Ryu-type extended Marshall-Olkin model with implicit shocks and joint life insurance applications

Published Version:
Ryu-type extended Marshall-Olkin model with implicit shocks and joint life insurance applications / Gobbi F.; Kolev N.; Mulinacci S.. - In: INSURANCE MATHEMATICS & ECONOMICS. - ISSN 0167-6687. - ELETTRONICO. - 101:November 2021(2021), pp. 342-358. [10.1016/j.insmatheco.2021.08.007]

Availability:
This version is available at: https://hdl.handle.net/11585/834299 since: 2021-12-21

Published:
DOI: http://doi.org/10.1016/j.insmatheco.2021.08.007

Ryu-type Extended Marshall-Olkin Model with Implicit Shocks and Joint Life Insurance Applications

Fabio Gobbi[1], **Nikolai Kolev**[2] **and Sabrina Mulinacci**[3]
[1] Department of Economics and Statistics, University of Siena, Italy
[2] Department of Statistics, University of Sao Paulo, Brazil
[3] Department of Statistics, University of Bologna, Italy

Abstract: In this paper we suggest an improvement of the Extended Marshall-Olkin methodology by allowing an implicit effect of the common shocks affecting the elements of the system. Properties of this new model are studied. We propose an empirical application to a sample of censored residual lifetimes of couples of insureds extracted from a data set of annuities contracts of a large Canadian life insurance company. We obtain estimation of the model parameters using a two-stage maximum likelihood technique and discuss the obtained results.

JEL classification: C34, C46, G22

Keywords: Extended Marshall-Olkin model, Implicit common shocks, Joint life insurance pricing, Mortality intensities, Singularity.

1 Introduction and preliminaries

The classical bivariate Marshall-Olkin (MO) shock model has a long history since the seminal paper of Marshall and Olkin (1967). It is specified by the stochastic representation

$$(X_1, X_2) = (\min(T_1, T_3), \min(T_2, T_3)), \tag{1}$$

where non-negative continuous random variables T_1 and T_2 identify the occurrence of independent "individual shocks" affecting two devices and T_3 is their "common shock" arrival time under the assumption that the shocks are governed by independent homogeneous Poisson processes, i.e., T_i's in (1) are exponentially distributed. The random vector (X_1, X_2) represents the joint distribution of both lifetimes and let us denote its joint survival function by $S_{X_1,X_2}(x_1,x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2)$ for all $x_1, x_2 \geq 0$.

In general, the MO construction (1) implies that the distribution of (X_1, X_2) has a singularity along the line $\{x_1 = x_2\}$ generated by the occurrence of the simultaneous default of both elements in the system, due to the fact that $\mathbb{P}(X_1 = X_2) > 0$.

The stochastic relation (1) can be equivalently rewritten as

$$S_{X_1,X_2}(x_1 + t, x_2 + t) = S_{X_1,X_2}(x_1, x_2)\, S_{X_1,X_2}(t,t) \quad \text{for all} \quad x_1, x_2, t \geq 0, \tag{2}$$

characterizing the bivariate lack of memory property (BLMP). The only solution with exponential marginals of the functional equation (2) is given by

$$S_{MO}(x_1,x_2) = S_{X_1,X_2}(x_1,x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda \max(x_1,x_2)\}, \tag{3}$$

for all $x_1, x_2 \geq 0$ and $\lambda_1, \lambda_2, \lambda > 0$, see Marshall and Olkin (1967).

The MO bivariate exponential distribution (3) has exponential marginals with parameters $\lambda_1 + \lambda$ and $\lambda_2 + \lambda$ and hence, constant marginal failure (hazard) rates. This restricts its usefulness for practical needs. As a response, other solutions of (2) with non-exponential marginals have been introduced. Let us mention Block and Basu (1974), Proschan and Sullo (1974), Friday and Patil (1977). An important contribution to the bivariate lack of memory notion is offered by Kulkarni (2006) who suggested a class of bivariate distributions possessing BLMP specified by (2), but having increasing or/and decreasing marginal failure rates which should satisfy a set of restrictions.

Many textbooks use as a base and give a special attention to the BLMP and related bivariate exponential distributions, see Barlow and Proschan (1981), Balakrishnan and Lai (2009), Gupta et al. (2010), McNeil et al. (2015) and Joe (2015) among others. More than 2000 articles complement and extend Marshall-Olkin's bivariate exponential distribution (3), justifying advantages in analysis of various data sets from engineering, medicine, insurance, finance, biology, etc. For example, Li and Pellerey (2011) launched the Generalized Marshall-Olkin (GMO) model considering non-exponential independent random variables T_i in (1), $i = 1,2,3$. The corresponding joint distributions do not possess BLMP, but Denuit et al. (2006) show that

$$\mathbb{P}(X_2 > x_2 \,|\, X_1 > x_1, X_2 > x_1) = \mathbb{P}(X_2 > x_2 \,|\, X_2 > x_1), \text{ for } x_2 > x_1$$

meaning that the survival of X_1 to time x_1 is irrelevant for the survival of X_2 to time x_2 if $X_2 > x_1$. A multidimensional version of the GMO model is studied by Lin and Li (2014).

As a further step, Pinto and Kolev (2015) introduced the Extended MO (EMO) model generated through (1) by assuming dependence between arbitrary non-negative random variables T_1 and T_2, but keeping T_3 independent of them. The motivation is that the individual shocks might be dependent if the items share a common environment. Thus, the EMO model is specified by the joint survival function

$$S_{EMO}(x_1, x_2) = S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) S_{T_3}(\max(x_1, x_2)) \tag{4}$$

for all $x_1, x_2 \geq 0$. New properties of the EMO model (4) are provided by Gobbi et al. (2019), where the authors justify its utility in joint life insurance pricing. Indeed, in joint life insurance, the dependence of lives X_1 and X_2 arises from exogenous events that are individual to each life in the couple (represented by T_1 and T_2) and a common (fatal) one, identified by T_3. For example, the common shock may be an accident or the onslaught of a contagious disease, see Denuit et al. (2006) for a relevant interpretation and discussion.

All MO-type models and their generalizations listed above assume that the shocks (identified by random variables T_1, T_2 and T_3) are *explicit*, i.e., they have immediate killing effect. A notable exceptions are the papers of Ghurye and Marshall (1984) and Ryu (1993). The practice shows that such a fatal scenario is not always true. For instance, a general financial crisis affects first the weaker finance institutions and has a delayed impact on stronger ones, see examples in Cherubini et al. (2015). Therefore, it would be natural and valuable to investigate MO-type models with *implicit* shocks, i.e., when the fatal event is registered later than the shock occurrence. We refer the reader to the recent book of Cha and Finkelstein (2018) where one can find applications of Generalized Polya and shot noise processes, being able to model the possible delay of the shocks (see Chapter 9).

An immediate extension of model (1) with implicit impact of the common shock can be described by relation

$$(X_1, X_2) = (\min(T_1, T_3), \min(T_2, f(T_3))), \tag{5}$$

where $f(.)$ is some appropriate increasing continuous function in the first quadrant ensuring that the corresponding $S_{X_1,X_2}(.,.)$ is a proper bivariate survival function. Let us give a reliability interpretation of (5). Denote by X_i the lifetime of a component i, $i = 1, 2$. The stochastic relation (5) tells us that a common "fatal shock" destroys immediately the first component and has a delayed effect on the second one. Kolev and Pinto (2018) studied a special case of (5) when $f(x) = \alpha x$ for some $\alpha > 1$. Therefore, an important characteristic of construction (5), is that it permits a "late" failure of one component when a "fatal shock" strikes both components (as a counterpart of MO models generated by (1) where both components fail simultaneously if occurs a common "fatal shock" distinguished by the random variable T_3).

In order to model a real practical scenario with implicit breakdowns, Ryu (1993) considers a two component system which is subject to common shocks governed by a homogeneous Poisson process $\{N(t)\}_{t\geq 0}$ causing a delayed effect, improving the MO model generated by (1) as follows: A realization of $N(t)$ can be equivalently represented through a sequence of shock arrival times $\tau_1, \tau_2, \ldots$. Let $w_i = const$ be the impact size (magnitude) of those Poisson shocks affecting the $i-$th component, $i = 1, 2$. Then, given a realization of the homogeneous Poisson process $N(t)$, the hazard rate of the corresponding duration variable Z_i at time t for the i-th element is given by $w_i N(t)$ for $i = 1, 2$.

Under this setting, Ryu (1993) investigates the *MO model with implicit common shocks* generated by the stochastic representation

$$(X_1, X_2) = (\min(T_1, Z_1), \min(T_2, Z_2)), \tag{6}$$

where T_1, T_2, Z_1 and Z_2 are non-negative random variables, T_1 and T_2 being independent and exponentially distributed with parameters λ_1 and λ_2. In fact, T_1 and T_2 represent the occurrence of independent individual shocks (governed by two complementary homogeneous Poisson processes). The lifetime vector (T_1, T_2) is assumed independent of the common shocks represented by associated duration random variables Z_i, $i = 1, 2$ causing delayed (implicit) effects under the condition that Z_1 and Z_2 are conditionally independent given realization of the process $N(t)$.

In this case,

$$\mathbb{P}(Z_i > t|N) = \exp\Big\{ -w_i \int_0^t N(u)du \Big\}, \quad i = 1, 2,$$

consult Ryu (1993). Taking the expectation of this equation with respect to the stochastic nature of $N(t)$, the unconditional survival function is given by

$$\mathbb{P}(Z_i > t) = \mathbb{E}[\mathbb{P}(Z_i > t|N)].$$

The conditional joint distribution of $(X_1, X_2|N)$ can be represented as

$$\mathbb{P}(X_1 > x_1, X_2 > x_2|N) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}\mathbb{P}(Z_1 > x_1, Z_2 > x_2|N),$$

where

$$\mathbb{P}(Z_1 > x_1, Z_2 > x_2|N) = \exp\Big\{ -w_1 \int_0^{x_1} N(u)du - w_2 \int_0^{x_2} N(u)du \Big\}.$$

In this paper we suggest an improvement of the Extended Marshall-Olkin methodology embodying the ideas of Ryu (1993), i.e., by allowing an implicit effect of the common shocks affecting the elements of the system. In Section 2 we provide an explicit formula for the joint distribution of (X_1, X_2) in a general EMO model generated by (6), where the vector (T_1, T_2) is independent of a bivariate stochastic processes governing the common shocks that might be non-fatal and represented by the associated random vector (Z_1, Z_2). In Section 3 we simplify the model assuming that the common shocks are conducted by a homogeneous Poisson process being fatal if their magnitude is greater than a pre-specified threshold. The influence of the parameters involved on the bivariate lifetime and corresponding mortality intensities is studied. We find convenient to test the model (6) in joint life insurance context, since it allows a delayed effect identified by (Z_1, Z_2). For example, the common shock might involve both spouses, but only one of them dies. We apply the Ryu-type EMO model specified by (6) to a sample of censored residual lifetimes of couples of insureds extracted from a data set of annuities contracts of a large Canadian life insurance company[1] in Section 4. We obtain the two-stage maximum likelihood estimates of the parameters and compare the results with other inspections on the same data set. Concluding remarks are given in Section 5.

