Ricerche di Matematica (2026) 75:561–579
https://doi.org/10.1007/s11587-025-01029-w

Local boundedness for weak solutions to strongly degenerate orthotropic parabolic equations

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Received: 6 October 2025 / Accepted: 11 November 2025 / Published online: 20 November 2025

Abstract

We prove the local boundedness of local weak solutions to the parabolic equation

$$\partial_t u = \sum_{i=1}^{n} \partial_{x_i} \left[(|u_{x_i}| - \delta_i)_+^{p-1} \frac{u_{x_i}}{|u_{x_i}|} \right] \qquad \text{in } \Omega_T = \Omega \times (0, T],$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $p \geq 2$, $\delta_1, \ldots, \delta_n$ are non-negative numbers and $(\,\cdot\,)_+$ denotes the positive part. The main novelty here is that the above equation combines an orthotropic structure with a strongly degenerate behavior. The core result of this paper thus extends a classical boundedness theorem, originally proved for the parabolic p-Laplacian, to a widely degenerate anisotropic setting. As a byproduct, we also obtain the local boundedness of local weak solutions to the isotropic counterpart of the above equation.

Keywords Degenerate parabolic equations · Anisotropic equations · Local boundedness · De Giorgi iteration

Mathematics Subject Classification 35B45 · 35B65 · 35K10 · 35K65 · 35K92

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $T \in (0, \infty)$. We are interested in the local boundedness of local weak solutions to the following parabolic equation

Dedicated to Antonia Passarelli di Napoli on the occasion of her 60th birthday, celebrated during the “Workshop on Variational Problems and PDEs” (University of Naples “Parthenope”, June 19-20, 2025).

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$$\partial_t u \;=\; \sum_{i=1}^{n} \partial_{x_i}\left[(|u_{x_i}|-\delta_i)_+^{p-1}\frac{u_{x_i}}{|u_{x_i}|}\right] \qquad \text{in } \;\Omega_T=\Omega\times(0,T]\,, \tag{1.1}$$

where $p \geq 2$, $\delta_1,\ldots,\delta_n$ are non-negative numbers and $(\,\cdot\,)_+$ stands for the positive part. Evolutionary equations of this form have been studied since the 1960s, with significant contributions from the Soviet school; see, for instance, the work [29] by Vishik. Equation (1.1) with all δ_i set to zero is also presented explicitly in several monographs, including [26], [27, Example 4.A, Chapter III] and [30, Example 30.8], among others.

Let us first observe that (1.1) looks quite similar to the parabolic p-Laplace equation

$$\partial_t u \;=\; \sum_{i=1}^{n} (|Du|^{p-2}\,u_{x_i})_{x_i} \qquad \text{in } \;\Omega_T\,. \tag{1.2}$$

However, the main novelty of equation (1.1) is that it couples the following two features:

orthotropic structure and *strongly degenerate behavior* .

Indeed, unlike the parabolic p-Laplace equation, for which the loss of ellipticity of the operator $\operatorname{div}(|Du|^{p-2}Du)$ is restricted to a single point, equation (1.1) becomes degenerate on the larger set

$$\bigcup_{i=1}^{n}\{|u_{x_i}|\leq\delta_i\}\,.$$

A more recent work in which equation (1.1) appears with all δ_i equal to zero is [8]. There, the authors establish local L^∞ estimates for the spatial gradient of local weak solutions to (1.1), but confining their analysis to the case $p \geq 2$ and $\max\{\delta_i\}=0$. In this special case, as already noted in [8], the basic regularity theory equally applies to both (1.1) and (1.2). A classical reference in the field is DiBenedetto's monograph [15], which provides boundedness results for the solution u (see [15, Chapter V]), Hölder continuity estimates for u (see [15, Chapter III]), as well as Harnack inequalities for non-negative solutions (see [15, Chapter VI]). From a technical standpoint, there is no distinction to be made between (1.2) and (1.1) with all δ_i set to zero. Consequently, the results in [8] and [15, Chapter V] imply that, in the case $p \geq 2$ and $\max\{\delta_i\}=0$, the local weak solutions of (1.1) are locally Lipschitz continuous in the spatial variable, uniformly in time.

