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(Looking for) the heart of abelian Polish groups ☆

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A B S T R A C T

We prove that the category $\mathcal{M}$ of abelian groups with a Polish cover introduced in collaboration with Bergfalk and Panagiotopoulos is the left heart of (the derived category of) the quasi-abelian category $\mathcal{A}$ of abelian Polish groups in the sense of Beilinson–Bernstein–Deligne and Schneiders. Thus, $\mathcal{M}$ is an abelian category containing $\mathcal{A}$ as a full subcategory such that the inclusion functor $\mathcal{A} \to \mathcal{M}$ is exact and finitely continuous. Furthermore, $\mathcal{M}$ is uniquely characterized up to equivalence by the following universal property: for every abelian category $\mathcal{B}$, a functor $\mathcal{A} \to \mathcal{B}$ is exact and finitely continuous if and only if it extends to an exact and finitely continuous functor $\mathcal{M} \to \mathcal{B}$. In particular, this provides a description of the left heart of $\mathcal{A}$ as a concrete category.
We provide similar descriptions of the left heart of a number of categories of algebraic structures endowed with a topology, including: non-Archimedean abelian Polish groups; locally compact abelian Polish groups; totally disconnected locally compact abelian Polish groups; Polish R-modules, for a given Polish group or Polish ring R; and separable Banach spaces

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and separable Fréchet spaces over a separable complete non-Archimedean valued field.

1. Introduction

The category of abelian groups with a Polish cover and Borel-definable group homomorphisms was recently introduced in collaboration with Bergfalk and Panagiotopoulos [6,7]. In this work, we showed that several classical invariants from homological algebra and algebraic topology, including Ext of countable groups, Steenrod homology of compact metrizable spaces, and Čech cohomology of locally compact metrizable spaces can be seen as functors to the category of abelian groups with a Polish cover. These provide *definable* refinements of such invariants that are *finer*, *richer*, and *more rigid* than the purely algebraic ones.

In this paper we prove that the category $\mathcal{M}$ of abelian groups with a Polish cover and Borel-definable group homomorphisms is an abelian category. The category $\mathcal{A}$ of abelian Polish groups is a full subcategory of $\mathcal{M}$, such that the inclusion functor $\mathcal{A} \to \mathcal{M}$ is finitely continuous and exact. Furthermore, $\mathcal{M}$ is characterized up to equivalence by the following universal property: a functor from the category $\mathcal{A}$ of abelian Polish groups to an abelian category is finitely continuous and exact if and only if it is isomorphic to a functor that extends to a finitely continuous exact functor on $\mathcal{M}$, in which case such an extension is unique up to isomorphism. In other words, $\mathcal{M}$ together with the inclusion $\mathcal{A} \to \mathcal{M}$ is a *universal arrow* [36, Section III.1] from $\mathcal{A}$ to the forgetful functor from the category of abelian categories and finitely continuous exact functors (identified up to isomorphism) to the category of quasi-abelian categories and finitely continuous exact functors (identified up to isomorphism). This universal property also identifies $\mathcal{M}$ as the *left heart* $\mathrm{LH}(\mathcal{A})$ of the quasi-abelian category $\mathcal{A}$ (where "left heart of $\mathcal{A}$" stands for "the heart of the derived category of $\mathcal{A}$ with respect to its canonical left truncation structure") as constructed in [44,46]; see also [5,10,45].

The core of the proof consists in showing that $\mathcal{M}$ is indeed an abelian category, which is far from obvious. This is obtained by means of tools from descriptive set theory, including a selection theorem for Borel relations of Kechris and Macdonald [26], and a dichotomy theorem for coset equivalence relations of Solecki [48]. After it is established that $\mathcal{M}$ is abelian, this category can be recognized as the left heart of $\mathcal{A}$ by means of the characterization of the left heart provided in [46, Proposition 1.2.36].

It is natural to consider the *left* heart of $\mathcal{A}$, rather than the dual notion of right heart. Indeed, $\mathrm{LH}(\mathcal{A})$ has the property that the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ preserves finite limits, and in particular maps monomorphisms to monomorphisms (but does not map epimorphisms to epimorphisms, in general). This is desirable, since $\mathcal{A}$ already has the "right" monomorphisms, which are the injective continuous group homomorphisms. However,

$\mathcal{A}$ has in some sense "too many" epimorphisms, being these the continuous group homomorphisms *with dense image*. This is corrected in LH($\mathcal{A}$), where the epimorphisms are precisely the *surjective* Borel-definable homomorphisms. The fact that the forgetful functor from $\mathcal{A}$ to the category **Ab** of discrete abelian groups is not finitely cocontinuous can be seen as a manifestation of the fact that $\mathcal{A}$ has "too many" epimorphisms. In particular, the forgetful functor $\mathcal{A} \to \mathbf{Ab}$ does not extend to a functor on the right heart of $\mathcal{A}$. However, being exact and finitely continuous, it extends to a forgetful functor $\mathrm{LH}(\mathcal{A}) \to \mathbf{Ab}$. This shows that the objects of $\mathrm{LH}(\mathcal{A})$ can be regarded as groups with additional structure, but the same cannot be said for the objects of the right heart.

We provide similar descriptions of the left heart of several naturally occurring quasi-abelian categories, including:

- the category of non-Archimedean abelian Polish groups;
- the category of locally compact abelian Polish groups;
- the category of totally disconnected locally compact Polish groups;
- the category of Polish R-modules, for a given Polish group or Polish ring R;
- the categories of separable Fréchet spaces and separable Banach spaces over a Polish non-Archimedean valued field K.

The left heart of a quasi-abelian category is also described in [5,46] as a category of formal quotients; see also the work of Waelbroeck and Vasilescu on spaces of formal quotients of Banach, Fréchet, or bornological spaces [10,52–60]. However, in that context the morphisms are defined abstractly by formally inverting certain arrows. In this context, we identify the morphisms as a concrete collection of group homomorphisms that satisfy the natural requirement of being Borel-definable. This provides for each of the quasi-abelian categories mentioned above a description of the left heart as a *concrete category*.

These results make available to the study of abelian Polish groups and groups with a Polish cover tools from category theory and homological algebra. Furthermore, they provide the foundation stone for the study of homological functors on abelian Polish groups and their derived functors. Homological algebra in the context of locally compact abelian Polish groups has been studied in [12,21,39,43]. The concrete description of the left heart of abelian Polish groups provided in this paper has already found a number of applications. These include the characterization of the injective and projective objects in the left heart of locally compact Polish abelian groups [8], and the calculation of the potential Borel complexity of the classification problem for extensions of countable abelian groups [32,33].

In this paper we begin by recalling in Section 2 the notions of abelian category, quasi-abelian category, and the left heart of a quasi-abelian category. We then present in Section 3 the notions of Borel-definable set and Borel-definable group from [6,30,31], and the more restrictive notion of *group with a Polish cover* from [7]. In Section 4.1 we introduce the notion of Polishable subgroup of a group with a Polish cover. The main result here is that images and preimages of Polishable subgroups of abelian groups with a Polish cover are Polishable; see Proposition 4.4. In Section 4.2 we reformulate in this context some results concerning the Borel complexity of Polishable subgroups from [34]. In Section 4.3 we describe a canonical chain of Polishable subgroups of a given abelian group with a Polish cover, which we call *Solecki subgroups*. These were originally defined by Solecki in [47], and have also been considered in [13,48]. Section 4.4 explains how all the results obtained up to that point apply more generally to *Polish R-spaces* for a fixed Polish group or Polish ring R, and in particular to Polish K-vector spaces for a Polish field K.

In Section 5 we show that in certain circumstances a Borel-definable R-homomorphism has a lift that is well-behaved with respect to the algebraic structure. Finally, in Section 6.1 we prove the characterization of the left heart of the category of Polish R-modules (Theorem 6.3), and in Section 6.1 a more general result describing the left heart of a strictly full quasi-abelian subcategory of the category of Polish R-modules (Theorem 6.14). The latter is applied in Section 6.3 to describe the left heart of a number of categories of algebraic structures endowed with a topology, including: non-Archimedean Polish R-modules, locally compact Polish R-modules, locally bounded vector spaces over a Polish field, separable Banach spaces and separable Fréchet spaces over a separable non-Archimedean valued field; see Theorem 6.18.

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2. Category theory background

In this section we recall some notions and results from category theory that are needed in the rest of the paper. For an introduction to category theory, see [3,36].

2.1. Additive categories

Recall that a *preadditive category*, also called an Ab-category, is a category $\mathcal{C}$ in which each hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects A and B is an abelian group, in such a way that composition of morphisms is bilinear [36, Section I.8]. Thus, for morphisms $f_0, f_1 : A \to B$ and $g_0, g_1 : B \to C$ in $\mathcal{C}$, one has that

$$(g_0 + g_1) \circ (f_0 + f_1) = g_0 \circ f_0 + g_0 \circ f_1 + g_1 \circ f_0 + g_1 \circ f_1.$$

In a preadditive category, binary products and binary coproducts coincide, and are called *biproducts*. Furthermore, an object X is initial if and only if it is terminal if and only if 1_X is the zero element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(X, X)$, in which case X is called a zero object; see [36, Section VIII.2] and [35, Section IX.1]. An *additive category* is a preadditive category that has a *zero object*, denoted by 0, and such that every pair of objects A, B has a biproduct, denoted by $A \oplus B$; see [36, Section VIII.2]. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is called additive if satisfies $F(f_0 + f_1) = F(f_0) + F(f_1)$ whenever $f_0, f_1 : A \to B$ are morphisms in $\mathcal{C}$. This is equivalent to the assertion that F preserves biproducts of pairs of objects of $\mathcal{C}$; see [36, Section VIII.2, Proposition 3].

An *additive subcategory* $\mathcal{B}$ of an additive category $\mathcal{A}$ is a (not necessarily full) subcategory of $\mathcal{A}$ that is also an additive category, and such that the inclusion functor $\mathcal{A} \to \mathcal{B}$ is additive.

2.2. Quasi-abelian categories

A *quasi-abelian category* [9, Definition 4.1] (called *almost abelian* in [44]) is an additive category such that:

(1) every morphism has a kernel and a cokernel;
(2) the class of kernels is stable under push-out along arbitrary morphisms, and the class of cokernels is stable under pull-back along arbitrary morphisms;

see also [46].

The first half of the latter requirement means that if in a pull-back diagram

$$\begin{array}{ccc} A & \stackrel{\eta}{\to} & B \\ \downarrow & & \downarrow \\ A' & \stackrel{\eta'}{\to} & B' \end{array}$$

the arrow η is a kernel, then the arrow η' is also a kernel. The second half is the dual assertion obtained by reversing all the arrows, and thus exchanging pull-backs with push-outs and monics with epics.

In a quasi-abelian category, one defines the image $\mathrm{im}(f)$ of an arrow $f : A \to B$ to be the subobject $\ker(\mathrm{coker}(f))$ of B, and the coimage $\mathrm{coim}(f)$ to be the quotient $\mathrm{coker}(\ker(f))$ of A [9, Definition 4.6]. Then f induces a unique arrow $\widehat{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ such that

$$\mathrm{im}(f) \circ \widehat{f} \circ \mathrm{coim}(f) = f.$$

Such an arrow $\widehat{f}$ is both monic and epic [9, Proposition 4.8]. By definition, the arrow f is *strict* if $\widehat{f}$ is an isomorphism [46, Definition 1.1.1]. One has that:

- an arrow is a kernel if and only if it is monic and strict;

- an arrow is a cokernel if and only if it is epic and strict;
- an arrow f is strict if and only if it has a factorization $f = me$ where m is a strict monomorphism and e is a strict epimorphism;
- the composition of strict epic arrows is strict [46, Proposition 1.1.7];
- the composition of strict monic arrows is strict.

Considering the expression of limits in terms of products and equalizers [3, Proposition 5.21], we have that a quasi-abelian category is *finitely complete*, i.e. it has all finite limits. Since the opposite of a quasi-abelian category is also quasi-abelian, by duality a quasi-abelian category also has all finite colimits.

2.3. The left heart of a quasi-abelian category

An *abelian category* is a quasi-abelian category $\mathcal{M}$ such that every monic arrow is a kernel, and every epic arrow is a cokernel or, equivalently, every arrow is strict [36, Section VIII.3]; see also [35, Section IX.2].

A sequence

$$0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$$

in a quasi-abelian category is *short-exact* or a *kernel-cokernel pair* if f is a kernel of g and g is a cokernel of f. A sequence

$$0 \to A \overset{f}{\to} B \overset{g}{\to} C$$

is left short-exact if f is a kernel of g, while a sequence

$$A \overset{f}{\to} B \overset{g}{\to} C \to 0$$

is right short-exact if g is a cokernel of f.

A functor $F : \mathcal{A} \to \mathcal{B}$ from a quasi-abelian category $\mathcal{A}$ to an abelian category $\mathcal{B}$ is:

- *left exact* if for every short-exact sequence

  $$0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0,$$

  the sequence

  $$0 \to F(A) \overset{F(f)}{\to} F(B) \overset{F(g)}{\to} F(C)$$

  is left short-exact or, equivalently, F preserves the kernels of strict arrows;

- *strongly left exact* if for every left short-exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C$$

the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is left short-exact or, equivalently, F preserves the kernels of arbitrary arrows;
- *right exact* if for every short-exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$$

is right short-exact or, equivalently, F preserves the cokernel of strict arrows;
- *strongly right exact* if for every short-exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$$

is right short-exact or, equivalently, F preserves the cokernel of arbitrary arrows
- *exact* if it is both left and right exact;

see [44, Section 1] and [49, Section 1.5].

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be quasi-abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ be a functor. The following assertions are equivalent:*

(1) F is finitely continuous;
(2) F is additive, left exact, and preserves monomorphisms;
(3) F is additive and strongly left exact.

If $\mathcal{A}$ is abelian, these are also equivalent to:

(4) F is additive and left exact.

Proof. (1)⇒(2) If F preserves all finite limits, then in particular it preserves kernels and biproducts. Thus, it preserves monomorphisms, since an arrow f is monic if and only if the kernel of f is zero. Furthermore, considering that F preserves the kernel of strict epimorphisms, we have that F is left exact.

(2)⇒(3) Suppose that F is additive, left exact, and preserves monomorphisms. We claim that F preserves kernels of arbitrary morphisms. Suppose that $f : A \to B$ is a morphism in $\mathcal{A}$ and let $e : E \to A$ be a kernel of f. Consider the canonical decomposition $f = k \circ j$ as in [46, Proposition 1.1.14], where $j : A \to \mathrm{coim}\,(f)$ is a cokernel and $k : \mathrm{coim}\,(f) \to B$ is a *monomorphism* (which is not necessarily a kernel). Since F preserves monomorphisms, we have that $F\,(k)$ is a monomorphism.

Since k is monic, we have that e is a kernel of j. Since j is a cokernel and F is left exact, we have that $F\,(e)$ is a kernel of $F\,(j)$. Since $F\,(k)$ is monic, $F\,(e)$ is a kernel of $F\,(k) \circ F\,(j) = F\,(f)$. This concludes the proof that F preserves kernels of arbitrary morphisms.

(3)⇒(1) Since F is additive, it preserves biproducts and the zero object. Hence, by induction, it preserves finite products. Since F is strongly left exact, it preserves kernels. Hence, being additive, it also preserves equalizers. Considering the expression of finite limits in terms of finite products and equalizers, we have that F preserves all finite limits [3, Proposition 5.21].

Finally, if $\mathcal{A}$ is abelian, then every arrow in $\mathcal{A}$ is strict, and a left exact functor is also strongly exact. □

Corollary 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be quasi-abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ be a functor. The following assertions are equivalent:*

(1) F is exact and finitely continuous;
(2) F is additive, exact, and preserves monomorphisms;
(3) F is additive, right exact, and strongly left exact.

If $\mathcal{A}$ is abelian, these are also equivalent to:

(4) F is additive and exact.

If $\mathcal{A}$ is a abelian category, then an *abelian subcategory* of $\mathcal{A}$ is a (not necessarily full) subcategory $\mathcal{B}$ that is also an abelian category and such that the inclusion functor is additive and exact. Similarly, if $\mathcal{A}$ is a quasi-abelian category, then a *quasi-abelian subcategory* of $\mathcal{A}$ is a (not necessarily full) subcategory $\mathcal{B}$ that is also a quasi-abelian category, and such that the inclusion functor $\mathcal{B} \to \mathcal{A}$ is finitely continuous and finitely cocontinuous.

Let $\mathcal{A}$ be a quasi-abelian category. Then there exists an essentially unique (left) “completion” of $\mathcal{A}$ to an abelian category. This is constructed:

- in [46, Section 1.2.4], building on [5], and more generally for *additive regular categories* in [19]—see also [10, Chapter III]—where it is called the *left heart* (*coeur*) of (the derived category of) $\mathcal{A}$ and denoted by $\mathcal{LH}(\mathcal{A})$;
- in [44, Section 3], under the weaker assumption that $\mathcal{A}$ is *left quasi-abelian*, where it is called the *left abelian cover* of $\mathcal{A}$ and denoted by $\mathrm{Q}_l(\mathcal{A})$;
- in [4], in the more general context of *exact categories*, where it is called the *right abelian envelope* of $\mathcal{A}$ and denoted by $\mathcal{A}_r(\mathcal{A})$;
- in [45, Section 3], in the more general context of *left exact categories*, where it is denoted by $\mathbf{Q}_\ell(\mathcal{A})$ and called the *left quotient category* of $\mathcal{A}$, as in this case $\mathbf{Q}_\ell(\mathcal{A})$ is left abelian but not necessarily abelian.

Following [46], we will call such a category the *left heart* of $\mathcal{A}$, and denote it by $\mathrm{LH}(\mathcal{A})$. We collect in the following proposition the main properties of this category and the inclusion functor $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$.

