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A Five-Step Block Method Coupled with Symmetric Compact Finite Difference Scheme for Solving Time-Dependent Partial Differential Equations

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Abstract: In this article, we present a five-step block method coupled with an existing fourth-order symmetric compact finite difference scheme for solving time-dependent initial-boundary value partial differential equations (PDEs) numerically. Firstly, a five-step block method has been designed to solve a first-order system of ordinary differential equations that arise in the semi-discretisation of a given initial boundary value PDE. The five-step block method is derived by utilising the theory of interpolation and collocation approaches, resulting in a method with eighth-order accuracy. Further, characteristics of the method have been analysed, and it is found that the block method possesses A-stability properties. The block method is coupled with an existing fourth-order symmetric compact finite difference scheme to solve a given PDE, resulting in an efficient combined numerical scheme. The discretisation of spatial derivatives appearing in the given equation using symmetric compact finite difference scheme results in a tridiagonal system of equations that can be solved by using any computer algebra system to get the approximate values of the spatial derivatives at different grid points. Two well-known test problems, namely the nonlinear Burgers equation and the FitzHugh-Nagumo equation, have been considered to analyse the proposed scheme. Numerical experiments reveal the good performance of the scheme considered in the article.

Keywords: PDEs; block methods; compact finite difference scheme; stability

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Citation: Kaur, K.; Singh, G.; Ritelli, D. A Five-Step Block Method Coupled with Symmetric Compact Finite Difference Scheme for Solving Time-Dependent Partial Differential Equations. *Symmetry* **2024**, *16*, 307. https://doi.org/10.3390/sym16030307

Academic Editors: Youssef N. Raffoul, Calogero Vetro and Sergei D. Odintsov

Received: 22 January 2024
Revised: 21 February 2024
Accepted: 1 March 2024
Published: 5 March 2024

1. Introduction

Nonlinear partial differential equations are used to model many important physical phenomena that appear in real-world applications of sciences and engineering [1]. It is well-known that the availability of analytical methods for solving nonlinear partial differential equations is limited to a specific class of problems; considering numerical approximations to the solution is one possible way to approach the given problem in this instance [2–7]. One ongoing objective in this field is to solve these problems by creating new, efficient numerical schemes or modifying existing ones. Our work in this article examines the approximate solution of time-dependent initial-boundary value PDEs of one dimension in the following form

$$u_t = F(x,t,u,u_x,u_{xx}), \quad \text{with} \quad a \le x \le b, \quad t \ge t_0, \tag{1}$$

with initial condition

$$u(x,t_0) = \psi(x),$$

and boundary conditions as

$$u(a,t) = \phi_1(t), \quad u(b,t) = \phi_2(t),$$

where u represents the exact solution of the problem whereas x and t are space and time variables, respectively. In this paper, we actually consider only a special type of Equation (1) in which the second order partial derivative is linear. The spatial semi-discretisation of the above problem (1) converts it into a system of first-order ordinary differential equations in t as follows

$$\frac{dU}{dt} = f(t, U) \quad \text{and} \quad U(t_0) = U_0, \tag{2}$$

which can be solved by various existing time integration techniques, for instance, Runge–Kutta or linear multi-step methods [8]. In this article, our focus is on employing a time-marching numerical method of block nature to address the resulting system of first-order ordinary differential equations (ODEs) (2). To assess its performance, we examine two well-known nonlinear time-dependent partial differential equations (PDEs) found in the scientific literature, each with numerous real-world applications. FitzHugh–Nagumo and Burgers' equations have been analysed numerically using a block approach with a compact finite difference scheme. A nonlinear reaction-diffusion equation, the FitzHugh–Nagumo equation originated in science and technology, particularly in neurophysiology and population growth models, flame propagation, logistic population growth, nuclear reactor theory and catalytic chemical reactions. Some researchers have looked at finite difference and compact difference methods, as in [9–11], to obtain numerical solutions of FitzHugh–Nagumo equations.

Another important nonlinear partial differential equation that primarily appears in shock theory and turbulence modelling is the Burgers' equation. Applications of the Burgers' equation can be found in many different domains, including quantum fields, traffic flow, fluid dynamics, gas dynamics, shock theory, viscous flow and turbulence. Numerous numerical techniques based on the finite element method, finite difference method, compact finite difference method, MacCormack method, quadrature method, Haar wavelet quasilinearisation approach, splitting methods, etc., have been used in the past to study the Burgers' equation [12–21].

Due to their smaller stencil size and increased accuracy, compact finite difference schemes have been more widely used than standard finite difference schemes over the past 50 years [22,23]. Firstly, we develop a five-step block method and combine it with a fourth-order compact finite difference scheme to solve a given problem (1). Block methods are good alternatives for linear multi-step methods that require no starting values to get an approximate solution to a given problem. These methods are self-starting and were first proposed by Milne [24]. With these techniques, multiple points can be approximated simultaneously, saving computation time without sacrificing accuracy [25]. Here, we want to increase the applicability of block methods for solving time-dependent PDEs by combining them with compact finite difference schemes.