2 General Ryu-type EMO model

Our aim is to investigate an EMO-type model with implicit common shocks generated by the stochastic representation (6). Following EMO methodology developed by Pinto and Kolev (2015) and incorporating Ryu's (1993) approach we assume that

A1. The random variables T_1 and T_2 represent the occurrence of individual shocks which are supposed to be dependent. The distribution of the pair (T_1, T_2) is defined by their joint survival function $S_{T_1,T_2}(x_1, x_2)$ that we assume to be absolutely continuous;

A2. The variables Z_i causing delayed (implicit) common effect are conditionally independent given the realizations of a bivariate stochastic process $\mathbf{H} = (H_1(t), H_2(t))_{t\geq 0}$, where the marginal processes $(H_i(t))_{t\geq 0}$ are increasing, right-continuous, such that $H_i(0) = 0$ and $\lim_{t\to\infty} H_i(t) = +\infty$ a.s. for $i = 1, 2$;

A3. The lifetime vector (T_1, T_2) is independent of the the underlying bivariate stochastic processes $\mathbf{H}$ and of the associated random vector (Z_1, Z_2);

A4. We suppose that

$$\mathbb{P}\left(Z_i > x_i|\mathbf{H}\right) = \mathbb{P}\left(Z_i > x_i|H_i(x_i)\right) = \exp\{-H_i(x_i)\}, \quad x_i \geq 0, \quad i = 1, 2.$$

Remark 2.1. *To justify assumption A4, note that* $\exp\{-H_i(x_i)\}$ *is the survival function of exponentially distributed random variable* E_i *with parameter 1 evaluated at* $H_i(x_i)$, $i = 1, 2$. *Hence, the last equation can be rewritten as*

$$\mathbb{P}\left(Z_i > x_i|H_i(x_i)\right) = \mathbb{P}\left(E_i > H_i(x_i)\right),$$

[1]We wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano Valdez, for allowing the use of the data in this paper.

or equivalently

$$\mathbb{P}\left(Z_i \le x_i | H_i(x_i)\right) = \mathbb{P}\left(E_i \le H_i(x_i)\right) = \mathbb{P}\left(H_i^{-1}(E_i) \le x_i\right).$$

Therefore, the time to delayed effect Z_i coincides with the time at which the process H_i crosses the random threshold E_i, i.e., $Z_i = H_i^{-1}(E_i)$, $i = 1, 2$.

Of course, one can postulate an appropriate absolutely continuous distribution different than the unit exponential one, see Theorem 2 in Singpurwalla (2006).

Under this setting, the unconditional survival distribution of Z_i is given by

$$\mathbb{P}\left(Z_i > x_i\right) = \mathbb{E}\left[\mathbb{P}\left(Z_i > x_i | H_i(x_i)\right)\right] = \mathbb{E}\left[\exp\{-H_i(x_i)\}\right] = \mathcal{L}_{H_i(x_i)}(1), \quad i = 1, 2,$$

where $\mathcal{L}_{H_i(x_i)}(1)$ denotes the Laplace transform of $H_i(x_i)$ evaluated at 1.

The conditional joint survival distribution of the random variables X_1 and X_2 specified by (6) can be written as

$$\begin{aligned}\mathbb{P}\left(X_1 > x_1, X_2 > x_2 | \mathbf{H}\right) &= S_{T_1,T_2}(x_1, x_2)\mathbb{P}\left(Z_1 > x_1, Z_2 > x_2 | \mathbf{H}\right) \\ &= S_{T_1,T_2}(x_1, x_2)\mathbb{P}\left(Z_1 > x_1 | \mathbf{H}\right)\mathbb{P}\left(Z_2 > x_2 | \mathbf{H}\right) \\ &= S_{T_1,T_2}(x_1, x_2)\exp\{-H_1(x_1) - H_2(x_2)\}\end{aligned}$$

and therefore,

$$S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2)\mathbb{E}\left[\exp\{-H_1(x_1) - H_2(x_2)\}\right]$$

Thus, we can formulate our main statement as follows.

Theorem 2.1. *Under assumptions A1-A4, the unconditional joint survival function $S_{X_1,X_2}(x_1, x_2)$ of the model generated by (6) can be represented by*

$$S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2)\mathcal{L}_{(H_1(x_1),H_2(x_2))}(1, 1) \tag{7}$$

where $\mathcal{L}_{(H_1(x_1),H_2(x_2))}$ denotes the joint Laplace transform of $(H_1(x_1), H_2(x_2))$.

Example 2.1. *Let $(W_k^1, W_k^2)_{k=1,2,\ldots}$ be a sequence of i.i.d. random vectors with positive components and $N = (N(t))_{t\ge 0}$ be a homogeneous Poisson process with intensity $\lambda > 0$ independent of $(W_k^1, W_k^2)_{k=1,2,\ldots}$. We consider the bivariate stochastic process*

$$\mathbf{H} = \left(\sum_{k=1}^{N(t)} W_k^1, \sum_{k=1}^{N(t)} W_k^2\right).$$

In this case,

$$\mathbb{P}\left(Z_1 > x_1, Z_2 > x_2 | \mathbf{H}\right) = \exp\left(-\sum_{k=1}^{N(x_1)} W_k^1 - \sum_{k=1}^{N(x_2)} W_k^2\right)$$

which corresponds to the bivariate survival distribution of two dicrete random variables taking values in $\{\tau_1, \tau_2, \ldots\}$ *where* τ_j *is the* j*-th jump time of the Poisson process* N *with*

$$\mathbb{P}\left(Z_i = \tau_j | \mathbf{H}\right) = \exp\left(-\sum_{k=1}^{j-1} W_k^i\right) - \exp\left(-\sum_{k=1}^{j} W_k^i\right), \quad i = 1, 2.$$

Notice that

$$\lim_{x_i \to \infty} \mathbb{P}\left(Z_i > x_i | \mathbf{H}\right) = \exp\left(-\sum_{k=1}^{\infty} W_k^i\right),$$

might be positive, allowing for the possibility that the fatal shock never occurs.

Straightforward computations imply that, for $x_1 \leq x_2$,

$$\mathcal{L}_{H_1(x_1), H_2(x_2)}(1,1) = \exp\left\{\lambda x_1 \left(\mathcal{L}_{W_1^1 + W_1^2}(1) - 1\right) + \lambda(x_2 - x_1)\left(\mathcal{L}_{W_1^2}(1) - 1\right)\right\}.$$

A similar expression can be obtained when $x_1 > x_2$.

Under the knowledge of $S_{T_1,T_2}(x_1, x_2)$ *one can apply Theorem 2.1 to get the joint survival function* $S_{X_1,X_2}(x_1, x_2)$. *For example, if we assume that* W_i^1 *are independent and Gamma distributed with shape parameter* α_i *and rate parameter* μ *for* $i = 1, 2$, *then from (7) we obtain*

$$S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) \exp\left\{\lambda x_1 \left[\left(1 + \mu^{-1}\right)^{-\alpha_1 - \alpha_2} - 1\right] + \lambda(x_2 - x_1)\left[\left(1 + \mu^{-1}\right)^{-\alpha_2} - 1\right]\right\}$$

for $x_1 \leq x_2$.

Remark 2.2. *A general scenario, very close to the model based on assumptions A1-A4 is considered by Mercier and Pham (2017). Rewritten in terms of our notations, the authors assume that random variables* T_1 *and* T_2 *are independent and the dependence is induced by the random vector* (Z_1, Z_2) *in (6). On the other side, in Mercier and Pham (2017) the random variables* Z_1 *and* Z_2 *are not, in general, conditionally independent.*

The general formula (7) might be useful for practical needs under simplifying assumptions. In the next section we will study a particular case, assigning a predetermined threshold for the amplitude of common shocks governed by a homogeneous Poisson process.

3 Ryu-type EMO model with threshold

Consider a homogeneous Poisson process $N = (N(t))_{t \geq 0}$ governing the common shock arrival times of a two components system with lifetimes (X_1, X_2) generated by stochastic representation (6). We will believe that the shock corresponding to the first jump of the Poisson process is fatal if its magnitude is larger than a given threshold $w > 0$. Otherwise, the shock is implicit (non-fatal) and its impact on the residual lifetimes results in an increment of the corresponding stochastic hazard rate.

To proceed, we postulate hereafter the following assumptions:

B1. The random variables T_1 and T_2 represent the occurrence of individual shocks and are supposed to be dependent with joint absolutely continuous survival function $S_{T_1,T_2}(x_1, x_2)$;

B2. Common shock arrival times are modeled by a homogeneous Poisson process $N = (N(t))_{t\geq 0}$ with intensity $\lambda > 0$ and the magnitude of implicit common shocks are represented by two independent random variables Y_1 and Y_2 being independent of the Poisson process N. We assign a threshold $w > 0$ and constants $w_1, w_2 \in [0, w]$, such that

$$\mathbb{P}(Y_i > w) = p_i \in [0,1] \quad \text{and} \quad \mathbb{P}(Y_i = w_i \leq w) = 1 - p_i = \bar{p}_i, \quad i = 1, 2;$$

B3. The variables Z_i, $i = 1, 2$ modeling the delayed (implicit) effects are conditionally independent given the realizations of the Poisson process N and of the random variables Y_1 and Y_2.

B4. The lifetime vector (T_1, T_2) is independent of the Poisson process N, of the random variables Y_1 and Y_2 and of the corresponding random vector (Z_1, Z_2);

B5. Following Ryu (1993), we assume that

$$\mathbb{P}(Z_i > t | Y_i = w_i \leq w) = \mathbb{E}\left[\exp\left(-w_i \int_0^t N(u)du\right)\right], \quad i = 1, 2.$$

The model specified by (6) under assumptions B1-B5 will be referred as a REMO model hereafter.

The assumption B2 means that, when the magnitude of Y_i is larger than a given threshold w, we treat the shock as fatal and this event happens with a probability p_i, $i = 1, 2$. In such a case, without loss of generality, we may assume that $w_i \to +\infty$. In fact, Y_i is a discrete random variable with mass at $+\infty$ and w_i with probabilities p_i and $1 - p_i$, respectively. Note that when $w_i = 0$, then the common shock does not have influence on i-th lifetime.

