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# ANALYTIC HYPOELLIPTICITY FOR SUMS OF SQUARES IN THE PRESENCE OF SYMPLECTIC NON TREVES STRATA

ANTONIO BOVE AND MARCO MUGHETTI

Abstract. In [1], [2] it was shown that Treves conjecture for the real analytic hypoellipticity of sums of squares operators does not hold. Models were proposed where the critical points causing a non analytic regularity might be interpreted as strata. We stress that up to now there is no notion of stratum which could replace the original Treves stratum. In the proposed models such "strata" were non symplectic analytic submanifolds of the characteristic variety.

In this note we modify one of those models in such a way that the critical points are a symplectic submanifold of the characteristic variety while still not being a Treves stratum. We show that the operator is analytic hypoelliptic.

## 1. Introduction and Statement of the Result

The purpose of this paper is to emphasize the importance of the concept of stratification in the study of real analytic hypoellipticity of sums of squares operators. In 1999 F. Treves stated a conjecture according to which a sum of squares of vector fields is analytic hypoelliptic if and only if the characteristic variety can be stratified as a disjoint union of certain symplectic, real analytic manifolds, called the Poisson stratification (see [17] and [5] for precise definitions.)

In [1] it was shown that the Treves conjecture does not hold by producing an example contradicting the conjecture. The example is the operator in (1.3). Up to now there is no new conjecture for the analytic hypoellipticity of sums of squares.

The characteristic variety of the operator in [1] however can be stratified in a different way, so that the associated strata are not all symplectic. We remark that still these strata are real analytic manifolds such

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that the restriction to the strata of the symplectic form has constant rank.

In this paper we study an example similar to that of [1], where the characteristic variety can be stratified with symplectic real analytic manifolds in a way different from that of the Treves conjecture, but very close to the aforementioned stratification of the model in [1]. It is shown that the operator is analytic hypoelliptic.

The examples discussed seem to suggest that real analytic hypoellipticity still depends of the geometric properties of a suitably defined stratification of the characteristic variety, even though we have no precise definition at the moment.

Let $r$, $p$, $q$ and $k$ be positive integers such that $r < p < q$. Consider the sum of squares operator in $\mathbb{R}^4$

$$\begin{aligned} P(x, D) &= D_1^2 + D_2^2 + x_1^{2(r-1)} D_3^2 + x_1^{2(r-1)} D_4^2 + x_2^{2(p-1)} D_3^2 \\ &\quad + x_2^{2(p-1)} x_3^{2k} D_4^2 + x_2^{2(q-1)} D_4^2 \\ &= \sum_{j=1}^{7} X_j(x, D)^2. \end{aligned} \tag{1.1}$$

The characteristic variety of $P$ is actually the real analytic manifold

$$\mathrm{Char}(P) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\}, \tag{1.2}$$

which is a symplectic manifold.

In [1] the operator

$$\begin{aligned} Q(x, D) &= \sum_{\substack{j=1 \\ j \neq 6}}^{7} X_j(x, D)^2 \\ &= D_1^2 + D_2^2 + x_1^{2(r-1)} D_3^2 + x_1^{2(r-1)} D_4^2 + x_2^{2(p-1)} D_3^2 \\ &\quad + x_2^{2(q-1)} D_4^2 \end{aligned} \tag{1.3}$$

was studied and it was shown that $Q$ is Gevrey $s_0$ hypoelliptic and not better, where

$$\frac{1}{s_0} = \frac{1}{r} + \frac{r-1}{r} \frac{p-1}{q-1}. \tag{1.4}$$

Here for $s \geq 1$ and $U$ an open subset of $\mathbb{R}^4$, we denote by $G^s(U)$ the class of all the Gevrey functions of order $s$ in $U$, i.e. the space of all functions $f$ in $C^\infty(U)$ such that for every compact $K$, $K \Subset U$, there is a positive constant $C_K$ such that, for every multiindex $\alpha \geq 0$ we have

$$\sup_K |\partial^\alpha f(x)| \leq C_K^{|\alpha|+1} \alpha!^s. \tag{1.5}$$

In particular $G^1(U) = C^\omega(U)$.