Proposition 2.3. *Let $\mathcal{A}$ be a quasi-abelian category, and let $\mathcal{A} \subseteq \mathrm{LH}(\mathcal{A})$ be its left heart as in [46, Definition 1.2.18]. Denote by $I : \mathcal{A} \to \mathrm{LH}(\mathcal{A})$ the inclusion functor, and by $U : \mathrm{LH}(\mathcal{A}) \to \mathcal{A}$ the functor given in [46, Definition 1.2.24 and Definition 1.2.26]. Then we have that:*

- *I is finitely continuous and exact [46, Corollary 1.2.28, Proposition 1.2.29];*
- *The essential image of I is closed under extensions, i.e. if*

$$0 \to A \to B \to C \to 0$$

  is a short exact sequence in $\mathrm{LH}(\mathcal{A})$ such that A and C are in $\mathcal{A}$, then B is isomorphic to an object of $\mathcal{A}$ [46, Definition 1.2.18];
- *The essential image of I is stable by subobject, i.e. if $A \to B$ is a monic arrow in $\mathrm{LH}(\mathcal{A})$ such that B is in $\mathcal{A}$, then A is isomorphic to an object of $\mathcal{A}$;*
- *Every object M of $\mathrm{LH}(\mathcal{A})$ has a* presentation *given by a short exact sequence $0 \to A_0 \to \hat{A} \to M \to 0$ in $\mathrm{LH}(\mathcal{A})$, where the arrow $A_0 \to \hat{A}$ is in $\mathcal{A}$, such that M is isomorphic to an object of $\mathcal{A}$ if and only if $A_0 \to \hat{A}$ is a strict monomorphism in $\mathcal{A}$ [44, Section 3];*
- *There exist a canonical isomorphism $i : U \circ I \to \mathrm{id}_{\mathcal{A}}$ and a canonical epimorphism $e : \mathrm{id}_{\mathrm{LH}(\mathcal{A})} \to I \circ U$ that establish an adjunction $U \dashv I$ witnessing that $\mathcal{A}$ is reflective subcategory of $\mathrm{LH}(\mathcal{A})$ [46, Proposition 1.2.27];*
- *For every abelian category $\mathcal{M}$, the functor I induces an equivalence of categories from the category of right exact functors $\mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ to the category of right exact functors $\mathcal{A} \to \mathcal{M}$, which restricts to an equivalence of categories from the category of finitely continuous exact functors $\mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ to the category of finitely continuous exact functors $\mathcal{A} \to \mathcal{M}$ [46, Proposition 1.2.34]. Thus, $\mathrm{LH}\,(-)$ is the left adjoint of the inclusion functor from the category of abelian categories and right exact func-*

tors (respectively, finitely continuous exact functors) to the category of quasi-abelian categories and right exact functors (respectively, finitely continuous exact functors);

- *Let $\mathcal{B}$ be an abelian category, and let $J : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor such that the essential image of J is closed under subobjects, and such that for every object B of $\mathcal{B}$ there exists an epimorphism $J(A) \to B$ in $\mathcal{B}$ for some object A of $\mathcal{A}$. Then J extends to an equivalence of categories* $\mathrm{LH}(\mathcal{A}) \to \mathcal{B}$ *[46, Proposition 1.2.36].*

3. Abelian groups with a Polish cover

3.1. Borel-definable groups

We present here the notion of Borel-definable set and Borel-definable group as in [6,30,31]. We begin by recalling the notion of *idealistic* equivalence relation from [22]; see also [14, Definition 5.4.9] and [26]. We will consider as in [31] a slightly more generous notion. Recall that a σ-filter on a set C is a nonempty family $\mathcal{F}$ of nonempty subsets of C that is closed under countable intersections and such that $A \subseteq B \subseteq C$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

Definition 3.1. Suppose that X is a standard Borel space, and E is an equivalence relation on X. We say that E is *idealistic* if there exist a Borel function $s : X \to X$ and an assignment $C \mapsto \mathcal{F}_C$ mapping each E-class C to a σ-filter $\mathcal{F}_C$ on C such that $s(x)\, Ex$ for every $x \in X$, and for every Borel subset $A \subseteq X \times X$,

$$A_{s,\mathcal{F}} := \left\{ x \in X : \{x' \in [x]_E : (s(x), x') \in A\} \in \mathcal{F}_{[x]_E} \right\}$$

is a Borel subset of X.

The term idealistic is due to the fact that the notion can be equivalently defined in terms of σ-ideals, in view of the duality between σ-ideals and σ-filters. If X is a Polish space endowed with a continuous action of a Polish group G, then the corresponding orbit equivalence relation E_G^X is idealistic [14, Proposition 5.4.10].

A *Borel-definable set* is a pair $(\hat{X}, E)$ where $\hat{X}$ is a Polish space and E is a Borel and idealistic equivalence relation on $\hat{X}$. We denote such a Borel-definable set by $X = \hat{X}/E$, as we think of it as an explicit presentation of the set X as a quotient of the Polish space $\hat{X}$ by the "well-behaved" equivalence relation E. We identify a Polish space $\hat{X}$ with the Borel-definable set $X = \hat{X}/E$ where E is the identity relation on $\hat{X}$. A subset Z of a Borel-definable set $X = \hat{X}/E$ is Borel if $\hat{Z} := \{x \in \hat{X} : [x]_E \in Z\}$ is a Borel subset of $\hat{X}$, in which case Z is itself a Borel-definable set. Similarly, we say that Z is $\mathbf{\Sigma}_1^1$ or *analytic* if $\hat{Z}$ is an analytic subset of $\hat{X}$. If $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are Borel-definable sets, then we define $X \times Y$ to be the Borel-definable set $(\hat{X} \times \hat{Y})/(E \times F)$, where $(x, y)(E \times F)(x', y') \Leftrightarrow xEx'$ and yFy'. (One can verify that $E \times F$ is Borel and idealistic whenever both E and F are Borel and idealistic.)

Suppose that $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are Borel-definable sets, and $f : X \to Y$ is a function. A *lift* $\hat{f}$ of f is a function $\hat{f} : \hat{X} \to \hat{Y}$ such that $f([x]_E) = [\hat{f}(x)]_F$ for every $x \in X$. In this case, we also say that f is induced by $\hat{f}$.

Definition 3.2. Suppose that $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are Borel-definable sets, and $f : X \to Y$ is a function. Then f is *Borel-definable* if it admits a Borel lift $\hat{f} : \hat{X} \to \hat{Y}$.

By the version of the selection theorem [23, Theorem 18.6] presented in the proof of [26, Lemma 3.7], the function $f : X \to Y$ is Borel-definable if and only if its graph is a Borel subset of $X \times Y$.

The category of Borel-definable sets has Borel-definable functions as morphisms. This category contains the category of standard Borel spaces and Borel functions as a full subcategory, and it satisfies natural generalizations of several good properties of the latter. We recall here the most salient ones; see [31, Proposition 1.10] and references therein.

Proposition 3.3 *(Kechris–Macdonald [26]). Suppose that X and Y are Borel-definable sets. If $f : X \to Y$ is a Borel-definable injection, and $A \subseteq X$ is Borel, then $f(A) \subseteq Y$ is Borel, and the inverse function $f^{-1} : f(A) \to X$ is Borel-definable.*

Proposition 3.4 *(Motto Ros [40]). Suppose that X and Y are Borel-definable sets. If there exist a Borel-definable injection $X \to Y$ and a Borel-definable injection $Y \to X$, then there exists a Borel-definable bijection $X \leftrightarrow Y$.*

If $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are Borel-definable sets, then the Borel-definable set $X \times Y$ as defined above is the product of X and Y in the category of Borel-definable sets.

More generally, one can consider sets that are presented as $X = \hat{X}/E$ where $\hat{X}$ is a Polish space and E is a analytic equivalence relation on $\hat{X}$ that is not necessarily Borel or idealistic. In this case, we say that X is a $\boldsymbol{\Sigma}^1_1$*-definable set*. The notions of *Borel* and *analytic* subset of a $\boldsymbol{\Sigma}^1_1$-definable set, and of *Borel-definable function* between $\boldsymbol{\Sigma}^1_1$-definable sets, can be formulated as in the case of Borel-definable sets. The category of $\boldsymbol{\Sigma}^1_1$-definable sets has $\boldsymbol{\Sigma}^1_1$-definable functions (which are the functions with analytic graph) as morphisms. The following result is established in [30, Corollary 1.14].

Lemma 3.5. *Suppose that X is a Borel-definable set and Y is a $\boldsymbol{\Sigma}^1_1$-definable set. If there exists a Borel-definable bijection $f : X \to Y$, then Y is Borel-definable.*

A *Borel-definable group* is simply a group object in the category of Borel-definable sets in the sense of [36, Section III.6]. Explicitly, a Borel-definable group is a Borel-definable set G that is also a group, and such that the group operation $G \times G \to G$ and the function $G \to G$ mapping each element to its inverse are Borel-definable. Notice that every standard Borel group is, in particular, a Borel-definable group. Naturally, a $\mathbf{\Sigma}_1^1$-definable group is a group object in the category of $\mathbf{\Sigma}_1^1$-definable sets.

An important example of $\mathbf{\Sigma}_1^1$-definable group that is not a Borel-definable group is $\mathbb{R}^\omega/E_1$, where $\mathbb{R}^\omega$ is the product of countably many copies of the Polish group $\mathbb{R}$, and E_1 is the tail-equivalence relation on $\mathbb{R}^\omega$, obtained by setting $(\boldsymbol{x}_i)\, E_1\, (\boldsymbol{y}_i) \Leftrightarrow \exists n \forall i \geq n$, $x_i = y_i$. Notice that, if $\mathbb{R}^{(\omega)} \subseteq \mathbb{R}^\omega$ is the subgroup consisting of sequences that are eventually zero, then E_1 is the coset equivalence relation associated with $\mathbb{R}^{(\omega)}$. Thus, $\mathbb{R}^\omega/E_1$ can be seen as the quotient group $\mathbb{R}^\omega/\mathbb{R}^{(\omega)}$. It is proved in [25, Theorem 4.1] that if X is a Borel-definable set, then there is no Borel-definable injection $\mathbb{R}^\omega/\mathbb{R}^{(\omega)} \to X$.

3.2. Groups with a Polish cover

We now recall the notion of abelian group with a Polish cover introduced in [7].

Definition 3.6. An abelian group with a Polish cover is a Borel-definable abelian group given as a quotient $\hat{G}/N$ where $\hat{G}$ is an abelian Polish group and $N \subseteq \hat{G}$ is a Polishable subgroup. This means that N is a Borel subgroup of $\hat{G}$ such that there is a Polish group topology on N whose open sets are Borel in $\hat{G}$ or, equivalently, there exist a Polish group H and a continuous homomorphism $\psi : H \to \hat{G}$ with image N. For $x, y \in \hat{G}$, we write $x \equiv y \bmod N$ if $x - y \in N$.

We term here abelian groups with a Polish cover what in [6] are called groups with an abelian Polish cover. We opt for this terminology for uniformity with the more general setting of modules—in which case the terminology of [6] is difficult to adapt—and to adhere to the spirit of the main results of this work, showing that under natural assumptions the left heart of categories of Polish spaces endowed with an algebraic structure of a certain kind has as objects formal quotients of those.

We regard an abelian Polish group G as an abelian group with a Polish cover $\hat{G}/N$ where $G = \hat{G}$ and $N = \{0\}$. If $G = \hat{G}/N$ and $H = \hat{H}/M$ are abelian groups with a Polish cover, then we define $G \oplus H$ to be the abelian group with a Polish cover $(\hat{G} \oplus \hat{H})/(N \oplus M)$. Similarly, if $G_k = \hat{G}_k/N_k$ is an abelian group with a Polish cover for $k \in \omega$, then we define $\prod_{k \in \omega} G_k$ to be the abelian group with a Polish cover $\hat{G}/N$ where $\hat{G} = \prod_k \hat{G}_k$ and $N = \prod_k N_k$.

Recall that a Polish group is *non-Archimedean* if it has a basis of neighborhoods of the identity consisting of *open subgroups*; see [11, Proposition 2.1] for other characterizations. A Polish group is *locally compact* if it has a basis of neighborhoods of the identity consisting of *compact* subsets.

Definition 3.7. Suppose that $G = \hat{G}/N$ is an abelian group with a Polish cover. Then we say that G is:

- an abelian group with a non-Archimedean Polish cover if $\hat{G}$ and N are non-Archimedean Polish groups;
- an abelian group with a locally compact Polish cover if $\hat{G}$ and N are locally compact Polish groups.

As an abelian group with a Polish cover is, in particular, a Borel-definable group, the notion of Borel-definable homomorphism between abelian groups with a Polish cover is a particular instance of the notion of Borel-definable group homomorphism between Borel-definable groups.

Definition 3.8. Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are abelian groups with a Polish cover. A group homomorphism $f : G \to H$ is:

- *Borel-definable* if it has a Borel lift $\hat{f} : \hat{G} \to \hat{H}$;
- *Baire-definable* if it has a Baire-measurable lift $\hat{f} : \hat{G} \to \hat{H}$ [23, Definition 8.37];
- *continuously definable* if it has a continuous lift $\hat{f} : \hat{G} \to \hat{H}$;
- *locally continuously definable* if it has a Borel lift $\hat{f} : \hat{G} \to \hat{H}$ that is *locally continuous*, in other words continuous on a zero neighborhood in $\hat{G}$;
- *liftable* if it has a lift $\hat{f} : \hat{G} \to \hat{H}$ that is a continuous group homomorphism.

If G is an abelian group with a locally compact Polish cover, then we say that $f : G \to H$ is:

- *Haar-definable* if it has a Haar-measurable lift $\hat{f} : \hat{G} \to \hat{H}$.

One can analogously define the notions from Definition 3.8 in the more general context of $\boldsymbol{\Sigma}^1_1$-definable groups.

Definition 3.9. Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are abelian groups with a Polish cover. Let $f : \hat{G} \to \hat{H}$ be lift of a group homomorphism $G \to H$. We let $\delta f : \hat{G} \times \hat{G} \to M$ be the corresponding 2-cocycle, defined by $\delta f(x, y) = f(y) - f(x + y) + f(x)$.

Remark 3.10. It follows from Proposition 3.3 that if a Borel-definable homomorphism $G \to H$ is bijective, then its inverse $H \to G$ is also a Borel-definable homomorphism.

In what follows, we consider groups with a Polish cover as objects of a category that has Borel-definable homomorphisms as morphisms.

Remark 3.11. It follows from [7, Proposition 4.6] that, *when $\hat{G}$ is non-Archimedean*, a group homomorphism $\varphi : \hat{G}/N \to \hat{H}/M$ between abelian groups with a Polish cover is Borel-definable if and only if it is continuously definable.

Lemma 3.12. *Let $f : \hat{G}/N \to \hat{H}/M$ be a group homomorphism between abelian groups with a Polish cover. Let V be a zero neighborhood in $\hat{G}$ and let $\hat{f} : V \to \hat{H}$ be a continuous function such that $\hat{f}(x) + M = f(x + N)$ for every $x \in V$. Then there exists a Borel lift for f whose restriction to V is equal to $\hat{f}$.*

Proof. Let $\{a_n : n \in \omega\}$ be a countable dense subset of $\hat{G}$ with $a_0 = 0$. For $n \in \omega$, let $b_n \in \hat{H}$ be such that $f(a_n + N) = b_n + M$, where $b_0 = 0$. By [23, Theorem 18.10] there exists a Borel function $\hat{G} \to \omega$, $x \mapsto n(x)$ such that $x \in V + a_{n(x)}$ for every $x \in \hat{G}$ and $n(x) = 0$ for $x \in V$. We can thus extend $\hat{f}$ to a Borel function on $\hat{G}$ by setting $\hat{f}(x) := \hat{f}\left(x - a_{n(x)}\right) + b_{n(x)}$ for $x \in \hat{X}$. □

4. Subgroups of groups with a Polish cover

4.1. Subgroups with a Polish cover

We now introduce in the context of abelian groups with a Polish cover the notion of Borel subgroup and subgroup with a Polish cover.

Definition 4.1. Suppose that $G = \hat{G}/N$ is an abelian group with a Polish cover, and $H \subseteq G$ is a subgroup. Define $\hat{H} = \{x \in \hat{G} : x + N \in H\} \subseteq \hat{G}$. Then we say that:

- H is a *Borel* (respectively, *analytic*) *subgroup* of G if $\hat{H}$ is a Borel (respectively, analytic) subgroup of $\hat{G}$;
- H is a *subgroup with a Polish cover* of G if $\hat{H}$ is a Polishable subgroup of $\hat{G}$;
- H is a *subgroup with a non-Archimedean Polish cover* of G if $\hat{H}$ is a non-Archimedean Polishable subgroup of $\hat{G}$.

If $H = \hat{H}/N$ is a subgroup with a Polish cover of an abelian group with a Polish cover $G = \hat{G}/N$, where $\hat{H}$ is a Polishable subgroup of $\hat{G}$, then we regard H as the abelian group with a Polish cover $\hat{H}/N$, and G/H as the abelian group with a Polish cover $\hat{G}/\hat{H}$.

If $G = \hat{G}/N$ is group with a Polish cover and H is a subgroup with a Polish cover of G, then we let $\overline{H}^G$ be the closed subgroup of G obtained as the closure of H in G with respect to the quotient topology induced by $\hat{G}$. We say that H is *dense in* G if $\overline{H}^G = G$.

Lemma 4.2. *Suppose that G is an abelian group with a Polish cover. Let $(G_n)_{n\in\omega}$ be a sequence of subgroups with a Polish cover of G. Then G_0+G_1 and $\bigcap_{n\in\omega} G_n$ are subgroups with a Polish cover of G.*

Proof. Write $G = \hat{G}/N$. For every $n \in \omega$, we have that $G_n = \hat{G}_n/N$ for some Polishable subgroup $\hat{G}_n$ of $\hat{G}$. We have that

$$\{x \in \hat{G} : x + N \in G_0 + G_1\} = \hat{G}_0 + \hat{G}_1 + N$$

is the image of the Polish group $\hat{G}_0 \oplus \hat{G}_1 \oplus N$ under the continuous homomorphism $\hat{G}_0 \oplus \hat{G}_1 \oplus N \to \hat{G}$, $(x, y, z) \mapsto x + y + z$.