The primary goal of this paper is to prove that the local weak solutions of (1.1) are locally bounded even when $\max\{\delta_i\}>0$, thus extending DiBenedetto's result [15, Chapter V, Theorem 4.1] to our anisotropic and more degenerate setting. More precisely, our main result reads as follows. For notation and definitions we refer to Section 2.

Theorem 1.1 *Let $n \geq 2$ and $p \geq 2$. Moreover, assume that*

$$u\,\in\,C^0_{loc}\left(0,T;L^2_{loc}(\Omega)\right)\cap L^p_{loc}\left(0,T;W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (1.1). Then $u \in L^{\infty}_{loc}(\Omega_T)$. *More precisely, for every cylinder* $[(x_0, t_0) + Q(\theta, \rho)] \subset \Omega_T$ *and every* $\sigma \in (0, 1)$, *we have that:*

(a) if $p > 2$, *the estimate*

$$\operatorname*{ess\,sup}_{[(x_0,t_0)+Q(\sigma\theta,\sigma\rho)]} |u| \le \max\left\{\rho, \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}}, \frac{C}{(1-\sigma)^{\frac{n+p}{2}}} \sqrt{\frac{\theta}{\rho^p}} \left(\fint\!\!\fint_{[(x_0,t_0)+Q(\theta,\rho)]} |u|^p \,dx\,dt\right)^{\frac{1}{2}}\right\} \tag{1.3}$$

holds true for some positive constant C depending only on n, p and $\max\{\delta_1, \ldots, \delta_n\}$;

(b) if $p = 2$, *the estimate*

$$\operatorname*{ess\,sup}_{[(x_0,t_0)+Q(\sigma\theta,\sigma\rho)]} |u| \le \max\left\{\rho, \frac{C}{(1-\sigma)^{\frac{n+2}{2}}} \sqrt{\left(\frac{\rho^2}{\theta}\right)^{\frac{n}{2}} + \frac{\theta}{\rho^2}} \left(\fint\!\!\fint_{[(x_0,t_0)+Q(\theta,\rho)]} |u|^2 \,dx\,dt\right)^{\frac{1}{2}}\right\} \tag{1.4}$$

holds true for some positive constant C depending only on n and $\max\{\delta_1, \ldots, \delta_n\}$.

The proof of Theorem 1.1 relies on an adaptation of De Giorgi's iteration technique; see, for instance, [24] for the nondegenerate case. In Section 3, we first establish a local energy estimate (Proposition 3.1), which allows us to control the superlevel sets of the local weak solution u through suitable cut-off functions. We then construct a sequence of shrinking cylinders Q_j and increasing levels $k_j > 0$, and use the energy estimate to derive recursive inequalities for the integral quantities

$$Y_j := \fint\!\!\fint_{Q_j} (u - k_j)_+^p \,dx\,dt\,.$$

These inequalities fall within the scope of a general iteration lemma (Lemma 2.4), ensuring that $Y_j \to 0$ as $j \to \infty$. Consequently, the measure of the limiting superlevel set of u vanishes, which yields the local boundedness of u and the local L^{∞} estimates (1.3)–(1.4).

It is worth recalling that, in the elliptic setting, the local boundedness of solutions to anisotropic problems has been extensively investigated, as can be seen, for instance, in [4, 6, 9, 11–14, 17–19, 23, 25, 28]. Pioneering contributions are due to Kolodiĭ [22]. We further mention the more recent work by DiBenedetto, Gianazza and Vespri [16], where precise a priori L^{∞} estimates for the solutions are established (see Section 6 there). For an insight into the parabolic anisotropic setting, we recall that the local boundedness of weak solutions is indeed a direct consequence of Caccioppoli's estimates; see, for example, [10, Section 6] and the references therein.

In the final part of this paper, we will focus on the local weak solutions of the parabolic equation

$$\partial_t u - \operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|}\right) = 0 \qquad \text{in } \Omega_T\,, \tag{1.5}$$

where $p \ge 2$ and $\lambda > 0$ is a fixed parameter. Indeed, by slightly modifying the proof of Theorem 1.1, we can establish the following L^{∞}-regularity result.