It is also immediate to see that

$$\mathrm{Char}(Q) = \mathrm{Char}(P).$$

In this note we prove

**Theorem 1.1.** *The operator $P$ in* (1.1) *is analytic hypoelliptic, i.e. if $u$ is a distribution on the open set $\Omega \subset \mathbb{R}^4$ such that $Pu = f$, where $f \in C^\omega(\Omega)$ then $u \in C^\omega(\Omega)$.*

The theorem above as well as the choice of the operator $P$ are worth some explanation.

The operator $Q$ in (1.3) is a counterexample to Treves conjecture. Actually the stratification associated to $Q$ in the statement of the conjecture is made of the sole stratum

$$\mathrm{Char}(Q) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\} = \mathrm{Char}\, P.$$

An inspection of the proof though, which basically consisted in the construction of a solution to $Qu = f$, $f$ real analytic in a neighborhood of the origin, being of class $G^{s_0}$ and not better than that, shows that the real analytic submanifold

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}$$

is important for the Gevrey regularity of $Q$ because of the presence of the vector field $X_5 = x_2^{p-1} D_3$. This remark would lead us to consider the characteristic set $\mathrm{Char}(Q)$ as the disjoint union of the following two analytic strata

$$\Sigma_0 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 \neq 0\},$$
$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}.$$

Actually $\Sigma_1$ is non symplectic and has Hamilton leaves which are the $x_3$ lines where the propagation of the Gevrey–$s_0$ wave front set occurs. Hence we might think of $\Sigma_1$ as a “non Treves stratum” where the existence of Hamilton leaves implies non analytic regularity.

We must make it clear though that, to our knowledge, there is neither a replacement conjecture nor an alternative definition of stratification.

The model operator $P$ is such that, even though almost all the properties of $Q$ as far as the Treves stratification is concerned are retained, the manifold $\Sigma_1$ is replaced by

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, 3, \xi_4 \neq 0\}, \tag{1.6}$$

due to the presence in $P$ of both vector fields $X_5 = x_2^{p-1} D_3$ and $X_6 = x_2^{p-1} x_3^k D_3$. We point out that in this case $\Sigma_1$ is a symplectic submanifold and hence has no Hamilton leaves.

In other words it seems that the analytic regularity of a sum of squares should depend on a suitable stratification of the characteristic variety of the operator and on the fact that its strata are analytic symplectic manifolds.

Unfortunately we cannot be more precise on this at the moment.

## 2. Proof of the Theorem

This section is devoted to the proof of Theorem 1.1. First of all we point out that the operator $P$ satisfies the so called Hörmander condition (see [8]), i.e. by taking the commutators of the vector fields $X_j$ we generate a Lie algebra of dimension 4. On the other hand a microlocal approach to the regularity properties of $P$ is more illuminating.
The characteristic manifold of $P$ is given by

$$\mathrm{Char}(Q) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\}.$$

Let us denote by $U$ a neighborhood of the origin in $\mathbb{R}^4$ and denote by $\Gamma$ a cone around the direction $e_3 = (0, 0, 1, 0)$, so that $U \times \Gamma$ is a conical neighborhood of the point $(0, e_3)$. If the cone $\Gamma$ is close enough to the direction $e_3$ we have that $(x, \xi) \in U \times \Gamma$ implies $\xi_3 \neq 0$. Hence, in the symbol of $P$,

$$\xi_1^2 + \xi_2^2 + x_1^{2(r-1)}(\xi_3^2 + \xi_4^2) + x_2^{2(p-1)}(\xi_3^2 + x_3^{2k}\xi_4^2) + x_2^{2(q-1)}\xi_4^2$$

the terms $x_2^{2(p-1)} x_3^{2k} \xi_4^2$ and $x_2^{2(q-1)} \xi_4^2$ are just harmless perturbations if $x_2$ is close to zero and $x_3$ is bounded, so that the operator is very similar to a sum of anharmonic oscillators in two different variables. It is then well known and not too difficult to show that $P$ is microhypoanalytic at $(0, e_3)$, i.e. that $(0, e_3) \notin WF_a(u)$ if $(0, e_3) \notin WF_a(Pu)$, where $WF_a$ denotes the analytic wave front set as defined in e.g. [10] Def. 8.4.3.

Pursuing further the microlocal analysis above, we are then left with the case $\xi_3 = 0$ and hence $\xi_4 \neq 0$. In order to show that $P$ is analytic hypoelliptic it is enough to prove that $P$ is microhypoanalytic at $(0, e_4)$, i.e. that $(0, e_4) \notin WF_a(u)$ if $(0, e_4) \notin WF_a(Pu)$.