Similarly, we have that

$$\{x \in \hat{G} : x + N \in \bigcap_{n\in\omega} G_n\} = \bigcap_{n\in\omega} \hat{G}_n$$

is the image of the Polish group

$$Z := \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\omega} \hat{G}_n : \forall n \in \omega, x_n = x_{n+1} \right\} \subseteq \prod_{n\in\omega} \hat{G}_n$$

under the continuous homomorphism $Z \to \hat{G}$, $(x_n)_{n\in\omega} \mapsto x_0$. □

Suppose that L is a Borel subgroup of an abelian group with a Polish cover G. Then one can consider the quotient G/L as a $\mathbf{\Sigma}^1_1$-definable group. The implication (1)⇒(3) in the following proposition can be seen as a reformulation of [48, Theorem 1.1].

Proposition 4.3. *Suppose that L is a Borel subgroup of an abelian group with a Polish cover G. Consider the corresponding $\mathbf{\Sigma}^1_1$-definable group G/L. The following assertions are equivalent:*

(1) there does not exist a Borel-definable injection $\mathbb{R}^\omega/\mathbb{R}^{(\omega)} \to G/L$;
(2) G/L is a Borel-definable group;
(3) L is a subgroup with a Polish cover of G.

Proof. Write $G = \hat{G}/N$ and let $\hat{L} = \{x \in \hat{G} : x + N \in L\}$. Since $G/L = \hat{G}/\hat{L}$, after replacing L with $\hat{L}$ and G with $\hat{G}$, we can assume that G is in fact a Polish group.

The implication (3)⇒(2) follows from the fact that a group with a Polish cover is, in particular, a Borel-definable group in view of [14, Proposition 5.4.10]. The implication (2)⇒(1) follows from [25, Theorem 4.1]. Finally, the implication (1)⇒(3) is the content of [48, Theorem 1.1]. □

We now show that images and preimages of subgroups with a Polish cover under Borel-definable homomorphisms are subgroups with a Polish cover.

Proposition 4.4. *Suppose that G, H are abelian groups with a Polish cover, and $f : G \to H$ is a Borel-definable group homomorphism.*

(1) If H_0 is a subgroup with a Polish cover of H, then $f^{-1}(H_0)$ is a subgroup with a Polish cover of G;
(2) If G_0 is a subgroup with a Polish cover of G, then $f(G_0)$ is a subgroup with a Polish cover of H.

Proof. (1) After replacing H with H/H_0, we can assume that $H_0 = \{0\}$, in which case

$$f^{-1}(H_0) = \ker(f) := \{g \in G : f(g) = 0\}.$$

We have that f induces a Borel-definable injective group homomorphism $G/\ker(f) \to H$. Notice that $\ker(f)$ is a Borel subgroup of G. Since H is a group with a Polish cover, we have that there does not exist a Borel-definable injection $\mathbb{R}^{\omega}/\mathbb{R}^{(\omega)} \to H$ by Proposition 4.3. Thus, there does not exist a Borel-definable injection $\mathbb{R}^{\omega}/\mathbb{R}^{(\omega)} \to G/\ker(f)$. Thus, $\ker(f)$ is a subgroup with a Polish cover of G by Proposition 4.3 again.

(2) After replacing G with G_0 and f with its restriction to G_0, we can assume that $G = G_0$. By the first item, $\ker(f)$ is a subgroup with a Polish cover of G. Thus, after replacing G with $G/\ker(f)$, we can assume that f is injective. In this case, we have that $f(G)$ is a Borel subgroup of the Borel-definable group H by Proposition 3.3. By Proposition 4.3, to conclude the proof it suffices to prove that $H/f(G)$ is a Borel-definable group.

Write $G = \hat{G}/N$ and $H = \hat{H}/M$, where $\hat{G}, \hat{H}$ are Polish groups and $N \subseteq \hat{G}$ and $M \subseteq \hat{H}$ are Polishable subgroups. Suppose that $\varphi : \hat{G} \to \hat{H}$ is a Borel lift of f. Define the Borel function $\delta\varphi : \hat{G} \times \hat{G} \to M$ as in Definition 3.9. We need to prove that the equivalence relation E on $\hat{H}$ defined by setting $xEy \Leftrightarrow \exists (g,h) \in \hat{G} \oplus M, \varphi(g) + h + x = y$ is idealistic. The argument is similar to the one from [14, Proposition 5.4.10]. We adopt the notation of category quantifiers as in [23, Section 16]. For $x \in \hat{H}$ and $A \subseteq [x]_E$, we set $A \in \mathcal{F}_{[x]_E} \Leftrightarrow \forall^* g \in \hat{G}, \forall^* h \in M, \varphi(g) + h + x \in A$. Observe that $\mathcal{F}_{[x]}$ does not depend on the choice of the representative x for the equivalence class $[x]_E$. Indeed, if $x_0 E x$ then there exists $(g_0, h_0) \in \hat{G} \times M$ such that $\varphi(g_0) + h_0 + x_0 = x$. If $A \in \mathcal{F}_{[x]}$ then $\forall^* g \in \hat{G}$, $\forall^* h \in M$, $\varphi(g) + h + x \in A$. We have that

$$\begin{aligned}\varphi(g) + h + x &= \varphi(g) + h + \varphi(g_0) + h_0 + x_0 \\ &= \varphi(g + g_0) + h + h_0 + \delta\varphi(g, g_0) + x_0.\end{aligned}$$

For a fixed $\tilde{g} \in \hat{G}$, if $\forall^* h \in M$, $\varphi(\tilde{g}) + h + x \in A$ then $\forall^* h \in M$, $\varphi(\tilde{g} + g_0) + h + h_0 + \delta\varphi(\tilde{g}, g_0) + x_0 \in A$. Since $h_0 + \delta\varphi(\tilde{g}, g_0) \in M$, this implies that $\forall^* h \in M$, $\varphi(\tilde{g} + g_0) + h + x_0 \in A$. Therefore, we have that $\forall^* g \in \hat{G}$, $\forall^* h \in M$, $\varphi(g + g_0) + h + x_0 \in A$ and hence $\forall^* g \in \hat{G}$, $\forall^* h \in M$, $\varphi(g) + h + x_0 \in A$. This shows that $\mathcal{F}_{[x]_E}$ is well-defined. It is easy to verify that $\mathcal{F}_{[x]_E}$ is a σ-filter on $[x]_E$. It remains to prove that if $A \subseteq \hat{H} \times \hat{H}$ is Borel, then

$$A_{\mathcal{F}} := \left\{x \in \hat{H} : \left\{y \in \hat{H} : (x,y) \in A\right\} \in \mathcal{F}_{[x]_E}\right\}$$

is a Borel subset of $\hat{H}$. The argument is the same as in the proof of [14, Theorem 3.3.3]. We have that $x \in A_{\mathcal{F}} \Leftrightarrow \forall^* g \in \hat{G}, \forall^* h \in M, (x, \varphi(g) + h + x) \in A$. Define the Borel set

$$B := \left\{(g,h,x) \in \hat{G} \times M \times \hat{H} : (x, \varphi(g) + h + x) \in A\right\}.$$

Then we have that

$$x \in A_{\mathcal{F}} \Leftrightarrow \forall^* g \in \hat{G}, \forall^* h \in M, (g, h, x) \in B \Leftrightarrow \forall^* (g, h) \in \hat{G} \times M, (g, h, x) \in B.$$

Since B is Borel, this shows that $A_{\mathcal{F}}$ is Borel by [23, Theorem 16.1]. □

The following corollary can be seen as a generalization of the existence of Borel right inverses for surjective continuous group homomorphisms between Polish groups; see [23, Theorem 12.17].

Corollary 4.5. *Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are abelian groups with a Polish cover. Let $f : G \to H$ be a surjective Borel-definable group homomorphism. Then there exists a Borel function $\psi : \hat{H} \to \hat{G}$ such that $f(\psi(h) + N) = h + M$ for every $h \in \hat{H}$.*

Proof. We can write f as the composition

$$G \xrightarrow{p} \frac{G}{\ker(f)} \xrightarrow{\bar{f}} H$$

where p is the quotient map and $\bar{f}$ is the Borel-definable group isomorphism induced by f. It thus suffices to prove that the conclusion holds for p and $\bar{f}$. In the case of p the conclusion is obvious. In the case of $\bar{f}$, it is a consequence of Proposition 3.3. □

Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are abelian groups with a Polish cover. Let $\varphi : G \to H$ be a group homomorphism. Define the *graph* $\Gamma(\varphi)$ of φ to be the subgroup

$$\{(x, y) \in G \oplus H : \varphi(x) = y\} \subseteq G \oplus H.$$

By Proposition 4.4, if φ is Borel-definable, then $\Gamma(\varphi)$ is a subgroup with a Polish cover of $G \oplus H$, being the kernel of the Borel-definable homomorphism $G \oplus G \to H$, $(x, y) \mapsto \varphi(x) - y$. When G and H are abelian groups with a non-Archimedean Polish cover, $\Gamma(\varphi)$ is also an abelian group with a non-Archimedean Polish cover by Theorem 6.18(1) below. The function $\pi_G : \Gamma(\varphi) \to G$, $(x, y) \mapsto x$ is a bijective liftable group homomorphism, and the function $\pi_H : \Gamma(\varphi) \to H$, $(x, y) \mapsto y$ is a liftable group homomorphism; see Definition 3.8. Furthermore, we have that $\varphi = \pi_H \circ (\pi_G)^{-1}$.

Theorem 4.6. *Suppose that $\varphi : G \to H$ is a group homomorphism between abelian groups with a Polish cover. The following assertions are equivalent:*

(1) φ is Borel-definable;
(2) the graph $\Gamma(\varphi)$ is a subgroup with a Polish cover of $G \oplus H$;
(3) there exist an abelian group with a Polish cover L, a bijective liftable group homomorphism $\sigma : L \to G$, and a liftable group homomorphism $\psi : L \to H$ such that $\varphi = \psi \circ \sigma^{-1}$.

Proof. The implication (1)⇒(2) follows from Proposition 4.4 as observed above.

(2)⇒(3) In order to verify that (3) holds, it suffices to take L to be the graph $\Gamma(\varphi)$ of φ, $\sigma : \Gamma(\varphi) \to G$ to be the Borel-definable homomorphism defined by $(x, y) \mapsto x$, and $\psi : \Gamma(\varphi) \to H$ to be the Borel-definable homomorphism defined by $(x, y) \mapsto y$.

(3)⇒(1) This follows by observing that a liftable group homomorphism is, in particular, Borel-definable, and that the inverse of a bijective Borel-definable group homomorphism is Borel-definable by Remark 3.10. □

Proposition 4.7. *Let $G = \hat{G}/N$ be an abelian group with a Polish cover, and let $H = \hat{H}/E_H$ be a $\mathbf{\Sigma}^1_1$-definable group. Suppose that $\varphi : G \to H$ is a group homomorphism. The following assertions are equivalent:*

(1) φ is Borel-definable;
(2) the graph $\Gamma(\varphi)$ is an analytic subgroup of $G \oplus H$;
(3) φ is Baire-definable.

If furthermore G is an abelian group with a locally compact Polish cover, then the above conditions are equivalent to:

(4) φ is Haar-definable.

Proof. Consider the lift

$$\hat{\Gamma}(\varphi) = \{(x, y) \in \hat{G} \oplus \hat{H} : \varphi(x + N)\, E_H y\} \subseteq \hat{G} \oplus \hat{H}$$

of $\Gamma(\varphi) \subseteq G \oplus H$.

(1)⇒(2) By assumption, we have that φ has a Borel lift $f : \hat{G} \to \hat{H}$. We have that $(x, y) \in \hat{\Gamma}(\varphi)$ if and only if there exist $z \in E_H \subseteq \hat{H} \times \hat{H}$ such that $\pi_0(z) = f(x)$ and $\pi_1(z) = y$, where π_0, π_1 are the canonical projections from $\hat{H} \times \hat{H}$ to $\hat{H}$. Since by assumption E_H is an analytic equivalence relation on $\hat{H}$, it follows that $\hat{\Gamma}(\varphi)$ is an analytic subset of $\hat{G} \oplus \hat{H}$, and hence by definition $\Gamma(\varphi)$ is an analytic subgroup of $G \oplus H$.

(2)⇒(3) By the Jankov–von Neumann Uniformization Theorem [23, Theorem 18.1] applied to $\hat{\Gamma}(\varphi) \subseteq \hat{G} \oplus \hat{H}$, we have that there exists a $\sigma(\mathbf{\Sigma}^1_1)$-measurable lift $f : \hat{G} \to \hat{H}$ for φ. Since analytic sets are Baire-measurable [23, Theorem 21.6], we have that f is Baire-measurable, and φ is Baire-definable.

(3)⇒(1) Suppose that φ is Baire-definable. Let $f : \hat{G} \to \hat{H}$ be a Baire-measurable lift of φ. Then there exists a dense G_δ subset C of G such that $f|_C$ is continuous. Consider the relation

$$P = \{(x, y) \in \hat{G} \times \hat{G} : y \in (x - C) \cap C\}.$$

Then by [23, Theorem 18.6]—see also [26, proof of Lemma 3.7]—there exists a Borel function $\sigma : \hat{G} \to \hat{G}$ such that $\sigma(x) = x$ for $x \in C$ and $(x, \sigma(x)) \in P$ for $x \in \hat{G} \setminus C$. The hypotheses of [23, Theorem 18.6] are satisfied by [23, Theorem 16.1], where one sets $\mathcal{I}_x$ to be the σ-ideal of meager subsets of $\hat{G}$ for every $x \in \hat{G}$. Then we have that defining

$$g(x) := f(\sigma(x)) + f(x - \sigma(x))$$

for $x \in \hat{G}$ yields a Borel lift for φ. This shows that φ is Borel-definable.

Suppose now that G is an abelian group with a locally compact Polish cover.

(4)⇒(1) Let $f : \hat{G} \to \hat{H}$ be a Haar-measurable lift of φ. Then there exists a Borel set $C \subseteq \hat{G}$ such that $\hat{G} \setminus C$ is null and $f|_C$ is Borel. Define

$$P = \{(x, y) \in \hat{G} \times \hat{G} : y \in (x - C) \cap C\}.$$

By [23, Corollary 18.7] there exists a Borel function $\sigma : \hat{G} \to \hat{G}$ such that $(x, \sigma(x)) \in P$ for every $x \in \hat{G}$. Then we have that defining

$$h(x) := f(\sigma(x)) + f(x - \sigma(x))$$

for $x \in \hat{G}$ yields a Borel lift for φ. □

Corollary 4.8. *Suppose that G is an abelian group with a Polish cover, and H is an abelian $\mathbf{\Sigma}^1_1$-definable group. If there exists a group isomorphism $\varphi : G \to H$ with analytic graph, then H is a Borel-definable group, and φ is a Borel-definable group isomorphism.*

Proof. This follows immediately from Lemma 3.5 and Proposition 4.7. □

4.2. Complexity of subgroups

In this section, we consider the complexity of subgroups of groups with a Polish cover. We reformulate in this context some results from [13,18,34]. Recall that a Borel *complexity class* Γ is an assignment $X \mapsto \Gamma(X)$ from Polish spaces to classes of Borel sets such that for every continuous function $f : X \to Y$ between Polish spaces X, Y and for every $A \in \Gamma(Y)$, $A \subseteq Y$ and $f^{-1}(A) \in \Gamma(X)$. Given such a complexity class, its dual class $\check{\Gamma}$ is defined by setting $\check{\Gamma}(X) = \{X \setminus A : A \in \Gamma(X)\}$ for every Polish space X. A complexity class Γ is *not self-dual* if it is different from $\check{\Gamma}$. We will be mostly concerned with the complexity classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $D(\mathbf{\Sigma}^0_\alpha)$, $\mathbf{\Delta}^0_\alpha$ for $1 \leq \alpha < \omega_1$ and their duals; see [23, Section 11.B]. Recall that $D(\mathbf{\Sigma}^0_\alpha)$ comprises those sets that can be written as a set-theoretic difference of $\mathbf{\Sigma}^0_\alpha$ sets or, equivalently, as intersection of a $\mathbf{\Sigma}^0_\alpha$ set and a $\mathbf{\Pi}^0_\alpha$ set.

Definition 4.9. Suppose that H is a subgroup of an abelian group with a Polish cover $G = \hat{G}/N$. Set $\hat{H} = \{x \in \hat{G} : x + N \in H\}$. Let Γ be a complexity class. We say that H belongs to $\Gamma(G)$ or that H is Γ in G if and only if $\hat{H} \in \Gamma(\hat{G})$. If Γ is not self-dual, then we say that Γ is the complexity class of H in G if and only if $\hat{H} \in \Gamma(\hat{G})$ and $\hat{H} \notin \check{\Gamma}(\hat{G})$.

Given a Borel-definable set $X = \hat{X}/E$, we denote by $=_X$ the equivalence relation E. In particular, if $G = \hat{G}/N$ is a group with a Polish cover, and $H = \hat{H}/N$ is a subgroup with a Polish cover of G, then $=_{G/H}$ is the coset relation of $\hat{H}$ inside of $\hat{G}$. Recall that an equivalence relation E on a Polish space X is *potentially* Γ if it is Borel-reducible to an equivalence relation F on a Polish space Y such that $F \in \Gamma(Y \times Y)$.

As a consequence of [34, Theorem 1.1 and Theorem 1.2] we have the following results.

Proposition 4.10. *The following is a complete list of possible Borel complexity classes of subgroups with a Polish cover of abelian groups with a Polish cover:* $\mathbf{\Pi}^0_{1+\lambda}$, $\mathbf{\Sigma}^0_{1+\lambda+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n+1})$, *and* $\mathbf{\Pi}^0_{1+\lambda+n+2}$ *for* $\lambda < \omega_1$ *either zero or a limit ordinal, and* $n < \omega$.