Corollary 1.2 *Let $n \geq 2$, $p \geq 2$ and $\lambda > 0$. Moreover, assume that*

$$u \in C^0_{loc}\left(0, T; L^2_{loc}(\Omega)\right) \cap L^p_{loc}\left(0, T; W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (1.5). Then $u \in L^\infty_{loc}(\Omega_T)$. More precisely, for every cylinder $[(x_0, t_0) + Q(\theta, \rho)] \subset \Omega_T$ and every $\sigma \in (0, 1)$, we have that:

(a) if $p > 2$, the estimate

$$\operatorname*{ess\,sup}_{[(x_0,t_0)+Q(\sigma\theta,\sigma\rho)]} |u| \leq \max\left\{\rho, \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}}, \frac{C}{(1-\sigma)^{\frac{n+p}{2}}} \sqrt{\frac{\theta}{\rho^p}} \left(\iint_{[(x_0,t_0)+Q(\theta,\rho)]} |u|^p \, dx\, dt\right)^{\frac{1}{2}}\right\} \tag{1.6}$$

holds true for some positive constant C depending only on n, p and λ;

(b) if $p = 2$, the estimate

$$\operatorname*{ess\,sup}_{[(x_0,t_0)+Q(\sigma\theta,\sigma\rho)]} |u| \leq \max\left\{\rho, \frac{C}{(1-\sigma)^{\frac{n+2}{2}}} \sqrt{\left(\frac{\rho^2}{\theta}\right)^{\frac{n}{2}} + \frac{\theta}{\rho^2}} \left(\iint_{[(x_0,t_0)+Q(\theta,\rho)]} |u|^2 \, dx\, dt\right)^{\frac{1}{2}}\right\} \tag{1.7}$$

holds true for some positive constant C depending only on n and λ.

When $\lambda > 0$, the main feature of equation (1.5) is that it exhibits a strong degeneracy, coming from the fact that its modulus of ellipticity vanishes in the region $\{|Du| \leq \lambda\}$, and hence its principal part behaves like a p-Laplace operator only for large values of $|Du|$.

The gradient regularity of weak solutions to (1.5) has been recently studied in [1–3, 5, 7, 20]. In particular, in [3] the authors establish local L^∞ bounds for the spatial gradient of solutions to equations and systems of the form (1.5) in the whole range $p > 1$. Therefore, by combining the results of [3] with Corollary 1.2, one obtains that local weak solutions of (1.5) are locally Lipschitz continuous in the spatial variable, uniformly in time.

Remark 1.3 If the intrinsic relation $\theta = \rho^p$ is imposed, the explicit geometric ratios on the right-hand side of (1.3), (1.4), (1.6) and (1.7) reduce to dimension-free constants and, in this sense, the aforementioned estimates are "dimensionless". However, if $\theta = \rho^p$, (1.3) and (1.6) are not homogeneous in u.

1.1 Plan of the paper

The paper is organized as follows. Section 2 is devoted to the preliminaries: after a list of classical notations and some essential lemmas, we recall the basic properties of Steklov averages. In Section 3, we establish a local energy estimate for the local weak solutions of (1.1). This estimate is then used in Section 4 to derive a family of local iterative inequalities. In Section 5, these inequalities are employed to complete the proof of Theorem 1.1. Finally, in the same section, we include the proof of Corollary 1.2 for completeness.

2 Notation and preliminaries

In this paper we shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^n$ will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^n$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

In what follows, we use the notation

$$K_\rho := (-\rho, \rho)^n, \qquad \rho > 0,$$

for the n-dimensional open cube centered at the origin with side length 2ρ. If $x_0 \in \mathbb{R}^n$, we denote by $[x_0 + K_\rho]$ the cube of center x_0 and side length 2ρ which is congruent to K_ρ, i.e.,

$$[x_0 + K_\rho] := \left\{ x \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - x_{0,i}| < \rho \right\}.$$

Moreover, for a positive number θ, we consider the cylinder

$$Q(\theta, \rho) := K_\rho \times (-\theta, 0),$$

and if $(x_0, t_0) \in \mathbb{R}^{n+1}$, we let $[(x_0, t_0) + Q(\theta, \rho)]$ denote the cylinder with vertex at (x_0, t_0) congruent to $Q(\theta, \rho)$, i.e.,

$$[(x_0, t_0) + Q(\theta, \rho)] := [x_0 + K_\rho] \times (t_0 - \theta, t_0).$$

For a general cylinder $Q = B \times (t_0, t_1)$, where $B \subset \mathbb{R}^n$ and $t_0 < t_1$, we denote by

$$\partial_{\text{par}} Q := (\overline{B} \times \{t_0\}) \cup (\partial B \times (t_0, t_1))$$

the usual *parabolic boundary* of Q, which is nothing but its standard topological boundary without the upper cap $\overline{B} \times \{t_1\}$.