We replace the classical Hörmander condition with its microlocal version, i.e. taking commutators (or Poisson brackets) between the vector fields) we satisfy the condition when we get a vector field microelliptic at the point $(0, e_4)$. We see that this occurs with brackets of length $r$, by just taking $(\mathrm{ad}\, X_1)^{r-1} X_4$.

We may, without loss of generality assume that $u$ is $C^\infty$ at $(0, e_4)$, or that $u$ is smooth due to the Hörmander theorem [8].

The basic idea for the proof of Theorem 1.1 is to use the subelliptic estimate (see e. g. [8] and [4] for the method of proof)

$$\|u\|^2_{\frac{1}{r}} + \sum_{j=1}^{7} \|X_j(x, D)u\|^2 \leq C \left( \langle P(x, D)u, u\rangle + \|u\|^2 \right), \tag{2.1}$$

where $C$ is a positive constant, $\|\cdot\|_{\frac{1}{r}}$ is the Sobolev norm of order $\frac{1}{r}$ and $u \in C_0^\infty(\mathbb{R}^4)$.

Actually we are going to use (2.1) but not the Sobolev subelliptic term, except in the bootstrap stage of our proof, since we want to obtain analytic regularity, so that we may say that the estimate that matters to us is the maximal estimate

$$\sum_{j=1}^{7} \|X_j(x, D)u\|^2 \leq C \left( \langle P(x, D)u, u\rangle + \|u\|^2 \right). \tag{2.2}$$

A further remark is that we may assume $\xi_4 \geq 1$: in fact denoting by $\psi$ a cutoff function such that $\psi \geq 0$, $\psi(\xi_4) = 1$ if $\xi_4 \geq 2$ and $\psi(\xi_4) = 0$ if $\xi_4 \leq 1$, we may apply $\psi(D_4)$ to the equation $Pu = f$ getting $P\psi u = \psi f$, since $\psi$ commutes with $P$. On the other hand $\psi f \in G^s$ if $f \in G^s$, for $s \geq 1$, and we are interested in the microlocal regularity of $u$ at the point $(0; e_4)$. We write $u$ instead of $\psi u$.
Moreover if $\theta = \theta(x_1, x_2)$ is a smooth compactly supported function which is identically 1 in a neighborhood of the origin, we have that $Pu = f$ implies $P(\theta u) = \theta f - [P, \theta]u$. Now the estimates of the wave front set of $\theta f$ near the origin are the same as those of $f$ and the term containing the commutator is by definition identically zero in a neighborhood of the origin. Thus $(0, e_4) \notin WF_a(\theta f - [P, \theta]u)$. We keep writing $u$ instead of $\theta u$.

Throughout the proof we will use a particular type of cutoff functions defined e.g. in Ehrenpreis [7] (see also Hörmander [10]). For the sake of completeness, we include the definition.

**Definition 2.1.** *For any $N$ natural number, denote by $\varphi_N = \varphi_N(y)$ a function in $C_0^\infty(\mathbb{R}^m)$. We say that $\varphi_N$ is an Ehrenpreis sequence of cutoff functions if there is a positive constant $R$ such that for $|\alpha| \leq RN$ we have, for every $N$*

$$|\partial_y^\alpha \varphi_N(y)| \leq C_\varphi^{|\alpha|+1} N^{|\alpha|},$$

*where $C_\varphi > 0$ and independent of $N$.*

We remark that in [10] an explicit construction of such type of cutoffs is exhibited. Let $\varphi_N = \varphi_N(x_3, x_4) \in C_0^\infty(\mathbb{R}^2)$ denote an Ehrenpreis type cutoff function with support near the origin, $N \in \mathbb{N}$. Our aim is to prove the estimates, for any $s, \ell \in \mathbb{N} : s + \ell \leq N$

$$\|X_j \varphi_N^{(s)} D_4^\ell u\| \leq C^{s+\ell+1} N^{s+\ell}, \quad j = 1, \ldots, 7, \tag{2.3}$$

where $\varphi_N^{(s)}$ denotes a self-adjoint derivative of order $s$ of $\varphi_N$ and $C$ is a positive constant independent of $s, \ell, j, N$.
From (2.3) it follows that $u$ is analytic microlocally near the point $(0, e_4)$. Indeed, summing up such estimates for $j = 1, ..., 7$ we get