2. Development of a Five-Step Block Method

We discretize the time domain $[t_0, t_f]$ with equal step size $k = t_{i+1} - t_i$ for finding the approximate solution of a problem (2). The method for solving a scalar problem $u' = f(t, u), u(t_0) = u_0$ could be applied using a component implementation to solve a system like the one in problem (2). Consider the following polynomial to provide the approximate solution to this problem on an interval $[t_n, t_{n+5}]$ as

$$u(t) \approx p(t) = \sum_{n=0}^{n=8} a_n t^n \tag{3}$$

where a_n are the constants that must be determined. To differentiate (3) w.r.t.t two times, we get

$$u'(t) \approx p'(t) = \sum_{n=1}^{n=8} n a_n t^{n-1} \tag{4}$$

and

$$u''(t) \approx p''(t) = \sum_{n=2}^{n=8} n(n-1)a_n t^{n-2} \tag{5}$$

where a_n are the unknown coefficients. To determine the values of nine unknown coefficients, the following interpolatory and collocation conditions are imposed

$$p(t_n) = u_n, p'(t_n) = f_n, p'(t_{n+1}) = f_{n+1}, p'(t_{n+2}) = f_{n+2}$$

$$p'(t_{n+3}) = f_{n+3}, p'(t_{n+4}) = f_{n+4}, p'(t_{n+5}) = f_{n+5}$$

$$p''(t_n) = f'_n, p''(t_{n+5}) = f'_{n+5}$$

Here, u_{n+j}, f_{n+j} and f'_{n+j} are respectively approximations to $u(t_{n+j}), u'(t_{n+j})$ and $u''(t_{n+j})$. The above equations can be written in a matrix form as

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 & t_n^8 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 & 8t_{n+1}^7 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^5 & 7t_{n+2}^6 & 8t_{n+2}^7 \\ 0 & 1 & 2t_{n+3} & 3t_{n+3}^2 & 4t_{n+3}^3 & 5t_{n+3}^4 & 6t_{n+3}^5 & 7t_{n+3}^6 & 8t_{n+3}^7 \\ 0 & 1 & 2t_{n+4} & 3t_{n+4}^2 & 4t_{n+4}^3 & 5t_{n+4}^4 & 6t_{n+4}^5 & 7t_{n+4}^6 & 8t_{n+4}^7 \\ 0 & 1 & 2t_{n+5} & 3t_{n+5}^2 & 4t_{n+5}^3 & 5t_{n+5}^4 & 6t_{n+5}^5 & 7t_{n+5}^6 & 8t_{n+5}^7 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 \\ 0 & 0 & 2 & 6t_{n+5} & 12t_{n+5}^2 & 20t_{n+5}^3 & 30t_{n+5}^4 & 42t_{n+5}^5 & 56t_{n+5}^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} u_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f'_n \\ f'_{n+5} \end{bmatrix}$$

Using the Mathematica system, the values of the nine unknowns that appear in the above system of equations have been determined. By substituting these values and changing the variable t to $t_n + mk$, the polynomial in (3) can be re-written as

$$p(t_n + mk) = b_0 u_n + k(b_1 f_n + b_2 f_{n+1} + b_3 f_{n+2} + b_4 f_{n+3} + b_5 f_{n+4} + b_6 f_{n+5}) + k^2(b_7 f'_n + b_8 f'_{n+5}) \tag{6}$$

where the coefficients $b_j's$ are continuous functions of variable m. After evaluating the above polynomial for $m = 1, 2, 3, 4, 5$, we get a complete structure of the block method that consists of the following five formulas

$$\begin{aligned} u_{n+1} &= u_n + \frac{k}{6,048,000}(3,068,391 f_n + 3,678,525 f_{n+1} - 1,128,100 f_{n+2} + 663,900 f_{n+3} \\ &\quad - 353,475 f_{n+4} + 118,759 f_{n+5}) + \frac{60k^2}{6,048,000}(7899 f'_n - 731 f'_{n+5}) \\ u_{n+2} &= u_n + \frac{k}{189,000}(844,431 f_n + 214,650 f_{n+1} + 85,900 f_{n+2} - 8600 f_{n+3} \\ &\quad + 2025 f_{n+4} - 418 f_{n+5}) + \frac{k^2}{189,000}(11,220 f'_n + 120 f'_{n+5}) \\ u_{n+3} &= u_n + \frac{k}{224,000}(106,213 f_n + 231,975 f_{n+1} + 230,100 f_{n+2} + 118,100 f_{n+3} \\ &\quad - 20,025 f_{n+4} + 5637 f_{n+5}) + \frac{k^2}{224,000}(15,420 f'_n - 1980 f'_{n+5}) \\ u_{n+4} &= u_n + \frac{k}{23,625}(10,686 f_n + 26,100 f_{n+1} + 20,600 f_{n+2} + 27,600 f_{n+3} \\ &\quad + 10,350 f_{n+4} - 836 f_{n+5}) + \frac{240k^2}{23,625}(6 f'_n + f'_{n+5}) \\ u_{n+5} &= u_n + \frac{k}{48,384}(22,835 f_n + 50,625 f_{n+1} + 47,500 f_{n+2} + 47,500 f_{n+3} \\ &\quad + 50,625 f_{n+4} + 22,835 f_{n+5}) + \frac{3300k^2}{48,384}(f'_n - f'_{n+5}) \end{aligned} \tag{7}$$

The above method is a five-step second derivative block method that will simultaneously yield approximations of solutions to the initial-value problems (2) at the nodal points $t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4}$ and t_{n+5}.

3. Basic Characteristics of the Method and Stability Analysis

This section discusses various characteristics of the block method (7).

3.1. Order of Accuracy and Consistency

Consider a difference operator $\mathcal{L}_j$ related to the five-step block method given by (7)

$$\mathcal{L}[u(t),k] = u(t+jk)-$$

$$F_j[k, u(t), u'(t), u'(t+k), u'(t+2k), u'(t+3k), u'(t+4k), u'(t+5k), u''(t), u''(t+5k)] \tag{8}$$

with $j = 1,2,3,4,5$ and F_j is the corresponding right-hand side of each formula. Expanding the expression (8) using Taylor's series about the point t and combining the like terms in k, the local truncation errors of each formula given in (7) are obtained as

$$\mathcal{LTE}_1 = \frac{-3061u^9(t)k^9}{6350400} + O(k^{10})$$
$$\mathcal{LTE}_2 = \frac{-113u^9(t)k^9}{1587600} + O(k^{10})$$
$$\mathcal{LTE}_3 = \frac{-33u^9(t)k^9}{78400} + O(k^{10})$$
$$\mathcal{LTE}_4 = \frac{-u^9(t)k^9}{9925} + O(k^{10})$$
$$\mathcal{LTE}_5 = \frac{-125u^9(t)k^9}{254016} + O(k^{10})$$

The above expressions for local truncation errors conclude that the proposed method has eighth-order accuracy, implying that the proposed block method is consistent.

