Robust Global Asymptotic Stabilization of Linear Cascaded Systems With Hysteretic Interconnection

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***Abstract*—We address the problem of setpoint regulation for cascaded minimum-phase linear systems interconnected through a scalar hysteresis, modeled as a Prandtl-Ishlinskii operator. Employing well-posed constrained differential inclusions to represent the hysteretic dynamics, we formulate the control problem in terms of stabilization of a compact set of equilibria depending on the hysteresis states. For our design, we firstly consider a proportional-integral controller for linear systems with hysteretic input, and provide model-free sufficient conditions based on high-gain arguments for closed-loop stability. Then, the controller is dynamically extended to obtain an inversion-free stabilizer of the overall cascade. For the presented schemes, we prove robust global asymptotic stability of a compact set that ensures setpoint regulation, regardless of the hysteresis states.**

***Index Terms*—Hysteresis, Lyapunov methods, Prandtl-Ishlinskii operator, switched systems.**

I. Introduction

SMART materials are becoming widespread nowadays due to their unique sensing and actuation capabilities [1], leading to innovative solutions in several domains such as next-generation mechatronic technologies. For control systems involving smart materials, one of the major challenges is the typical presence of hysteretic behaviors, causing performance deterioration if not correctly addressed. In this context, the control-theoretical literature has dedicated several works to hystereses represented with the Prandtl-Ishlinskii (PI) operator [2], particularly useful given the existence of its analytical inverse [3]. Numerous control strategies have been developed based on the explicit inversion of the PI operator [4]–[6].

Manuscript received 21 March 2022; revised 1 June 2022; accepted 20 June 2022. Date of publication 6 July 2022; date of current version 18 July 2022. Recommended by Senior Editor J. Daafouz. *(Corresponding author: Alessandro Bosso.)*

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Digital Object Identifier 10.1109/LCSYS.2022.3188748

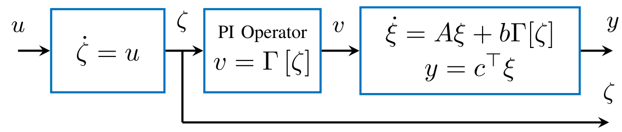

Fig. 1. Block representation of system (1).

However, the computational burden or the inversion inaccuracy has led to inversion-free or implicit inversion solutions, involving integral control [7], [8], adaptive control [9], or adaptive conditional servo-compensation [10].

All the above works consider systems with hysteretic inputs, while [8] also considers a feedback-interconnected hysteresis. In this note, instead, we address inversion-free setpoint regulation for linear cascaded systems where a finite-dimensional PI operator affects the interconnection. This problem is inspired by operator-based models of thermal shape memory alloys (SMAs), comprising a thermal and a mechanical subsystem where the temperature influences the elastic behavior through a hysteresis [11], [12]. Although highly nonlinear and coupled models typically describe thermal SMAs, we believe that the methodologies developed here are instrumental in addressing those systems, paving the way towards robust controllers for a large class of smart actuators. Specifically, we consider a cascade where the system affecting the hysteresis input is an integrator (see Fig. 1). This choice is made to streamline the presentation, as the proposed strategy is straightforwardly extended to any linear system that can be robustly stabilized by output feedback. On the other hand, we require that the system with hysteretic input has minimum phase and relative degree 1, while we aim to relax this assumption in future works.

Our design is based on equivalently writing the PI operator as a sum of stop operators, modeled as a well-posed differential inclusion constrained in a compact set [13, Ch. 2]. In this setting, we develop a setpoint regulation framework based on the analysis tools for hybrid dynamical systems [14]. We firstly address the scenario where the hysteresis input can be assigned as a control input. In that context, we show that a proportional controller ensures practical regulation, whereas including an integral action enables global exponential stabilization. As compared to the integrator-based approaches of [7], [8], we do not rely on model-based LMIs to tune the controller. Instead, we provide high-gain arguments ensuring

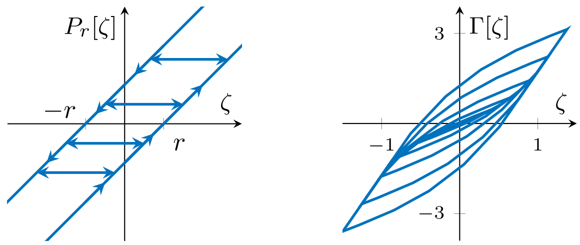

Fig. 2. Left: representation of a play operator. Right: behavior of a PI operator implemented as in (5), (6) with input $\zeta(t) = 0.04t\sin(t)$.

quadratic stability without any knowledge of the system matrices and of the hysteresis. The proportional-integral controller is then combined with a high-gain filter of the regulation error to obtain the design for the overall cascade. For the closed-loop dynamics, we prove the existence of a globally robustly asymptotically stable attractor ensuring setpoint regulation, regardless of the hysteresis states.

This letter is organized as follows. In Section II, we introduce the considered class of systems, present the state-space hysteresis model, and define the control problem. Section III is dedicated to the control design for linear systems with hysteretic input. Then, Section IV provides the overall controller and the main stability result. Finally, Section V reports numerical results and Section VI concludes this letter.