$$\begin{aligned}\langle P(x,D)\varphi_N^{(s)} D_4^\ell u, \varphi_N^{(s)} D_4^\ell u\rangle = \sum_{j=1}^{7} \|X_j(x,D)\varphi_N^{(s)} D_4^\ell u\|^2 \\ \leq C^{2(s+\ell+1)} N^{2(s+\ell)},\end{aligned}$$

whence, in view of (2.1), we have that

$$\|\varphi_N^{(s)} D_4^\ell u\|_{1/r} \leq C^{s+\ell+1} N^{s+\ell}, \quad \text{for any } s + \ell \leq N,$$

and finally we get that, for a new constant $C > 0$ independent of $N$,

$$\|D_4^\ell(\varphi_N u)\|_{1/r} \leq C^{\ell+1} N^\ell, \quad \text{for every } N \text{ with } \ell \leq N.$$

In view of the Sobolev imbeddings, it turns out that $u$ is analytic microlocally near the point $(0, e_4)$ (see Def. 8.4.2 [10]).

Note that in the above argument, the role of $N$ is "meaningful" if $N$ is large; indeed, the inequality (2.3) for bounded values of $N$ is a trivial consequence of the $C^\infty$-smoothness of $u$.

Instead of bounding the quantity in (2.3), for technical reasons it is useful to prove the following more general estimate:

$$\|\varphi_N^{(s)} D_4^\ell u\|_{1/r} + \max_{j=1,\ldots,7} \|X_j \varphi_N^{(s)} D_4^\ell u\| \leq C_1^{\ell+1} C_2^s N^{s+\ell}, \tag{2.4}$$

with $s, \ell \in \mathbb{N} : s + \ell \leq N$ and $C_1, C_2$ are positive constants independent of $N$. We are going to prove (2.4) proceeding by induction on $s + \ell = M \leq N$.

The estimate for $s + \ell = 0$ is trivial since $\varphi_N u$ is $C_0^\infty$.

Assume now that (2.4) is true if $s + \ell < H$, we have to show that (2.4) holds for $s + \ell = H$.

To this end we proceed again by induction on $\ell$. If $\ell = 0$, the inequality (2.4) is a straightforward consequence of our choice of the cut-off functions $\varphi_N$ (see Def. 2.1). It is enough to show that (2.4) holds if $\ell = M \leq H, s = H - M$, i.e. once we have supposed that (2.4) is true for $\ell < M$ and $s + \ell \leq H$. In order to make the proof more readable, we choose $M = N$ so that $s = 0$; the general case $H < N$ does not present extra difficulties. Hence we are reduced to prove

$$\|\varphi_N D_4^N u\|_{1/r} + \max_{j=1,\dots,7} \|X_j \varphi_N D_4^N u\| \le C_1^{N+1} N^N, \tag{2.5}$$

once we know that, for $\ell < N$ and $s + \ell \le N$,

$$\|\varphi_N^{(s)} D_4^\ell u\|_{1/r} + \max_{j=1,\dots,7} \|X_j \varphi_N^{(s)} D_4^\ell u\| \le C_1^{\ell+1} C_2^s N^{s+\ell}. \tag{2.6}$$

To this purpose, for technical reasons, it is convenient to work with the square of the norms in (2.5). Using (2.1) we may write

$$\|\varphi_N D_4^N u\|_{1/r}^2 + \max_{j=1,\dots,7} \|X_j \varphi_N D_4^N u\|^2 \le C(\langle P\varphi_N D_4^N u, \varphi_N D_4^N u\rangle + \|\varphi_N D_4^N u\|^2). \tag{2.7}$$

We start off by showing that the error term $\|\varphi_N D_4^N u\|^2$ can be actually absorbed in the l.h.s. of (2.7), modulo a term with an analytic growth estimate. To this end, denote by $\chi$ a smooth cutoff function such that $\chi(t) = 1$ if $|t| \ge 2$ and $\chi(t) = 0$ if $|t| \le 1$. It turns out that $\chi(N^{-1}D_4) \in OPS^0_{0,0}$ (see Def. 3.1 in Appendix) and then

$$\|\varphi_N D_4^N u\| \le \|(1 - \chi(N^{-1}D_4))\varphi_N D_4^N u\| + \|\chi(N^{-1}D_4)\varphi_N D_4^N u\|. \tag{2.8}$$

