 Concluding remarks

We are working on generalizing the proof of Conjecture 4 for all classical Coxeter groups. Our approach is based on a case by case analysis that uses the combinatorial description of any group. We are also interested in finding a uniform proof, possibly using properties of the associated root system or of the Coxeter arrangement.

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