II. Linear Cascaded Systems With Hysteretic Interconnection

A. Model Description

We consider a class of systems comprising two linear subsystems, having states $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$, interconnected through a scalar hysteresis $\Gamma[\cdot]$, as shown in Fig. 1:

$$\begin{aligned}\dot{\xi} &= A\xi + b\Gamma[\zeta] \\ \dot{\zeta} &= u\end{aligned} \qquad y_a := \begin{bmatrix} y \\ \zeta \end{bmatrix} := \begin{bmatrix} c^\top \xi \\ \zeta \end{bmatrix}, \tag{1}$$

with control input $u \in \mathbb{R}$ and available output $y_a \in \mathbb{R}^2$. In (1), matrix A and vectors b and c have appropriate dimensions, while hysteresis $\Gamma[\cdot]$ has been indicated in square brackets to highlight its intrinsically dynamic behavior.

For plant (1), we design a controller ensuring asymptotic setpoint regulation of constant reference signals for the output y of the ξ-subsystem, without any knowledge of matrices A, b, and c (except for the sign of $c^\top b$) or of the hysteresis $\Gamma[\cdot]$. This objective is addressed under the following standard assumptions [15, Ch. 2].

Assumption 1: Pair (A, b) is reachable and pair $(c^\top, A)$ is observable.

Assumption 2: System $\dot{\xi} = A\xi + bv$ with output $y = c^\top\xi$ has relative degree 1 and is minimum phase. Furthermore, it holds that $c^\top b > 0$.

B. Prandtl-Ishlinskii Operator

We model hysteresis $\Gamma[\cdot]$ in (1) as a finite-dimensional Prandtl-Ishlinskii (PI) operator, given by a weighted sum of basic hysteresis functions known as play operators. Specifically, following [16] and [7], for any continuous input $\zeta(\cdot)$ that is monotone in each interval $t \in [t_{i-1}, t_i]$ of the partition $0 = t_0 < t_1 < \ldots < t_j = T$, the output $\phi(\cdot)$ of a play operator $P_r[\cdot]$, with constant radius $r \geq 0$, is given by:

$$\phi(t) = \max\{\min\{\zeta(t) + r, \phi(t_{i-1})\}, \zeta(t) - r\}. \tag{2}$$

See Fig. 2-Left for a depiction of the play operator. The finite-dimensional Prandtl-Ishlinskii operator is then defined as [7]:

$$\Gamma[\zeta] := \mu_0\zeta + \sum_{i=1}^{p} \mu_i P_{r_i}[\zeta], \tag{3}$$

where $P_{r_i}[\cdot]$, $i \in \{1, \ldots, p\}$, are play operators with constant radii r_i, while μ_i, $i \in \{0, \ldots, p\}$, are constant scalar weights. Note that the linear term multiplying μ_0 corresponds to the output of a play operator with radius $r_0 = 0$. For simplicity, we consider the case where the radii satisfy $0 < r_1 < \ldots < r_p$, while for weights μ_i we require the following property, ensuring controllability of the cascade in Fig. 1.

Assumption 3: The weights of the Prandtl-Ishlinskii operator in (3) satisfy $\mu_i > 0$, for all $i \in \{0, \ldots, p\}$.

Loosely speaking, Assumption 3 ensures that the slope of the input-output behavior of $\Gamma[\zeta]$ is positive, as shown in the example of Fig. 2-Right. This property is reasonable for hysteretic behaviors without saturations. However, since saturations might occur for large inputs in several applications, we are planning to relax Assumption 3 in future works.

C. Representation via Constrained Differential Inclusions

We provide a state-space representation of the PI operator (3) based on the reformulation proposed in [13, Sec. 2.3] of the play operator as a well-posed constrained differential inclusion. Specifically, we employ the complement of the play operator $P_r[\cdot]$, called stop operator and denoted by $S_r[\cdot]$. Indeed, for all $i \in \{1, \ldots, p\}$ and under an exact one-to-one mapping of the operators initial conditions, it holds that $P_{r_i}[\zeta] + S_{r_i}[\zeta] = \zeta$, where $\delta_i := S_{r_i}[\zeta]$ has a behavior modeled as the regularization of a differential equation with discontinuous right-hand side [13, eq. (2.8)]:

$$\begin{aligned}\dot{\delta}_i &\in \begin{cases} \overline{\text{co}}\{\dot{\zeta}, \max\{\dot{\zeta}, 0\}\}, & \text{if } \delta_i = -r_i, \\ \dot{\zeta}, & \text{if } \delta_i \in (-r_i, r_i), \\ \overline{\text{co}}\{\dot{\zeta}, \min\{\dot{\zeta}, 0\}\}, & \text{if } \delta_i = r_i, \end{cases} \\ \delta_i &\in \Delta_i := [-r_i, r_i],\end{aligned} \tag{4}$$

where $\overline{\text{co}}\{\cdot\}$ denotes the closed convex hull operation. System (4) can be rewritten in compact form as $\dot{\delta}_i \in F_{r_i}(\delta_i, \dot{\zeta})$, $\delta_i \in \Delta_i$, where the set-valued map $F_{r_i}(\delta_i, \dot{\zeta})$ is outer semicontinuous, locally bounded relative to $\Delta_i \times \mathbb{R}$, and for each $(\delta_i, \dot{\zeta}) \in \Delta_i \times \mathbb{R}$ it is nonempty and convex. These properties will lead to a closed-loop system whose data satisfy the so-called hybrid basic assumptions [14, Assumption 6.5]. Under these assumptions, global asymptotic stability of a compact set implies robust global $\mathcal{KL}$ asymptotic stability in the presence of fairly general perturbations of the dynamics [14, Definition 7.18]. As compared to a classical switched dynamics formulation of the stop operator [13, eq. (2.6)], the regularized version (4) does not generate additional solutions in the sense of [14, Definition 2.6]. From (4), $\Gamma[\zeta]$ in (3) can be equivalently defined through stop operators, leading to