If $E \subseteq \mathbb{R}^k$ is a Lebesgue-measurable set, then we will denote by $|E|$ its k-dimensional Lebesgue measure. When $0 < |E| < \infty$, the mean value of a function $v \in L^1(E)$ is defined by

$$\fint_E v(y)\, dy := \frac{1}{|E|} \int_E v(y)\, dy.$$

Now let $F : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}$ be the functions defined respectively by

$$F(\xi) := \sum_{i=1}^n \frac{1}{p} (|\xi_i| - \delta_i)_+^p \qquad \text{and} \qquad G(\xi) := \frac{1}{p} (|\xi| - \lambda)_+^p. \tag{2.1}$$

In this work, we define a local weak solution of (1.1) and of (1.5) as follows.

Definition 2.1 A function u is a *local weak solution* of equation (1.1) if

$$u \in C^0_{loc}\left(0,T;L^2_{loc}(\Omega)\right)\cap L^p_{loc}\left(0,T;W^{1,p}_{loc}(\Omega)\right)$$

and, for every test function $\varphi\in C_0^\infty(\Omega_T)$,

$$\iint_{\Omega_T}\left(u\,\partial_t\varphi-\langle D_\xi F(Du),D\varphi\rangle\right)dx\,dt\;=\;0\,. \tag{2.2}$$

Definition 2.2 A function u is a *local weak solution* of equation (1.5) if

$$u \in C^0_{loc}\left(0,T;L^2_{loc}(\Omega)\right)\cap L^p_{loc}\left(0,T;W^{1,p}_{loc}(\Omega)\right)$$

and, for every test function $\varphi\in C_0^\infty(\Omega_T)$,

$$\iint_{\Omega_T}\left(u\,\partial_t\varphi-\langle D_\xi G(Du),D\varphi\rangle\right)dx\,dt\;=\;0\,. \tag{2.3}$$

We now recall some tools that will be useful to prove our results. We begin with the following interpolation inequality, whose proof can be found in [15, Proposition 3.1, Chapter I].

Lemma 2.3 *Let $n\geq 2$ and $1\leq r,s<\infty$. Then, there exists a positive constant C, depending only on n, r and s, such that for every $v\in L^\infty(0,T;L^r(\Omega))\cap L^s\left(0,T;W_0^{1,s}(\Omega)\right)$ we have*

$$\iint_{\Omega_T}|v(x,t)|^q\,dx\,dt\;\leq\;C^q\left(\iint_{\Omega_T}|Dv(x,t)|^s\,dx\,dt\right)\left(\operatorname*{ess\,sup}_{0<t<T}\int_\Omega|v(x,t)|^r\,dx\right)^{\frac{s}{n}},$$

where $q=s\,\dfrac{n+r}{n}$.

The next lemma is crucial to establish our main result; see [21, Lemma 7.1] for a proof.

Lemma 2.4 *Let $\alpha>0$ and let $\{Y_j\}_{j\in\mathbb{N}_0}$ be a sequence of positive real numbers, satisfying the recursive inequalities*

$$Y_{j+1}\;\leq\;C\,b^j\,Y_j^{1+\alpha}$$

where $C>0$ and $b>1$. If $Y_0\leq C^{-\frac{1}{\alpha}}\,b^{-\frac{1}{\alpha^2}}$, then

$$\lim_{j\to\infty}Y_j=0\,.$$

2.1 Steklov averages

In this section, we recall the definition and some elementary properties of Steklov averages. Let us denote a domain in space-time by $Q' := \Omega' \times I$, where $\Omega' \subseteq \Omega$ is a bounded domain and $I := (t_0, t_1) \subseteq (0, T)$. For every $h \in (0, t_1 - t_0)$ and $v \in L^1(\Omega' \times I, \mathbb{R}^k)$, where $k \in \mathbb{N}$, the *Steklov average* $[v]_h(\cdot, t)$ is defined by

$$[v]_h(x,t) := \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} v(x,s)\,ds & \text{if } t \in (t_0, t_1 - h], \\ 0 & \text{if } t \in (t_1 - h, t_1), \end{cases}$$

for $x \in \Omega'$. This definition implies, for almost every $(x, t) \in \Omega' \times (t_0, t_1 - h)$,

$$\frac{\partial [v]_h}{\partial t}(x,t) = \frac{v(x,t+h) - v(x,t)}{h}\,.$$

The proof of the following result is straightforward from the theory of Lebesgue spaces (see [15, Lemma 3.2, Chapter I]).