Proposition 4.11. *The following is a complete list of possible Borel complexity classes of subgroups with a* non-Archimedean *Polish cover of abelian groups with a Polish cover:* $\mathbf{\Pi}^0_{1+\lambda}$, $\mathbf{\Sigma}^0_{1+\lambda+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n+2})$, *and* $\mathbf{\Pi}^0_{1+\lambda+n+2}$ *for* $\lambda < \omega_1$ *either zero or a limit ordinal, and* $n < \omega$.

The following can be seen as a consequence of [34, Lemma 3.2 and Theorem 3.3] and Proposition 4.11; see also [18, Section 5].

Proposition 4.12. *Suppose that* G *is an abelian group with a Polish cover, and* H *is a subgroup with a Polish cover of* G.

- *Let* Γ *be one of the following complexity classes:* $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\beta$, $D(\mathbf{\Pi}^0_\alpha)$, *for* $1 \leq \alpha, \beta < \omega_1$ *and* $\beta \neq 2$. *Then* $H \in \Gamma(G)$ *if and only if* $=_{G/H}$ *is potentially* Γ.
- $=_{G/H}$ *is potentially* $\mathbf{\Sigma}^0_2$ *if and only if* H *is* $D(\mathbf{\Pi}^0_2)$ *in* G.
- *If* H *is a subgroup with a non-Archimedean Polish cover, then* $=_{G/H}$ *is potentially* $\mathbf{\Sigma}^0_2$ *if and only if* H *is* $\mathbf{\Sigma}^0_2$ *in* G;
- *If* H *is* $\check{D}(\mathbf{\Pi}^0_\alpha)$ *in* G *for some* $1 \leq \alpha < \omega_1$, *then* $H \in \mathbf{\Pi}^0_\alpha(G)$ *or* $H \in \mathbf{\Sigma}^0_\alpha(G)$.

As a consequence of Proposition 4.10, Proposition 4.12, Remark 3.10, and Theorem 6.18(1) below, one has the following.

Proposition 4.13. *Suppose that* G, H *are groups with a Polish cover, and* $f : G \to H$ *is a Borel-definable homomorphism. Let* Γ *be one of the following Borel complexity classes:* $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\beta$, $D(\mathbf{\Pi}^0_\alpha)$, $\check{D}(\mathbf{\Pi}^0_\alpha)$, $\mathbf{\Delta}^0_\alpha$ *for* $1 \leq \alpha, \beta < \omega_1$ *and* $\beta \neq 2$. *Suppose that* H_0 *is a subgroup with a Polish cover of* H.

- *If* H_0 *is* Γ *in* H, *then the subgroup with a Polish cover* $f^{-1}(H_0)$ *of* G *is* Γ *in* G. *The converse holds if* f *is surjective.*
- *Suppose that* G, H, H_0 *are abelian groups with a non-Archimedean Polish cover. If* H_0 *is* $\mathbf{\Sigma}^0_2$ *in* H, *then* $f^{-1}(G)$ *is* $\mathbf{\Sigma}^0_2$ *in* G. *The converse holds if* f *is surjective.*

Recall that E_0 denotes the $\boldsymbol{\Sigma}_2^0$ equivalence relation on the space $\mathcal{C} := \{0,1\}^\omega$ of infinite binary sequences obtained by setting $(x_i)\, E_0\, (y_i) \Leftrightarrow \exists n \in \omega \forall i \geq n,\ x_i = y_i$, and E_∞ denotes the orbit equivalence relation for the shift action of the free group $\mathbb{F}_2$ on 2 generators on $\{0,1\}^{\mathbb{F}_2}$. The $\boldsymbol{\Pi}_3^0$ equivalence relation E_0^ω on $\mathcal{C}^\omega$ is defined by setting $(\boldsymbol{x}_i)\, E_0^\omega\, (\boldsymbol{y}_i) \Leftrightarrow \forall i,\ \boldsymbol{x}_i E_0 \boldsymbol{y}_i$. The $\boldsymbol{\Pi}_3^0$ equivalence relation $=^+$ on $\mathbb{R}^\omega$ is defined by setting $(x_i) =^+ (y_i)$ if and only if (x_i) and (y_i) are enumerations of the same countable set of reals.

Proposition 4.14. *Suppose that $G = \hat{G}/N$ is a group with a Polish cover, and that H is a subgroup of G with a non-Archimedean Polish cover. Then:*

(1) $=_{G/H}$ is smooth if and only if H is $\boldsymbol{\Pi}_1^0$ in G;
(2) $=_{G/H}$ is Borel reducible to E_0 if and only if $=_{G/H}$ is Borel reducible to E_∞ if and only if H is $\boldsymbol{\Sigma}_2^0$ in G, and $=_{G/H}$ is Borel bireducible with E_0 if and only if $\boldsymbol{\Sigma}_2^0$ is the complexity class of H in G;
(3) $=_{G/H}$ is Borel reducible to E_0^ω if and only if $=_{G/H}$ is Borel reducible to $=^+$ if and only if H is $\boldsymbol{\Pi}_3^0$ in G, and $=_{G/H}$ is Borel bireducible with E_0^ω if and only if $\boldsymbol{\Pi}_3^0$ is the complexity class of H in G.

Proof. Without loss of generality we can assume that $H = \{0\}$.

(1) We have that $\{0\}$ is $\boldsymbol{\Pi}_1^0$ in G if and only if N is a closed subgroup of $\hat{G}$, which is equivalent to the assertion that $=_G$ is smooth; see [48, page 574].

(2) By [14, Theorem 12.5.7 and Theorem 7.3.8], $\{0\}$ is $\boldsymbol{\Sigma}_2^0$ in G if and only if $=_G \leq_B E_\infty$, which holds if and only if $=_G \leq_B E_0$ by [11, Theorem 6.1]. Furthermore, by Item (1), Pettis' Theorem [23, Theorem 9.9], and the Glimm–Effros dichotomy [17], we have that $\{0\}$ is not $\boldsymbol{\Pi}_2^0$ if and only if $\{0\}$ is not $\boldsymbol{\Pi}_1^0$ if and only if N is not a closed subgroup of $\hat{G}$, if and only if $E_0 \leq_B =_G$.

(3) By [11, Corollary 6.3] and Proposition 4.12 and we have that $\{0\}$ is not $\boldsymbol{\Sigma}_3^0$ if and only if $\{0\}$ is not $\boldsymbol{\Sigma}_2^0$ if and only if $E_0^\omega \leq_B =_G$. By [1, Corollary 6.11], $\{0\}$ is $\boldsymbol{\Pi}_3^0$ if and only if $=_G \leq_B E_0^\omega$, and by [14, Theorem 12.5.5], $\{0\}$ is $\boldsymbol{\Pi}_3^0$ if and only if $=_G \leq_B =^+$. □

4.3. The Solecki subgroups

Every abelian group with a Polish cover admits a canonical sequence of subgroups indexed by countable ordinals. As these were originally described by Solecki in [47], we call them *Solecki subgroups*. They have also been considered in [13,48].

Suppose that $G = \hat{G}/N$ is an abelian group with a Polish cover. Then [47, Lemma 2.3] implies that G has a smallest $\boldsymbol{\Pi}_3^0$ subgroup, which we denote by $s_1(G) = s_1^N(\hat{G})/N$. One can explicitly describe $s_1^N(\hat{G})$ as the subgroup of $\hat{G}$ defined by

$$\bigcap_V \bigcup_{z \in N} \overline{z + V}^G$$

where V ranges among the open zero neighborhoods in N and $\overline{z+V}^{\hat{G}}$ is the closure of $z+V$ inside of $\hat{G}$. It is proved in [47, Lemma 2.3] that $s_1(G)$ satisfies the following properties:

- $s_1(G)$ is a subgroup with a Polish cover;
- $\{0\}$ is dense in $s_1(G)$;
- a basis of zero neighborhoods in $s_1^N(\hat{G})$ consists of sets of the form $\overline{W}^{\hat{G}} \cap s_1^N(\hat{G})$ where W is an open zero neighborhood in N;
- if $A \subseteq \hat{G}$ is $\mathbf{\Pi}_3^0$ and contains N, then $A \cap s_1^N(\hat{G})$ is comeager in the Polish group topology of $s_1^N(\hat{G})$.

It follows that if H is a $\mathbf{\Pi}_3^0$ subgroup with a Polish cover of G, then $s_1(G) \subseteq \overline{\{0\}}^H \subseteq H$. We recall the following characterization of $s_1(G)$ from [34, Lemma 4.2].

Lemma 4.15. *Suppose that $G = \hat{G}/N$ is an abelian group with a Polish cover. Let $H = \hat{H}/N$ be a subgroup with a Polish cover of G such that:*

(1) $\{0\}$ is dense in H;
(2) for every open neighborhood V of zero in N, $\overline{V}^{\hat{G}} \cap \hat{H}$ contains an open neighborhood of zero in $\hat{H}$.

If $A \subseteq \hat{G}$ is $\mathbf{\Pi}_3^0$ and contains N, then $A \cap \hat{H}$ is comeager in $\hat{H}$. In particular, $H \subseteq s_1(G)$. If H is furthermore $\mathbf{\Pi}_3^0$, then $H = s_1(G)$.

The sequence of Solecki subgroups $s_\alpha(G)$ for $\alpha < \omega_1$ of the group with a Polish cover G is defined recursively by setting:

- $s_0(G) = \overline{\{0\}}^G$;
- $s_{\alpha+1}(G) = s_1(s_\alpha(G))$ for $\alpha < \omega_1$;
- $s_\lambda(G) = \bigcap_{\beta<\lambda} s_\beta(G)$ for a limit ordinal $\lambda < \omega_1$.

We also let $s_\alpha^N(\hat{G})$ be the Polishable subgroup of $\hat{G}$ such that $s_\alpha(G) = s_\alpha^N(\hat{G})/N$. One can prove by induction on $\alpha < \omega_1$ that $\{0\}$ is dense in $s_\alpha(G)$ for every $\alpha < \omega_1$, and if $\{0\}$ is a subgroup with a non-Archimedean Polish cover of G, then $s_\alpha(G)$ is a subgroup with a non-Archimedean Polish cover for every $1 \leq \alpha < \omega_1$; see [34, Section 4]. It is proved in [47, Theorem 2.1] that there exists $\alpha < \omega_1$ such that $s_\alpha(G) = \{0\}$. We call the least countable ordinal α such that $s_\alpha(G) = \{0\}$ the *Solecki rank* of G.

The following is an immediate consequence of [34, Theorem 5.4].

Theorem 4.16. *Suppose that G is an abelian group with a Polish cover, and $\alpha < \omega_1$. Then $s_\alpha(G)$ is the smallest $\mathbf{\Pi}_{1+\alpha+1}^0$ subgroup of G.*

Proof. For $\alpha = 0$ this is a consequence of the definition of $s_0^N(\hat{G})$ as the closure of N in $\hat{G}$, while for $\alpha = 1$ the claim is proved in [47, Lemma 2.3]. The proof of [13, Theorem 3.1] shows that $s_\alpha^N(\hat{G})$ is $\mathbf{\Pi}^0_{1+\alpha+1}$ in $\hat{G}$. The minimality assertion is proved by induction on α. At the inductive step, one needs to use the following relation between the complexity classes for $s_\alpha^N(\hat{G})$ and those of $\hat{G}$, which is obtained for $\beta = 0$ in the proof of [13, Theorem 3.1] and can be proved in general by induction on β: a $\mathbf{\Sigma}^0_\beta$ subset of $s_\alpha^N(\hat{G})$ can be written as the intersection with $s_\alpha^N(\hat{G})$ of a $\mathbf{\Sigma}^0_{1+\alpha+\beta}$ subset of $\hat{G}$. Conversely, if A is a $\mathbf{\Sigma}^0_{1+\alpha+\beta}$ subset of $\hat{G}$ and U is an open subset of N then, adopting the notation of the Vaught transform for the translation action $N \curvearrowright \hat{G}$ as in [14, Section 3.2], $A^{\Delta U} \cap s_\alpha^N(\hat{G})$ is $\mathbf{\Sigma}^0_{1+\beta}$ in $s_\alpha^N(\hat{G})$. By taking complementaries, one also simultaneously proves by induction on β the dual assertions concerning the complexity classes $\mathbf{\Pi}^0_\alpha$. □

As a consequence of Proposition 4.13 and Theorem 4.16 one obtains the following. Recall that if F is a functor on a category $\mathcal{C}$, then a *subfunctor* of F is a functor F' on $\mathcal{C}$ together with a natural transformation $\eta : F' \Rightarrow F$ whose components are monic. A subfunctor of the identity is simply a subfunctor of the identity functor.

Theorem 4.17. *Fix $\alpha < \omega_1$. If $f : G \to H$ is a Borel-definable homomorphism between groups with a Polish cover, then f maps $s_\alpha(G)$ to $s_\alpha(H)$. Thus, $G \mapsto s_\alpha(G) \subseteq G$ is a* subfunctor of the identity *on the category of abelian groups with a (non-Archimedean) Polish cover.*

We also have the following consequence of [34, Theorem 6.1].

Theorem 4.18. *Suppose that $G = \hat{G}/N$ is an abelian group with a Polish cover. Let $\alpha = \lambda + n$ be the Solecki rank of G, where $\lambda < \omega_1$ is either zero or a limit ordinal and $n < \omega$.*

(1) Suppose that $n = 0$. Then $\mathbf{\Pi}^0_{1+\lambda}$ is the complexity class of $\{0\}$ in G;
(2) Suppose that $n \geq 1$. Then:
   - *(a) if $\{0\} \in \mathbf{\Pi}^0_3(s_{\lambda+n-1}(G))$ and $\{0\} \notin D(\mathbf{\Pi}^0_2)(s_{\lambda+n-1}(G))$, then $\mathbf{\Pi}^0_{1+\lambda+n+1}$ is the complexity class of $\{0\}$ in G;*
   - *(b) if $n \geq 2$ and $\{0\} \in D(\mathbf{\Pi}^0_2)(s_{\lambda+n-1}(G))$, then $D(\mathbf{\Pi}^0_{1+\lambda+n})$ is the complexity class of $\{0\}$ in G;*
   - *(c) if $n = 1$, $\{0\} \in D(\mathbf{\Pi}^0_2)(s_\lambda(G))$, and $\{0\} \notin \mathbf{\Sigma}^0_2(s_\lambda(G))$, then $D(\mathbf{\Pi}^0_{1+\lambda+1})$ is the complexity class of $\{0\}$ in G;*
   - *(d) if $n = 1$ and $\{0\} \in \mathbf{\Sigma}^0_2(s_\lambda(G))$, then $\mathbf{\Sigma}^0_{1+\lambda+1}$ is the complexity class of $\{0\}$ in G.*

Furthermore, if $\{0\}$ is a subgroup of G with a non-Archimedean Polish cover, then the case (2c) *is excluded.*

4.4. Polish modules

In this section, we observe how all the results that we have obtained so far apply more generally in the context of Polish G-modules.

Suppose that G is a (multiplicatively denoted) Polish group, and R is a Polish ring. We say that A is a Polish G-module if A is an abelian Polish group endowed with a continuous action $G \curvearrowright A$ by automorphism of A, denoted by $(g, a) \mapsto g \cdot a$ [37, Section 3]. We say that A is a Polish R-module if it is an abelian Polish group that is also an R-module, such that the scalar multiplication operation is continuous. We now recall some automatic continuity results for modules; see also [37, Proposition 11].

The following lemma guarantees that a separately continuous function is continuous on a large set; see [23, Theorem 8.51].

Lemma 4.19. *Let X, Y, Z be Polish spaces and $f : X \times Y \to Z$ be a function that is separately continuous. Then there exists a dense G_δ set $C \subseteq X \times Y$ such that, for all $y \in Y$, $C^y = \{x \in X : (x, y) \in C\}$ is a dense G_δ in X, and f is continuous at every point of C.*

As an application of the automatic continuity of Borel group homomorphisms between Polish groups [23, Theorem 9.10] and Lemma 4.19 one has the following.

Lemma 4.20. *Suppose that G is a Polish group, and A is an abelian Polish group. If an action $G \curvearrowright A$ by automorphisms of A is Borel separately in each variable when seen as a function $G \times A \to A$, then it is continuous.*

Proof. For $g_0 \in G$, the map $A \to A$, $a \mapsto g_0 \cdot a$ is a Borel automorphism of A, and hence continuous [23, Theorem 9.10]. For $a_0 \in A$ we have that the map $\varepsilon_{a_0} : G \to A$, $g \mapsto g \cdot a_0$ is Borel. Thus there exists a dense G_δ subset D of G such that $\varepsilon_{a_0}|_D$ is continuous [23, Theorem 8.38]. We now prove that ε_{a_0} is continuous at an arbitrary $g_\infty \in G$. Suppose that $(g_n)_{n\in\mathbb{N}}$ is a sequence converging to g_∞. Consider $h \in Dg_\infty^{-1} \cap \bigcap_{n\in\mathbb{N}} Dg_n^{-1}$ Thus, $hg_n \in D$ for every $n \in \mathbb{N} \cup \{\infty\}$, and $hg_n \to hg_\infty$. Thus, $(hg_n \cdot a_0)_{n\in\mathbb{N}}$ converges to $hg_\infty \cdot a_0$. Since the function $a \mapsto h^{-1} \cdot a$ is a continuous automorphism of A, we have that $(g_n \cdot a_0)_{n\in\mathbb{N}}$ converges to $g_\infty \cdot a_0$. This shows that ε_{a_0} is continuous at g_∞.