$$\dot{\delta} \in F_\Gamma(\delta, \dot{\zeta}) := \begin{bmatrix} F_{r_1}(\delta_1, \dot{\zeta}) \\ \vdots \\ F_{r_p}(\delta_p, \dot{\zeta}) \end{bmatrix}, \qquad \delta \in \Delta := \prod_{i=1}^{p} \Delta_i, \tag{5}$$

having state $\delta := (\delta_1, \dots, \delta_p)$, which constitutes the hysteresis internal memory, and output map

$$\Gamma[\zeta] := \sum_{i=0}^{p} \mu_i \zeta - \sum_{i=1}^{p} \mu_i \delta_i = \gamma\zeta - \mu^\top \delta, \quad \gamma := \sum_{i=0}^{p} \mu_i, \tag{6}$$

where $\mu := [\mu_1 \cdots \mu_p]^\top$ is a constant vector of positive parameters. Note that $\gamma > 0$ in (6) by Assumption 3.

The state-space representation of system (1) is given by the following well-posed constrained differential inclusion:

$$\begin{aligned} \dot\xi &= A\xi + b(\gamma\zeta - \mu^\top\delta) \\ \dot\delta &\in F_\Gamma(\delta, u) \\ \dot\zeta &= u \end{aligned} \qquad \begin{bmatrix} \xi \\ \delta \\ \zeta \end{bmatrix} \in \mathbb{R}^n \times \Delta \times \mathbb{R}. \tag{7}$$

D. Problem Statement

For systems expressed in the form (7), we consider the problem of output-feedback setpoint regulation of constant references for the output $y := c^\top\xi$. More specifically, our objective is to design an output feedback control law for u, based on the available signals $y_a := (c^\top\xi, \zeta)$ and under Assumptions 1, 2, and 3, such that:

$$\lim_{t\to\infty} y(t) = y^\star, \tag{8}$$

for any constant reference $y^\star \in \mathbb{R}$. We intend to achieve this objective through robust asymptotic stabilization of a compact set depending on $y^\star$, so that suitable stability properties will be ensured in addition to (8). Among other things, stability will imply uniformity of the convergence (8). To this aim, we inspect the effect of δ on the equilibria of (7) ensuring $y = y^\star$. By Assumptions 1 and 2, the system of equations

$$A\xi + bv = 0, \quad c^\top\xi - y^\star = 0, \tag{9}$$

admits a unique equilibrium pair $(\xi^\star, v^\star)$. To evaluate the steady-state value for ζ, we consider then equation

$$\gamma\zeta - \mu^\top\delta = v^\star, \tag{10}$$

which can be solved because $\gamma > 0$ by Assumption 3, but whose solution is not unique due to the presence of the hysteresis state δ. More precisely, all equilibria of (7) satisfying (10) correspond to elements of the compact set

$$\mathcal{E}_\delta := \{(\delta, \zeta) \in \Delta \times \mathbb{R} : \gamma\zeta = \mu^\top\delta + v^\star\}. \tag{11}$$

Note that $\mathcal{E}_\delta$ is a set of equilibria for the (δ, ζ)-subsystem because, by Assumption 3 and $\dot\delta_i \in \overline{\mathrm{co}}\{0, \dot\zeta\}$, the definitions of γ and μ in (6) (with $\mu_0 > 0$) imply that $(\delta, \zeta) \in \mathcal{E}_\delta$ (satisfying $\gamma\dot\zeta = \mu^\top\dot\delta$) holds only if $\dot\delta = 0$ and $\dot\zeta = 0$. Therefore, we can reformulate goal (8) by ensuring appropriate attractivity properties of the following compact *regulation set*:

$$\begin{aligned} \mathcal{E} := \{&(\xi, \delta, \zeta) \in \mathbb{R}^n \times \Delta \times \mathbb{R} : \\ &A\xi + b(\zeta - \mu^\top\delta) = 0, c^\top\xi = y^\star\} = \{\xi^\star\} \times \mathcal{E}_\delta, \end{aligned} \tag{12}$$

for which we will also achieve a suitable stability property. We are ready to precisely state the control problem.

Problem 1: Under Assumptions 1, 2, 3, design a controller for the input u such that the closed-loop system admits a compact attractor that is robustly globally $\mathcal{KL}$ asymptotically stable in the sense of [14, Definition 7.18] and whose projection in the plant state directions coincides with the regulation set (12).