Observe that if τ_1 is the first jump of the Poisson process and $Y_i > w$, then $Z_i = \tau_1$ and therefore

$$\mathbb{P}(Z_i > t | Y_i > w) = \mathbb{P}(\tau_1 > t), \quad i = 1, 2.$$

Under assumption B2, the bivariate stochastic process $\mathbf{H}$ defined in A2 can be represented as

$$\mathbf{H} = \left(Y_1 \int_0^t N(u)du,\, Y_2 \int_0^t N(u)du \right)_{t\geq 0}$$

with $Y_i = +\infty$ when $Y_i > w$, $i = 1, 2$. In other words,

$$\mathbb{P}\left(Z_1 > x_1, Z_2 > x_2 | \mathbf{H}\right) = \mathbb{P}\left(Z_1 > x_1, Z_2 > x_2 | Y_1, Y_2, N\right) = \\ = \exp\left(-Y_1 \int_0^{x_1} N(u)du - Y_2 \int_0^{x_2} N(u)du \right).$$

Finally, applying Theorem 2.1 for the REMO model we arrive to

$$S_{X_1,X_2}\left(x_1, x_2\right) = S_{T_1,T_2}(x_1, x_2) S_{Z_1,Z_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) \mathcal{L}_{\left(Y_1 \int_0^{x_1} N(u)du, Y_2 \int_0^{x_2} N(u)du\right)}(1, 1).$$

We will present in the next an explicit expression for the joint survival function of the REMO model subject to common implicit shocks governed by a homogeneous Poisson process. The corresponding copula function will be derived as well. We will compute and analyze associated bivariate hazard rate intensities introduced by Cox (1972) in consequence.

3.1 Joint survival function

To compute the joint survival function of the pair (Z_1, Z_2), we first need to obtain an expression for the conditional probability $\mathbb{P}\left(Z_1 > x_1, Z_2 > x_2 |\, Y_1 = w_1,\, Y_2 = w_2\right)$. Under assumptions B2 and B5, thanks to Proposition 2 in Ryu (1993), we have

$$G(x_1, x_2) = P(Z_1 > x_1, Z_2 > x_2 \,|\, Y_1 = w_1, Y_2 = w_2) = \mathbb{E}\left[e^{-\left(w_1 \int_0^{x_1} N(u)du + w_2 \int_0^{x_2} N(u)du\right)} \right]$$
$$= \begin{cases} \exp\left[-\lambda x_2 + \frac{\lambda}{w_2}\left(1 - e^{-w_2(x_2 - x_1)}\right) + \frac{\lambda}{w_1 + w_2}\left(e^{-w_2(x_2 - x_1)} - e^{-w_1 x_1 - w_2 x_2}\right)\right], & x_2 \geq x_1; \\ \exp\left[-\lambda x_1 + \frac{\lambda}{w_1}\left(1 - e^{-w_1(x_1 - x_2)}\right) + \frac{\lambda}{w_1 + w_2}\left(e^{-w_1(x_1 - x_2)} - e^{-w_1 x_1 - w_2 x_2}\right)\right], & x_2 < x_1. \end{cases} \quad (8)$$

According to hypothesis B2, if at lest one of Y_i's is above the threshold w, the corresponding probability can be obtained from (8) when w_i tends to $+\infty$, $i = 1, 2$. Therefore,

- When $w_i \to +\infty$, $i = 1, 2$, we have

$$\lim_{w_1 \to +\infty, w_2 \to +\infty} G(x_1, x_2) = G_{11}(x_1, x_2) = \begin{cases} \exp(-\lambda x_2), & x_2 \geq x_1, \\ \exp(-\lambda x_1), & x_2 < x_1; \end{cases} \quad (9)$$

- When $w_1 \to +\infty$ and $w_2 < w \in (0, +\infty)$, the expression is

$$\lim_{w_1 \to +\infty} G(x_1, x_2) = G_{10}(x_1, x_2) = \begin{cases} \exp\left[-\lambda x_2 + \frac{\lambda}{w_2}\left(1 - e^{-w_2(x_2 - x_1)}\right)\right], & x_2 \geq x_1, \\ \exp(-\lambda x_1), & x_2 < x_1; \end{cases} \quad (10)$$

- If $w_1 < w \in (0, +\infty)$ and $w_2 \to +\infty$, then

$$\lim_{w_2 \to +\infty} G(x_1, x_2) = G_{01}(x_1, x_2) = \begin{cases} \exp(-\lambda x_2), & x_2 \geq x_1, \\ \exp\left[-\lambda x_1 + \frac{\lambda}{w_1}\left(1 - e^{-w_1(x_1 - x_2)}\right)\right], & x_2 < x_1. \end{cases} \tag{11}$$

Thus, we arrive to the following statement.

Proposition 3.1. *Under assumptions* B1-B5*, the joint survival function of the vector* (Z_1, Z_2) *can be represented as*

$$S_{Z_1,Z_2}(x_1, x_2) = \exp\{-\lambda \max(x_1, x_2)\} A(x_1, x_2),$$

where

$$A(x_1, x_2) = p_{\delta_{1,2}} + \bar{p}_{\delta_{1,2}} p_{3-\delta_{1,2}} \exp\left(\frac{\lambda}{w_{\delta_{1,2}}}\left(1 - e^{-w_{\delta_{1,2}}|x_2 - x_1|}\right)\right) +$$
$$+ \bar{p}_{\delta_{1,2}} \bar{p}_{3-\delta_{1,2}} \exp\left(\frac{\lambda}{w_{\delta_{1,2}}}\left(1 - e^{-w_{\delta_{1,2}}|x_2 - x_1|}\right) + e^{-w_{\delta_{1,2}}(|x_2 - x_1|)} \frac{\lambda}{w_1 + w_2}\left(1 - e^{-(w_1 + w_2)\cdot \min(x_1, x_2)}\right)\right) \tag{12}$$

with $\delta_{1,2} = \delta(x_1, x_2) = 1 \cdot \mathbf{1}_{\{x_1 > x_2\}} + 2 \cdot \mathbf{1}_{\{x_1 \leq x_2\}}$.

Proof. Let $x_2 \geq x_1$. Then,

$$\begin{aligned} S_{Z_1,Z_2}(x_1, x_2) &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-Y_1 \int_0^{x_1} N(u)du - Y_2 \int_0^{x_2} N(u)du\right) \middle| Y_1, Y_2\right]\right] \\ &= \mathbb{E}\left[\exp\left(-Y_1 \int_0^{x_1} N(u)du - Y_2 \int_0^{x_2} N(u)du\right) \middle| Y_1 > w, Y_2 > w\right] p_1 p_2 \\ &+ \mathbb{E}\left[\exp\left(-w_1 \int_0^{x_1} N(u)du - Y_2 \int_0^{x_2} N(u)du\right) \middle| Y_1 = w_1, Y_2 > w\right] \bar{p}_1 p_2 \\ &+ \mathbb{E}\left[\exp\left(-Y_1 \int_0^{x_1} N(u)du - w_2 \int_0^{x_2} N(u)du\right) \middle| Y_1 > w, Y_2 = w_2\right] p_1 \bar{p}_2 \\ &+ \mathbb{E}\left[\exp\left(-w_1 \int_0^{x_1} N(u)du - w_2 \int_0^{x_2} N(u)du\right) \middle| Y_1 = w_1, Y_2 = w_2\right] \bar{p}_1 \bar{p}_2, \end{aligned}$$

i.e.,

$$S_{Z_1,Z_2}(x_1, x_2) = G_{11}(x_1, x_2) p_1 p_2 + G_{01}(x_1, x_2) \bar{p}_1 p_2 + G_{10}(x_1, x_2) p_1 \bar{p}_2 + G(x_1, x_2) \bar{p}_1 \bar{p}_2.$$

Substituting the expressions of functions G, G_{11}, G_{10} and G_{01} from (8), (9), (10) and (11), correspondingly, we get

$$
\begin{aligned}
S_{Z_1,Z_2}(x_1,x_2) &= \exp\left(-\lambda x_2\right) p_2 + \exp\Big\{-\lambda x_2 + \frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right)\Big\} p_1\bar{p}_2 + \\
&+ \exp\Big\{-\lambda x_2 + \frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right) + \frac{\lambda}{w_1+w_2}\left(e^{-w_2(x_2-x_1)} - e^{-w_1x_1-w_2x_2}\right)\Big\}\bar{p}_1\bar{p}_2 = \\
&= \exp\left(-\lambda x_2\right) p_2 + \exp\Big\{-\lambda x_2 + \frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right)\Big\} p_1\bar{p}_2 + \\
&+ \exp\Big\{-\lambda x_2 + \frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right) + \frac{e^{-w_2(x_2-x_1)}\lambda}{w_1+w_2}\left(1-e^{-(w_1+w_2)x_1}\right)\Big\}\bar{p}_1\bar{p}_2.
\end{aligned}
$$

When $x_2 < x_1$, we obtain

$$
\begin{aligned}
S_{Z_1,Z_2}(x_1,x_2) &= \exp\left(-\lambda x_1\right) p_1 + \exp\Big\{-\lambda x_1 + \frac{\lambda}{w_1}\left(1-e^{-w_1(x_1-x_2)}\right)\Big\} p_2\bar{p}_1 + \\
&+ \exp\Big\{-\lambda x_1 + \frac{\lambda}{w_1}\left(1-e^{-w_1(x_1-x_2)}\right) + \frac{e^{-w_1(x_1-x_2)}\lambda}{w_1+w_2}\left(1-e^{-(w_1+w_2)x_2}\right)\Big\}\bar{p}_1\bar{p}_2,
\end{aligned}
$$

which completes the proof. □

In the following two remarks we offer a probability interpretation of the components of the function $A(x_1,x_2)$ from (12) and provide a decomposition of the survival function of marginals $Z_i, i = 1,2$.

Remark 3.1. *The function $A(x_1,x_2)$ given by (12) represents the contribution caused by implicit shocks to the joint survival function of the EMO model generated by (4). Indeed, if the common shock is fatal, then $p_i = 1, \bar{p}_i = 0$ for $i = 1,2$, and hence $A(x_1,x_2) = 1$ for all $x_1, x_2 > 0$. Remind that $Z_1 = Z_2 = \tau_1$ in this case, where τ_1 is the first jump of the common Poisson process.*

When p_1 and p_2 are not both equal to 1, then $A(x_1,x_2)$ can be represented as a weighted sum. When $x_2 \geq x_1$ one gets

$$A(x_1,x_2) = p_2 + p_1\bar{p}_2 L(x_1,x_2) + \bar{p}_1\bar{p}_2 M(x_1,x_2),$$

where

$$L(x_1,x_2) = \exp\Big\{\frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right)\Big\}$$

and

$$M(x_1,x_2) = \exp\Big\{\frac{\lambda}{w_2}\left(1-e^{-w_2(x_2-x_1)}\right) + e^{-w_2(x_2-x_1)}\frac{\lambda}{w_1+w_2}\left(1-e^{-(w_1+w_2)x_1}\right)\Big\}.$$

The term $L(x_1,x_2)$ corresponds to the case in which the shock is fatal only for lifetime 1. It can be easily shown that

$$\mathbb{P}\left(x_1 < \tau_1 \leq x_2, Z_2 > x_2 | Y_2 = w_2\right) = \exp\left(-\lambda x_2\right)\left[L(x_1,x_2) - 1\right].$$

When the common shock is not fatal for both lifetimes, then the expression $M(x_1, x_2)$ governs the associated contribution and

$$\mathbb{P}\left(Z_1 > x_1, Z_2 > x_2, \tau_1 \le x_1 | Y_1 = w_1, Y_2 = w_2\right) = \exp\left(-\lambda x_2\right)\left[M(x_1, x_2) - L(x_1, x_2)\right].$$

Similar probability interpretations hold when $x_2 \le x_1$.