3.2. Zero Stability

The block method (7) is said to be zero-stable if the roots of its first characteristic equation $\rho(\lambda) = 0$ have modulus <1, and the roots of modulus one must be simple. By considering the limit as k tends to zero, from the method (7), we get

$$IU_n - RU_{n-1} = 0$$

where $U_n = (U_{n+1}, U_{n+2}, U_{n+3}, U_{n+4}, U_{n+5})^T$ and $U_{n-1} = (U_{n-4}, U_{n-3}, U_{n-2}, U_{n-1}, U_n)^T$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and I is the identity matrix of order five. The characteristic equation for the above method is $\rho(\lambda) = det(R - \lambda I) = \lambda^4(1-\lambda) = 0$. The roots are $\{0,0,0,0,1\}$. It implies that the five-step block method is zero-stable.

3.3. Linear Stability Analysis

The linear stability analysis of a numerical scheme is carried out by applying it to Dahlquist's test equation

$$u' = \lambda u, \quad Re(\lambda) < 0. \tag{9}$$

As t approaches ∞, any true solution $ce^{\lambda t}$ to the above Equation (9) will decay. For the numerical method to be stable, the behaviour of the numerical solution should match the nature of the true solution. After substituting $\lambda k = \bar{k}$ and appying the proposed block method to Equation (9), we obtain a difference system in matrix form as

$$L \begin{bmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \\ u_{n+4} \\ u_{n+5} \end{bmatrix} = M \begin{bmatrix} u_{n-4} \\ u_{n-3} \\ u_{n-2} \\ u_{n-1} \\ u_{n} \end{bmatrix}$$

where matrices L and M are respectively given as

$$L = \begin{bmatrix} 1 - \frac{3678525\bar{k}}{6048000} & \frac{1128100\bar{k}}{6048000} & \frac{-663900\bar{k}}{6048000} & \frac{353475\bar{k}}{6048000} & \frac{418\bar{k}+43860\bar{k}^2}{6048000} \\ \frac{-214650\bar{k}}{189000} & 1 - \frac{85900\bar{k}}{189000} & \frac{8600\bar{k}}{189000} & \frac{-2025\bar{k}}{189000} & \frac{418\bar{k}-120\bar{k}^2}{189000} \\ \frac{-231975\bar{k}}{224000} & \frac{-230100\bar{k}}{224000} & 1 - \frac{118100\bar{k}}{224000} & \frac{20025\bar{k}}{224000} & \frac{-5637\bar{k}+1980\bar{k}^2}{224000} \\ \frac{-26100\bar{k}}{23625} & \frac{-20600\bar{k}}{23625} & \frac{-27600\bar{k}}{23625} & 1 - \frac{10350\bar{k}}{23625} & \frac{836\bar{k}-240\bar{k}^2}{23625} \\ \frac{-50625\bar{k}}{48384} & \frac{-47500\bar{k}}{48384} & \frac{-47500\bar{k}}{48384} & \frac{-50625\bar{k}}{48384} & 1 + \frac{-22835\bar{k}+3300\bar{k}^2}{48384} \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 + \frac{3068391\bar{k}+3068391\bar{k}^2}{6048000} \\ 0 & 0 & 0 & 0 & 1 + \frac{84443\bar{k}+11220\bar{k}^2}{189000} \\ 0 & 0 & 0 & 0 & 1 + \frac{106213\bar{k}+15420\bar{k}^2}{224000} \\ 0 & 0 & 0 & 0 & 1 + \frac{10686\bar{k}+240\bar{k}^2}{23625} \\ 0 & 0 & 0 & 0 & 1 + \frac{22835\bar{k}+3300\bar{k}^2}{48384} \end{bmatrix}$$

Thus, we have

$$\begin{bmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \\ u_{n+4} \\ u_{n+5} \end{bmatrix} = N(\bar{k}) \begin{bmatrix} u_{n-4} \\ u_{n-3} \\ u_{n-2} \\ u_{n-1} \\ u_{n} \end{bmatrix}$$

where the matrix $N(\bar{k}) = L^{-1}M$ is the stability matrix. To find out the stability characteristics of the block method, we consider the spectral of the stability matrix $\{0, 0, 0, 0, P(\bar{k})\}$ where

$$P(\bar{k}) = \frac{3360 + 8400\bar{k} + 9600\bar{k}^2 + 6500\bar{k}^3 + 2798\bar{k}^4 + 745\bar{k}^5 + 100\bar{k}^6}{3360 - 8400\bar{k} + 9600\bar{k}^2 - 6500\bar{k}^3 + 2798\bar{k}^4 - 745\bar{k}^5 + 100\bar{k}^6}$$

The region of absolute stability [26] is given by

$$S = \{\bar{k} \in C : |P(\bar{k})| < 1\}$$

The given method (7) is said to be A-stable if the left half of the complex plane is contained within S. In Figure 1, the absolute stability region of method (7) has been plotted, indicating its A-stability.