III. Setpoint Regulation for Systems With Hysteretic Input

This section considers a simplified scenario wherein ζ of (7) can be assigned algebraically as a differentiable control input (so that $\dot\zeta$ is well defined). Namely, we address the setpoint regulation objective $y = c^\top\xi \to y^\star$ in (8) for system

$$\begin{aligned} \dot\xi &= A\xi + b(\gamma\zeta - \mu^\top\delta) \\ \dot\delta &\in F_\Gamma(\delta, \dot\zeta) \end{aligned} \qquad \begin{bmatrix} \xi \\ \delta \end{bmatrix} \in \mathbb{R}^n \times \Delta, \tag{13}$$

with control input $\zeta \in \mathbb{R}$. In the following, we adopt a robust control approach where no knowledge of matrices A, b, c, and hysteresis $\Gamma[\zeta]$ is required, beyond Assumptions 1–3.

Given the equilibrium pair $(\xi^\star, v^\star)$ of (9), define $\tilde\xi := \xi - \xi^\star$, $e := y - y^\star$. From $A\xi^\star + bv^\star = 0$ due to (9), we compute the error dynamics for the ξ-subsystem as

$$\dot{\tilde\xi} = A\tilde\xi + b(\gamma\zeta - \mu^\top\delta - v^\star), \quad e = c^\top\tilde\xi. \tag{14}$$

Due to the relative degree 1 assumption, under a suitable change of coordinates [15, Sec. 2.3], [17, Remark 4.3.1], system (14) can be rewritten as

$$\dot z = A_z z + b_z e, \quad \dot e = c_z^\top z + \alpha e + \beta\left(\zeta - \frac{\mu^\top\delta + v^\star}{\gamma}\right), \tag{15}$$

where $z \in \mathbb{R}^{n-1}$ is the state of the internal dynamics, A_z, b_z, c_z, and α are matrices of suitable dimensions, while $\beta := \gamma c^\top b > 0$ is the high-frequency gain. Due to Assumption 2, A_z is Hurwitz, thus there exists $P_z = P_z^\top > 0$ such that:

$$P_z A_z + A_z^\top P_z = -I_{n-1}. \tag{16}$$

A. Proportional Control Law

Consider the proportional controller

$$\zeta = -ke, \tag{17}$$

where k is a positive gain. The resulting closed-loop system, obtained from the interconnection between (13) and (17), is the following constrained differential inclusion

$$\begin{aligned} &\dot z = A_z z + b_z e, \quad \dot e = c_z^\top z - (\beta k - \alpha)e - \beta(\mu^\top\delta + v^\star)/\gamma \\ &\dot\delta \in F_\Gamma(\delta, -k(c_z^\top z - (\beta k - \alpha)e - \beta(\mu^\top\delta + v^\star)/\gamma)), \end{aligned} \tag{18}$$

with $(z, e, \delta) \in \mathbb{R}^n \times \Delta$. Since δ belongs to the compact set Δ, the (z, e)-subsystem is a linear system with bounded input disturbance $\mu^\top\delta + v^\star$. Due to its structure, it is possible to choose k sufficiently large to ensure that its solutions are bounded. To show this property, consider the Lyapunov function $V_k(z, e) := z^\top P_z z + e^2$, with $P_z = P_z^\top > 0$ selected as in (16). Choose the proportional gain

$$k > k^\star := \tfrac{1}{\beta}(\alpha + |P_z b_z + c_z|^2). \tag{19}$$

Then, along the solutions of (18), we obtain

$$\begin{aligned} \dot V_k &= -|z|^2 + 2z^\top(P_z b_z + c_z)e - 2(\beta k - \alpha)e^2 \\ &\quad - 2\beta e(\mu^\top\delta + v^\star)/\gamma \\ &\le -\tfrac{1}{2}|z|^2 - 2\beta|e|((k - k^\star)|e| - |\mu^\top\delta + v^\star|/\gamma), \end{aligned}$$

which implies, from the boundedness of $\delta \in \Delta$ and $v^\star$:

$$|e| > e_M := \frac{\max_{\delta\in\Delta}\{|\mu^\top\delta + v^\star|\}}{\gamma(k - k^\star)} \implies \dot V_k < 0,$$

$$|z| > \frac{2\max_{\delta\in\Delta}\{|\mu^\top\delta + v^\star|\}}{\gamma\sqrt{(k-k^\star)/\beta}} \implies \dot{V}_k < 0, \quad (20)$$

with the second inequality obtained by splitting the analysis in $|e| \geq e_M$ and $|e| \leq e_M$. From (20), (z, e) eventually converges to compact sublevel sets of V_k, thus the solutions of (18) are bounded. Moreover, (z, e) approaches an arbitrarily small neighborhood of $(0, 0)$ (i.e., ξ approaches an arbitrarily small neighborhood of $\xi^\star$) for $k > 0$ sufficiently large.

B. Proportional-Integral Control Law

We have shown that controller (17) ensures global boundedness of solutions but cannot achieve $(z, e) \to 0$. Thus, in place of (17), consider a controller with an integral action:

$$\dot\sigma = -he, \qquad \zeta = \sigma - ke, \quad (21)$$

where $\sigma \in \mathbb{R}$ is the integrator state and k, h are the proportional and integral gains. Define $\tilde\sigma := \sigma - (\mu^\top\delta + v^\star)/\gamma$. Using (15), the interconnection among (13) and (21) yields:

$$\begin{aligned}
&\dot z = A_z z + b_z e, \quad \dot e = c_z^\top z - (\beta k - \alpha)e + \beta\tilde\sigma\\
&\dot{\tilde\sigma} \in -he - \frac{\mu^\top}{\gamma}F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k - \alpha)e + \beta\tilde\sigma))\\
&\dot\delta \in F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k - \alpha)e + \beta\tilde\sigma)), \quad (22)
\end{aligned}$$