Finally, by the above and Lemma 4.19 there exists a dense G_δ subset C of $G \times A$ such that, for every $a \in A$, C^a is a dense G_δ subset of G and the function $(g, a) \mapsto g \cdot a$ is continuous at every point of C. Fix now $(g_0, a_0) \in G \times A$. Let $h_0 \in G$ such that $m : (g, a) \mapsto g \cdot a$ is continuous at (h_0, a_0). Then we can write the function m as $(g, a) \mapsto \left(g_0 h_0^{-1}\right) \cdot \left(h_0 g_0^{-1} g \cdot a\right)$. This realizes m as a composition of a function that is continuous at (g_0, a_0) with a continuous function. Thus, m is continuous at (g_0, a_0). Since (g_0, a_0) is an arbitrary element of $G \times A$, we have that m is continuous. $\square$

Lemma 4.21. *Suppose that R is a Polish ring, and A is an R-module and an abelian Polish group. If the scalar multiplication operation $R \times A \to A$, $(\lambda, x) \mapsto \lambda \cdot x$ is Borel separately in each variable, then it is continuous.*

Proof. As in the proof of Lemma 4.20, for every $\lambda_0 \in R$, the group homomorphism $A \to A$, $x \mapsto \lambda x$ of A is Borel, and hence continuous. For the same reason, for every $a_0 \in A$, the map $R \to A$, $\lambda \mapsto \lambda a_0$ is a Borel group homomorphism, and hence continuous. Thus, by Lemma 4.19 there exists a dense G_δ subset C of $R \times A$ such that for every $a \in A$, C^a is a dense G_δ subset of R and the function $m : (\lambda, a) \mapsto \lambda \cdot a$ is continuous at every point of C. Fix $(\lambda_0, a_0) \in R \times A$ and pick $\mu_0 \in A$ such that the function m is continuous at (μ_0, a_0). Then we can write m as the function $(\lambda, a) \mapsto (\mu_0 - \lambda_0 + \lambda) \cdot a + (\mu_0 - \lambda_0) \cdot a$. This witnesses that m is continuous at (μ_0, a_0). Being (μ_0, a_0) an arbitrary element of $R \times A$, we have that m is continuous. □

Suppose that R is a Polish group or a Polish ring. The notions of Polishable R-submodule of a Polish module, R-module with a Polish cover, R-submodule with a Polish cover, and Borel-definable, continuously definable, and liftable R-homomorphism between R-modules with a Polish cover are defined as in the group case. It follows easily from Lemma 4.20 and Lemma 4.21 that all the results that we have obtained so far about groups with a Polish cover apply more generally to Polish R-modules. Furthermore, it is not difficult to see that if M is a R-module with a Polish cover, then its Solecki subgroups are in fact R-submodules with a Polish cover. For the sake of illustration, we present a proof of the latter statement.

Proposition 4.22. *Suppose that R is a Polish ring or Polish group and $X = \hat{X}/N$ is an R-module with a Polish cover. Then, for every $\alpha < \omega_1$, $s_\alpha(X)$ is an R-submodule with a Polish cover of X.*

Proof. For $\alpha = 0$ this is immediate considering that $s_0(X) = \overline{N}^{\hat{X}}/N$. We prove that the conclusion holds for $1 \leq \alpha < \omega_1$ by induction. For $\alpha = 1$, suppose that $a \in R$ and $x \in s_1^N(\hat{X})$. We claim that $ax \in s_1^N(\hat{X})$. If V is a zero neighborhood in N, since N is a Polish R-module, there exists a zero neighborhood W in N such that $aW \subseteq V$. Since $x \in s_1^N(\hat{X})$, there exists $z \in N$ such that $x + z \in \overline{W}^{\hat{X}}$. Therefore,

$$ax + az \in a\overline{W}^{\hat{X}} \subseteq \overline{aW}^{\hat{X}} \subseteq \overline{V}^{\hat{X}}.$$

As this holds for every zero neighborhood V in N, $ax \in s_1^N(\hat{X})$. This shows that $s_1^N(\hat{X})$ is an R-submodule of $\hat{X}$. As it is also a Polishable subgroup of $\hat{X}$, it follows that $s_1^N(\hat{X})$ is in fact a Polishable R-submodule of $\hat{X}$ by Lemma 4.21 or Lemma 4.20, depending on whether R is a Polish ring or Polish group.

If the conclusion holds for α, then it follows from the identity $s_{\alpha+1}^{N}(\hat{X}) = s_1^N(s_\alpha^N(\hat{X}))$ that it holds for $\alpha + 1$. If λ is a limit ordinal and the conclusion holds for $\beta < \lambda$, then the identity

$$s_\lambda^N(\hat{X}) = \bigcap_{\beta<\lambda} s_\beta^N(\hat{X})$$

implies that the conclusion holds for λ. $\square$

5. Better lifts

The main goal of this paper is to provide explicit descriptions as categories of structures "with a Polish cover" of the left heart of categories of algebraic structures endowed with a Polish topology. This goal requires us to identify which maps between such "formal quotients" define morphisms in the left heart, and to obtain for such morphisms as precise a characterization as possible. We have already seen in Proposition 4.7 and Remark 3.11 the "bootstrap" phenomenon that the morphisms in the left heart of abelian Polish groups—the group homomorphisms that are $\mathbf{\Sigma}_1^1$-definable—are in fact automatically Borel-definable, even continuously-definable in the non-Archimedean case, and Haar-definable in the locally compact case. We will obtain in Corollary 6.19 and Corollary 6.20 other improved lifting results for locally compact Polish groups and Lie groups, respectively. The proofs of these results will hinge on the results of this section, which show that in some cases one can obtain lifts that are also "approximately additive" in the sense that we are about to define.

5.1. Approximately additive lifts

We begin by considering the existence of continuous lifts with additional properties for continuously definable group homomorphisms between groups with a Polish cover.

Definition 5.1. Suppose that G is an abelian Polish group and $H = \hat{H}/M$ is a group with a Polish cover. A Borel lift $f : G \to \hat{H}$ for a group homomorphism $G \to H$ is *approximately additive* if $f(0) = 0$ and the function $\delta f : G \times G \to M$, $(x, y) \mapsto f(y) - f(x + y) + f(x)$ is continuous at $(0, 0)$.

A similar proof as [7, Lemma 4.9] gives the following.

Lemma 5.2. *Suppose that G is an abelian Polish group, $H = \hat{H}/M$ is an abelian group with a Polish cover, and $f : G \to \hat{H}$ is a Borel lift of a group homomorphism $G \to H$ that is continuous on a zero neighborhood G_0 of G. Let M_1 be a zero neighborhood in M such that $M_1 = \overline{M}_1^{\hat{H}} \cap M$, where $\overline{M}_1^{\hat{H}}$ is the closure of M_1 in $\hat{H}$. Then there exist $x_0, y_0 \in G_0$, and a zero neighborhood G_1 in G contained in G_0 such that:*

- *for* $x, y \in G_1$,

$$f(x + x_0) + f(y + y_0) - f(x + y + x_0 + y_0) \in M_1;$$

- *if* $g : G \to \hat{H}$ *is defined by*

$$g(z) := f(x_0 + y_0 + z) - f(x_0 + y_0),$$

then, for every $x, y \in G_1$,

$$\delta g(x, y) \in M_1 + M_1 + M_1 + M_1.$$

Proof. Since M_1 is non-meager in M, there exists $m \in M$ such that

$$A := \{(x, y) \in G_0 \times G_0 : \delta f(x, y) \in m + M_1\}$$

is non-meager in $G_0 \times G_0$. After replacing f with $z \mapsto f(z) - m$, we can assume that $m = 0$. Since $f : G_0 \to \hat{H}$ is continuous and $M_1 = \overline{M}_1^{\hat{H}} \cap M$, we have that $A \subseteq G_0 \times G_0$ is closed. Thus, A is somewhere dense, and there exists a zero neighborhood G_1 in G_0 and $x_0, y_0 \in G_0$ such that $(x_0, y_0) + (G_1 \times G_1) \subseteq A$. Thus, for $x, y \in G_1$ we have that

$$f(x + x_0) + f(y + y_0) - f(x + y + x_0 + y_0) \in M_1. \tag{1}$$

Define thus $g : G \to \hat{H}$ by

$$g(z) := f(x_0 + y_0 + z) - f(x_0 + y_0).$$

Then we have that, for $x, y \in G_1$,

$$\begin{aligned}\delta g(x, y) &= g(x + y) - g(x) - g(y) \\ &= f(x_0 + y_0 + x + y) - f(x_0 + y_0 + x) - f(x_0 + y_0 + y) + f(x_0 + y_0).\end{aligned}$$

By (1), we have

$$f(x_0 + y_0 + x + y) - f(x_0 + y_0 + x) - f(x_0 + y_0 + y) + f(x_0 + y_0) \in M_1 + M_1 + M_1 + M_1$$

This concludes the proof. □

One can infer from Lemma 5.2, as in the proof of [7, Theorem 4.5], the following lemma.

Proposition 5.3. *Suppose that* G *is an abelian Polish group,* $H = \hat{H}/M$ *is an abelian group with a Polish cover, and* $\varphi : G \to H$ *is a locally continuously definable group* *homomorphism. Suppose that M has a basis of zero neighborhoods that are closed in the subspace topology inherited from H. Then φ has an* approximately additive *locally continuous Borel lift.*

Proof. By assumption, we have that M has a basis $(M_k)_{k\in\omega}$ of zero neighborhoods such that $\overline{M}_k^{\hat{H}} \cap M = M_k$ for every $k \in \omega$. Let $(\hat{H}_k)$ be a basis of zero neighborhoods in $\hat{H}$ such that $\hat{H}_0 = \hat{H}$. Without loss of generality, we can assume that $M_0 = M$ and $M_k \subseteq \hat{H}_k$ for $k \in \omega$. Since φ is locally continuously definable, it has a Borel lift $f : G \to \hat{H}$ that is continuous on a zero neighborhood U in G. Let also d_G be a compatible complete invariant metric on G such that $\{z \in G_0 : d_G(z, 0) \leq 2\} \subseteq U$. Let G_0 be the zero neighborhood $\{z \in G_0 : d_G(z, 0) \leq 1/4\}$ in G, and set $f_0 := f$.

Applying Lemma 5.2, one can define by recursion on $k \in \omega$:

- a zero neighborhood G_{k+1} of G contained in $\left\{x \in G_k : d_G(x, 0) \leq 2^{-(k+2)}\right\}$;
- elements $x_k, y_k \in G_k$;
- a Borel function $f_k : G \to \hat{H}$,

such that, for every $k \in \omega$:

(1) for every $z \in G$,

$$f_{k+1}(z) = f_k(x_k + y_k + z) - f_k(x_k + y_k);$$

(2) for every $x, y \in G_k$,

$$f_k(x + y + x_k + y_k) \equiv f_k(x + x_k) + f_k(y + y_k) \mod M_{k+1};$$

(3) for every $x, y \in G_{k+1}$,

$$\delta f_{k+1}(x, y) \in M_{k+1}.$$

Indeed, suppose that $k \geq 0$, and G_{i+1}, f_{i+1}, and $x_i, y_i \in G_i$ have been defined for $i < k$. We apply Lemma 5.2 to obtain elements $x_k, y_k \in G_k$ and a zero neighborhood G_{k+1} in G contained in $\left\{x \in G_k : d_G(x, 0) \leq 2^{-(k+2)}\right\}$ such that, setting

$$f_{k+1}(z) := f_k(x_k + y_k + z) - f_k(x_k + y_k)$$

for $z \in G$, we have that

$$f_k(x + y + x_k + y_k) \equiv f_k(x + x_k) + f_k(y + y_k) \mod M_{k+1}$$

and

$$\delta f_{k+1}(x, y) \in M_{k+1}.$$

For $k \in \omega$, set

$$z_k := (x_0 + y_0) + \cdots + (x_k + y_k).$$

We prove by induction on $k \in \omega$ that, for every $z \in G$,

$$f_{k+1}(z) = f(z_k + z) - f(z_k).$$

For $k = 0$ this holds by definition. Suppose that it holds for k. Then we have that, by definition and the inductive hypothesis,

$$\begin{aligned} f_{k+2}(z) &= f_{k+1}(x_{k+1} + y_{k+1} + z) - f_{k+1}(x_{k+1} + y_{k+1}) \\ &= (f(z_k + x_{k+1} + y_{k+1} + z) - f(z_k)) - (f(z_k + x_{k+1} + y_{k+1}) - f(z_k)) \\ &= f(z_{k+1} + z) - f(z_{k+1}). \end{aligned}$$

Notice that

$$d_G(x_i + y_i, 0) \leq 2^{-(i+1)}$$

for every $i < \omega$, and hence the sequence $(z_k)_{k \in \omega}$ converges to some $z_\infty \in G$ such that $d_G(z_\infty, 0) \leq 1$.

Set $W := \{z \in G : d_G(z, 0) \leq 1\}$. Notice that, for every $i \in \omega$ and $z \in W$, $z_i + z \in U$ and $z_\infty + z \in U$. Define, for $z \in W$,

$$g(z) := f(z + z_\infty) - f(z_\infty) = \lim_{i \to \infty}(f(z + z_i) - f(z_i)) = \lim_{i \to \infty} f_i(z).$$

Since the family of functions $(f_i)_{i \in \omega}$ is uniformly equicontinuous on W, we have that the function $g : W \to \hat{H}$ is continuous. Furthermore, g satisfies $g(x) + M = \varphi(x)$ for $x \in W$ and, for every $k \in \omega$ and $x, y \in G_k$,

$$\delta g(x, y) \in M_k.$$

Finally, one can extend g to a Borel lift $g : G \to \hat{H}$ for φ using Lemma 3.12. □

We record here an analogous result, under the assumptions that the Polish groups involved are Lie groups. Recall that the abelian Lie groups are precisely the Polish groups of the form $\mathbb{T}^d \oplus \mathbb{R}^k \oplus D$ where d and k are nonnegative integers and D is countable.

Remark 5.4. A similar proof as Proposition 5.3 gives the following result: suppose that G is an abelian real Lie group, $H = \hat{H}/M$ is an abelian group with a Polish cover where $\hat{H}$ and M are Lie groups, and $\varphi : G \to H$ is a group homomorphism that has a Borel lift $G \to \hat{H}$ that is analytic on an open zero neighborhood in G. Then φ has an *approximately additive* Borel lift that is analytic on an open zero neighborhood in G.

5.2. Approximately R-linear lifts

A *non-Archimedean Polish ring* is a Polish ring that has a basis of (open) zero neighborhoods consisting of subrings. Let R be a non-Archimedean Polish ring. A subset A of R is *bounded* if for every zero neighborhood U in R there exists a zero neighborhood V of R such that $V \cdot A \subseteq U$ [61]. We say that R is *locally bounded* if it has a bounded zero neighborhood. A Polish R-module X is *non-Archimedean* if for every zero neighborhood U of X there exists an open subring O of R and an open O-submodule V of X contained in U.

Definition 5.5. Suppose that R is a non-Archimedean Polish ring and that $X = \hat{X}/N$ and $Y = \hat{Y}/M$ are R-modules with a non-Archimedean Polish cover. An *approximately R-linear continuous lift* of an R-homomorphism $\varphi : X \to Y$ is an approximately additive continuous lift $f : \hat{X} \to \hat{Y}$ such that for every zero neighborhood U of $\hat{Y}$ there exists an open subring O of R and an open O-submodule W of $\hat{X}$ such that $f(\lambda x) \equiv \lambda f(x) \mod U$ for every $\lambda \in O$ and $x \in W$.

Lemma 5.6. *Suppose that R is a non-Archimedean Polish ring, X is non-Archimedean Polish R-module, and $Y = \hat{Y}/M$ is an R-module with a non-Archimedean Polish cover. Let $g : X \to \hat{Y}$ be a continuous lift for an R-homomorphism $X \to Y$. Let M_1 be an open subgroup of M that is closed in the subspace topology inherited from $\hat{Y}$. Suppose that X_1 is an open subgroup of X such that, for $x, y \in X_1$,*

$$g(x + y) \equiv g(x) + g(y) \mod M_1.$$

Then there exist an open subring O of R and an open O-submodule $X_1' \subseteq X$ such that for every $x \in X_1'$ and $\lambda \in O$,

$$g(\lambda x) \equiv \lambda g(x) \mod M_1.$$

Furthermore, for every bounded open subring S of R there exists an open S-submodule $X_1'' \subseteq X_1$ such that for every $x \in X_1''$ and $\lambda \in S$,

$$g(\lambda x) \equiv \lambda g(x) \mod M_1.$$

Proof. Consider the Borel function

$$\nabla g : R \times X \to M, \ (\lambda, x) \mapsto g(\lambda x) - \lambda g(x)$$

Since M_1 is closed in the subspace topology on M inherited from Y, we have that there exists $x_0 \in X_0$, $\lambda_0 \in R$, an open subring O of R, an open subgroup X_1' in X and $m \in M$ such that

$$g\left(\left(\lambda+\lambda_0\right)\left(x+x_0\right)\right)\equiv\left(\lambda+\lambda_0\right)g\left(x+x_0\right)+m \bmod M_1$$

for every $\lambda \in O$ and $x \in X_1'$. In particular for $\lambda = 0$ and $x = 0$ we obtain

$$g\left(\lambda_0 x_0\right)\equiv\lambda_0 g\left(x_0\right)+m \bmod M_1.$$

Thus, in particular for $x = 0$ and $\lambda \in O$ we obtain

$$\begin{aligned} g\left(\lambda x_0\right)+\lambda_0 g\left(x_0\right)+m &\equiv g\left(\lambda x_0\right)+g\left(\lambda_0 x_0\right) \\ &\equiv g\left(\left(\lambda+\lambda_0\right)x_0\right) \\ &\equiv \left(\lambda+\lambda_0\right)g\left(x_0\right)+m \\ &\equiv \lambda g\left(x_0\right)+\lambda_0 g\left(x_0\right)+m \bmod M_1. \end{aligned}$$

Thus, we have that

$$g\left(\lambda x_0\right)\equiv\lambda g\left(x_0\right) \mod M_1$$

for every $\lambda \in O$.