Lemma 2.5 *Let $q, r \geq 1$ and $v \in L^r\left(t_0, t_1; L^q(\Omega')\right)$. Then, as $h \to 0$, $[v]_h$ converges to v in $L^r\left(t_0, t_1 - \varepsilon; L^q(\Omega')\right)$ for every $\varepsilon \in (0, t_1 - t_0)$. If $v \in C^0\left(t_0, t_1; L^q(\Omega')\right)$, then as $h \to 0$, $[v]_h(\cdot, t)$ converges to $v(\cdot, t)$ in $L^q(\Omega')$ for every $t \in (t_0, t_1 - \varepsilon)$, $\forall\, \varepsilon \in (0, t_1 - t_0)$.*

A very useful formulation, equivalent to (2.2), is the one involving Steklov averages. Assume that $u \in C^0_{loc}\left(0, T; L^2_{loc}(\Omega)\right) \cap L^p_{loc}\left(0, T; W^{1,p}_{loc}(\Omega)\right)$ is a local weak solution of (1.1) in Ω_T and let $h \in (0, T)$. Then, the Steklov average $[u]_h$ satisfies

$$\int_{\mathcal{K} \times \{\tau\}} \left(\frac{\partial [u]_h}{\partial t} \cdot \varphi + \langle [D_\xi F(Du)]_h, D\varphi \rangle \right) dx = 0 \tag{2.4}$$

for every compact subset $\mathcal{K}$ of Ω, for all $\tau \in (0, T - h]$ and all test functions

$$\varphi \in C^0_{loc}\left(0, T; L^2(\mathcal{K})\right) \cap L^p_{loc}\left(0, T; W^{1,p}_0(\mathcal{K})\right).$$

If u is a local weak solution of (1.5), then the Steklov average formulation of (2.3) is obtained by simply replacing $D_\xi F(Du)$ with $D_\xi G(Du)$ in (2.4).

3 A local energy estimate

The proof of Theorem 1.1 is based on the following local energy estimate. Throughout this section and the sequel, $(x_0, t_0) \in \Omega_T$ and $\theta, \rho > 0$ are such that

$[(x_0, t_0) + Q(\theta, \rho)] \subset \Omega_T$, while ζ denotes a piecewise smooth cut-off function in $[(x_0, t_0) + Q(\theta, \rho)]$ satisfying

$$\begin{aligned} &0 \le \zeta \le 1, \qquad \|D\zeta\|_\infty < +\infty, \\ &\zeta \equiv 0 \quad \text{on the parabolic boundary of } [(x_0, t_0) + Q(\theta, \rho)]. \end{aligned}$$

Proposition 3.1 *Let $n \ge 2$ and $p \ge 2$. Moreover, assume that u is a local weak solution of equation (1.1). Then, for every level $k > 0$ we have*

$$\begin{aligned} &\sup_{t_0-\theta<\tau<t_0} \int_{[x_0+K_\rho]} (u-k)_+^2\, \zeta^p(x,\tau)\, dx \\ &\quad + \sum_{i=1}^n \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (|u_{x_i}| - \delta_i)_+^p\, \zeta^p\, \mathbb{1}_{\{u>k\}}\, dx\, dt \\ &\quad \le p \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (u-k)_+^2\, \zeta^{p-1}\, \partial_t \zeta\, dx\, dt \\ &\quad + C \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (u-k)_+^p\, |D\zeta|^p\, dx\, dt \end{aligned} \tag{3.1}$$

for a positive constant C depending only on n and p.