The first summand can be easily estimated because of the support of the cutoff $\chi$. In doing that, we cannot use the standard composition formula in the classes $S^m_{0,0}$, so we proceed in a slight indirect way. We apply the transposed Leibniz formula (see [16] (0.3))

$$\varphi_N D_4^N u = \sum_{s=0}^{N} (-1)^s \binom{N}{s} D_4^{N-s}\big(\varphi_N^{(s)} u\big)$$

whence we get

$$\|(1 - \chi(N^{-1}D_4))\varphi_N D_4^N u\| \le \sum_{s=0}^{N} \binom{N}{s} \|(1 - \chi(N^{-1}D_4)) D_4^{N-s}\big(\varphi_N^{(s)} u\big)\|. \tag{2.9}$$

A direct computation shows that

$$\sigma\big((1 - \chi(N^{-1}D_4))D_4^{N-s}\big) = (1 - \chi(N^{-1}\xi_4))\xi_4^{N-s} \in S^0_{0,0}$$

since $N^{-1}|\xi_4| \le 2$ on the support of $1 - \chi$; furthermore, it is straightforward to see that its $S^0_{0,0}$-semi-norms (see (3) in Appendix) satisfy

$$|\sigma\big((1 - \chi(N^{-1}D_4))D_4^{N-s}\big)|_\ell^{(0)} \le C^{N-s+1} N^{N-s},$$

where $C > 0$ is a suitable constant independent of $N$. It is important to note that the integer $\ell$ , in the semi-norms above, does not depend on $N$.
From the Calderón-Vaillancourt theorem (see Thm. (3.1)) it readily follows that, for a new positive constant $C > 0$,

$$\|(1 - \chi(N^{-1}D_4))D_4^{N-s}\|_{\mathcal{L}(L^2,L^2)} \leq C^{N-s+1}N^{N-s},$$

whence, in view of the growth properties of the cutoff $\varphi_N$, in (2.9) we obtain

$$\begin{aligned}\|(1 - \chi(N^{-1}D_4))\varphi_N D_4^N u\| &\leq \sum_{s=0}^{N} \binom{N}{s} C^{N-s+1}N^{N-s}\|\varphi_N^{(s)}u\| \\ &\leq C^{N+2}N^N\|u\| \sum_{s=0}^{N} \binom{N}{s} \leq C_1^{N+1}N^N,\end{aligned}$$

which is an analytic growth estimate.
Thus we are left with the estimate of the second summand in the r.h.s. of (2.8). We have that

$$\|\chi(N^{-1}D_4)\varphi_N D_4^N u\| = N^{-1/r}\|N^{1/r}\chi(N^{-1}D_4)D^{-1/r} \circ D^{1/r}\varphi_N D_4^N u\|$$

where $D^s = Op\big((1 + |\xi|^2)^{s/2}\big)$ for any $s \in \mathbb{R}$. Due to the support of the cutoff $\chi$, we see that

$$\sigma\big(N^{1/r}\chi(N^{-1}D_4)D^{-1/r}\big) = N^{1/r}\chi(N^{-1}\xi_4)(1 + |\xi|^2)^{-1/2r} \in S^0_{0,0}$$

with the $S^0_{0,0}$-semi-norms uniformly bounded on $N$; thus from the Calderón-Vaillancourt theorem it follows that

$$\|N^{1/r}\chi(N^{-1}D_4)D^{-1/r}\|_{\mathcal{L}(L^2,L^2)} \leq C$$

$C$ being a positive constant independent on $N$. Finally, we obtain

$$\begin{aligned}\|\chi(N^{-1}D_4)\varphi_N D_4^N u\| &\leq CN^{-1/r}\|D^{1/r}\varphi_N D_4^N u\| \\ &\leq CN^{-1/r}\|\varphi_N D_4^N u\|_{1/r}.\end{aligned}$$

Summing up we get that

$$\|\varphi_N D_4^N u\| \leq CN^{-1/r}\|\varphi_N D_4^N u\|_{1/r} + C^{N+1}N^N. \tag{2.10}$$