For $\lambda = 0$ and $x \in X_1'$ we obtain

$$\begin{aligned} g\left(\lambda_0 x\right)+\lambda_0 g\left(x_0\right)+m &\equiv g\left(\lambda_0 x\right)+g\left(\lambda_0 x_0\right) \\ &\equiv g\left(\lambda_0\left(x+x_0\right)\right) \\ &\equiv \lambda_0 g\left(x+x_0\right)+m \\ &\equiv \lambda_0 g\left(x\right)+\lambda_0 g\left(x_0\right)+m \bmod M_1. \end{aligned}$$

Thus we have that

$$g\left(\lambda_0 x\right)\equiv\lambda_0 g\left(x\right) \mod M_1$$

for $x \in X_1'$.

Finally for $\lambda \in O$ and $x \in X_1'$ we have that

$$\begin{aligned} & g\left(\lambda x\right)+\lambda_0 g\left(x\right)+\lambda g\left(x_0\right)+\lambda_0 g\left(x_0\right)+m \\ \equiv\ & g\left(\lambda x\right)+g\left(\lambda_0 x\right)+g\left(\lambda x_0\right)+g\left(\lambda_0 x_0\right) \\ \equiv\ & g\left(\lambda x+\lambda_0 x+\lambda x_0+\lambda_0 x_0\right) \\ \equiv\ & \lambda g\left(x\right)+\lambda_0 g\left(x\right)+\lambda g\left(x_0\right)+\lambda_0 g\left(x_0\right)+m \bmod M_1. \end{aligned}$$

Thus, we have that for every $\lambda \in O$ and $x \in X_1'$,

$$g\left(\lambda x\right)\equiv\lambda g\left(x\right) \mod M_1.$$

If S is a bounded open subring of R then there exists an open zero neighborhood V of R such that $V \cdot S \subseteq O$. Thus, for every $\lambda \in S$ and $v \in V$ we have that

$$g(\lambda v x) \equiv \lambda v g(x) \mod M_1$$

and in particular

$$g(vx) \equiv vg(x) \mod M_1$$

Define now $X_1'' = V \cdot X_1'$. Then we have that X_1'' is an open S-submodule of X contained in X_1. Furthermore, for $\lambda \in S$ and $x \in X_1''$ we have that $x = vy$ for some $y \in X_1'$ and $v \in V$ and hence

$$g(\lambda x) = g(\lambda v y) \equiv \lambda v g(y) \equiv \lambda g(vy) = \lambda g(x) \mod M_1.$$

This concludes the proof. □

Proposition 5.7. *Let R be a non-Archimedean Polish ring. Suppose that $\varphi : X \to Y$ is a Borel-definable R-homomorphism between R-modules with a non-Archimedean Polish cover. Suppose that $Y = \hat{Y}/M$ where M has a basis of zero neighborhoods that are closed in the subspace topology inherited from $\hat{Y}$. Then φ has an approximately R-linear continuous lift.*

Proof. This follows immediately from Lemma 5.6 and Proposition 5.3. □

6. Left hearts of categories of Polish modules

In this section, we provide explicit descriptions of the heart of categories of Polish modules.

6.1. The left heart of the category of Polish modules

Fix a Polish group or Polish ring R. We let $\mathcal{A}_R$ be the category whose objects are the Polish R-modules and whose morphisms are the continuous R-homomorphisms. Recall that a category is *countably complete* if it has all countable limits. A quasi-abelian category is countably complete if and only if it has countable products. A functor between countably complete categories is *countably continuous* if it commutes with countable limits.

Proposition 6.1. *Let $\mathcal{A}$ be a quasi-abelian category. If $\mathcal{A}$ is countably complete, then* $\mathrm{LH}(\mathcal{A})$ *is countably complete, and the inclusion functor $I : \mathcal{A} \to \mathrm{LH}(\mathcal{A})$ is countably continuous.*

Suppose that $\mathcal{M}$ is a countably complete abelian category. Let $F : \mathcal{A} \to \mathcal{M}$ be a finitely continuous exact functor, and let $\hat{F} : \mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ be the (unique up to isomorphism) finitely continuous exact functor such that $\hat{F} \circ I$ is isomorphic to F. If F is countably continuous, then $\hat{F}$ is countably continuous.

Lemma 6.2. *$\mathcal{A}_R$ is a countably complete quasi-abelian category.*

Proof. It is clear that $\mathcal{A}_R$ is an additive category. If $\varphi : X \to Y$ is a continuous R-homomorphism, then its kernel in $\mathcal{A}_R$ is $\ker(\varphi) = \{x \in X : \varphi(x) = 0\}$, and its cokernel in $\mathcal{A}_R$ is the quotient of Y by the *closure* of the image of φ. Thus, the kernels in $\mathcal{A}_R$ are the continuous injective R-homomorphisms *with closed image*, and the cokernels in $\mathcal{A}_R$ are the continuous surjective R-homomorphisms. It remains to prove that the class of kernels is stable under push-out along arbitrary morphisms, and the class of cokernels is stable under pull-back along arbitrary morphisms.

Suppose that $\varphi_i : X \to Y_i$ are continuous R-homomorphisms for $i \in \{0, 1\}$. Let $p_i : Y_i \to Z$ for $i \in \{0, 1\}$ be their pushout. Thus we have that Z is the quotient of $Y_0 \oplus Y_1$ by the closure M of the R-submodule $\{(-\varphi_0(x), \varphi_1(x)) : x \in X\}$. The map $p_0 : Y_0 \to Z$ is defined by $y \mapsto (y, 0) + M$, and the map $p_1 : Y_1 \to Z$ is defined by $y \mapsto (0, y) + M$. Suppose that φ_0 is a kernel, namely it is injective and it has closed range. We need to prove that p_1 is also a kernel.

Suppose that $y \in Y_1$ is such that $p_1(y) = 0$. Thus, we have that $(0, y) \in M$. Hence, there exists a sequence (x_n) in X such that $\varphi_0(x_n) \to 0$ and $\varphi_1(x_n) \to y$. Since φ_0 is injective with closed range, it is a homeomorphism onto its image. Therefore, we have that $x_n \to 0$ and hence $y = \lim_n \varphi_1(x_n) = 0$. This shows that p_1 is injective.

Suppose now that (y_n) is a sequence in Y such that $(p_1(y_n))$ converges in Z to $(z_0, z_1) + M$. Thus we can find a sequence (x_n) in X such that $(\varphi_0(x_n), p_1(y_n) - \varphi_1(x_n))$ converges in $Y_0 \oplus Y_1$ to (z_0, z_1). Since φ_0 is a kernel, this implies that (x_n) converges in X to some $x \in X$ such that $\varphi_0(x) = z_0$. Thus, we have that $(p_1(y_n))$ converges to $z_1 + \varphi_1(x)$ and hence

$$(z_0, z_1) + M = (z_0 - \varphi_0(x), z_1 + \varphi_1(x)) + M = \lim_n ((0, p_1(y_n)) + M).$$

This shows that p_1 has closed range, concluding the proof that p_1 is a kernel.

Suppose now that $\varphi_i : Y_i \to Z$ for $i \in \{0, 1\}$ are continuous R-homomorphisms. Let $\eta_i : X \to Y_i$ for $i \in \{0, 1\}$ be their pullback. Then we have that

$$X = \{(y_0, y_1) \in Y_0 \oplus Y_1 : \varphi_0(y_0) = \varphi_1(y_1)\}.$$

The map $\eta_0 : X \to Y_0$ is given by $(y_0, y_1) \mapsto y_0$, and the map $\eta_1 : X \to Y_1$ is given by $(y_0, y_1) \mapsto y_1$. Suppose that φ_1 is a cokernel, i.e. surjective. We need to show that η_0 is also a cokernel. Suppose that $y_0 \in Y_0$. Consider $\varphi_0(y_0) \in Z$. Since φ_1 is surjective, there exists $y_1 \in Y_1$ such that $\varphi_1(y_1) = \varphi_0(y_0)$. Thus we have that $x := (y_0, y_1) \in X$ is such that $\eta_0(x) = y_0$. This concludes the proof that η_0 is a cokernel.

Finally, the fact that $\mathcal{A}_R$ is countably complete follows from the fact that it has countable products. □

We let $\mathcal{M}_R$ be the category whose objects are the R-modules with a Polish cover, and whose morphisms are the Borel-definable R-homomorphisms. We regard $\mathcal{A}_R$ as a full subcategory of $\mathcal{M}_R$, by identifying a Polish R-module X with the R-module with a Polish cover X/N where N is the trivial submodule of X.

Theorem 6.3. *The category $\mathcal{M}_R$ is abelian. The inclusion functor $\mathcal{A}_R \to \mathcal{M}_R$ is exact and countably continuous, and it extends to an equivalence of categories* $\mathrm{LH}(\mathcal{A}_R) \to \mathcal{M}_R$*.*

Proof. Suppose that $\varphi : X \to Y$ is a Borel-definable R-homomorphism between R-modules with a Polish cover. Then by Proposition 4.4 and the results of Subsection 4.4, we have that

- $\ker(\varphi) := \{x \in X : \varphi(x) = 0\}$ is an R-submodule with a Polish cover of X, and
- $\varphi(X)$ is an R-submodule with a Polish cover of Y.

It is easy to see that $\ker(\varphi) \to X$ is the kernel of φ and $Y \to Y/\varphi(X)$ is the cokernel of φ. This easily implies that every monic arrow is a kernel and every epic arrow is a cokernel, and hence $\mathcal{M}_R$ is an abelian category.

It follows from the characterization of the left heart provided by the last item in Proposition 2.3 that the inclusion $J : \mathcal{A}_R \to \mathcal{M}_R$ extends to an equivalence of categories $\mathrm{LH}(\mathcal{A}_R) \to \mathcal{M}_R$. It is also easy to see that $\mathcal{M}_R$ has countable products, and that J preserves countable products. Since J is finitely continuous and preserves countable products, it is also countably continuous. □

6.2. Left hearts of subcategories of the category of Polish modules

Recall that a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is *strictly full* if its collection of objects is closed under isomorphism in $\mathcal{D}$, and for objects x, y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{D}}(x, y)$. Let $\mathcal{B}$ be a strictly full quasi-abelian subcategory of the category $\mathcal{A}_R$ of Polish R-modules. We notice that this implies that, if X, Y are objects of $\mathcal{B}$, and $f : X \to Y$ is a continuous R-homomorphism, then f is a morphism in $\mathcal{B}$, $\ker(f)$ is an object of $\mathcal{B}$. This follows from the assumption that $\mathcal{B}$ is a quasi-abelian subcategory of $\mathcal{A}_R$, which in particular means that $\mathcal{B}$ has kernels and the inclusion $\mathcal{B} \to \mathcal{A}_R$ preserves kernels (as a particular instance of limits). Furthermore, $f(X)$ endowed with the Polish R-module topology induced via f by the Polish R-module topology on X is an object of $\mathcal{B}$ (being isomorphic to $\mathrm{coker}(\ker(f))$).

Definition 6.4. An *R-module with a $\mathcal{B}$-cover* is an R-module X explicitly presented as a quotient $\hat{X}/N$ where $\hat{X}$ and N are objects of $\mathcal{B}$, N is an R-submodule of $\hat{X}$, and the inclusion $N \to \hat{X}$ is continuous.

If $X = \hat{X}/N$ is an R-module with a $\mathcal{B}$-cover, then an R-submodule with a $\mathcal{B}$-cover of X is an R-submodule $Y = \hat{Y}/N$ where $\hat{Y}$ is an R-submodule of $\hat{X}$, $\hat{Y}$ is an object of $\mathcal{B}$, and the inclusion $\hat{Y} \to \hat{X}$ is continuous.

Definition 6.5. An R-homomorphism $\varphi : X \to Y$ between R-modules with a $\mathcal{B}$-cover is:

- $\mathcal{B}$-*definable* if its graph $\Gamma(\varphi)$ is an R-submodule with a $\mathcal{B}$-cover of $X \oplus Y$;
- *liftable* if has a lift to a continuous R-homomorphism $\hat{X} \to \hat{Y}$, where $X = \hat{X}/N$ and $Y = \hat{Y}/M$.

Remark 6.6. If $\varphi : X \to Y$ is a $\mathcal{B}$-definable bijective R-homomorphism, then φ^{-1} is $\mathcal{B}$-definable, since we have that $(x, y) \in \Gamma(\varphi) \Leftrightarrow (y, x) \in \Gamma\left(\varphi^{-1}\right)$.

Remark 6.7. By Theorem 4.6, an R-homomorphism is $\mathcal{A}_R$-definable if and only if it is Borel-definable.

Lemma 6.8. *A liftable R-homomorphism is $\mathcal{B}$-definable.*

Proof. Suppose that $\varphi : \hat{X}/N \to \hat{Y}/M$, where $X = \hat{X}/N$ and $Y = \hat{Y}/M$, is induced by a continuous R-homomorphism $f : \hat{X} \to \hat{Y}$. Then we have that

$$W := \{(x, y, z) \in \hat{X} \oplus \hat{Y} \oplus M : f(x) = y + z\}$$

is an object of $\mathcal{B}$ being the kernel of the morphism

$$\hat{X} \oplus \hat{Y} \oplus M \to \hat{Y},\ (x, y, z) \mapsto f(x) - y - z$$

in $\mathcal{B}$. Thus, we have that

$$\{(x, y) \in \hat{X} \oplus \hat{Y} : f(x) \equiv y \mod M\}$$

is a Polishable R-submodule of $\hat{X} \oplus \hat{Y}$ that belongs to $\mathcal{B}$, being the image of W under the continuous R-homomorphism

$$W \to \hat{X} \oplus \hat{Y},\ (x, y, z) \mapsto (x, y). \quad \square$$

Lemma 6.9. *Suppose that $\varphi : X \to Y$ and $\psi : Y \to Z$ are $\mathcal{B}$-definable R-homomorphisms. Then $\psi \circ \varphi : X \to Z$ is $\mathcal{B}$-definable.*

Proof. Suppose that φ, ψ are $\mathcal{B}$-definable. By assumption, we have that $\Gamma(\varphi) \subseteq X \oplus Y$ and $\Gamma(\psi) \subseteq Y \oplus Z$ are R-submodules with a Polish cover. Then we have that

$$W := \{(x, y_0, y_1, z) : (x, y_0) \in \Gamma(\varphi), (y_1, z) \in \Gamma(\psi), y_0 = y_1\}$$

is an R-submodule with a $\mathcal{B}$-cover of $\Gamma(\varphi) \oplus \Gamma(\psi)$, being the kernel of the liftable R-homomorphism

$$\Gamma(\varphi) \oplus \Gamma(\psi) \to Y, \ (x, y_0, y_1, z) \mapsto y_0 - y_1.$$

It follows that $\Gamma(\psi \circ \varphi)$ is an R-submodule with a $\mathcal{B}$-cover of $X \oplus Z$, being the image of W under the liftable R-homomorphism

$$W \to X \oplus Z, \ (x, y, y, z) \mapsto (x, z). \quad \square$$

Lemma 6.10. *Suppose that $\varphi : X \to Y$ is a $\mathcal{B}$-definable R-homomorphism. Then* $\ker(\varphi) \subseteq X$ *and $\varphi(X) \subseteq Y$ are R-submodules with a $\mathcal{B}$-cover.*

Proof. We have that

$$W := \{(x, y) \in \Gamma(\varphi) : y = 0\}$$

is an R-submodule with a $\mathcal{B}$-cover of $\Gamma(\varphi)$, and hence $\mathrm{Ker}(\varphi) \subseteq X$ is an R-submodule with a $\mathcal{B}$-cover, being the image of the liftable R-module homomorphism $W \to X$, $(x, y) \mapsto x$.

Similarly, we have that $\varphi(X)$ is the image of the liftable R-module homomorphism $\Gamma(\varphi) \to Y$, $(x, y) \mapsto y$. $\square$

Lemma 6.11. *If $\varphi : X \to Y$ is an R-homomorphism between R-modules with a $\mathcal{B}$-cover, the following are equivalent:*

(1) φ is $\mathcal{B}$-definable;
(2) there exist an R-module with a $\mathcal{B}$-cover Z and liftable R-homomorphisms $\psi : Z \to Y$ and $\sigma : Z \to X$ such that σ is a bijection and $\varphi = \varphi' \circ \sigma^{-1}$.

Proof. Suppose that $\varphi : X \to Y$ is an R-module homomorphism between R-modules with a $\mathcal{B}$-cover. Suppose that φ is $\mathcal{B}$-definable. Then we can set $Z = \Gamma(\varphi)$, $\sigma : Z \to X$, $(x, y) \mapsto x$, and $\psi : Z \to Y$, $(x, y) \mapsto y$. This shows that (1) implies (2). Suppose now that φ satisfies (2). Then we have that ψ and σ are $\mathcal{B}$-definable, being liftable. Hence $\varphi = \psi \circ \sigma^{-1}$ is $\mathcal{B}$-definable, and (1) holds. $\square$

Corollary 6.12. *If $\varphi : X \to Y$ is a $\mathcal{B}$-definable R-homomorphism between R-modules with a $\mathcal{B}$-cover, then it is Borel-definable.*

Proof. By Lemma 6.11, adopting the notation as in (2) from its statement, we have that $\varphi = \psi \circ \sigma^{-1}$. Since ψ and σ are liftable, we have that ψ and (by Remark 3.10) σ^{-1} are Borel-definable. Hence, φ is Borel-definable. $\square$

Lemma 6.13. *If $\varphi_0, \varphi_1 : X \to Y$ are $\mathcal{B}$-definable R-homomorphisms, then $\varphi_0 + \varphi_1$ is $\mathcal{B}$-definable.*

Proof. Suppose that $\varphi_0, \varphi_1 : X \to Y$ are $\mathcal{B}$-definable R-homomorphism. Then there exist objects X', X'' of $\mathcal{B}$ and continuous R-homomorphisms $\varphi_0' : X' \to Y$, $\varphi_1' : X'' \to Y$, $\sigma : X' \to X$, and $\tau : X'' \to X$ such that σ, τ are bijective, $\varphi_0 = \varphi_0' \circ \sigma^{-1}$, and $\varphi_1 = \varphi_1' \circ \tau^{-1}$. We can thus consider the continuous R-homomorphism $\psi : X' \oplus X'' \to Y$, $(x, y) \mapsto \varphi_0'(x) + \varphi_1'(y)$, and the continuous bijective R-homomorphism $\lambda : X' \oplus X'' \to X \oplus X$, $(x, y) \mapsto (\sigma(x), \tau(y))$. Then we have that $\varphi_0 + \varphi_1 = \psi \circ \lambda^{-1} \circ \Delta$ where $\Delta : X \to X \oplus X$, $x \mapsto (x, x)$. This shows that $\varphi_0 + \varphi_1$ is $\mathcal{B}$-definable. □

Define $\mathcal{M}_{\mathcal{B}}$ to be the (not necessarily full) subcategory of $\mathcal{M}_R$ whose objects are the R-modules with a $\mathcal{B}$-cover and whose morphisms are the $\mathcal{B}$-definable R-homomorphisms.