Proof After a translation, we may assume that $(x_0, t_0) = (0, 0)$. Hence, it suffices to prove (3.1) for the cylinder $Q(\theta, \rho)$. In (2.4) we take the test functions

$$\varphi = ([u]_h - k)_+\, \zeta^p$$

and integrate with respect to time over $(-\theta, \tau)$, with $\tau \in (-\theta, 0)$. We thus obtain

$$\int_{-\theta}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+\, \zeta^p\, dx\, dt + \iint_{Q^\tau} \langle [A(Du)]_h, D[([u]_h - k)_+\, \zeta^p] \rangle\, dx\, dt = 0, \tag{3.2}$$

where, for convenience of notation, we have set

$$Q^\tau := K_\rho \times (-\theta, \tau) \qquad \text{and} \qquad A(\eta) := D_\xi F(\eta), \quad \eta \in \mathbb{R}^n.$$

The first term in (3.2) can be rewritten as

$$\int_{-\theta}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+\, \zeta^p\, dx\, dt = \frac{1}{2} \int_{-\theta}^{\tau} \int_{K_\rho} \frac{\partial ([u]_h - k)_+^2}{\partial t}\, \zeta^p\, dx\, dt.$$

Therefore, integrating by parts, using that $\zeta \equiv 0$ on $\partial_{\text{par}} Q(\theta, \rho)$ and letting $h \to 0$, by Lemma 2.5 we have

$$\int_{-\theta}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+ \, \zeta^p \, dx \, dt \longrightarrow \\ \frac{1}{2} \int_{K_\rho} (u-k)_+^2 \, \zeta^p(x,\tau) \, dx - \frac{p}{2} \iint_{Q^\tau} (u-k)_+^2 \, \zeta^{p-1} \, \partial_t \zeta \, dx \, dt . \tag{3.3}$$

Now observe that $|A(Du)| \leq \sqrt{n}\, |Du|^{p-1}$. Then, taking the limit as $h \to 0$ in the second term of (3.2), we can apply Lemma 2.5 again. Thus we get

$$\iint_{Q^\tau} \langle [A(Du)]_h, D[([u]_h - k)_+ \, \zeta^p] \rangle \, dx \, dt \longrightarrow \\ \iint_{Q^\tau} \langle A(Du), D(u-k)_+ \rangle \, \zeta^p \, dx \, dt + p \iint_{Q^\tau} \langle A(Du), D\zeta \rangle \, (u-k)_+ \, \zeta^{p-1} \, dx \, dt . \tag{3.4}$$

From (3.2)–(3.4), we then obtain

$$\frac{1}{2} \int_{K_\rho} (u-k)_+^2 \, \zeta^p(x,\tau) \, dx + \iint_{Q^\tau} \langle A(Du), D(u-k)_+ \rangle \, \zeta^p \, dx \, dt \\ = \frac{p}{2} \iint_{Q^\tau} (u-k)_+^2 \, \zeta^{p-1} \, \partial_t \zeta \, dx \, dt - p \iint_{Q^\tau} \langle A(Du), D\zeta \rangle \, (u-k)_+ \, \zeta^{p-1} \, dx \, dt .$$

We now estimate

$$\iint_{Q^\tau} \langle A(Du), D(u-k)_+ \rangle \, \zeta^p \, dx \, dt = \sum_{i=1}^{n} \iint_{Q^\tau \cap \{u>k\}} (|u_{x_i}| - \delta_i)_+^{p-1} \, |u_{x_i}| \, \zeta^p \, dx \, dt \\ \geq \sum_{i=1}^{n} \iint_{Q^\tau \cap \{u>k\}} (|u_{x_i}| - \delta_i)_+^{p} \, \zeta^p \, dx \, dt .$$

Furthermore, applying Young's inequality with $\varepsilon > 0$, we have

$$- p \iint_{Q^\tau} \langle A(Du), D\zeta \rangle \, (u-k)_+ \, \zeta^{p-1} \, dx \, dt \\ \leq p \sum_{i=1}^{n} \iint_{Q^\tau} (|u_{x_i}| - \delta_i)_+^{p-1} \, |\zeta_{x_i}| \, (u-k)_+ \, \zeta^{p-1} \, dx \, dt \\ \leq \varepsilon(p-1) \sum_{i=1}^{n} \iint_{Q^\tau \cap \{u>k\}} (|u_{x_i}| - \delta_i)_+^{p} \, \zeta^p \, dx \, dt \\ + \frac{1}{\varepsilon^{p-1}} \sum_{i=1}^{n} \iint_{Q^\tau} |\zeta_{x_i}|^p \, (u-k)_+^p \, dx \, dt .$$