with $(z, e, \tilde\sigma, \delta) \in \mathbb{R}^{n+1} \times \Delta$. We now expand the dynamics of $\tilde\sigma$ in terms of the components δ_i that lie in the linear region. Namely, inspired by (4), introduce

$$q_i(\delta_i, \dot\zeta) := \begin{cases} 0, & \text{if } |\delta_i| = r_i \text{ and } \delta_i\dot\zeta \geq 0,\\ 1, & \text{otherwise,}\end{cases} \quad (23)$$

so that we can write $\dot\delta_i = q_i(\delta_i, \dot\zeta)\dot\zeta$, then define

$$q(\delta, \dot\zeta) := \frac{1}{\gamma}\sum_{i=1}^{p}\mu_i q_i(\delta_i, \dot\zeta). \quad (24)$$

Note that, by Assumption 3 and from (6),

$$0 \leq q \leq q_{\max} := \frac{1}{\gamma}\sum_{i=1}^{p}\mu_i < 1. \quad (25)$$

Using (22) and (24), we obtain

$$\begin{aligned}
\dot{\tilde\sigma} &= -he - \mu^\top\dot\delta/\gamma = -he - q\dot\zeta\\
&= -he - q(-he - k(c_z^\top z - (\beta k - \alpha)e + \beta\tilde\sigma))\\
&= -(h(1-q) + kq(\beta k - \alpha))e + k\beta q\tilde\sigma + qkc_z^\top z. \quad (26)
\end{aligned}$$

Thus, the $(z, e, \tilde\sigma)$-subsystem in (22) can be written as

$$\begin{bmatrix}\dot z\\ \dot e\\ \dot{\tilde\sigma}\end{bmatrix} = \underbrace{\begin{bmatrix} A_z & b_z & 0\\ c_z^\top & -(\beta k - \alpha) & \beta\\ qkc_z^\top & \begin{pmatrix}-h(1-q)+\\ -kq(\beta k - \alpha)\end{pmatrix} & k\beta q\end{bmatrix}}_{:=A_s(q)}\begin{bmatrix} z\\ e\\ \tilde\sigma\end{bmatrix}, \quad (27)$$

with the time-varying input $q \in [0, q_{\max}] \subset [0, 1)$ due to (25). For system (22), we provide the following relevant result.

Theorem 1: Let $k = g\bar k$ and $h = g^2\bar h$, with gains $g > 0$, $\bar h > 0$, and $\bar k$ such that

$$\bar k \geq \frac{\alpha + \sqrt{\beta\bar h(2 - q_{\max})}}{\beta(1 - q_{\max})}. \quad (28)$$

Then, there exists $g^\star > 0$ such that, for all $g > g^\star$, the compact attractor $\mathcal{A}_\sigma := \{(z, e, \tilde\sigma, \delta) \in \mathbb{R}^{n+1} \times \Delta : (z, e, \tilde\sigma) = 0\}$ is globally exponentially stable for (22).

Proof: Define $\bar\sigma := \tilde\sigma/g$ and $\chi := (e, \bar\sigma) \in \mathbb{R}^2$. Then, using (27), system (22) can be rewritten as

$$\begin{aligned}
&\dot z = A_z z + [b_z \;\; 0]\chi\\
&\dot\chi = gA(q, g^{-1})\chi + B(q)z, \quad q \in [0, q_{\max}]\\
&\dot\delta \in F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k - \alpha)e + \beta g\bar\sigma)), \quad (29)
\end{aligned}$$

with $(z, \chi, \delta) \in \mathbb{R}^{n+1} \times \Delta$, $B(q) := [c_z \;\; \bar k q c_z]^\top$ and

$$A(q, g^{-1}) := \begin{bmatrix} -(\beta\bar k - \alpha g^{-1}) & \beta\\ -\bar h(1-q) - \bar k q(\beta\bar k - \alpha g^{-1}) & \bar k q\beta\end{bmatrix}. \quad (30)$$

Consider the auxiliary switching system:

$$\dot\psi = A(q, g^{-1})\psi, \qquad q(t) \in \mathcal{Q}, \quad (31)$$

where $\mathcal{Q} := \{q \in \mathbb{R} : q = \sum_{i\in\mathcal{I}}\mu_i/\gamma, \mathcal{I} \subset \{1, \ldots, p\}\}$. Below, we show that the origin of system (31) is GES for all $g \geq 1$ and for any switching sequence $t \mapsto q(t)$. Notice that the characteristic polynomial of $A(q, g^{-1})$ is $\lambda^2 + \lambda(\beta\bar k(1-q) - \alpha g^{-1}) + \beta\bar h(1-q)$, therefore, by (28), $A(q, g^{-1})$ is Hurwitz for all $(q, g) \in [0, q_{\max}] \times [1, \infty)$ because $q_{\max} < 1$ and

$$\bar k > \frac{\alpha}{\beta(1 - q_{\max})} \geq \frac{\alpha g^{-1}}{\beta(1-q)}, \qquad \bar h > 0. \quad (32)$$