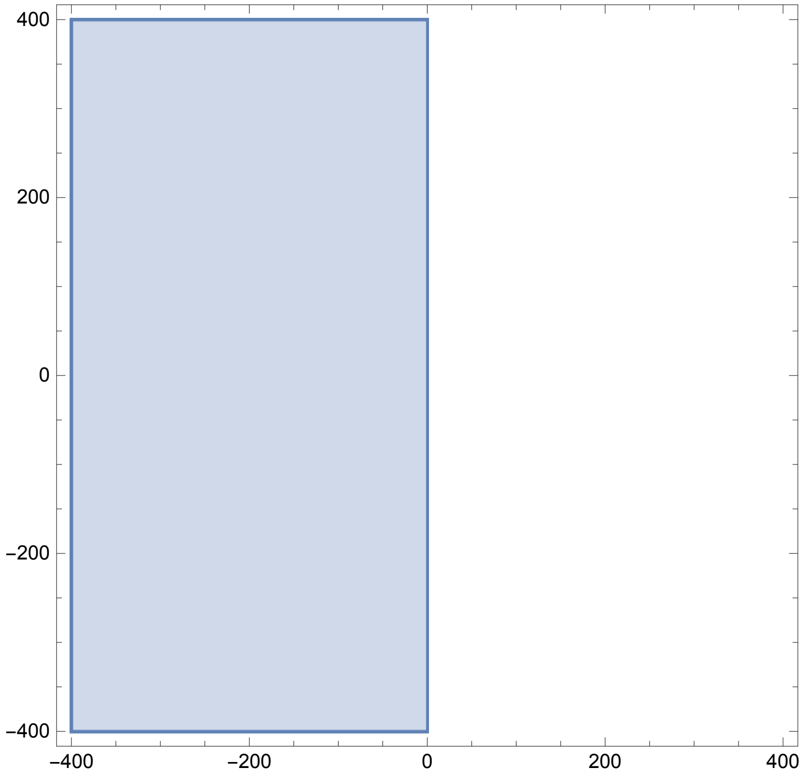

Figure 1. Stability region for the proposed block method. It shows that the method is A-stable.

4. Symmetric Compact Finite Difference Scheme

To discretise the spatial derivatives appearing in a given PDE (1), we have used symmetric compact finite difference schemes rather than conventional finite difference approximations to spatial derivatives—the reason for considering this because due to their better accuracy and smaller stencils compared to the traditional finite difference scheme. Note that discretisation of spatial derivatives present in the given PDE using symmetric compact finite difference schemes results in a tridiagonal system of equations that can be easily handled by Mathematica software. Many researchers have developed compact schemes with various boundary conditions and different orders, as in [27,28].

Discretise the space variable $a < x < b$ into N subintervals of equal length $h = x_{i+1} - x_i$ where $i = 1, 2, 3....., N + 1$.

Consider a fourth-order compact finite difference scheme for discretising the first-order spatial derivative appearing in (1) at the interior nodes $i = 2, 3, 4......., N$.

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{3}{4h}(u_{i+1} - u_{i-1})$$

where the prime denotes the derivative concerning the space variable and the following one-sided boundary scheme to obtain approximations at boundary points given for $i = 1$

$$u'_1 + 3u'_2 = \frac{1}{h}\left(-\frac{17}{6}u_1 + \frac{3}{2}u_2 + \frac{3}{2}u_3 - \frac{1}{6}u_4\right)$$

and for $i = N + 1$

$$u'_{N+1} + 3u'_N = \frac{1}{h}\left(\frac{17}{6}u_{N+1} - \frac{3}{2}u_N - \frac{3}{2}u_{N-1} + \frac{1}{6}u_{N-2}\right).$$

The above equations can be written in a matrix form given by

$$A_1 U' = B_1 U \tag{10}$$

$$A_1 = \begin{bmatrix} 1 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 1/4 & 1 & 1/4 & 0 & \cdots & 0 & 0 \\ 0 & 1/4 & 1 & 1/4 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1/4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1/4 \\ 0 & 0 & 0 & 0 & \cdots & 3 & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$B_1 = \frac{1}{2h}\begin{bmatrix} 17/3 & 3 & 3 & -1/3 & 0 & \cdots & 0 & 0 \\ -3/2 & 0 & 3/2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -3/2 & 0 & 3/2 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -3/2 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -3/2 & 0 & 3/2 \\ 0 & 0 & 0 & \cdots & 1/3 & -3 & -3 & -17/3 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ . \\ . \\ . \\ U_N \\ U_{N+1} \end{bmatrix}_{(N+1)\times 1}$$

Solving the above system of equations, we can obtain approximations to first-order space derivatives at the discrete points of interest.

Similarly, to approximate the second spatial derivative appearing in the equation, we consider the following fourth-order compact finite difference scheme for interior nodes $i = 2, 3, ...N$ given by

$$\frac{1}{10}u''_{i-1} + u''_i + \frac{1}{10}u''_{i+1} = \frac{6}{5h^2}(u_{i+1} - 2u_i + u_{i-1}).$$

For boundary points, we have:
for $i = 1$,

$$u''_1 + 10u''_2 = \frac{1}{h^2}\left(\frac{145}{12}u_1 - \frac{76}{3}u_2 + \frac{29}{2}u_3 - \frac{4}{3}u_4 + \frac{1}{12}u_5\right)$$

and for $i = N + 1$

$$u''_{N+1} + 10u''_N = \frac{1}{h^2}\left(\frac{145}{12}u_{N+1} - \frac{76}{3}u_N + \frac{29}{2}u_{N-1} - \frac{4}{3}u_{N-2} + \frac{1}{12}u_{N-3}\right)$$

The complete matrix system for the tridiagonal fourth-order compact scheme for approximating the second derivative can be written as follows

$$A_2U'' = B_2U \tag{11}$$

$$A_2 = \begin{bmatrix} 1 & 10 & 0 & 0 & \cdots & 0 & 0 \\ 1/10 & 1 & 1/10 & 0 & \cdots & 0 & 0 \\ 0 & 1/10 & 1 & 1/10 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1/10 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1/10 \\ 0 & 0 & 0 & 0 & \cdots & 10 & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$B_2 = \frac{1}{h^2}\begin{bmatrix} 145/12 & -76/3 & 29/2 & -4/3 & 1/12 & 0 & \cdots & 0 & 0 \\ 6/5 & -12/5 & 6/5 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 6/5 & -12/5 & 6/5 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 6/5 & -12/5 & 6/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6/5 & -12/5 & 6/5 \\ 0 & 0 & 0 & \cdots & 1/12 & -4/3 & 29/2 & -76/3 & 145/12 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ . \\ . \\ . \\ U_N \\ U_{N+1} \end{bmatrix}_{(N+1)\times 1}$$

Solving the above system of equations, one can get an approximation to the second-order space derivatives appearing in the equation at the discrete points of interest.