Theorem 6.14. *Let $\mathcal{B}$ be a strictly full quasi-abelian subcategory of the category $\mathcal{A}_R$ of Polish R-modules and continuous R-module homomorphisms. Let $\mathcal{M}_{\mathcal{B}}$ be the category of R-modules with a $\mathcal{B}$-cover and $\mathcal{B}$-definable R-homomorphisms. Then we have that:*

(1) $\mathcal{M}_{\mathcal{B}}$ is an abelian category;
(2) the inclusion $\mathcal{B} \to \mathcal{M}_{\mathcal{B}}$ extends to an equivalence of category $\mathrm{LH}(\mathcal{B}) \to \mathcal{M}_{\mathcal{B}}$;
(3) if $\mathcal{B}$ is countably complete and the inclusion $\mathcal{B} \to \mathcal{A}_R$ is countably continuous, then $\mathcal{M}_{\mathcal{B}}$ is countably complete and the inclusions $\mathcal{B} \to \mathcal{M}_{\mathcal{B}} \to \mathcal{M}_R$ are countably continuous.

Proof. (1) We begin with showing that $\mathcal{M}_{\mathcal{B}}$ is an additive subcategory of $\mathcal{M}_R$. It is clear that the zero object for $\mathcal{M}_R$ is also the zero object for $\mathcal{M}_{\mathcal{B}}$. By Lemma 6.13 and Corollary 6.12, the set of $\mathcal{B}$-definable R-homomorphisms $X \to Y$ is a subgroup of the set of Borel-definable R-homomorphisms. It remains to prove that, for objects X, Y in $\mathcal{M}_{\mathcal{B}}$, their biproduct $X \oplus Y$ in $\mathcal{M}_R$ is also their coproduct in $\mathcal{M}_{\mathcal{B}}$. Since $\mathcal{B}$ is a quasi-abelian subcategory of $\mathcal{A}_R$, we have that $X \oplus Y$ is an R-module with a $\mathcal{B}$-cover. Since every liftable R-homomorphism is $\mathcal{B}$-definable, we have that the canonical maps $X \to X \oplus Y$ and $Y \to X \oplus Y$ are $\mathcal{B}$-definable. It remains to prove that $X \oplus Y$ satisfies the universal property of the coproduct. Suppose that $\varphi : Z \to X$ and $\psi : Z \to Y$ are $\mathcal{B}$-definable R-homomorphisms. Let $\varphi \oplus \psi : Z \to X \oplus Y$ be the corresponding Borel-definable R-homomorphism. We need to prove that $\varphi \oplus \psi$ is $\mathcal{B}$-definable. We have that $\varphi = \varphi' \circ \sigma^{-1}$ and $\psi = \psi' \circ \tau^{-1}$ for some R-modules with a $\mathcal{B}$-cover Z', Z'' and liftable homomorphisms $\sigma : Z' \to Z$, $\varphi' : Z' \to X$, $\tau : Z'' \to Z$, $\psi' : Z'' \to Y$ such that σ, τ are bijective. Thus we have that

$$\varphi \oplus \psi = (\varphi' \oplus \psi') \circ (\sigma \oplus \tau)^{-1}$$

is $\mathcal{B}$-definable since $\varphi' \oplus \psi'$ and $\sigma \oplus \tau$ are liftable and $\sigma \oplus \tau$ is bijective. This shows that $\mathcal{M}_{\mathcal{B}}$ is an additive subcategory of $\mathcal{M}_R$.

We now prove that $\mathcal{M}_{\mathcal{B}}$ is an abelian category, which is furthermore an abelian subcategory of $\mathcal{M}_R$. Suppose that $\varphi : X \to Y$ is a $\mathcal{B}$-definable R-homomorphism. We have that $\ker(\varphi)$ is an R-module with a $\mathcal{B}$-cover. Let $\iota : \ker(\varphi) \to X$ be the inclusion map. We now show that $\ker(\varphi)$ is the kernel of φ in $\mathcal{M}_{\mathcal{B}}$. Suppose that $\psi : Z \to X$ is a $\mathcal{B}$-definable R-homomorphism such that $\varphi \circ \psi = 0$. We can write $\psi = \psi' \circ \sigma^{-1}$ where $\sigma : Z' \to Z$ and $\psi' : Z' \to X$ are liftable R-homomorphism such that σ is bijective and Z' is an R-module with a $\mathcal{B}$-cover. Since $\varphi \circ \psi = 0$ we have that $\varphi(Z) \subseteq \ker(\varphi)$ and hence $\psi'(W) \subseteq \ker(\varphi)$. Thus, we can write $\psi' = \iota \circ \psi''$ for a liftable R-homomorphism $\psi'' : W \to \ker(\varphi)$. Hence, we have that $\psi'' \circ \sigma^{-1} : Z \to \ker(\varphi)$ is a $\mathcal{B}$-definable R-homomorphism such that $\iota \circ \left(\psi'' \circ \sigma^{-1}\right) = \psi$. This concludes the proof that $\ker(\varphi)$ is the kernel of φ in $\mathcal{M}_{\mathcal{B}}$.

Suppose that $\varphi : X \to Y$ is a $\mathcal{B}$-definable R-homomorphism. We show that $Y/\varphi(X)$ is the cokernel of φ in $\mathcal{M}_{\mathcal{B}}$. We can write $\varphi = \varphi' \circ \sigma^{-1}$ for some liftable R-homomorphisms $\varphi' : X' \to Y$ and $\sigma : X' \to X$ where σ is bijective and X' is an R-module with a $\mathcal{B}$-cover. Thus, we have that $\varphi(X) = \varphi'(X') \subseteq Y$ is a R-submodule with a $\mathcal{B}$-cover since φ' is liftable and $\mathcal{B}$ is closed under taking quotients by closed R-submodules. Hence, $Y/\varphi(X)$ is an R-module with a $\mathcal{B}$-cover. Suppose now that $\psi : Y \to Z$ is a $\mathcal{B}$-definable R-homomorphism such that $\psi \circ \varphi = 0$. We can write $\psi = \psi' \circ \tau^{-1}$, where $\psi' : Y' \to Z$ and $\tau : Y' \to Y$ are liftable R-homomorphisms, τ is bijective, and Y' is an R-module with a $\mathcal{B}$-cover. Then we have that $\left(\tau^{-1} \circ \varphi\right)(X) \subseteq Y'$ is an R-submodule with a $\mathcal{B}$-cover of Y'. Furthermore $0 = \psi \circ \varphi = \psi' \circ \left(\tau^{-1} \circ \varphi\right)$ and hence $\left(\tau^{-1} \circ \varphi\right)(X) \subseteq \ker(\psi')$. Since ψ' is liftable, it induces a liftable R-homomorphism $\bar{\psi}' : Y'/(\tau^{-1} \circ \varphi)(X) \to Z$. Similarly, we have that $\tau\left(\left(\tau^{-1} \circ \sigma\right)(X)\right) = \sigma(X)$ since $\tau : Y' \to Y$ is a bijective R-linear homomorphism. Hence, being liftable, it induces a liftable bijective R-linear homomorphism $\bar{\tau} : Y'/\left(\tau^{-1} \circ \varphi\right)(X) \to Y/\varphi(X)$. Thus, we have that $\bar{\psi}' \circ \bar{\tau}^{-1} : Y/\varphi(X) \to Z$ is a $\mathcal{B}$-definable R-homomorphism such that, letting $\pi_Y : Y \to Y/\varphi(X)$ and $\pi_{Y'} : Y' \to Y'/\left(\tau^{-1} \circ \varphi\right)(X)$ be the canonical quotient mappings,

$$\bar{\psi}' \circ \bar{\tau}^{-1} \circ \pi_Y = \bar{\psi}' \circ \pi_{Y'} \circ \tau^{-1} = \psi' \circ \tau^{-1} = \psi.$$

This concludes the proof that $Y/\varphi(X)$ is the cokernel of φ in $\mathcal{M}_{\mathcal{B}}$.

(2) This follows immediately from (1) and the characterization of the left heart of a quasi-abelian category from the last item in Proposition 2.3.

(3) Suppose that $(X_n)_{n\in\omega}$ is a sequence of R-modules with a $\mathcal{B}$-cover. It suffices to prove that their product $X_\omega := \prod_{n\in\omega} X_n$ in $\mathcal{M}_R$ is also their product in $\mathcal{M}_{\mathcal{B}}$. We have that X_ω is an R-module with a $\mathcal{B}$-cover, since $\mathcal{B}$ is countably complete and the inclusion $\mathcal{B} \to \mathcal{A}_R$ is countably continuous. Furthermore the projection maps $\pi_i : X_\omega \to X_i$ for $i \in \omega$ are liftable, and hence $\mathcal{B}$-definable. Suppose now that Z is an R-module with a $\mathcal{B}$-cover, and $\varphi_i : X_i \to Z$ are $\mathcal{B}$-definable R-homomorphisms for $i \in \omega$. Then we have that there exist R-modules with a $\mathcal{B}$-cover X_i', liftable R-homomorphisms $\sigma_i : X_i' \to X_i$ and $\varphi_i' : X_i' \to Z$ such that σ_i is bijective and $\varphi_i = \varphi_i' \circ \sigma_i^{-1}$. Define then $X_\omega' := \prod_{i\in\omega} X_\omega'$.

Then we have that the sequences (σ_i) and (φ'_i) induce liftable R-homomorphisms $\sigma : X'_\omega \to X_\omega$ and $\varphi' : X'_\omega \to X$. Setting $\varphi := \varphi' \circ \sigma^{-1} : X_\omega \to Y$ we obtain a $\mathcal{B}$-definable R-homomorphism such that $\varphi \circ \pi_i = \varphi_i$ for every $i \in \omega$. This shows that X_ω is the product of (X_n) in $\mathcal{M}_{\mathcal{B}}$, concluding the proof. □

6.3. Examples

In this section we apply Theorem 6.14 to describe the left heart of a number of important categories of Polish R-modules as a full subcategory of the category of R-modules with a Polish cover.

Definition 6.15. Suppose that $\mathcal{B}$ is a strictly full quasi-abelian subcategory of $\mathcal{A}_R$. We say that $\mathcal{B}$ is a *thick subcategory* of $\mathcal{A}_R$ [27, Definition 8.3.21(iv)] if it is closed under extensions, i.e., for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of Polish R-modules, we have that if A and C are in $\mathcal{B}$, then B is in $\mathcal{B}$ as well.

Proposition 6.16. *Suppose that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}_R$. An R-homomorphism between R-modules with a $\mathcal{B}$-cover is $\mathcal{B}$-definable if and only if it is Borel-definable.*

Proof. Suppose that $\varphi : \hat{X}/N \to \hat{Y}/M$ is Borel-definable, where $\hat{X}/N$ and $\hat{Y}/M$ are R-modules with a $\mathcal{B}$-cover. Then we have a short exact sequence

$$0 \to \{0\} \oplus M \to \hat{\Gamma}(\varphi) \to \hat{X} \to 0$$

where $\hat{\Gamma}(\varphi)$ is the lift to $\hat{X} \oplus \hat{Y}$ of the graph of φ. We have that $\hat{\Gamma}(\varphi)$ is Polish by Theorem 4.6. Since $\mathcal{B}$ is a thick subcategory of $\mathcal{A}_R$, this implies that $\hat{\Gamma}(\varphi)$ is in $\mathcal{B}$, and hence φ is $\mathcal{B}$-definable. □

The same argument as in the previous proposition shows the following.

Proposition 6.17. *Suppose that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}_R$. Let $\varphi : \hat{X}/N \to \hat{Y}/M$ be a Borel-definable R-homomorphism between R-modules with a Polish cover, where $\hat{X}$, N, and M are in $\mathcal{B}$. Then $\hat{Y}$ is in $\mathcal{B}$.*

Proof. Adopting the notation of Proposition 6.16, we have that $\hat{\Gamma}(\varphi)$ is in $\mathcal{B}$, being an extension of objects of $\mathcal{B}$. Considering the short exact sequence

$$0 \to N \oplus M \to \hat{\Gamma}(\varphi) \to Y \to 0$$

shows that Y is in $\mathcal{B}$ as well. □

As an application of Proposition 6.16, we obtain as a particular instance of Theorem 6.14 a description of the left heart of a number of categories of Polish modules. We refer to [42] for the theory of Fréchet and Banach spaces over a non-Archimedean valued field. Recall that a Polish abelian group G:

- is *compactly generated* if it has a compact generating set;
- is a *topological torsion group* if for every $x \in G$, $\lim_{n\to\infty} n!x = 0$—see [2, Chapter 3];
- a *topological p-group*, for some prime p, if for every $x \in G$, $\lim_{n\to\infty} p^n x = 0$—see [2, Chapter 2].

We let the *dimension* of a locally compact Polish space to be its *covering dimension*. Recall also the notion of locally compact abelian Polish group that *has finite ranks* according to [21, Definition 2.6]. It is proved in [21] that a locally compact Polish group A can be written uniquely as an extension $F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$ where $A_{\mathbb{Z}}$ is a countable torsion-free discrete group, and $F_{\mathbb{Z}}$ is in turn an extension $A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$ where $A_{\mathbb{S}^1}$ is compact connected and $A_{\mathbb{A}}$ is the direct sum of a topological torsion group A_{t} and $\mathbb{R}^n$ for some $n \geq 0$. One then says that A has finite ranks if $A_{\mathbb{S}^1}$ is finite-dimensional, $A_{\mathbb{Z}}$ has finite rank, and for every prime number p, the p-component of A_{t} is isomorphic to a finite direct sum of groups isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ for some $n \in \mathbb{N}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, or $\mathbb{Q}_p/\mathbb{Z}_p$.

Theorem 6.18. *Let R be a Polish ring. Let $\mathcal{B}$ one of the following full subcategories of $\mathcal{A}_R$:*

(1) non-Archimedean Polish R-modules;
(2) locally compact Polish R-modules;
(3) finite-dimensional locally compact Polish R-modules;
(4) if R is a field, locally bounded Polish vector spaces over R;
(5) if R is a separable non-Archimedean valued field, separable Fréchet spaces over R;
(6) if R is a separable non-Archimedean valued field, separable Banach spaces over R.

For $R = \mathbb{Z}$, let $\mathcal{B}$ one of the following subcategories of the category $\mathcal{A}_{\mathbb{Z}}$ of Polish abelian groups:

(7) abelian real Lie groups;
(8) totally disconnected locally compact Polish abelian groups;
(9) compactly generated Polish abelian groups;
(10) locally compact Polish topological torsion abelian groups;
(11) locally compact Polish topological abelian p-groups, for a given prime number p;
(12) locally compact Polish abelian groups that have finite ranks.

Then $\mathcal{B}$ is a thick subcategory of $\mathcal{A}_R$, and the inclusion $\mathcal{B} \to \mathcal{A}_R$ extends to a fully faithful functor $\mathrm{LH}(\mathcal{B}) \to \mathrm{LH}(\mathcal{A}_R) = \mathcal{M}_R$. Thus, the left heart of $\mathcal{B}$ is equivalent *to the category that has R-modules with a $\mathcal{B}$-cover as objects and Borel-definable R-homomorphisms as morphisms.*

Proof. We show that each of these categories is a thick subcategory of $\mathcal{A}_R$, and apply Proposition 6.16 and Theorem 6.14.

(1) Let

$$0 \to A \to X \to C \to 0$$

be an extension of Polish R-modules, where A, C are non-Archimedean. We identify A with a closed submodule of X. By [7, Proposition 4.6], there exists a continuous function $\phi : C \to X$ that is right inverse for the quotient map $\pi : X \to C$; see Remark 3.11. Define $\kappa(x, y) = \delta\phi(x, y) = \phi(x + y) - \phi(x) - \phi(y)$. By Proposition 5.7 one can choose ϕ such that for every open zero neighborhood U of A there exist an open subring O of R and an open O-submodule V of C such that $\kappa(x, y) \in U$ and $\phi(\lambda x) - \lambda\phi(x) \in U$ for $x, y \in V$ and $\lambda \in O$.

Consider the abelian Polish group $A \oplus_\kappa C$ that is equal to $A \times C$ as a Polish space, with group operation defined by $(a, c) + (a', c') = (a + a' + \kappa(c, c'), c + c')$. Then the function $X \to A \oplus_\kappa C$, $x \mapsto (x - \phi\pi(x), \pi(x))$ is a Borel group isomorphism with inverse $A \oplus_\kappa C \to X$, $(a, c) \mapsto a + \phi(c)$, and hence it is a homeomorphism. This shows that $\phi(C)$ is a closed subset of X, and the sets of the form $U_A + \phi(U_C)$, where U_A is a zero neighborhood in A and U_C is a zero neighborhood in C, form a basis of zero neighborhoods for X. Let U_X be a zero neighborhood in X. Consider an open subring O of R and open O-submodules U_A of A and U_C of C such that $U_A + \phi(U_C) \subseteq U_X$ and for every $x, y \in V$ and $\lambda \in P$, $\kappa(x, y) \in U_A$ and $\phi(\lambda x) - \lambda\phi(x) \in U_A$. Then we have that $U_A + \phi(U_C)$ is an open O-submodule of X contained in U. This proves that X is non-Archimedean.