By [7] and [18], (31) admits a Common Quadratic Lyapunov Function (CQLF) if there exists $P_\chi = P_\chi^\top > 0$ such that:

$$\begin{aligned}
&P_\chi A(0, g^{-1}) + A(0, g^{-1})^\top P_\chi \leq -I_2\\
&P_\chi A(q_{\max}, g^{-1}) + A(q_{\max}, g^{-1})^\top P_\chi \leq -I_2. \quad (33)
\end{aligned}$$

Since $\operatorname{rank}(A(0, g^{-1}) - A(q_{\max}, g^{-1})) = 1$, from [19], we have that matrix P_χ exists if and only if the matrix pencil $\Pi(\varsigma) := \beta\bar h A(0, g^{-1})^{-1} + \varsigma A(q_{\max}, g^{-1})$ is invertible, for all $\varsigma \geq 0$. Straightforward computations show that

$$\begin{aligned}
&\det(\Pi(\varsigma)) = \beta\bar h(1 - q_{\max})\varsigma^2 + \beta\bar h\\
&+((\beta\bar k - \alpha g^{-1})(\beta\bar k(1 - q_{\max}) - \alpha g^{-1}) - \beta\bar h(2 - q_{\max}))\varsigma.
\end{aligned}$$

As $\det(\Pi(0)) > 0$, it is sufficient to verify that $(\beta\bar k - \alpha g^{-1})(\beta\bar k(1 - q_{\max}) - \alpha g^{-1}) - \beta\bar h(2 - q_{\max}) > 0$, which is ensured for any $g \geq 1$ by choosing $\bar h > 0$ and $\bar k$ according to (28). Returning to system (29), define

$$V_\sigma(z, \chi) := z^\top P_z z + \chi^\top P_\chi\chi, \quad (34)$$

which is positive definite and radially unbounded with respect to $\mathcal{A}_\sigma$. Along the solutions of (29), it holds that

$$\begin{aligned}
\dot V_\sigma &\leq -|z|^2 - g|\chi|^2 + 2z^\top(P_z b_z[1 \;\; 0] + B(q)^\top P_\chi)\chi\\
&\leq -\tfrac{1}{2}|z|^2 - (g - 4|P_z b_z|^2 - 4|P_\chi B(q)|^2)|\chi|^2.
\end{aligned}$$

Therefore, selecting g according to

$$g > g^\star := 4|P_z b_z|^2 + 4|P_\chi|^2|c_z|^2(1 + \bar k^2 q_{\max}^2) \quad (35)$$

ensures that V_σ is a CQLF for $\mathcal{A}_\sigma$, which implies GES. ∎

Remark 1: Compared with [7], [8], our tuning approach involves high-gain arguments instead of LMIs. In fact, for any $\bar h > 0$, it is possible to choose $\bar k > 0$ sufficiently large to satisfy (28), which then implies the existence of $g^\star$ in (35).

We present a direct consequence of the proof of Theorem 1.

Corollary 1: Choose gains k and h so that $\mathcal{A}_\sigma$ in Theorem 1 is GES. Then, there exists a CQLF for (22). Namely, there exists $P_s = P_s^\top > 0$ such that $P_s A_s(q) + A_s(q)^\top P_s \leq -I_{n+1}$, for all $q \in [0, q_{\max}] \subset [0, 1)$, with $A_s(q)$ defined in (27).

IV. Setpoint Regulation for Cascaded Systems With Hysteretic Interconnection

We return to the problem of setpoint regulation for system (1), i.e., we address the problem of designing an input u such that $y \to y^\star$. The key idea is to first generate a reference $\zeta^\star$ that solves the regulation problem for the reduced plant (13), then impose $\zeta \to \zeta^\star$ by selecting u for the complete plant (7). To this aim, we require $\dot{\zeta}^\star$ be well defined and available for control, which implies that (21) cannot be employed directly as $\dot{e}$ is unknown. Thus, we assign $\zeta^\star$ through a modified version of (21) comprising a filter of the tracking error:

$$\dot{\sigma} = -h\eta, \qquad \dot{\eta} = -\ell(\eta - e), \quad \zeta^\star = \sigma - k\eta, \tag{36}$$

where $\sigma \in \mathbb{R}$ is the integrator state, $\eta \in \mathbb{R}$ is the filter state, while ℓ, h, and k are positive scalar gains. For convenience in the analysis, we define the error variables

$$\tilde{\sigma} := \sigma - (\mu^\top \delta + v^\star)/\gamma, \quad \tilde{\eta} := \eta - e. \tag{37}$$

Then, using (15), (24), and $\zeta = \sigma - k\eta + \zeta - \zeta^\star$, we can write the interconnection between plant (7) and controller (36) as

$$\begin{aligned}
\dot{z} &= A_z z + b_z e\\
\dot{e} &= -(\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z - \beta k\tilde{\eta} + \beta(\zeta - \zeta^\star)\\
\dot{\tilde{\sigma}} &= -h\eta - q\dot{\zeta}\\
\dot{\tilde{\eta}} &= -(\ell\tilde{\eta} - (\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z - \beta k\tilde{\eta} + \beta(\zeta - \zeta^\star))\\
\dot{\zeta} &= u, \qquad \dot{\delta} \in F_\Gamma(\delta, \dot{\zeta}),
\end{aligned} \tag{38}$$

with $(z, e, \tilde{\sigma}, \tilde{\eta}, \zeta, \delta) \in \mathbb{R}^{n+3} \times \Delta$.