By using the above estimate in (2.7), the term $CN^{-1/r}\|\varphi_N D_4^N u\|_{1/r}$ can be absorbed in the l.h.s. of (2.7) provided $N$ be chosen large enough and this yields, for a new constant $C > 0$,

$$\|\varphi_N D_4^N u\|^2_{1/r} + \max_{j=1,\dots,7} \|X_j \varphi_N D_4^N u\|^2 \leq C \langle P\varphi_N D_4^N u, \varphi_N D_4^N u\rangle + C^{2(N+1)} N^{2N}. \tag{2.11}$$

We are thus left with the term containing the scalar product. It can be written as

$$\begin{aligned} \langle P\varphi_N D_4^N u, \varphi_N D_4^N u\rangle &= \langle \varphi_N D_4^N Pu, \varphi_N D_4^N u\rangle + \langle [P, \varphi_N D_4^N] u, \varphi_N D_4^N u\rangle \\ &= \langle \varphi_N D_4^N Pu, \varphi_N D_4^N u\rangle + \sum_{j=1}^{7} \langle [X_j^2, \varphi_N] D_4^N u, \varphi_N D_4^N u\rangle \\ &\leq \frac{1}{\delta^2} \|\varphi_N D_4^N Pu\|^2 + \delta^2 \|\varphi_N D_4^N u\|^2 + \sum_{j=1}^{7} \langle [X_j^2, \varphi_N] D_4^N u, \varphi_N D_4^N u\rangle. \end{aligned} \tag{2.12}$$

The second term above can be again absorbed in the l.h.s. of (2.11) if $\delta$ is sufficiently small, whereas the first one is easily bound because we know that $(0, e_4) \notin WF_a(Pu)$; it yields an estimate by $C^{2(N+1)} N^{2N}$, for a suitable positive constant independent of $N$. Therefore from (2.11) and (2.12) it follows that

$$\|\varphi_N D_4^N u\|^2_{1/r} + \max_{j=1,\dots,7} \|X_j \varphi_N D_4^N u\|^2 \leq C \sum_{j=1}^{7} \langle [X_j^2, \varphi_N] D_4^N u, \varphi_N D_4^N u\rangle + C^{2(N+1)} N^{2N}. \tag{2.13}$$

We have to examine then the term containing the commutators.
We have

$$\begin{aligned} & \langle [X_j^2, \varphi_N] D_4^N u, \varphi_N D_4^N u\rangle \\ = \;& \langle [X_j, \varphi_N] D_4^N u, X_j \varphi_N D_4^N u\rangle - \langle X_j D_4^N u, [X_j, \varphi_N] \varphi_N D_4^N u\rangle \\ = \;& \langle [X_j, \varphi_N] D_4^N u, X_j \varphi_N D_4^N u\rangle - \langle X_j \varphi_N D_4^N u, [X_j, \varphi_N] D_4^N u\rangle \\ & + \langle [X_j, \varphi_N] D_4^N u, [X_j, \varphi_N] D_4^N u\rangle \\ = \;& 2i \operatorname{Im} \langle [X_j, \varphi_N] D_4^N u, X_j \varphi_N D_4^N u\rangle + \|[X_j, \varphi_N] D_4^N u\|^2 \\ = \;& \|[X_j, \varphi_N] D_4^N u\|^2. \end{aligned} \tag{2.14}$$

Here we used the fact that the scalar product $\langle [X_j, \varphi_N] D_4^N u, X_j \varphi_N D_4^N u\rangle$ is real.

Therefore, from (2.13) it follows that

$$\|\varphi_N D_4^N u\|_{1/r}^2 + \max_{h=1,\dots,7} \|X_h \varphi_N D_4^N u\|^2 \leq C \sum_{j=1}^{7} \|[X_j, \varphi_N] D_4^N u\|^2 + C^{2(N+1)} N^{2N}. \tag{2.15}$$

For $j = 1, 2$ the commutators in the r.h.s. of the above inequality is zero since $\varphi_N$ does not depend on the $(x_1, x_2)$ variables.