Remark 3.2. *The marginal distribution of Z_i is given by*

$$S_{Z_i}(x_i) = p_i \exp(-\lambda x_i) + \bar{p}_i \exp\Big\{ -\lambda x_i + \frac{\lambda}{w_i}\left(1 - e^{-w_i x_i}\right)\Big\}, \quad i = 1, 2.$$

It is a mixture of the exponential distribution with parameter λ (when the shock is fatal for the i-th component) and the second term is the survival distribution obtained by Chiang and Conforti (1989), governing the delay effect.

We summarize the above facts in the following Theorem.

Theorem 3.1. *Under Assumptions* B1-B5*, the joint survival function of the vector (X_1, X_2) of the REMO model defined by the stochastic relation (6) is given by*

$$S_{REMO}(x_1, x_2) = S_{EMO}(x_1, x_2) A(x_1, x_2) \tag{13}$$

where

$$S_{EMO}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) \exp\{-\lambda \max(x_1, x_2)\}$$

is the survival function of the EMO model generated by (4) when the common shock arrival time T_3 is exponentially distributed with parameter λ and $A(x_1, x_2)$ has the representation (12).

Proof. Using (7) and Proposition 3.1 we conclude that

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= S_{T_1,T_2}(x_1, x_2) S_{Z_1,Z_2}(x_1, x_2) \\ &= S_{T_1,T_2}(x_1, x_2) \exp\{-\lambda \max(x_1, x_2)\} A(x_1, x_2). \end{aligned}$$

Thanks to (4), we know that $S_{T_1,T_2}(x_1, x_2) \exp\{-\lambda \max(x_1, x_2)\}$ is the survival function of the EMO model in the particular case when the common fatal shock arrival time T_3 coincides with the first jump τ_1 of the Poisson process. Thus, we get (13). □

Note that the REMO model exhibits singularity along the line $\{x_1 = x_2\}$ since

$$\begin{aligned} \mathcal{S}_{REMO}(t) = \mathbb{P}\left(X_1 = X_2 > t\right) &= \mathbb{P}\left(t < Z_1 \le T_1, t < Z_2 \le T_2 | Y_1 > w, Y_2 > w\right) p_1 p_2 \\ &= \mathbb{P}\left(T_1 \ge \tau_1, T_2 \ge \tau_1 > t\right) p_1 p_2 \\ &= p_1 p_2 \int_t^{+\infty} \mathbb{P}\left(T_1 \ge z, T_2 \ge z\right) \lambda e^{-\lambda z} dz \\ &= \lambda p_1 p_2 \int_t^{+\infty} S_{T_1,T_2}(z, z) e^{-\lambda z} dz \\ &= p_1 p_2 \mathcal{S}_{EMO}(t) \end{aligned}$$

where $\mathcal{S}_{EMO}(t) = \lambda \int_t^{+\infty} S_{T_1,T_2}(z,z)e^{-\lambda z}dz$ is the probability that the simultaneous end occurs after time t in the EMO model when the common shock arrival time is exponentially distributed with parameter λ. As a consequence, the singularity mass is

$$\mathcal{S}_{REMO} = \mathbb{P}(X_1 = X_2) = p_1 p_2 \mathcal{S}_{EMO},$$

where $\mathcal{S}_{EMO} = \lambda \int_0^{+\infty} S_{T_1,T_2}(t,t)e^{-\lambda t}dt$ is the singularity mass in the EMO case.

For a pre-specified joint distribution of vector (T_1, T_2), one can obtain a joint survival function of the REMO model, generated by stochastic relation (6).

Example 3.1. *Let us assume that (T_1, T_2) has bivariate exponential type I distribution introduced by Gumbel (1960), that is,*

$$S_{T_1,T_2}(x_1,x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \theta\lambda_1\lambda_2 x_1 x_2\} \quad \textit{for} \quad \lambda_1, \lambda_2 > 0,\ \theta \in [0,1].$$

Hence, using (13) the resulting REMO joint survival function is given by

$$S_{REMO}(x_1,x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \theta\lambda_1\lambda_2 x_1 x_2 - \lambda \max(x_1,x_2)\}A(x_1,x_2),$$

where $A(x_1,x_2)$ is specified by (12). Thus, we got a singular version of the absolutely continuous Gumbel's bivariate exponential distribution.

Observe that, when $\theta = 0$ and $p_1 = p_2 = 0$, we recover the model considered in Ryu (1993), while if $p_1 = p_2 = 1$ we obtain the classical MO bivariate exponential distribution (3).

Since REMO model incorporates the EMO model (see (13)), we are interested in analyzing the function $A(x_1,x_2)$ specified by (12) when $p_i < 1$ for at least one $i = 1,2$.

Proposition 3.2. *The lower and upper bounds of the function $A(x_1,x_2)$ from (12) are given by*

$$\min_{(x_1,x_2)\in[0,+\infty)^2} A(x_1,x_2) = A(0,0) = 1 \tag{14}$$

and

$$\begin{aligned} \sup_{(x_1,x_2)\in[0,+\infty)^2} A(x_1,x_2) &= \max[\lim_{x_1\to\infty} A(x_1,0), \lim_{x_2\to\infty} A(0,x_2)] \\ &= \max\left[p_1 + \bar{p}_1 \exp\left(\frac{\lambda}{w_1}\right), p_2 + \bar{p}_2 \exp\left(\frac{\lambda}{w_2}\right)\right]. \end{aligned} \tag{15}$$

Proof. Let $x_1 > x_2$ and set $B(x_1,x_2) = \exp\{-w_1(x_1 - x_2)\}$ and $D(x_1,x_2) = \exp\{-w_1 x_1 - w_2 x_2\}$. Then, (12) can be rewritten as

$$A(x_1,x_2) = p_1 + \bar{p}_1 e^{\frac{\lambda}{w_1}[1-B(x_1,x_2)]}\Big\{p_2 + \bar{p}_2 e^{\frac{\lambda}{w_1+w_2}[B(x_1,x_2)-D(x_1,x_2)]}\Big\}.$$

Since $B(x_1,x_2) > D(x_1,x_2)$, then $e^{\frac{\lambda}{w_1+w_2}[B(x_1,x_2)-D(x_1,x_2)]} \geq 1$. Moreover, $B(x_1,x_2) \leq 1$ implying $e^{\frac{\lambda}{w_1}[1-B(x_1,x_2)]} \geq 1$, so the relation (14) is established.

The case $x_1 < x_2$ leads to the same conclusion.

In order to obtain the upper bound (15), we analyze the partial derivatives of $A(x_1,x_2)$. When $x_1 > x_2$, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}A(x_1,x_2) = \bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\lambda B(x_1,x_2)\left[p_2 + \bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\right] \\
&+ \bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\frac{\lambda}{w_1+w_2}[-w_1B(x_1,x_2)+w_1D(x_1,x_2)] \\
&= \lambda\bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\left[B(x_1,x_2)p_2 + \bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\right. \\
&\left.\times\left\{\frac{w_2}{w_1+w_2}B(x_1,x_2) + \frac{w_1}{w_1+w_2}D(x_1,x_2)\right\}\right] \geq 0.
\end{aligned}$$

By analogy,

$$\begin{aligned}
&\frac{\partial}{\partial x_2}A(x_1,x_2) = -\bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\lambda B(x_1,x_2)\left[p_2 + \bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\right] \\
&+ \bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\frac{\lambda}{w_1+w_2}[w_1B(x_1,x_2)+w_2D(x_1,x_2)] \\
&= \lambda\bar{p}_1 e^{\frac{\lambda}{w_1}(1-B(x_1,x_2))}\left[-B(x_1,x_2)p_2 - \bar{p}_2 e^{\frac{\lambda}{w_1+w_2}(B(x_1,x_2)-D(x_1,x_2))}\frac{w_2}{w_1+w_2}\left(B(x_1,x_2)-D(x_1,x_2)\right)\right]
\end{aligned}$$

Since $B(x_1,x_2) > D(x_1,x_2)$, then $\frac{\partial}{\partial x_2}A(x_1,x_2) \leq 0$.

Similarly, for $x_2 > x_1$ we have $\frac{\partial}{\partial x_1}A(x_1,x_2) \leq 0$ and $\frac{\partial}{\partial x_2}A(x_1,x_2) \geq 0$.

From the signs of the partial derivatives we conclude that the upper bound is reached on the axes. Since

$$A(x_1,0) = p_1 + \bar{p}_1 \exp\left\{\frac{\lambda}{w_1}\left[1-\exp(-w_1x_1)\right]\right\}$$

and

$$A(0,x_2) = p_2 + \bar{p}_2 \exp\left\{\frac{\lambda}{w_2}[1-\exp(-w_2x_2)]\right\}$$

are both increasing in their argument functions, we arrive to relation (15). □

Graphs of the function $A(x_1,x_2)$ are shown in Figure 1 (varying p_1 and p_2) and Figure 2 (varying w_1 and w_2), confirming the lower and upper bounds (14) and (15).

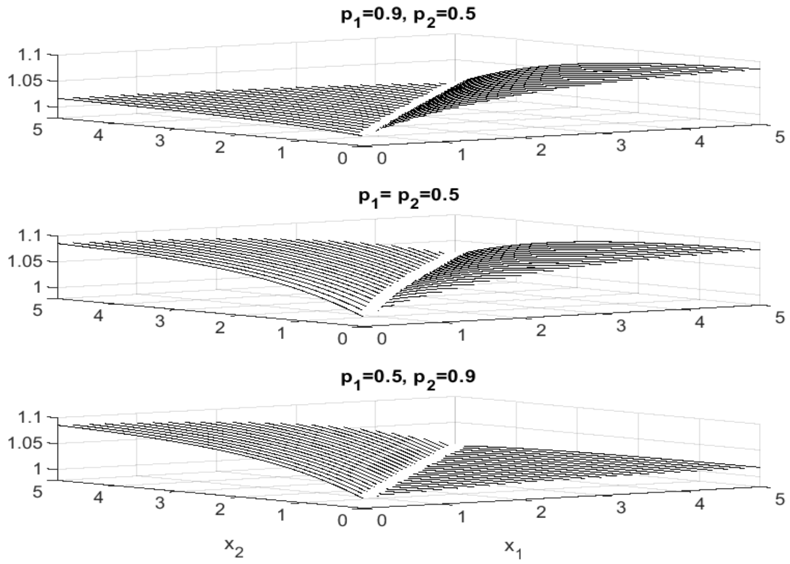

Figure 1: Shapes of the function $A(x_1, x_2)$ for various values of (p_1, p_2) for fixed $w_1 = w_2 = 0.5$ and $\lambda = 0.1$.