A. Stability Analysis of the Reduced System

We study system (38) under the assumption that

$$\zeta = \zeta^\star = \sigma - k\eta \tag{39}$$

is satisfied at all times. This reduction argument is instrumental in introducing the controller of Section IV-B. Define:

$$x_f := \ell\tilde{\eta} - (\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z. \tag{40}$$

With (36), (37) we have $\dot{\zeta} = \dot{\zeta}^\star = -h\eta + k\ell\tilde{\eta}$, and then using (39), (40) we can follow similar computations to (26) exploiting $\mu^\top\dot{\delta}/\gamma = q\dot{\zeta}$ to show that

$$\begin{aligned}
\dot{\tilde{\sigma}} &= -(h(1-q) + kq(\beta k - \alpha))e + k\beta q\tilde{\sigma} + qkc_z^\top z\\
&\quad - qkx_f - \ell^{-1}h(1-q)(x_f + (\beta k - \alpha)e - \beta\tilde{\sigma} - c_z^\top z).
\end{aligned} \tag{41}$$

Define $x_s := (z, e, \tilde{\sigma}) \in \mathbb{R}^{n+1}$. Then, in the coordinates $x := (x_s, x_f)$, system (38) with condition (39) reads as

$$\begin{aligned}
\dot{x}_s &= A_s(q)x_s + B_s(q)x_f + \ell^{-1}D_s(q)x\\
\dot{x}_f &= -\ell x_f + D_f(q, \ell^{-1})x
\end{aligned} \qquad \dot{\delta} \in F_\Gamma(\delta, \dot{\zeta}), \tag{42}$$

where $A_s(q)$ is found in (27), $B_s(q) := [0^\top\, 0\, -qk]^\top$, while $D_s(q)$ and $D_f(q, \ell^{-1})$ are matrices of appropriate dimensions. In particular, $D_f(q, \ell^{-1})$ is an affine function of ℓ^{-1}, so that its entries are bounded as $\ell \to \infty$. Using ℓ, the analysis of (42) is performed via timescale separation, where the x_f-subsystem (the fast subsystem) is made arbitrarily fast with respect to the x_s-subsystem (the slow subsystem) by selecting $\ell > 0$ sufficiently large. These arguments lead to the following result.

Proposition 1: Choose gains k and h so that $\mathcal{A}_\sigma$ in Theorem 1 is GES. Then, there exists $\ell^\star > 0$ such that, for all $\ell > \ell^\star$, attractor $\mathcal{A}_\eta := \{(x_s, x_f, \delta) \in \mathbb{R}^{n+2} \times \Delta : (x_s, x_f) = 0\}$ is GES for system (42).

Proof: Pick P_s from Corollary 1, then define $d := 2|P_s|^2 k^2 q_{\max}^2 \geq 2|P_s B_s(q)|^2$ for all $q \in [0, q_{\max}]$ and

$$V_\eta(x) := x_s^\top P_s x_s + d\ell^{-1}x_f^2, \tag{43}$$

which is positive definite and radially unbounded with respect to $\mathcal{A}_\eta$. Then, we obtain:

$$\begin{aligned}
\dot{V}_\eta &\leq -|x_s|^2 + 2x_s^\top P_s B_s x_f - 2dx_f^2 + 2\ell^{-1}(x_s^\top P_s D_s + dx_f D_f)x\\
&\leq -\min\{\tfrac{1}{2}, d\}|x|^2 + 2\ell^{-1}(|P_s D_s| + d|D_f|)|x|^2.
\end{aligned}$$

Recalling that $\lim_{\ell\to\infty} \ell^{-1}(|P_s D_s(q)| + d|D_f(q, \ell^{-1})|) = 0$, for $\ell > 0$ sufficiently large, $\dot{V}_\eta$ can be made quadratically negative definite, ensuring that V_η is a CQLF for $\mathcal{A}_\eta$. ■

B. Main Result

Differently from (39), we now address the general case of the augmented plant (38) by removing the assumption that $\zeta = \zeta^\star$. Thus, we define the error $\tilde{\zeta} := \zeta - \zeta^\star$, whose dynamics are given, by (38) with (36) and (37), as $\dot{\tilde{\zeta}} = u - \dot{\zeta}^\star = u + h\eta - k\ell(\eta - e)$. Therefore, we can ensure $\tilde{\zeta} \to 0$ by selecting $u = \dot{\zeta}^\star - k_\zeta\tilde{\zeta}$, with gain k_ζ, leading to:

$$\dot{\tilde{\zeta}} = -k_\zeta\tilde{\zeta}. \tag{44}$$

Using (36), this yields the following controller for plant (7):

$$\begin{aligned}
&\dot{\eta} = -\ell(\eta - e), \qquad \dot{\sigma} = -h\eta,\\
&u = -h\eta + k\ell(\eta - e) - k_\zeta(\zeta - \sigma + k\eta),
\end{aligned} \tag{45}$$

with positive gains ℓ, h, k, and k_ζ. The following statement, which confirms that controller (45) provides a solution for Problem 1, is the main result of this letter.