Choosing $\varepsilon = \frac{1}{2(p-1)}$ and collecting the three previous estimates, we get

$$\int_{K_\rho} (u-k)_+^2 \, \zeta^p(x,\tau)\, dx \; + \sum_{i=1}^{n} \iint_{Q^\tau \cap \{u>k\}} (|u_{x_i}| - \delta_i)_+^p \, \zeta^p \, dx\, dt$$
$$\leq \; p \iint_{Q^\tau} (u-k)_+^2 \, \zeta^{p-1}\, \partial_t \zeta \, dx\, dt \; + C \iint_{Q^\tau} (u-k)_+^p \, |D\zeta|^p \, dx\, dt\,,$$

where C is a positive constant depending only on n and p. Recalling that $\tau \in (-\theta, 0)$ is arbitrary, from the above inequality we obtain

$$\sup_{-\theta<\tau<0} \int_{K_\rho} (u-k)_+^2 \, \zeta^p(x,\tau)\, dx \; + \sum_{i=1}^{n} \iint_{Q(\theta,\rho)} (|u_{x_i}| - \delta_i)_+^p \, \zeta^p \, \mathbb{1}_{\{u>k\}} \, dx\, dt$$
$$\leq \; p \iint_{Q(\theta,\rho)} (u-k)_+^2 \, \zeta^{p-1}\, \partial_t \zeta \, dx\, dt \; + \; C \iint_{Q(\theta,\rho)} (u-k)_+^p \, |D\zeta|^p \, dx\, dt\,.$$

This concludes the proof. □

4 Local iterative inequalities

An essential ingredient in the proof of Theorem 1.1 is a family of iterative inequalities. We shall now derive them, starting from the energy estimate (3.1). After a translation, we may assume that (x_0, t_0) coincides with the origin. Fixed $\sigma \in (0,1)$, we consider the sequences

$$\rho_j := \sigma\rho + \frac{(1-\sigma)}{2^j}\,\rho\,, \qquad \theta_j := \sigma\theta + \frac{(1-\sigma)}{2^j}\,\theta\,, \qquad j \in \mathbb{N}_0\,,$$

and the corresponding cylinders $Q_j := Q(\theta_j, \rho_j)$. From the definitions it follows that

$$Q_0 = Q(\theta,\rho) \qquad \text{and} \qquad Q_\infty = Q(\sigma\theta, \sigma\rho)\,.$$

We also consider the family of boxes

$$\widetilde{Q}_j := Q(\tilde{\theta}_j, \tilde{\rho}_j)\,,$$

where, for $j \in \mathbb{N}_0$,

$$\tilde{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2} = \sigma\rho + \frac{3(1-\sigma)}{2^{j+2}}\,\rho\,, \qquad \tilde{\theta}_j := \frac{\theta_j + \theta_{j+1}}{2} = \sigma\theta + \frac{3(1-\sigma)}{2^{j+2}}\,\theta\,.$$

For these boxes, we have the inclusions

$$Q_{j+1} \subset \widetilde{Q}_j \subset Q_j\,, \qquad j \in \mathbb{N}_0\,. \tag{4.1}$$

We now introduce the sequence of increasing levels

$$k_j := k - \frac{k}{2^j}\,, \tag{4.2}$$

where k is a positive number to be chosen later. We shall work with inequality (3.1) written for the functions $(u-k_{j+1})_+$, over the cylinders Q_j. The piecewise smooth cut-off function ζ is taken to satisfy

$$\begin{cases} 0 \le \zeta \le 1\,, \quad \zeta \equiv 0 \quad \text{on } \partial_{\text{par}} Q_j\,, \quad \zeta \equiv 1 \quad \text{in } \widetilde{Q}_j\,, \\ |D\zeta| \le \dfrac{2^{j+2}\,c}{(1-\sigma)\rho}\,, \quad 0 \le \partial_t \zeta \le \dfrac{2^{j+2}\,c}{(1-\sigma)\theta}\,. \end{cases} \tag{4.3}$$