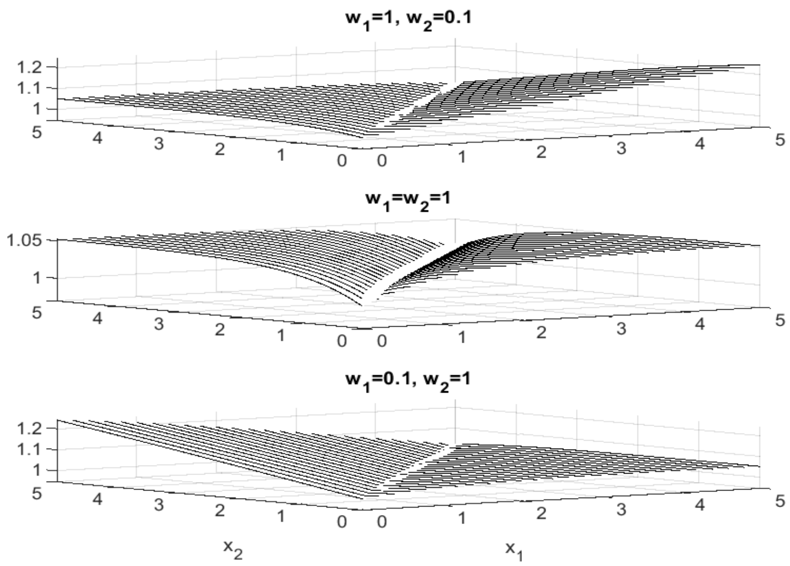

Figure 2: Shapes of the function $A(x_1, x_2)$ for various values of (w_1, w_2) for fixed $p_1 = p_2 = 0.5$ and $\lambda = 0.1$.

Remark 3.3. *Many joint life actuarial products depend on the residual lifetimes joint survival distribution values on the straight line $x_1 = x_2$. For instance, the continuous n-years joint life annuity net premium is defined by*

$$\bar{a}_{y_1 y_2;n\rceil} = \int_0^n e^{-ru} S_{X_1,X_2}(u,u)\, du,$$

where $r > 0$ is the instantaneous interest rate and y_1 and y_2 are the entry ages of the two individuals. Let us denote by $\bar{a}^{REMO}_{y_1 y_2;n\rceil}$ and $\bar{a}^{EMO}_{y_1 y_2;n\rceil}$ the net premium in REMO and EMO models correspondingly. Applying (13), one concludes that the cost of the possible delayed effect of the shock is given by

$$\bar{a}^{REMO}_{y_1 y_2;n\rceil} - \bar{a}^{EMO}_{y_1 y_2;n\rceil} = \int_0^n e^{-ru} S^{EMO}_{X_1,X_2}(u,u) \left[A(u,u) - 1\right] du.$$

The last difference is positive since (14) is valid.

Clearly, the opposite result holds with respect to the first death policy.

3.2 Associated copula function

Here we will obtain the copula function $C_{REMO}(u,v)$ corresponding to the joint survival function of the REMO model specified by (13). First, using the Sklar's theorem we rewrite $S_{EMO}(x_1,x_2) = S_{T_1,T_2}(x_1,x_2)\exp\{-\lambda \max(x_1,x_2)\}$ as

$$S_{EMO}(x_1,x_2) = C(S_{T_1}(x_1), S_{T_1}(x_2)) \exp\{-\lambda \max(x_1,x_2)\},$$

where $C(u,v)$ is a given copula function associated to the vector (T_1,T_2) with marginal survival functions $S_{T_1}(x_1)$ and $S_{T_1}(x_2)$. The marginals of $S_{EMO}(x_1,x_2)$ are given by $S_{EMO,i}(x_i) = S_{T_i}(x_i) exp\{-\lambda x_i\}$ and their inverse functions $S^{-1}_{EMO,i}(.)$ can be computed for $i = 1,2$. Therefore the associated copula $C_{EMO}(u,v)$ can be obtained via relation

$$C_{EMO}(u,v) = S_{EMO}(S^{-1}_{EMO,1}(u), S^{-1}_{EMO,2}(u)), \quad (u,v) \in [0,1]^2.$$

The reader can find its properties and examples in Gobbi et al. (2019).

Using (13), the marginal survival functions of the REMO model can be written as

$$S_{REMO,1}(x_1) = S_{EMO,1}(x_1)A(x_1,0) \quad \text{and} \quad S_{REMO,2}(x_2) = S_{EMO,2}(x_2)A(0,x_2).$$

Hence, for $i = 1,2$, we get

$$S_{REMO,i}(x_i) = S_{T_i}(x_i)\exp\left(-\lambda x_i\right)\left[p_i + \bar{p}_i \exp\left(\frac{\lambda}{w_i}\left(1 - \exp\left(-w_i x_i\right)\right)\right)\right]. \tag{16}$$

Applying again Sklar's theorem in (13) one can obtain the associated copula function

$$C_{REMO}(u,v) = C_{EMO}\left(\frac{u}{A\left(S^{-1}_{REMO,1}(u),0\right)}, \frac{v}{A\left(0,S^{-1}_{REMO,2}(v)\right)}\right) A\left(S^{-1}_{REMO,1}(u), S^{-1}_{REMO,2}(v)\right)$$

where $C_{EMO}(u,v)$ is the copula function associated to the EMO model with $S^{-1}_{REMO,1}(u)$ and $S^{-1}_{REMO,2}(v)$ being the inverse functions of the marginal survival functions given above.

Notice that the copula $C_{REMO}(u,v)$ is not absolutely continuous and it admits a singularity along the curve

$$\left\{(u,v) \in [0,1]^2 : v = S_{REMO,2} \circ S^{-1}_{REMO,1}(u)\right\}.$$

3.3 Mortality intensities

A multivariate hazard (mortality) rate concept has been introduced in a classical paper by Cox (1972). In the bivariate case we have the following four components (conditional hazard functions) of the hazard vector:

$$\lambda_{i0}(x) = \lim_{h\to 0^+} \frac{\mathbb{P}\left(x < X_i \le x+h | X_1 > x, X_2 > x\right)}{h}, \quad i = 1,2,$$

$$\lambda_{1|2}(x_1|X_1 > x_1, X_2 = x_2) = \lim_{h\to 0^+} \frac{\mathbb{P}\left(x_1 < X_1 \le x_1+h | X_1 > x_1, X_2 = x_2\right)}{h} \quad \text{for} \quad x_2 < x_1$$

and

$$\lambda_{2|1}(x_2|X_1 = x_1, X_2 > x_2) = \lim_{h\to 0^+} \frac{\mathbb{P}\left(x_2 < X_2 \le x_2+h | X_1 = x_1, X_2 > x_2\right)}{h} \quad \text{for} \quad x_2 > x_1.$$

A reliability interpretation of these quantities is as follows: an expert can specify the functions $\lambda_{i0}(x)$ for $i = 1,2$, based on aging characteristic on an item, while the functions $\lambda_{1|2}(x_1|X_1 > x_1, X_2 = x_2)$ and $\lambda_{2|1}(x_2|X_1 = x_1, X_2 > x_2)$ take into account the aging. The above conditional hazard functions completely specify the joint distribution of (X_1, X_2), see Singpurwalla (2006) for the corresponding relations.

In what follows we will use $\partial_i^+ f(x_1,x_2)$ and $\partial_i f(x_1,x_2)$ for the right-partial derivative and partial derivative of a differentiable function $f(x_1,x_2)$ with respect to $x_i, i = 1,2$, correspondingly.

In terms of joint survival function $S_{X_1,X_2}(x_1,x_2)$, it is readily seen that

$$\lambda_{i0}(x) = -\frac{\partial_i^+ S_{X_1,X_2}(x,x)}{S_{X_1,X_2}(x,x)}, \quad i = 1,2,$$

$$\lambda_{1|2}(x_1|X_1 > x_1, X_2 = x_2) = -\frac{\partial_1\partial_2 S_{X_1,X_2}(x_1,x_2)}{\partial_2 S_{X_1,X_2}(x_1,x_2)}, \quad \text{if} \quad x_1 > x_2$$

and

$$\lambda_{2|1}(x_1|X_1 > x_1, X_2 = x_2) = -\frac{\partial_1\partial_2 S_{X_1,X_2}(x_1,x_2)}{\partial_1 S_{X_1,X_2}(x_1,x_2)}, \quad \text{if} \quad x_2 > x_1.$$

Our aim is to obtain explicit expressions of these conditional mortality rates associated to the REMO model specified by (13), to compare them with EMO case and to display their dynamics for different values of parameters involved.

Denote by $s_{REMO}(x_1,x_2)$ and $s_{EMO}(x_1,x_2)$ the bivariate densities of the absolutely continuous part of the REMO and EMO models, respectively. After some algebra using (13), we get the following relations.