5. Test Problems

The two well-known nonlinear problems, Burgers' and FitzHugh–Nagumo equations, will be solved using the above combined numerical scheme based on the block method in conjunction with a compact finite difference scheme. Additionally, special consideration has been given to the stability of the resulting differential systems.

5.1. Burgers' Equation

Consider the one-dimensional Burgers' equation:

$$u_t + uu_x = vu_{xx} \tag{12a}$$

with the initial condition

$$u(x,0) = \psi(x); \quad a \le x \le b, \tag{12b}$$

and two boundary conditions are given as

$$u(a,t) = \phi_1(t) = u_1(t) \quad \textit{and} \quad u(b,t) = \phi_2(t) = u_{N+1}(t), \quad t \ge 0. \tag{12c}$$

where u represents fluid's velocity, v is the kinematic viscosity and $xandt$ are the space and time variables, respectively. The system of first-order ODEs derived from (12a)–(12c) can be expressed as follows after semi-discretisation:

$$\begin{bmatrix} u_1' \\ u_2' \\ \cdots \\ u_N' \\ u_{N+1}' \end{bmatrix} = v(A_2^{-1}B_2)\begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \\ u_{N+1} \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \\ u_{N+1} \end{bmatrix} \circ (A_1^{-1}B_1)\begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \\ u_{N+1} \end{bmatrix}$$

where "$\circ$" indicates the elementwise product of two matrices of the same dimensions. The above system can be expressed in a compact form as

$$U' = CU + D \tag{13}$$

where $C = v(A_2^{-1}B_2)$ is an $(N+1)\ times(N+1)$ matrix and the D matrix contains nonlinear terms.

5.2. FitzHugh–Nagumo Equation

Consider the FitzHugh–Nagumo equation:

$$u_t = u_{xx} + u(1-u)(u-\mu) \tag{14a}$$

with initial condition

$$u(x,0) = \psi(x); \quad a \le x \le b, \tag{14b}$$

and boundary conditions

$$u(a,t) = \phi_1(t) = u_1(t) \quad and \quad u(b,t) = \phi_2(t) = u_{N+1}(t), \quad t \ge 0. \tag{14c}$$

where x and t are space and time variables, respectively. The spatial derivatives in this equation will be approximated using a fourth-order compact finite difference scheme. After semi-discretisation, the system of first-order ODEs obtained from (14a)–(14c) can be expressed as follows

$$\begin{bmatrix} u_1' \\ u_2' \\ \dots \\ u_N' \\ u_{N+1}' \end{bmatrix} = (A_2^{-1}B_2 - \mu I)\begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_N \\ u_{N+1} \end{bmatrix} + (1+\mu)\begin{bmatrix} u_1^2 \\ u_2^2 \\ \dots \\ u_N^2 \\ u_{N+1}^2 \end{bmatrix} - \begin{bmatrix} u_1^3 \\ u_2^3 \\ \dots \\ u_N^3 \\ u_{N+1}^3 \end{bmatrix}$$

The above system can be written as

$$U' = CU + D \tag{15}$$

where $C = (A_2^{-1}B_2 - \mu I)$ is an $(N+1)\times(N+1)$ matrix and the D matrix contains nonlinear terms.

5.3. Stability of Differential System

To examine the stability of the differential systems for the considered PDEs, semidiscretise the problem by applying a compact finite difference scheme to the spatial derivatives in the Equation (1). This will result in a system of ODEs of the form

$$U' = CU + D \tag{16}$$

where C is an $(N+1)$ square matrix and D is an $(N+1)\times 1$ vector containing nonhomogeneous parts.

The matrix for the Burgers' equation can be written as

$$C = vC_2$$

and the matrix for the FitzHugh–Nagumo equation is

$$C = (C_2 - \mu I)$$

For both of the PDEs, the matrix C_2 is given by $C_2 = A_2^{-1}B_2$.

To find out the stability of differential system (16), we linearise the non-linear terms appearing in both PDEs by assuming the value of $u(x,t) = U_j$ is constant. Thus, the stability of the ensuing linear differential system will imply the stability of the non-linear differential system. The stability of the differential system is related to the eigenvalues of matrix C. It is said to be stable if the real part of each eigenvalue is either zero or negative. This has been validated for the two differential systems under investigation for the various spatial grid points depicted in Figures 2 and 3. The differential system is stable in both cases.

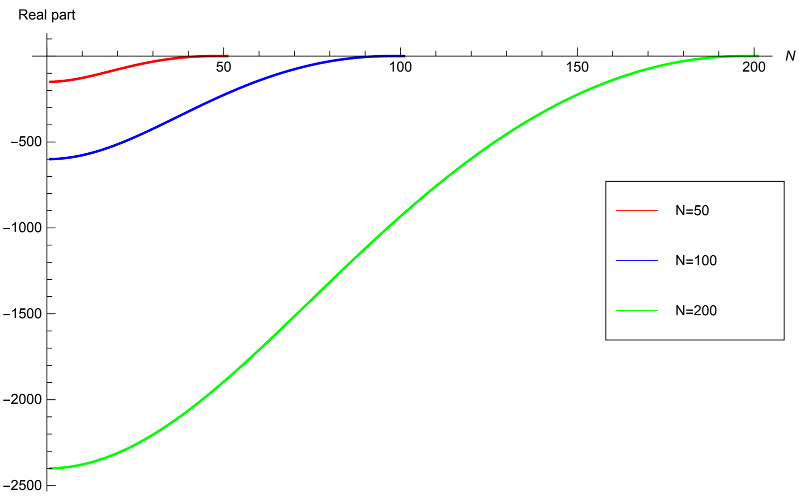

Figure 2. Real part of eigenvalues for Burgers' equation with $v = 0.01$.