Let us start by considering the terms with $j = 3, 4$; we have

$$[X_j, \varphi_N] D_4^N u = x_1^{r-1} \varphi'_N D_4^N u.$$

For $j = 7$ we also get

$$[X_7, \varphi_N] D_4^N u = x_2^{q-1} \varphi'_N D_4^N u$$

while when $j = 5, 6$ we have

$$[X_5, \varphi_N] D_4^N u = x_2^{p-1} \varphi'_N D_4^N u$$

and

$$[X_6, \varphi_N] D_4^N u = x_2^{p-1} x_3^k \varphi'_N D_4^N u.$$

Let us consider the terms corresponding to $j = 3, 4$ first

$$\|[X_j, \varphi_N] D_4^N u\|^2 = \|x_1^{r-1} \varphi'_N D_4^N u\|^2$$

In order to apply the inductive hypothesis (2.6) to the identity above we use the formula

$$\varphi'_N D_4^N = \sum_{j=0}^{N-1} (-1)^j D_4 \varphi_N^{(j+1)} D_4^{N-j-1} + (-1)^N \varphi_N^{(N+1)}. \tag{2.16}$$

and we get

$$\begin{aligned} \varphi'_N x_1^{r-1} D_4^N &= \sum_{j=0}^{N-1} (-1)^j x_1^{r-1} D_4 \varphi_N^{(j+1)} D_4^{N-j-1} + (-1)^N x_1^{r-1} \varphi_N^{(N+1)}, \\ &= \sum_{j=0}^{N-1} (-1)^j X_4 \varphi_N^{(j+1)} D_4^{N-j-1} + (-1)^N x_1^{r-1} \varphi_N^{(N+1)}. \end{aligned}$$

Thus,

$$\|x_1^{r-1} \varphi'_N D_4^N u\| \ \leq \ \sum_{j=0}^{N-1} \|X_4 \varphi_N^{(j+1)} D_4^{N-j-1} u\| \ + \ C\|\varphi_N^{(N+1)} u\|, \tag{2.17}$$

where we used the fact that the field $X_4$ could be reconstructed by just "pulling back" one $x_4$-derivative and that $x_1^{r-1}$ is bounded on the support of $u$. Since $\varphi_N$ is an Ehrenpreis sequence of cutoff functions (see Def. 2.1) we get, possibly enlarging the constant $C$ in Definition 2.1, the bound

$$\|\varphi_N^{(N+1)}u\| \le C^{N+1}N^N,$$

and applying the inductive hypothesis (2.6) to the estimate (2.17) yields

$$\begin{aligned}
\|x_1^{r-1}\varphi_N' D_4^N u\| &\le \sum_{j=0}^{N-1} C_1^{N-j}C_2^{j+1}N^N + C^{N+1}N^N \\
&\le C_1^{N+1}N^N\left(\left(\frac{C}{C_1}\right)^{N+1} + \sum_{j=0}^{N-1}\left(\frac{C_2}{C_1}\right)^{j+1}\right) \\
&\le C_1^{N+1}N^N\sum_{j=0}^{N}\left(\frac{C_2}{C_1}\right)^{j+1} \le \frac{\varepsilon}{1-\varepsilon}C_1^{N+1}N^N \le 2\varepsilon C_1^{N+1}N^N.
\end{aligned}$$

Here we have chosen $C_1, C_2$ in such way that $C < C_2 < \varepsilon C_1$ for an arbitrary small positive constant $\varepsilon < 1/2$. This completes the analysis of the term on the r.h.s. of (2.14) if $j = 3, 4$.

A completely analogous treatment leads to an analogous conclusion when $j = 6, 7$. Let us briefly recall the main steps. We have that

$$\|[X_6, \varphi_N]D_4^N u\|^2 + \|[X_7, \varphi_N]D_4^N u\|^2 = \\ \|x_2^{p-1}x_3^k\varphi_N' D_4^N u\|^2 + \|x_2^{q-1}\varphi_N' D_4^N u\|^2,$$

and using again (2.16) yields

$$\begin{aligned}
\|x_2^{p-1}x_3^k\varphi_N' D_4^N u\| &\le \sum_{j=0}^{N-1}\|X_6\varphi_N^{(j+1)}D_4^{N-j-1}u\| + C\|\varphi_N^{(N+1)}u\|, \\
\|x_2^{q-1}\varphi_N' D_4^N u\| &\le \sum_{j=0}^{N-1}\|X_7\varphi_N^{(j+1)}D_4^{N-j-1}u\| + C\|\varphi_N^{(N+1)}u\|.
\end{aligned}$$

Furthermore it is clear that the terms on the right of the above inequalities yield a real analytic growth estimate, after using the properties of $\varphi_N$ and the inductive hypothesis (2.6) as done before.

We are thus left with the term for $j = 5$ in (2.15). This term requires a special treatment; precisely, we need to reconstruct a vector field in order to replace the vector field destroyed by the action of the commutator.