(2) It is the content of [20, Theorem 5.22, Theorem 5.25] that the category of locally compact Polish groups is thick.

(3) If X is a Polish R-module and Y is a closed R-submodule, then we have that

$$\dim(X) = \dim(Y) + \dim(X/Y);$$

see [41]. Thus, we have that X is finite-dimensional if and only if Y and X/Y are finite-dimensional.

(4) Let R be a field, and

$$0 \to M \to X \to Y \to 0$$

be an extension of Polish R-vector spaces, where M and Y are locally bounded. Since the quotient map $\pi : X \to Y$ is open, M is closed in X, and M, Y are locally bounded, we have that there exists an open zero neighborhood A in X such that $A \cap M$ and $\pi(A)$ are bounded in M and Y, respectively. Fix a decreasing neighborhood basis (W_i) for 0 in R. Fix $i_0 \in \mathbb{N}$ and an open zero neighborhood $B \subseteq A$ in X such that $B + W_{i_0}B \subseteq A$. We claim that B is bounded in X. Suppose that this is not the case. Then there exist a zero neighborhood V_0 in X, a vanishing sequence (λ_i) in R, and a sequence (b_i) in X such that $\lambda_i \in W_i$, $b_i \in B$, and $\lambda_i b_i \notin V_0$ for $i \in \mathbb{N}$.

Fix a zero neighborhood V_0 in X. Fix $i_1 \geq i_0$ and a zero neighborhood V_1 in X such that $V_1 + (W_{i_1}V_1 \cup V_1) \subseteq V_0$. Fix $i \geq i_1$. Since $\pi(A)$ is bounded in Y, there exists $\alpha_i \geq i$ such that $\lambda_{\alpha_i}\pi(b_{\alpha_i}) \in W_i \pi(V_1)$ and $\lambda_i^{-1}\lambda_{\alpha_i} \in W_{i_1}$. Thus there exist $v_i \in V_1$ such that $\lambda_{\alpha_i}b_{\alpha_i} - \lambda_i v_i \in M$. For every $i \geq i_1$, we can write

$$\lambda_{\alpha_i}b_{\alpha_i} - \lambda_i v_i = \lambda_i\left(\lambda_i^{-1}\lambda_{\alpha_i}b_{\alpha_i} - v_i\right)$$

where

$$\lambda_i^{-1}\lambda_{\alpha_i}b_{\alpha_i} - v_i \in (W_{i_1}B + B) \cap M \subseteq A \cap M.$$

Since $A \cap M$ is bounded in M, there exists $i_2 \geq i_1$ such that

$$\lambda_{i_2}^{-1}\lambda_{\alpha_{i_2}}b_{\alpha_{i_2}} - v_{i_2} = \lambda_{i_2}(\lambda_{i_2}^{-1}\lambda_{\alpha_{i_2}}b_{\alpha_{i_2}} - v_{i_2}) \in V_1$$

and hence

$$\lambda_{\alpha_{i_2}}b_{\alpha_{i_2}} = (\lambda_{\alpha_{i_2}}b_{\alpha_{i_2}} - \lambda_{i_2}v_{i_2}) + \lambda_{i_2}v_{i_2} \in V_1 + W_0V_1 \subseteq V_0.$$

This contradicts the fact that $\lambda_{\alpha_{i_2}}b_{\alpha_{i_2}}$ does not belong to V_0.

(5) This is a particular instance of (1), since for a separable non-Archimedean valued field R, the separable Fréchet spaces over R are precisely the non-Archimedean Polish vector spaces over R—see [42, Definition 3.5.1, Theorem 3.5.2].

(6) This follows from (10) and (11), since for a separable non-Archimedean valued field R, the separable Banach spaces over R are precisely the locally bounded separable Fréchet spaces over R; see [42, Definition 2.1.1, Theorem 3.6.2].

(7) Suppose that G is a locally compact Polish group, and $H \subseteq G$ is a closed subgroup. If G is a Lie group, then H is a Lie group by the Closed Subgroup Theorem for Lie groups [29, Theorem 20.12]. If H is normal in G, then G/H is a Lie group by [29, Theo7]. If G is abelian and both H and G/H are Lie groups, then G is a Lie group by [50]; see also [15,28]. Notice also that a continuous group homomorphism between real Lie groups is real analytic; see [51, Theorem 2.11.2].

This assertion also follows from [39, Theorem 2.6(1)], after noticing that a locally compact abelian Polish group is a Lie group if and only if it *has no small subgroups*, which means that it has a zero neighborhood that does not contain any nontrivial subgroups—see [39].

(8) This follows from (1) and (2), since a locally compact Polish group is totally disconnected if and only if it is non-Archimedean.
(9) Compactly generated abelian Polish groups form a thick subcategory of $\mathcal{A}_{\mathbb{Z}}$ by [39, Theorem 2.6(2)].
(10) Locally compact abelian Polish topological torsion groups form a thick subcategory of $\mathcal{A}_{\mathbb{Z}}$ by [2, 3.17].
(11) That locally compact abelian Polish topological p-groups form a thick subcategory of $\mathcal{A}_{\mathbb{Z}}$ follows easily from (10) after observing that a topological torsion locally compact abelian Polish group G is a topological p-group if and only if it is equal to its $\mathbb{Z}_p$-component in the sense of [2, Definition 4.12]; see [2, Remark 3.9(a) and Example 4.13(a)].
(12) Locally compact Polish abelian groups that have finite ranks form a thick subcategory of $\mathcal{A}_{\mathbb{Z}}$ by [21, Proposition 2.9]. □

A Borel function $f : X \to Y$ between locally compact Polish spaces is called *locally bounded* if, for every compact subset C of X, $f(C)$ has compact closure in Y.

Corollary 6.19. *Suppose that $\varphi : X \to Y$ is a R-homomorphism between R-modules with a Polish cover $X = \hat{X}/N$ and $Y = \hat{Y}/M$, where $\hat{X}$ and M are locally compact. Then the following assertions are equivalent:*

(1) φ is Borel-definable;
(2) φ has a locally bounded *Borel lift $\hat{X} \to \hat{Y}$.*

If furthermore $\hat{X}$ is finite-dimensional, then these conditions are equivalent to:

(3) φ has an approximately additive locally continuous *Borel lift.*

Proof. Suppose that (1) holds. By Theorem 6.18, we can write $\varphi = \psi \circ \sigma^{-1}$ where $Z = \hat{Z}/L$ is an R-module with a Polish cover, $\psi : Z \to Y$ and $\sigma : Z \to X$ are liftable R-homomorphisms such that $\hat{Z}$ is an extension of $\hat{X}$ by M (and, in particular, a locally compact Polish R-module) and σ lifts to a surjective continuous R-homomorphism $\hat{\sigma} : \hat{Z} \to \hat{X}$. By the main theorem in [24], we have that $\hat{\sigma}$ has a Borel locally bounded right inverse $g : \hat{X} \to \hat{Z}$. (Notice that [24] adopts the terminology of [16, Section 51] where the "Baire σ-algebra" on a topological space is the σ-algebra generated by the compact G_δ sets. This coincides with the Borel σ-algebra in the case of locally compact Polish spaces. Thus, a "Baire function" between locally compact Polish spaces in the sense of [24] is just a Borel function.) Thus, if $\hat{\psi} : \hat{Z} \to \hat{Y}$ is a continuous R-homomorphism that lifts ψ, then we have that $\hat{\psi} \circ g$ is a locally bounded Borel lift for φ. If furthermore $\hat{X}$ is finite-dimensional, then by [38, Theorem 8], we have that there exist a zero neighborhood V of $\hat{X}$ and a continuous function $g : V \to \hat{Z}$ that is a right inverse for $\hat{\sigma}|_{\hat{\sigma}^{-1}(V)} : \hat{\sigma}^{-1}(V) \to V$. Thus, if $\hat{\psi} : \hat{Z} \to \hat{Y}$ is a continuous R-homomorphism that lifts ψ, then we have that $\hat{\varphi} := \hat{\psi} \circ g$ is a continuous local lift for φ. We can extend $\hat{\varphi}$ to a Borel lift on $\hat{X}$ by Lemma 3.12. Since M is locally compact, we have that M has a basis of zero neighborhoods that are compact, and in particular closed in $\hat{Y}$. Therefore we have that φ has an approximately additive Borel lift that is continuous in a zero neighborhood of $\hat{X}$ by Proposition 5.3. □

Corollary 6.20. *Suppose that $\varphi : G \to H$ is a group homomorphism between abelian groups with a real Lie cover $G = \hat{G}/N$ and $H = \hat{H}/M$. Then the following assertions are equivalent:*

(1) φ is Borel-definable;
(2) φ has an approximately additive Borel lift that is analytic *on an open zero neighborhood in $\hat{G}$.*

Proof. The proof is the same as the proof of Corollary 6.19 together with the fact that a continuous group homomorphism between real Lie groups is real analytic [51, Theorem 2.11.2], and that if $\pi : X \to Y$ is a surjective continuous homomorphism between real Lie groups, then there exists a zero neighborhood V of X and an analytic right inverse $g : V \to X$ for $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ [51, Theorem 2.9.5]; see also Remark 5.4. □

References

[1] S. Allison, Non-Archimedean TSI Polish groups and their potential Borel complexity spectrum, arXiv:2010.05085, 2020.
[2] D.L. Armacost, The Structure of Locally Compact Abelian Groups, Monographs and Textbooks in Pure and Applied Mathematics, vol. 68, Marcel Dekker, Inc., New York, 1981.
[3] S. Awodey, Category Theory, Oxford Logic Guides, vol. 49, The Clarendon Press, Oxford University Press, New York, 2006.
[4] A. Bodzenta, A. Bondal, Abelian envelopes of exact categories and highest weight categories, arXiv:2012.15707, 2020.
[5] A.A. Beilinson, J.N. Bernstein, P. Deligne, Faisceaux pervers, in: Analysis and Topology on Singular Spaces, I, Luminy, 1981, in: Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.
[6] J. Bergfalk, M. Lupini, A. Panagiotopoulos, The definable content of homological invariants II: Čech cohomology and homotopy classification, Forum Math. Pi (2024), in press.
[7] J. Bergfalk, M. Lupini, A. Panagiotopoulos, The definable content of homological invariants I: Ext & $\lim^1$, arXiv:2008.08782, 2020.
[8] J. Bergfalk, M. Lupini, F. Sarti, Applications of Borel-definable homological algebra to locally compact groups, available at https://www.lupini.org/research.
[9] T. Bühler, Exact categories, Expo. Math. 28 (1) (2010) 1–69.
[10] T. Bühler, On the Algebraic Foundations of Bounded Cohomology, Memoirs of the American Mathematical Society, vol. 214, 2011, 1006, xxii+97, ISBN 9780821853115.
[11] L. Ding, S. Gao, Non-archimedean abelian Polish groups and their actions, Adv. Math. 307 (2017) 312–343.
[12] R.O. Fulp, P.A. Griffith, Extensions of locally compact abelian groups. I, II, Trans. Am. Math. Soc. 154 (1971) 357–363.
[13] I. Farah, S. Solecki, Borel subgroups of Polish groups, Adv. Math. 199 (2) (2006) 499–541.
[14] S. Gao, Invariant Descriptive Set Theory, Pure and Applied Mathematics, vol. 293, CRC Press, Boca Raton, FL, 2009.
[15] A.M. Gleason, Spaces with a compact Lie group of transformations, Proc. Am. Math. Soc. 1 (1950) 35–43.

[16] P.R. Halmos, Measure Theory, D. Van Nostrand Co., Inc., New York, N. Y., 1950.
[17] L.A. Harrington, A.S. Kechris, A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, J. Am. Math. Soc. 3 (4) (1990) 903–928.
[18] G. Hjorth, A.S. Kechris, A. Louveau, Borel equivalence relations induced by actions of the symmetric group, Ann. Pure Appl. Log. 92 (1) (1998) 63–112.
[19] R. Henrard, S. Kvamme, A.-C. van Roosmalen, S.-A. Wegner, The left heart and exact hull of an additive regular category, arXiv:2105.11483, 2021.
[20] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis. Vol. I, second ed., Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 115, Springer-Verlag, Berlin-New York, 1979.
[21] N. Hoffmann, M. Spitzweck, Homological algebra with locally compact abelian groups, Adv. Math. 212 (2) (2007) 504–524.
[22] A.S. Kechris, Countable sections for locally compact group actions. II, Proc. Am. Math. Soc. 120 (1) (1994) 241–247.
[23] A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.
[24] E.T. Kehlet, Cross sections for quotient maps of locally compact groups, Math. Scand. 55 (1) (1984) 152–160, Publisher: Mathematica Scandinavica.
[25] A. Kechris, A. Louveau, The classification of hypersmooth Borel equivalence relations, J. Am. Math. Soc. 10 (1) (1997) 215–242.
[26] A.S. Kechris, H.L. Macdonald, Borel equivalence relations and cardinal algebras, Fundam. Math. 235 (2) (2016) 183–198.
[27] M. Kashiwara, P. Schapira, Categories and Sheaves, Grundlehren der mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006.
[28] M. Kuranishi, On Euclidean local groups satisfying certain conditions, Proc. Am. Math. Soc. 1 (1950) 372–380.
[29] J.M. Lee, Introduction to Smooth Manifolds, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.
[30] M. Lupini, Definable Eilenberg–Mac Lane universal coefficient theorems, arXiv:2009.10805, 2020.
[31] M. Lupini, Definable K-homology of separable C*-algebras, arXiv:2008.13344, 2020.
[32] M. Lupini, The classification problem for extensions of torsion abelian groups, arXiv:2102.03477, 2022.
[33] M. Lupini, The classification problem for extensions of torsion-free abelian groups, I, arXiv:2204.05431, 2022.
[34] M. Lupini, Complexity classes of Polishable subgroups, arXiv:2202.01965, 2022.
[35] S. Mac Lane, Homology, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[36] S. Mac Lane, Categories for the Working Mathematician, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.
[37] C.C. Moore, Group extensions and cohomology for locally compact groups. III, Trans. Am. Math. Soc. 221 (1) (1976) 1–33.
[38] P.S. Mostert, Sections in principal fibre spaces, Duke Math. J. 23 (1956) 57–71.
[39] M. Moskowitz, Homological algebra in locally compact abelian groups, Trans. Am. Math. Soc. 127 (3) (1967) 361–404 (en).
[40] L. Motto Ros, On the complexity of the relations of isomorphism and bi-embeddability, Proc. Am. Math. Soc. 140 (1) (2012) 309–323.
[41] K. Nagami, Dimension-theoretical structure of locally compact groups, J. Math. Soc. Jpn. 14 (1962) 379–396.
[42] C. Perez-Garcia, W.H. Schikhof, Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, Cambridge, 2010.
[43] F. Prosmans, Derived projective limits of topological abelian groups, J. Funct. Anal. 162 (1) (1999) 135–177.
[44] W. Rump, Almost abelian categories, Cah. Topol. Géom. Différ. Catég. 42 (3) (2001) 163–225.
[45] W. Rump, The abelian closure of an exact category, J. Pure Appl. Algebra 224 (10) (2020) 106395.
[46] J.-P. Schneiders, Quasi-abelian categories and sheaves, Mém. Soc. Math. Fr. (1999) 76, vi+134.
[47] S. Solecki, Polish group topologies, in: Sets and Proofs, Leeds 1997, in: London Math. Soc. Lecture Note Ser., vol. 258, Cambridge Univ. Press, Cambridge, 1999, pp. 339–364.
[48] S. Solecki, The coset equivalence relation and topologies on subgroups, Am. J. Math. 131 (3) (2009) 571–605.

[49] N. Stein, On the Universal Abelian Category of an Exact Category, Ph.D. thesis, University of Stuttgart, 2014.

[50] T. Tao, Central extensions of Lie groups, and cocycle averaging, 2011.

[51] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984.

[52] F.-H. Vasilescu, Spectral theory in quotient Fréchet spaces. I, Rev. Roum. Math. Pures Appl. 32 (6) (1987) 561–579.

[53] F.-H. Vasilescu, Spectral capacities in quotient Fréchet spaces, in: Topics in Operator Theory, in: Oper. Theory Adv. Appl., vol. 32, Birkhäuser, Basel, 1988, pp. 243–263.

[54] F.-H. Vasilescu, Spectral theory in quotient Fréchet spaces. II, J. Oper. Theory 21 (1) (1989) 145–202.

[55] L. Waelbroeck, Quotient Banach spaces; multilinear theory, Banach Cent. Publ. 8 (1) (1982) 563–571.

[56] L. Waelbroeck, The category of quotient bornological spaces, in: Aspects of Mathematics and Its Applications, in: North-Holland Math. Library, vol. 34, North-Holland, Amsterdam, 1986, pp. 873–894.

[57] L. Waelbroeck, Quotient Fréchet spaces, Académie de la République Populaire Roumaine, Rev. Roum. Math. Pures Appl. 34 (2) (1989) 171–179.

[58] L. Waelbroeck, The category of quotient bornological spaces, in: Operator Extensions, Interpolation of Functions and Related Topics, Timişoara, 1992, in: Oper. Theory Adv. Appl., vol. 61, Birkhäuser, Basel, 1993, pp. 201–212.

[59] L. Waelbroeck, Bornological Quotients, Mémoire de la Classe des Sciences, Académie Royale de Belgique, Classe des Sciences, Brussels, 2005.

[60] S.-A. Wegner, The heart of the Banach spaces, J. Pure Appl. Algebra 221 (11) (2017) 2880–2909.

[61] E. Weiss, Boundedness in topological rings, Pac. J. Math. 6 (1956) 149–158.