Proposition 3.3. *For the REMO model we have*

$$\lambda_{i0}^{REMO}(x) = \lambda_{i0}^{EMO}(x) - \frac{\partial_i^+ A(x,x)}{A(x,x)}, \quad i = 1,2$$

where $A(x,x) = 1 - \bar{p}_1\bar{p}_2 + \bar{p}_1\bar{p}_2 \exp\left\{\frac{\lambda}{w_1+w_2}[1 - e^{-(w_1+w_2)x}]\right\}$ *and*

$$\partial_i^+ A(x,x) = \bar{p}_i\lambda\left\{p_{3-i} + \bar{p}_{3-i}e^{\frac{\lambda}{w_1+w_2}\left(1-e^{-(w_1+w_2)x}\right)}\left[\frac{w_{3-i}}{w_1+w_2} + \frac{w_i}{w_1+w_2}e^{-(w_1+w_2)x}\right]\right\}.$$

The conditional intensities $\lambda_{1|2}$ *and* $\lambda_{1|2}$ *are given by*

$$\lambda_{1|2}(x_1|X_1 > x_1, X_2 = x_2) = -\frac{s_{REMO}(x_1,x_2)}{\partial_2 S_{REMO}(x_1,x_2)} \quad \text{if} \quad x_1 > x_2,$$

and

$$\lambda_{2|1}(x_2|X_1 = x_1, X_2 > x_2) = -\frac{s_{REMO}(x_1,x_2)}{\partial_1 S_{REMO}(x_1,x_2)} \quad \text{if} \quad x_1 > x_2,$$

where

$$\begin{aligned} s_{REMO}(x_1,x_2) = {} & s_{EMO}(x_1,x_2)A(x_1,x_2) + \partial_2 S_{EMO}(x_1,x_2)\partial_1 A(x_1,x_2) + \\ & + \partial_1 S_{EMO}(x_1,x_2)\partial_2 A(x_1,x_2) + S_{EMO}(x_1,x_2)\partial_1\partial_2 A(x_1,x_2), \end{aligned}$$

with

$$\partial_i S_{REMO}(x_1,x_2) = \partial_i S_{EMO}(x_1,x_2)A(x_1,x_2) + S_{EMO}(x_1,x_2)\partial_i A(x_1,x_2), \quad i = 1,2.$$

Remark 3.4. *Notice that, since* $\partial_i^+ A(x,x) \geq 0$*, thus*

$$\lambda_{i0}^{REMO}(x) \leq \lambda_{i0}^{EMO}(x), \quad i = 1,2. \tag{17}$$

Remind that if substitute $p_1 = p_2 = 1$ in (12), then $A(x_1, x_2) = 1$, i.e., we recover the formulas of the corresponding intensities for the EMO model obtained in Gobbi et al. (2019). In this case we have

$$\lambda_{i0}^{REMO}(x) = \lambda_{i0}^{EMO}(x) = -\frac{\partial_i S_{T_1,T_2}(x,x)}{S_{T_1,T_2}(x,x)} + \lambda$$

$$\lambda_{1|2}(x_1|X_1 > x_1, X_2 = x_2) = -\frac{s_{EMO}(x_1,x_2)}{\partial_2 S_{EMO}(x_1,x_2)} = -\frac{s_{T_1,T_2}(x_1,x_2)}{\partial_2 S_{T_1,T_2}(x_1,x_2)} + \lambda$$

and

$$\lambda_{2|1}(x_2|X_1 = x_1, X_2 > x_2) = -\frac{s_{EMO}(x_1,x_2)}{\partial_1 S_{EMO}(x_1,x_2)} = -\frac{s_{T_1,T_2}(x_1,x_2)}{\partial_1 S_{T_1,T_2}(x_1,x_2)} + \lambda.$$

Unlike inequality (17), there is no dominance relationship between the EMO and REMO based conditional hazard rates $\lambda_{1|2}$ and $\lambda_{2|1}$. For example, Figures 3 and 4 show that under different values of the parameters one can observe different shapes of the conditional intensity $\lambda_{1|2}$. In both cases we assume that the dependence structure between T_1 and T_2 is given by the Frank copula with parameter $\alpha = 2.6$. The marginal distributions of T_1 and T_2 are exponential with parameter 0.01 and the intensity of the common shock is $\lambda = 0.0012$.

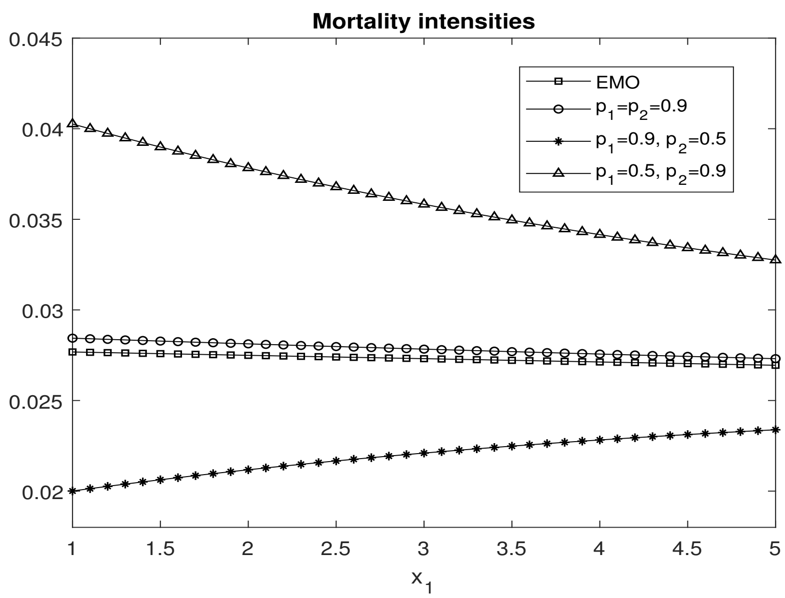

Figure 3: Dynamics of mortality intensity $\lambda_{1|2}$ when $x_1 > x_2 = 1$. Different curves refer to different pairs of values of p_1 and p_2, whereas $w_1 = w_2 = 0.2$.

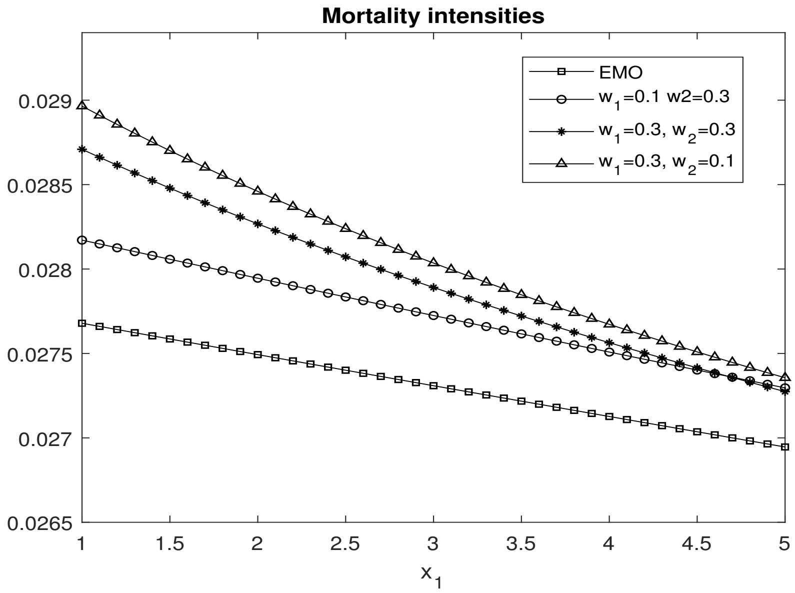

Figure 4: Dynamics of mortality intensity $\lambda_{1|2}$ when $x_1 > x_2 = 1$. Different curves refer to different pairs of values of w_1 and w_2 when $p_1 = p_2 = 0.9$.

4 Empirical applications

In this section we will fit the REMO model specified by (13) to a sample of censored residual lifetimes of couples of insureds extracted from a data set of annuities contracts of a Canadian life insurance company, registered in the period from December 29, 1988 to December 31, 1993. The data set is both left and right truncated. The available information provides the entry ages y_1 and y_2 of the two spouses and the corresponding censored residual lifetimes x_1 and x_2.

The Canadian data set has already been analysed in Frees et al. (1996), Carriere (2000), Shemyakin and Youn (2006), Ji et al. (2011), Dufrense et al. (2018), among the others. In Gobbi et al. (2019) the same data has been considered (where contracts involving insureds with the same sex and multiple contracts on the same couple have been removed and only entry ages greater than 60 considered, for a total number of observations equal to 9535) to fit the EMO model. We find convenient to apply the REMO model since it additionally includes a possibility of common external shocks with after-effects, see assumption B2. Following the same approach as in Gobbi et al. (2019), we apply the two-stage maximum likelihood technique of Shih and Louis (1995): we first compute the maximum likelihood estimates of the parameters of the marginal distributions, separately, and then we compute the maximum likelihood estimates of the remaining parameters assuming those already estimated as given. We will assume marginal distributions of Gompertz type (frequently used in actuarial practice) and we will compare the goodness of fit with that of the EMO one through the Bayesian Information Criteria (BIC).

Hereafter, treating data or random variables, we will assign index 1 or 2 referring to the male or to the female in the couple, correspondingly.

4.1 Model specification

Taking into account the analysis conducted on Canadian data set in Frees et al. (1996) and in Carriere (2000), we will assume that marginal residual lifetimes X_1 and X_2 of the REMO model are distributed according to the Gompertz law.

Specifically, we suppose that residual lifetime survival distributions from ages y_1 and y_2 are given by

$$S_{X_i}(x_i) = \exp\left\{a_i(y_i)\left(1 - e^{\frac{x_i}{\sigma_i}}\right)\right\}, \quad i = 1, 2, \tag{18}$$

with $a_i(y_i) = \exp\left(\frac{y_i - M_i}{\sigma_i}\right)$, where M_i and σ_i are the corresponding mode and dispersion parameters.

In order to simplify notations, we will drop in the sequel the dependence on the initial entry ages y_1 and y_2, that is, we will set $a_1 = a_1(y_1)$ and $a_2 = a_2(y_2)$.

Given the adopted Gompertz marginal distributions in (18), using (16) we obtain that

$$S_{T_i}(x_i) = \frac{\exp\left\{a_i\left(1 - e^{\frac{x_i}{\sigma_i}}\right) + \lambda x_i\right\}}{p_i + \bar{p}_i \exp\left\{\frac{\lambda}{w_i}\left(1 - e^{-w_i x_i}\right)\right\}}, \quad i = 1, 2. \tag{19}$$

The REMO model is well defined if expressions (19) are proper survival functions. The corresponding restrictions on parameters λ, p_i, w_i, M_i and σ_i, $i = 1, 2$, are summarized below.

Proposition 4.1. *For the REMO model with marginal Gompertz distributions the following constraints hold:*

C1. If $S_{T_i}(x_i)$ in (19) is a valid survival function, then

$$\lambda \leq \frac{a_i}{p_i \sigma_i}; \tag{20}$$

C2. If $p_i < 1$ and

$$\lambda \leq \frac{a_i}{\sigma_i} \min\left(\frac{1}{p_i}, 1 + \frac{1}{\sigma_i w_i}\right), \tag{21}$$

then $S_{T_i}(x_i)$ in (19) is a proper survival function, i = 1,2.

Proof. First notice, that the first derivative $S'_{T_i}(x_i)$ of the expression in (19) can be represented as

$$S'_{T_i}(x_i) = g_i(x_i)\Big\{ \left(\lambda - \frac{a_i}{\sigma_i} e^{\frac{x_i}{\sigma_i}}\right) [p_i + \bar{p}_i h_i(x_i)] - \bar{p}_i h_i(x_i)\lambda e^{-w_i x_i}\Big\}, \quad i = 1, 2, \quad (22)$$

where $g_i(x_i) = \exp\left\{a_i \left(1 - e^{\frac{x_i}{\sigma_i}}\right) + \lambda x_i\right\}$ and $h_i(x_i) = \exp\left\{\frac{\lambda}{w_i}\left(1 - e^{-w_i x_i}\right)\right\}$.

Moreover, $S_{T_i}(0) = 1$ and $\lim_{x_i \to \infty} S_{T_i}(x_i) = 0$, for $i = 1, 2$.