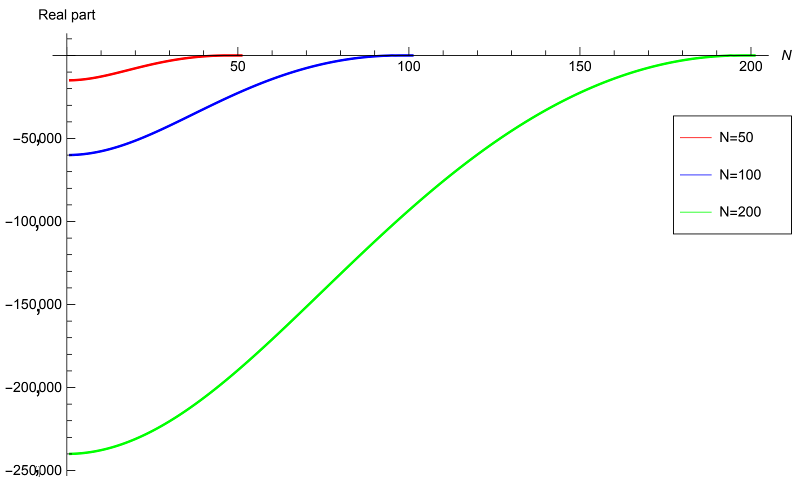

Figure 3. Real part of eigenvalues for FitzHugh–Nagumo equation using $\mu = 0.75$.

6. Numerical Experiments

In this section, some numerical experiments have been presented to illustrate the performance of the block method in conjunction with a compact finite difference scheme. We have used Wolfram Mathematicaversion 11.0 for performing numerical computations. The standard formulas are utilized to compute the L_∞ and L_{rms} errors [29,30]

$$L_\infty = \max_{1 \le i \le N+1} |e_i|$$

$$L_{rms} = \left(\sum_{i=1}^{N+1} \frac{e_i^2}{N+1} \right)^{1/2}$$

where

$$e_i = u(x_i, t) - U(x_i, t).$$

Here, $u(x_i, t)$ and $U(x_i, t)$ represent the analytical and numerical solutions at the point (x_i, t), respectively.

6.1. Nonlinear Burgers' Equation

6.1.1. Example 1

We consider the initial and boundary conditions for the Burgers' equation in (12a) given by

$$u(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1$$

$$u(0, t) = u(1, t) = 0, \quad t > 0$$

The exact solution is given as [16]

$$u(x, t) = 2v\pi \frac{\sum_{n=1}^{\infty} a_n exp(-n^2\pi^2 vt) n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} exp(-n^2\pi^2 vt) \cos n\pi x}$$

with the Fourier coefficients

$$a_0 = \int_0^1 exp(-2v\pi)^{-1}[1 - \cos(\pi x)]\, dx$$

$$a_n = 2\int_0^1 exp(-2v\pi)^{-1}[1 - \cos(\pi x)] \cos n\pi x\, dx, \quad n = 1, 2, 3...$$

In Figure 4, we have plotted the exact solution of the given PDE and its numerical solution obtained by the proposed method for a specific value of time $t = 0.5$ by considering various grid points as $x = 0.1, 0.3, 0.5, 0.7, 0.9$ with $N = 100, k = 0.001$ and $v = 0.01$. It shows that the physical behavior of both solutions is similar.

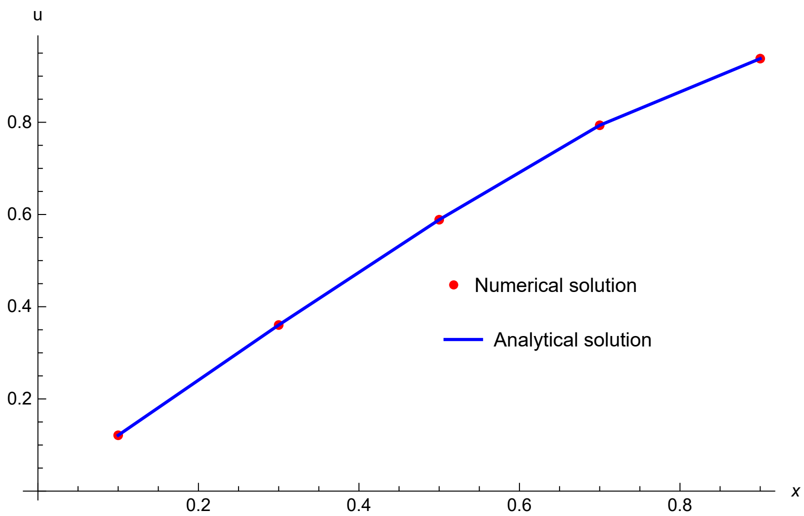

Figure 4. Numerical solution v/s Analytical solution at $t = 0.5$.

6.1.2. Example 2

Consider the test problem (12a) taking $v = 0.01$ and $a = 2$ subject to the initial and boundary conditions given by

$$u(x, 0) = \frac{2v\pi \sin(\pi x)}{a + \cos(\pi x)}, \quad 0 \leq x \leq 1$$

$$u(0, t) = u(1, t) = 0, \quad t > 0$$

gives the exact solution to the problem [20]

$$u(x,t) = \frac{2v\pi \sin(\pi x) \exp^{-\pi^2 vt}}{a + \cos(\pi x) \exp^{-\pi^2 vt}}$$

Table 1 presents absolute errors for $t = 0.1$ by applying the proposed scheme for $N = 20, k = 0.001$. It demonstrates that the proposed scheme integrates the given problem accurately.