Theorem 2: Choose any gain $k_\zeta > 0$ and positive gains k, h, ℓ such that $\mathcal{A}_\eta$ of Proposition 1 is GES. Then, attractor $\mathcal{A} := \{(\xi, \delta, \zeta, \eta, \sigma) \in \mathbb{R}^n \times \Delta \times \mathbb{R}^3 : (\xi, \delta, \zeta) \in \mathcal{E}, \eta = 0, \sigma = \zeta\}$, with $\mathcal{E}$ as in (12), is robustly globally $\mathcal{KL}$ asymptotically stable for the interconnection between system (7) and controller (45).

Proof: The closed-loop system can be seen as the cascade interconnection of (44) and (42) perturbed by $\tilde{\zeta}$. Namely, the x-subsystem can be rewritten as

$$\dot{x} = A_x(q)x + B_x(q)\tilde{\zeta}. \tag{46}$$

Consider the Lyapunov function $V := V_\eta + \rho\tilde{\zeta}^2$, where $V_\eta := x^\top P_x x$ is defined in (43) and $\rho > 0$. By Proposition 1, there exists a constant matrix $Q_x = Q_x^\top > 0$ such that

$$\dot{V} \leq -x^\top Q_x x + 2x^\top P_x B_x(q)\tilde{\zeta} - 2\rho k_\zeta\tilde{\zeta}^2. \tag{47}$$

Choosing $\rho > 0$ sufficiently large, we conclude that $\mathcal{A}$ is GES. Finally, we note that system (7), (45) can be regarded as a hybrid system, with empty jump set and jump map, satisfying the hybrid basic conditions. Thus, from [14, Th. 7.21], $\mathcal{A}$ is robustly globally $\mathcal{KL}$ asymptotically stable. ■

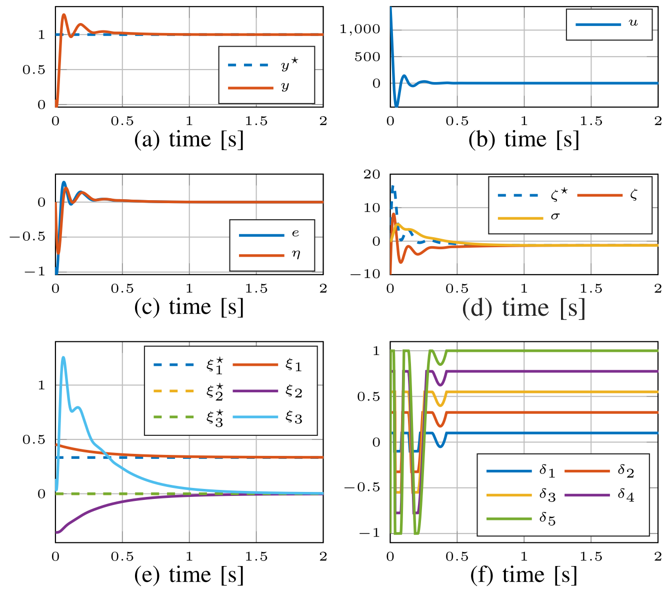

Fig. 3. Closed-loop simulation results. (a): reference $y^\star$ and output y. (b): input u. (c): error e and filter state η. (d): reference $\zeta^\star$, state ζ, and integrator state σ. (e): $\xi^\star$ and states ξ. (f): hysteresis states δ.

Remark 2: Controller (45) can be tuned following Remark 1 and selecting ℓ sufficiently large as per Proposition 1. On the other hand, $k_\zeta > 0$ can be chosen arbitrarily.

V. Numerical Results

We perform a numerical analysis to illustrate our theoretical results. Following the structure of (1) and Assumptions 1 and 2, we consider a ξ-subsystem having transfer function

$$G(s) = \frac{(s+3)(s+1)}{(s-5)(s^2+2s+4)}. \quad (48)$$

Matrices A, b, c are obtained as the minimal realization of (48) in controllability canonical form. The PI operator $\Gamma[\zeta]$, whose behavior is shown in Fig. 2-Right, is implemented according to (5) using $p = 5$ stop operators with weights $\mu = [0.1 \quad 0.325 \quad 0.55 \quad 0.775 \quad 1]^\top$, $\mu_0 = 1$, and radii $r_i = \mu_i$, $i \in \{1, \dots, 5\}$. The feedback law is the one in (45) with gains selected according to Remarks 1 and 2 as $g = 20$, $k = 20$, $h = 200$, $\ell = 75$, and $k_\zeta = 5$. Simulations have all the system states initialized randomly and $y^\star = 1$. The results obtained for a single run are shown in Fig. 3. Finally, although not shown here due to space constraints, we tested references where a sinusoid is added to $y^\star$. In this scenario, due to the semiglobally practically robust asymptotic stability [14, Definition 7.18] ensured by Theorem 2, the tracking error becomes arbitrarily small as the harmonic frequency tends to zero.

VI. Conclusion

We provided a robust control strategy for linear cascaded systems with a hysteretic interconnection. Employing a well-posed constrained differential inclusion to represent the PI operator, we formulated the regulation problem by defining a compact set of equilibria depending on the hysteresis states. Relying on high-gain arguments and the properties of well-posed hybrid dynamical systems, we proved that the closed-loop system admits a robustly globally asymptotically stable compact attractor for a selection of the controller gains not requiring parametric knowledge of the system. Future work comprises generalizing the cascade dynamics and the hysteresis model, tracking references generated by exosystems (e.g., sums of sinusoids), and applying the results to thermal SMAs.

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