To this end, we preliminarily point out that it is easier to choose $\varphi_N = \varphi_N(x_3, x_4)$ as a product of two functions in $x_3$ and $x_4$ respectively: $\varphi_N(x_3, x_4) = \omega_N(x_3)\omega_N(x_4)$, where $\omega_N$ is in turn an Ehrenpreis type cutoff function (see Def. 2.1 with $m = 1$) which is identically 1 in a neighborhood of the origin. Taking the same function in $x_3$ as well as in $x_4$ is not really a big deal since we may always shrink the neighborhood of the origin to a square region. Thus we have that

$$\begin{aligned}[X_5, \varphi_N] D_4^N u &= [x_2^{p-1} D_3, \omega_N(x_3)\omega_N(x_4)] D_4^N u \\ &= x_2^{p-1} \omega_N'(x_3)\omega_N(x_4) D_4^N u.\end{aligned}$$

To estimate the second term in the inequality above we are going to use formula (2.16) in order to pull back the $x_4$ derivative. We may write

$$\begin{aligned}\|x_2^{p-1} \omega_N'(x_3)\omega_N(x_4) D_4^N u\| &\leq C' \|x_2^{p-1} x_3^k \omega_N'(x_3)\omega_N(x_4) D_4^N u\| \\ &\leq C' \sum_{j=0}^{N-1} \|X_6 \omega_N'(x_3) \omega_N^{(j)}(x_4) D_4^{N-j-1} u\| \\ &\qquad\qquad + C'' \|\omega_N'(x_3)\omega_N^{(N)}(x_4) u\|.\end{aligned}$$

The first inequality is due to the fact that $\omega_N'(x_3)$ is identically zero in a neighborhood of the origin in $x_3$, so that dividing by a power of $x_3$ is estimated by a suitable constant. Note that this argument strongly depends on the fact that $x_3$ and $\xi_3$ is a couple of symplectically conjugated variables. This fact emphasizes the role played by the symplectic stratum $\Sigma_1$ proposed in (1.6).
We stress that all the terms above yield an analytic growth rate in view of the growth properties of $\omega_N$ (see Def. 2.1) and the inductive hypothesis (2.6).

Summing up we conclude that all terms coming from commutators in (2.15) have analytic growth rate. This achieves the proof of (2.4), which implies the microhypoanalyticity of $P$ at $(0, e_4)$ .
Theorem 1.1 is thus proved.

## 3. Appendix

For the sake of completeness we recall here some well-known facts used throughout the paper.

**Definition 3.1.** *For any $m \in \mathbb{R}$, $\rho, \delta \in \mathbb{R}$ with $0 \leq \delta \leq \rho \leq 1, \delta < 1$, we denote by $S^m_{\rho,\delta}$ the set of all the functions $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that for every multi-index $\alpha, \beta$ there exits a positive constant $C_{\alpha,\beta}$ for which*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

*where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.*

*We denote by $OPS^m_{\rho,\delta}$ the class of the corresponding pseudodifferential operators $P = p(x,D)$ .*

It is trivial to see that the symbol class $S^m_{\rho,\delta}$ equipped with the seminorms

$$|p|^{(m)}_\ell = \max_{|\alpha+\beta|\leq \ell} \sup_{(x,\xi)} \{ |\partial^\alpha_\xi \partial^\beta_x p(x,\xi)| \langle \xi \rangle^{-(m-\rho|\alpha|+\delta|\beta|)} \}, \quad \ell \in \mathbb{N}$$

is a Fréchet space.

The Calderón-Vaillancourt theorem shows the $L^2$-continuity properties of the pseudodifferential operators in the above classes (see [6] or, for a more general setting, [12] Chap. 7, Th.1.6). We state below a formulation of such a theorem for pseudodifferential operators of order zero.

**Theorem 3.1** (Calderón-Vaillancourt). *Let $P = p(x, D) \in OPS^0_{\rho,\delta}$ with $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. Then there exist a positive integer $\ell$ and a positive constant $M$ (depending only on $n$) such that*

$$\|Pu\| \leq M |p|^{(0)}_\ell \|u\|, \quad \textit{for every } u \in L^2(\mathbb{R}^n).$$

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Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna Italy

*E-mail address*: `bove@bo.infn.it`

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna Italy

*E-mail address*: `marco.mughetti@unibo.it`