Table 1. Absolute error at $t = 0.1$.

x	Absolute Error
0.1	4.6342×10^{-9}
0.2	1.4033×10^{-10}
0.3	1.9936×10^{-9}
0.4	1.2330×10^{-10}
0.5	4.9473×10^{-9}
0.6	2.3491×10^{-8}
0.7	3.8157×10^{-8}
0.8	6.0069×10^{-8}
0.9	1.5378×10^{-8}

In Tables 2 and 3, absolute error of the proposed scheme has been compared with [20] and [31] for various values of v and $t = 0.001$. We use the same number of time steps as in [20]. It shows that the proposed scheme offers better results.

Table 2. Comparison of results with $v = 1$ and $N = 40$.

x	Absolute Error (Asai [31])	Absolute Error (Mittal [20])	Absolute Error (The Proposed Scheme)
0.1	4.50×10^{-5}	7.40×10^{-5}	3.84×10^{-8}
0.2	7.70×10^{-5}	6.00×10^{-6}	5.45×10^{-9}
0.3	1.21×10^{-4}	1.20×10^{-5}	6.79×10^{-9}
0.4	2.40×10^{-5}	1.78×10^{-4}	1.23×10^{-9}
0.5	2.53×10^{-4}	3.90×10^{-5}	4.41×10^{-8}
0.6	3.56×10^{-4}	4.40×10^{-5}	1.63×10^{-7}
0.7	4.84×10^{-4}	1.00×10^{-5}	2.53×10^{-7}
0.8	3.32×10^{-4}	7.40×10^{-5}	3.48×10^{-7}
0.9	4.17×10^{-4}	2.81×10^{-4}	7.86×10^{-7}

Table 3. Comparison of results with $v = 0.5$ and $N = 40$.

x	Absolute Error (Asai [31])	Absolute Error (Mittal [20])	Absolute Error (The Proposed Scheme)
0.1	4.00×10^{-6}	0.000000	5.76×10^{-10}
0.2	9.00×10^{-6}	2.00×10^{-6}	1.47×10^{-9}
0.3	1.40×10^{-5}	3.00×10^{-6}	1.96×10^{-9}
0.4	2.20×10^{-5}	6.00×10^{-6}	2.82×10^{-10}
0.5	3.20×10^{-5}	1.00×10^{-5}	9.95×10^{-9}
0.6	4.90×10^{-5}	1.20×10^{-5}	4.02×10^{-9}
0.7	7.50×10^{-5}	7.50×10^{-5}	6.72×10^{-9}
0.8	4.50×10^{-5}	1.00×10^{-4}	1.58×10^{-9}
0.9	8.10×10^{-5}	7.40×10^{-5}	3.35×10^{-7}

Table 4 shows the Rate of Convergence *(ROC)* of the proposed scheme in the spatial direction for the values $v = 0.01$, $t = 0.001$ and $k = 0.0001$. It can be observed from Table 4 that the *ROC* agrees with the theoretical order of convergence of the proposed scheme in the spatial direction.

Table 4. $\mathcal{L}_\infty$-error and *ROC*.

N	$\mathcal{L}_\infty$-Error	ROC
40	8.08777×10^{-9}	
80	9.97993×10^{-10}	3.0186
160	3.83759×10^{-11}	4.7008
320	1.69230×10^{-12}	4.5031

6.2. Non-Linear FitzHugh–Nagumo Equation

6.2.1. Example 1

Consider the test problem (14a) using $\mu = 0.75$ along with initial condition as

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right), \quad -10 \leq x \leq 10.$$

The exact solution to the problem is [9]

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} - \frac{(2\mu - 1)t}{4}\right)$$

Table 5 compares the L_∞ error norm using the proposed scheme with some available data from Akkoyunlu [9] for different values of N at time $t = 0.2$. With the proposed scheme, we have almost the same or even better accuracy in the numerical approximation after just four applications, while the scheme presented in Akkoyunlu [9] has reached similar accuracy after 20 time steps. As a result, the proposed scheme produced similar errors in fewer iterations, saving computational effort.

Table 5. Comparison of L_∞-error norm at time $t = 0.2$.

N	L_∞-Error with CPU (sec.) (The Proposed Scheme)	L_∞ (Method in Akkoyunlu [9])
12	6.3673×10^{-4} (0.65)	3.9857×10^{-4}
24	4.7712×10^{-5} (2.34)	2.3475×10^{-5}
48	8.0504×10^{-6} (9.60)	8.3749×10^{-6}
64	4.7818×10^{-6} (17.75)	5.9363×10^{-6}
No. of iterations	**4**	**20**

In Table 6, we have compared the L_{rms} error for this problem using the proposed scheme with the results from Ahmad et al. [32] and Jiwari et al. [11] for $N = 100$ and $v = 0.75$ at different values of time. It must be mentioned here that the proposed scheme integrates the given problem with a large time step size and produces similar accuracy in just four iterations. In contrast, the approaches presented in [11,32] achieve similar accuracy with smaller time steps resulting in many iterations. Thus, the proposed method provides better or equal results for fewer iterations.