C1. Using (22) we obtain that $S'_{T_i}(0) = \lambda - \frac{a_i}{\sigma_i} - \bar{p}_i\lambda = p_i\lambda - \frac{a_i}{\sigma_i}$. Since $S_{T_i}(x_i)$ is a survival function, then necessarily $S'_{T_i}(0) \leq 0$ and inequality (20) is established.

C2. In fact, we want to show that when $p_i < 1$ and (21) holds, then $S'_{T_i}(x_i) \leq 0$ for all $x_i \in [0, +\infty), i = 1, 2$. If (22) is fulfilled, the condition $S'_{T_i}(x_i) \leq 0$ is equivalent to

$$e^{w_i x_i} - \frac{a_i}{\lambda\sigma_i} \exp\left\{\frac{1}{\sigma_i} + w_i x_i\right\} \leq \bar{p}_i \frac{h_i(x)}{p_i + \bar{p}_i h_i(x)}, \quad i = 1, 2. \quad (23)$$

After careful analysis of the last inequality, one can conclude that $S_{T_i}(x_i)$ is a proper survival function indeed.

□

Remark 4.1. *Unfortunately, Proposition 4.1 shows that we are unable to establish necessary and sufficient conditions on the parameters of the REMO model, such that* $S'_{T_i}(x_i) \leq 0$ *on* $[0, +\infty)$, $i = 1, 2$.

However, if $p_i = 1$ *in REMO model, then we recover the corresponding marginal distributions in the EMO case and the restriction (20) is a necessary and sufficient condition for* $S_{T_i}(x_i)$ *to be a valid survival function. Note that (23) is satisfied for all* $x_i \geq 0$ *in EMO model, when (20) holds with* $p_i = 1$ *for* $i = 1, 2$.

4.2 Estimation methodology and results

As we established, the REMO joint distribution is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. We will use the maximum likelihood estimation technique considering the REMO distribution density with respect to the dominating measure μ on $\mathbb{R}^2$ given by the sum of the Lebesgue measure on the plane and of the Lebesgue measure on the straight line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. We refer the reader to Gobbi et al. (2019) for details, where the same procedure has been applied to the EMO model.

We will model the dependence structure between random variables T_1 and T_2 in (6) using Frank and Clayton copulas with parameter α, defined as

$$C(u, v) = -\frac{1}{\alpha} \ln\left[1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right]$$

and

$$C(u,v)=\left(u^{-\alpha}+v^{-\alpha}-1\right)^{-\frac{1}{\alpha}}.$$

Thanks to Theorem 3.1 and expressions in the proof of Proposition 3.2, the REMO density with respect to the dominating measure μ on $\mathbb{R}^2$ is given by

$$s_{REMO}(x_1,x_2)=\begin{cases} e^{-\lambda x_2}\left[H(x_1,x_2)-\lambda K_1(x_1,x_2)\right], & 0\le x_1<x_2;\\ e^{-\lambda x_1}\left[H(x_1,x_2)-\lambda K_2(x_1,x_2)\right], & x_1>x_2\ge 0;\\ \lambda p_1p_2e^{-\lambda x}C\left(S_{T_1}(x),S_{T_2}(x)\right), & x_1=x_2=x\ge 0,\end{cases}$$

where

$$\begin{aligned}H(x_1,x_2)&=s_{T_1,T_2}(x_1,x_2)A(x_1,x_2)+\partial_1S_{T_1,T_2}(x_1,x_2)\partial_2A(x_1,x_2)+\\&\quad+\partial_2S_{T_1,T_2}(x_1,x_2)\partial_1A(x_1,x_2)+S_{T_1,T_2}(x_1,x_2)\partial_1\partial_2A(x_1,x_2)\end{aligned}$$

and

$$K_i(x_1,x_2)=\partial_iS_{T_1,T_2}(x_1,x_2)A(x_1,x_2)+S_{T_1,T_2}(x_1,x_2)\partial_iA(x_1,x_2),\ i=1,2.$$

Let $\boldsymbol{\theta}_i=(M_i,\sigma_i)$ be the vector of parameters of the marginal Gompertz survival functions $S_{X_i}(x_i)$, $i=1,2$. The joint survival distribution of the residual lifetimes from entry ages y_1 and y_2 can be written as

$$S_{REMO}(x_1,x_2|y_1,y_2)=C\left(\frac{S_{T_1}(x_1|y_1;\boldsymbol{\theta_1})}{e^{-\lambda x_1}A(x_1,0;\boldsymbol{\eta})},\frac{S_{T_2}(x_2|y_2;\boldsymbol{\theta_2})}{e^{-\lambda x_2}A(0,x_2;\boldsymbol{\eta})};\alpha\right)e^{-\lambda\max(x_1,x_2)}A(x_1,x_2;\boldsymbol{\eta})$$

where α is the parameter of the considered copula function C and $\boldsymbol{\eta}=(\lambda,p_1,p_2,w_1,w_2)$ are the parameters of the function $A(x_1,x_2)$. Denote by $\boldsymbol{\gamma}=(\alpha,\boldsymbol{\eta})$ the vector of REMO model parameters that are not involved in the marginal distributions. We will assume that the parameter vector $\boldsymbol{\gamma}$ is independent of the entry ages y_1 and y_2.

Let $(\hat{\boldsymbol{x}}_1,\hat{\boldsymbol{x}}_2)=\{(\hat{x}_{1i},\hat{x}_{2i}):i=1,\dots,n\}$ be a sample of n censored observed residual lifetimes pairs from ages $\{(y_{1i},y_{2i}):i=1,\dots,n\}$. If (C_{1i},C_{2i}) denote independent random censoring times for the male and the female individuals in the couple i then the i-th observation $(\hat{x}_{1i},\hat{x}_{2i})$ is defined as

$$\hat{x}_{1i}=\min(x_{1i},C_{1i})\quad\text{and}\quad\hat{x}_{2i}=\min(x_{2i},C_{2i}),\ i=1,\dots,n,$$

where x_{1i} and x_{2i} are the corresponding residual lifetimes. If $\delta_{ji}=\mathbf{1}_{\{\hat{x}_{ji}=x_{ji}\}}$ for $i=1,\dots,n$ and $j=1,2$, the likelihood function of the vector of parameters $(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma})$ is given by

$$\begin{aligned}L\left((\hat{\boldsymbol{x}}_1,\hat{\boldsymbol{x}}_2);\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma}\right)=\prod_{i=1}^{n}\Big\{&\left[s_{REMO}(x_{1i},x_{2i}|y_{1i},y_{2i};\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma})\right]^{\delta_{1i}\delta_{2i}}\\&\times\left[-\partial_1S_{REMO}\left(x_{1i},C_{2i}|y_{1i},y_{2i};\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma}\right)\right]^{\delta_{1i}(1-\delta_{2i})}\\&\times\left[-\partial_2S_{REMO}(C_{1i},x_{2i}|y_{1i},y_{2i};\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma})\right]^{(1-\delta_{1i})\delta_{2i}}\\&\left[S_{REMO}(C_{1i},C_{2i}|y_{1i},y_{2i};\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\gamma})\right]^{(1-\delta_{1i})(1-\delta_{2i})}\Big\}.\end{aligned}\tag{24}$$

$\hat{M}_1$	$\hat{\sigma}_1$	$\hat{M}_2$	$\hat{\sigma}_2$
86.1144	9.5642	92.0369	7.8195

Table 1: Estimators of the parameters of marginal Gompertz distributions.

We apply the two-stage parametric method for censored data introduced in Shih and Louis (1995). More precisely, the procedure consists in

Step 1: Compute the maximum likelihood estimators of $\boldsymbol{\theta}_j = (M_j, \sigma_j)$ with $j = 1, 2$, of the Gompertz type marginal distributions (this can be achieved assuming independence in the likelihood (24));

Step 2: Given $\left(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2\right)$ obtained in the previous step, compute the maximum likelihood estimator $\hat{\boldsymbol{\gamma}}$ of the remaining parameters $\boldsymbol{\gamma} = (\alpha, \lambda, p_1, p_2, w_1, w_2)$ as a solution of the following constrained maximization problem

$$\begin{cases} \max\limits_{\boldsymbol{\gamma}} L\left((\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2) ; \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \boldsymbol{\gamma}\right) \\ \text{under restrictions: } \lambda p_1 \leq \bar{\lambda}_1, \ \lambda p_2 \leq \bar{\lambda}_2, \end{cases} \tag{25}$$

where $\bar{\lambda}_j = \min\left\{\frac{\hat{a}_{ji}}{\hat{\sigma}_j} : i = 1, \ldots, n\right\}$ with $\hat{a}_{ji} = \exp\left(\frac{y_{ji} - \hat{M}_j}{\hat{\sigma}_j}\right)$ for $j = 1, 2$, due to (20) in Proposition 4.1.

It remains to check if the estimator $\hat{\boldsymbol{\gamma}}$ fulfills restriction (21):

1. If $\hat{\boldsymbol{\gamma}}$ is such that inequality (21) is satisfied, then $\hat{\boldsymbol{\gamma}}$ is the maximum likelihood estimator we are looking for;

2. If $\hat{\boldsymbol{\gamma}}$ doesn't satisfy (21) for some $i = 1, 2$, then further investigation is needed to ensure that the expressions in (19) are proper survival functions.

In order to take into account a delay in reporting the exact date of death, we consider as simultaneous deaths those occurring by a 5 days lag as in Ji et al. (2011).

The output of the first step for parameter estimates of Gompertz marginal distributions are listed in Table 1. We use them to obtain the following upper bounds in (25):

$$\bar{\lambda}_1 = 0.00681618 \quad \text{and} \quad \bar{\lambda}_2 = 0.00212564. \tag{26}$$

Then, we apply the second step of the estimation procedure, maximizing the likelihood in (25). The estimates of the REMO model parameters and relative standard errors are displayed in the first panel of Table 2.

REMO				**Frank copula**			
	$\hat{\alpha}_{REMO}$	$\hat{\lambda}_{REMO}$	$\hat{p_1}$	$\hat{w_1}$	$\hat{p_2}$	$\hat{w_2}$	BIC
	1.7551	0.001476	0.9999	0.0000	0.3128	1.0453	3008.09
	(0.3887)	(0.0002)	(0.0000)	(0.0000)	(0.1448)	(0.2269)	
				Clayton copula			
	$\hat{\alpha}_{REMO}$	$\hat{\lambda}_{REMO}$	$\hat{p_1}$	$\hat{w_1}$	$\hat{p_2}$	$\hat{w_2}$	BIC
	1.2793	0.001347	0.9999	0.0000	0.5199	0.9474	3022.36
	(0.3006)	(0.0002)	(0.0000)	(0.0000)	(0.2775)	(0.2958)	
EMO				**Frank copula**			
	$\hat{\alpha}_{EMO}$	$\hat{\lambda}_{EMO}$					BIC
	2.2518	0.001096					3035.478
	(0.0107)	(0.0000)					
				Clayton copula			
	$\hat{\alpha}_{EMO}$	$\hat{\lambda}_{EMO}$					BIC
	1.1678	0.001178					3039.03
	(0.0047)	(0.0000)					

Table 2: Maximum likelihood estimates and relative standard errors of the EMO and REMO models.