Table 6. Comparison with different approaches for Example1 with $\mu = 0.75$ and $N = 100$.

t	Ahmad [32] $\mathcal{L}_{rms}$-Error	Jiwari [11] $\mathcal{L}_{rms}$-Error	The Proposed Scheme $\mathcal{L}_{rms}$-Error with CPU (sec.)
0.2	2.1960×10^{-7}	1.5880×10^{-5}	4.0099×10^{-7} (47.89)
0.5	1.5696×10^{-6}	3.8433×10^{-5}	2.8629×10^{-6} (125.23)
1.0	7.1449×10^{-6}	8.1870×10^{-5}	1.3175×10^{-5} (268.56)
1.5	1.7262×10^{-5}	1.3387×10^{-4}	3.2282×10^{-5} (395.18)
2	3.1857×10^{-5}	1.9433×10^{-4}	6.1114×10^{-5} (527.95)
time step-size(k)	**0.0001**	**0.001**	**0.01**

6.2.2. Example 2

Consider the test problem given by (14a) taking $\mu = 0.5$ along with the initial condition as

$$u(x,0) = \frac{1}{1 + exp(\frac{-x}{\sqrt{2}})}, \quad 0 \le x \le 1.$$

where the exact solution is given by [33].

Table 7 compares errors produced by the proposed scheme and the approaches given in [33]. The errors produced by the schemes in Inan et al. [33] called ANM and ExpFDM are bigger than those obtained using the proposed scheme. Also, note that the proposed scheme uses only one iteration to integrate the problem.

Table 7. Comparison of maximum absolute error for Example 2 with $\mu = 0.5$ and $t = 0.04$.

x	The Proposed Scheme	ExpFDM [33]	ANM [33]
0.2	2.5011×10^{-7}	3.00×10^{-6}	2.00×10^{-7}
0.4	3.5686×10^{-7}	1.00×10^{-5}	5.00×10^{-7}
0.6	1.0422×10^{-7}	2.00×10^{-5}	7.00×10^{-7}
0.8	9.7036×10^{-7}	4.00×10^{-5}	6.00×10^{-7}
No. of iterations	**1**	**8**	**8**

6.3. PDE with Manufactured Solution

Consider the PDE

$$u_t - u_{xx} - u^2 = f(x,t) \tag{17}$$

whose exact solution will be manufactured. We will formulate its solution with the help of a technique called Method of manufactured solutions (MMS). In this method, we choose a function $u(x,t)$, which satisfies the initial and two boundary conditions. For the above PDE, one choice is $u(x,t) = x \sin t + 1 - x^2$. So, we substitute this value into the above differential equation to find $f(x,t)$.

Thus, we have the exact solution $u(x,t) = x \sin t + 1 - x^2$ to the initial-boundary value problem:

$$u_t - u_{xx} - u^2 = x \cos t + 2 + x^2 \sin^2 t + (1 - x^2)^2 + 2x(1 - x^2) \sin t \quad 0 \le t \le T, \quad 0 < x < 1 \tag{18}$$

with initial condition

$$u(x,0) = 1 - x^2$$

and boundary conditions

$$u(0,t) = 1, \quad u(1,t) = \sin t.$$

In Figure 5, we have plotted the manufactured solution of the given PDE and its numerical solution obtained by the proposed method, demonstrating the proposed method's good performance.

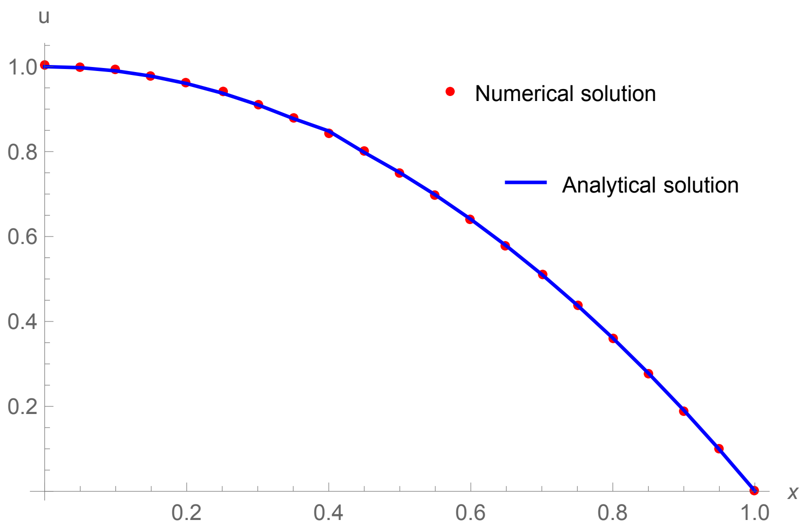

Figure 5. Numerical solution v/s Analytical solution for (17). We have plotted the numerical solution against the exact solution for the values $N = 20, k = 0.0001, t = 0.01$. It shows that the physical behavior of both the solutions is similar.

7. Conclusions

This article considers a five-step block method coupled with a compact finite difference scheme for numerically integrating time-dependent initial-boundary value PDEs. Theoretical development of the block method and its basic characteristics have been presented. The block method has very good stability characteristics with eighth-order accuracy. Further, a combined numerical scheme is obtained by coupling the block method with a compact finite difference scheme. The effectiveness of the presented approach has been demonstrated by applying it to two well-known test problems: Burgers' equation and FitzHugh–Nagumo equation. The approach considered in this article is a good alternative for solving the types of problems considered in the article.

Author Contributions: Conceptualization, K.K., G.S. and D.R.; methodology, K.K., G.S. and D.R.; software, K.K.; validation, K.K., G.S. and D.R.; formal analysis, K.K., G.S. and D.R.; investigation, K.K., G.S. and D.R.; resources, K.K., G.S. and D.R.; data curation, K.K., G.S. and D.R.; writing—original draft preparation, K.K.; writing—review and editing, G.S. and D.R.; visualization, K.K.; supervision, G.S. and D.R. ; project administration, K.K., G.S. and D.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data is associated with this manuscript.

Acknowledgments: We would like to thank the anonymous reviewers for their constructive comments that have greatly improved the quality of the article. Komalpreet Kaur would like to thank I.K. Gujral Punjab Technical University Jalandhar, Punjab (India) for providing research facilities for the present work.

Conflicts of Interest: The authors declare no conflicts of interest.

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