Remark 4.2. *Since the obtained log-likelihood is a strongly non-linear of six variables, the estimation has been conducted applying a procedure composed by several steps, in each of which, on the basis of a grid of initial values of the parameters, we have identified the solution that minimized the BIC. The importance of the choice of several initial values in the optimization procedure is discussed in Greene (2000).*

We used the maximum likelihood estimators $\boldsymbol{\theta}_j = (M_j, \sigma_j)$ with $j = 1, 2$ already obtained in of Gobbi et al. (2019) for the EMO model (Step 1). In Step 2 we got the minimum BIC within a grid of initial values relating to the four parameters of the REMO model p_1, w_1, p_2 and w_2. We have thus obtained a first significant evidence. The value of p_1 which minimized the BIC was very close to 1, which made the parameter w_1 irrelevant for the estimation. In consequence, a grid of initial values for the remaining parameters (p_2, w_2) relating to females has been built, obtaining the minimum BIC corresponding to the estimates reported in Table 2. The determination of the standard errors took place by calculating the hessian matrix numerically by approximating the partial derivatives and the second-order partial derivatives through finite differences.

In the second panel of Table 2 we give the maximum-likelihood estimators for parameters of the EMO model, reported by Gobbi et al. (2019).

The estimated values 0.001476 and 0.001347 of intensity parameter λ using Frank and Clayton copulas correspondingly, are smaller than the upper bounds given in

(26). Thus, condition (21) is satisfied in the whole data set, since $p_i \leq 1$ for $i = 1, 2$, ensuring that relations (19) represent proper survival functions.

In order to compare different models we have used the BIC expression for censored data as suggested in Volinsky and Raftery (2000). In this case, $BIC = -2\,LL + k \log m$, where LL is the maximum value of the log-likelihood, k is the number of parameters and m is the number of non-censored observations.

Let us analyze the the estimators listed in Table 2.

1. Comparing the estimators of common parameters in REMO and EMO models we have
$$\hat{\lambda}_{REMO} > \hat{\lambda}_{EMO} \quad \text{and} \quad \hat{\alpha}_{REMO} < \hat{\alpha}_{EMO}.$$
These relations are quite reasonable, because REMO and EMO models are fitted to the same data set. Really, all external shocks in EMO model are assumed to be fatal, while the REMO model additionally incorporates the possibility of implicit common shocks. Therefore, it is natural that the estimated intensity $\hat{\lambda}_{REMO}$ of the common shock in REMO model dominates the corresponding one in EMO model. However, an inverse relation holds for associated copula parameters measuring the degree of dependence between individual shocks represented by the random vector (T_1, T_2), being common for both models. Since the EMO model is more conservative, then the associated copula parameter $\hat{\alpha}_{EMO}$ should dominate those estimated in REMO case;

2. It was a real surprise for us, that for the REMO model we got $\hat{p_1} = 0.9999 \approx 1$ in Frank and Clayton cases, implying that the corresponding magnitudes $\hat{w_1} \approx 0$. This simply means that the common shock can be treated as a fatal for the man in a couple of considered data set. However, for women, the corresponding estimates using the Frank copula are $\hat{p_2} = 0.3128$ and $\hat{w_2} = 1.0453$. In other words, the chances of women to survive a fatal shock are about tree times higher than the men in a couple.

   This conclusion can be confirmed screening again the Canadian data set. One can observe that roughly three times more males as females died during the study period. It also suggests higher mortality rates for males than for females, see Dufresne at al. (2018) and Shemyakin and Youn (2006);

3. The best performance is achieved by the REMO model with the Frank copula connecting random variables T_1 and T_2 in (6), with BIC = 3008.09. A possible reason is that the Clayton copula exhibits a lower tail dependence, which is probably not appropriate for modeling the bivariate lifetimes of Canadian data set. Dufrense at al. (2018) arrived to the same conclusion.

5 Conclusions

In this paper, we introduce a Ryu-type Extended Marshall-Olkin model considering a delayed effect of the common shocks affecting the elements of the system. A general expression for the joint survival function of the model is given in Theorem 2.1. We examined in detail its particular version specified by (6) under assumptions B1-B5, called REMO model, when the common shocks are governed by a homogeneous Poisson process, causing different impact on components considering a "fatal" threshold level.

Using real insurance data, we develop an appropriate estimator of the joint distribution of the lifetimes of spouses with copula models. A goodness of fit procedure clearly shows that the REMO model outperform the models assuming explicit common shocks. The results of our illustrations, focusing on valuation of joint life insurance products, suggest that lifetimes dependence factors should be taken into account.

Finally, let us note that another versions of the Extended Marshall-Olkin model can be investigated. For example, following the methodology proposed by Marshall and Olkin (1988), one might consider a scenario where the duration variables Z_i in stochastic relation (6) are defined by $Z_i = V_0 + V_i$, $i = 1, 2$ where V_0, V_1 and V_2 are independent non-negative random variables. In this case, Z_1 and Z_2 are correlated since they contain the common element V_0. We left this problem for a future research.

Acknowledgments: The authors are grateful for the support of FAPESP Grants 2018/23185-0 and 2013/07375-0.

References

[1] Balakrishnan N. and Lai C.D. (2009): *Continuous Bivariate Distributions*, 2nd Edition, Springer

[2] Barlow R. and Proschan F. (1981): *Statistical Theory of Reliability and Life Testing*, Silver Spring.

[3] Block H.W. and Basu A.P. (1974): A continuous bivariate exponential extension, *Journal of the American Statistical Association*, 69, 1031-1037.

[4] Carriere J. (2000): Bivariate Survival Models of Coupled Lives, *Scandinavian Actuarial Journal*, 17-31.

[5] Cox D. (1972): Regression models and life tables (with discussion), *Journal of the Royal Statistical Society*, B, 34, 187-220.

[6] Cha J.H. and Finkelstein M. (2018): *Point Processes for Reliability Analysis*, Springer, New York.

[7] Chiang, C. and Conforti, P. (1989): A survival model and estimation of time to tumor. *Mathematical Biosciences*, 94, 1-29.

[8] Cherubini U., Durante F. and Mulinacci S. (2015): *Marshall-Olkin Distributions - Advances in Theory and Applications*, *Springer Series in Mathematics & Statistics*, 141, Springer, Heidelberg.

[9] Denuit M., Frostig E., Levikson B. (2006): Shift in Interest Rate and Common Shock Model for Coupled Lives. *Belgian Actuarial Bulletin* **6**, 1-4.

[10] Dufresne F. Hashrova E., Ratovomirija G. and Toukourou Y. (2018): On age difference in joint lifetime modelling with life insurance annuity applications, *Annals of Actuarial Sciences*, 12, 350-371.

[11] Frees E., Carriere J. and Valdez E. (1996): Annuity Valuation with Dependent Mortality, *Journal of Risk and Insurance*, 63, 229-261.

[12] Friday D.S. and Patil G.P. (1977): A bivariate exponential model with applications to reliability and computer generation of random variables, in: *The Theory and applications of Reliability*, C.P. Tsokos and I.N. Shimi (eds.), 527-549. Academic Press, New York.

[13] Gobbi F., Kolev N. and Mulinacci S. (2019): Joint life insurance pricing using extended Marshall-Olkin models, *ASTIN Journal*, 49(2), 409-432.

[14] Gumbel E. (1960): Bivariate exponential distribution, *Journal of the American Statistical Association*, 50, 698-707.

[15] Gupta A., Zeung W. and Hu Y. (2010): *Probability and Statistical Models: Foundations for Problems in Reliability and Financial Mathematics*, Birkhauser.

[16] Ghurye, S. and Marshall, A. (1984). Shock rocesses with aftere effects and multivariate lack of memory. *Journal of Applied Probability*, 21, 786-801.

[17] Greene W.H. (2000): *Econometric Analysis*, 4th Edition, Prentice Hall International Editions.

[18] Ji M., Hardy M.R. and Li J.S.-H. (2011): Markovian approaches to joit-life mortality, *North American Actuarial Journal*, 15, 2, 357-376.

[19] Joe H. (2015): *Dependence Modeling with Copulas*, CRC Press, Boca Raton.

[20] Kolev N. and Pinto J. (2018): A weak version of bivariate lack of memory property, *Brazilian Journal of Probability and Statistics*, 32, 873-906.

[21] Kulkarni H.V. (2006): Characterizations and modelling of multivariate lack of memory property, *Metrika*, 64, 167-180.

[22] Li X. and Pellerey F. (2011): Generalized Marshall-Olkin distributions and related bivariate aging properties, *Journal of Multivariate Analysis*, 102, 1399-1409.

[23] Lin J. and Li X. (2014): Multivariate generalized Marshall-Olkin distributions and copulas, *Methodology and Computing in Applied Probability*, 16, 53-78.

[24] McNeil A., Frey L. and Embrechts, P. (2015): *Quantitative Risk Management*, 2nd edition, Princeton University Press.

[25] Marshall A. and Olkin, I. (1967): A multivariate exponential distribution, *Journal of American Statistical Association*, 62, 30-44.

[26] Mercier S. and Pham H.H. (2017): A bivariate failure time model with random shocks and mixed effects, *Journal of Multivariate Analysis*, 153, 33-51.

[27] Pinto J. and Kolev N. (2015): Extended Marshall-Olkin model and its dual version, in: *Springer Series in Mathematics & Statistics*, 141, U. Cherubini, F. Durante and S. Mulinacci (eds.), 87-113.

[28] Proschan F. and Sullo, P. (1974): Estimating the parameters of a bivariate exponential distribution in several sampling situations, in: *Reliability and Biometry: Statistical Analysis of Life Lengths*, F. Proschan and R.J. Serfling (eds.), 423-440.

[29] Ryu K. (1993): An extension of Marshall and Olkin's bivariate exponential distribution, *Journal of the American Statistical Association*, 88, 1458-1465.

[30] Shemyakin A. and Youn H. (2006): Copula models of joint last survivor analysis, *Applied Stochastic Models in Business and Industry*, 22, 211-224.

[31] Shih J.H. and Louis T.A. (1995): Inferences on the association parameter in copula models for bivariate survival data, *Biometrics*, 51, 1384-1399.

[32] Singpurwalla N. (2006): The Hazard potential: introduction and overview, *Journal of the American Statistical Association*, 101, 1705-1717.

[33] Volinski C.T. and Raftery A.E. (2000): Bayesian information criterion for censored survival models. *Biometrics*, 56, 256-262.