With these choices, estimate (3.1) yields

$$\begin{aligned} &\sup_{-\theta_j<\tau<0} \int_{K_{\rho_j}} (u-k_{j+1})_+^2\, \zeta^p(x,\tau)\,dx + \sum_{i=1}^{n} \iint_{Q_j\cap\{u>k_{j+1}\}} (|u_{x_i}|-\delta_i)_+^p\, \zeta^p\,dx\,dt \\ &\quad\le \frac{C_1\,2^j}{(1-\sigma)\theta} \iint_{Q_j} (u-k_{j+1})_+^2\,dx\,dt + \frac{C_1\,2^{jp}}{(1-\sigma)^p\,\rho^p} \iint_{Q_j} (u-k_{j+1})_+^p\,dx\,dt\,, \end{aligned} \tag{4.4}$$

where $C_1 \equiv C_1(n,p) > 0$. From definition (4.2), we immediately have

$$\iint_{Q_j} (u-k_{j+1})_+^p\,dx\,dt \le \iint_{Q_j} (u-k_j)_+^p\,dx\,dt\,. \tag{4.5}$$

Now observe that, for all $s>0$,

$$\begin{aligned} \iint_{Q_j} (u-k_j)_+^s\,dx\,dt &\ge \iint_{Q_j\cap\{u>k_{j+1}\}} (u-k_j)_+^s\,dx\,dt \\ &\ge (k_{j+1}-k_j)^s\,|A_{j+1}| \\ &= \frac{k^s}{2^{(j+1)s}}\,|A_{j+1}|\,, \end{aligned} \tag{4.6}$$

where we have set

$$|A_{j+1}| := \text{meas}\left\{(x,t)\in Q_j : u(x,t) > k_{j+1}\right\}. \tag{4.7}$$

Then, using Hölder's inequality, (4.5) and (4.6), we get

$$\begin{aligned} \iint_{Q_j} (u-k_{j+1})_+^2\,dx\,dt &\le \left(\iint_{Q_j} (u-k_{j+1})_+^p\,dx\,dt\right)^{\frac{2}{p}} |A_{j+1}|^{1-\frac{2}{p}} \\ &\le \frac{2^{(p-2)(j+1)}}{k^{p-2}} \iint_{Q_j} (u-k_j)_+^p\,dx\,dt\,. \end{aligned} \tag{4.8}$$

Combining estimates (4.4), (4.5) and (4.8), and applying the properties $(4.3)_1$ of ζ, we obtain the following basic iterative inequalities:

$$\sup_{-\tilde{\theta}_j<\tau<0}\int_{K_{\tilde{\rho}_j}}(u(x,\tau)-k_{j+1})_+^2\,dx+\sum_{i=1}^{n}\iint_{\widetilde{Q}_j\cap\{u>k_{j+1}\}}(|u_{x_i}|-\delta_i)_+^p\,dx\,dt$$
$$\leq\frac{C_1\,2^{jp}}{(1-\sigma)^p}\left(\frac{1}{\theta\,k^{p-2}}+\frac{1}{\rho^p}\right)\iint_{Q_j}(u-k_j)_+^p\,dx\,dt\,. \tag{4.9}$$

To move forward, we construct a piecewise smooth cut-off function $\tilde{\zeta}_j$ in $\widetilde{Q}_j$ such that

$$\begin{cases}0\leq\tilde{\zeta}_j\leq 1\,, & \tilde{\zeta}_j\equiv 0\quad\text{on the lateral boundary of }\widetilde{Q}_j,\\ \tilde{\zeta}_j\equiv 1\quad\text{in }Q_{j+1}\,, & |D\tilde{\zeta}_j|\leq\dfrac{2^{j+2}}{(1-\sigma)\rho}\,.\end{cases}$$

Then the function $(u-k_{j+1})_+\,\tilde{\zeta}_j$ vanishes on the lateral boundary of $\widetilde{Q}_j$ and, by Lemma 2.3, we have

$$\iint_{\widetilde{Q}_j}(u-k_{j+1})_+^q\,\tilde{\zeta}_j^q\,dx\,dt\leq C_2\left(\iint_{\widetilde{Q}_j}|D\,(u-k_{j+1})_+|^p\,dx\,dt\right.$$
$$\left.+\iint_{\widetilde{Q}_j}(u-k_{j+1})_+^p\,|D\tilde{\zeta}_j|^p\,dx\,dt\right)$$
$$\times\left(\sup_{-\tilde{\theta}_j<\tau<0}\int_{K_{\tilde{\rho}_j}}(u(x,\tau)-k_{j+1})_+^2\,dx\right)^{\frac{p}{n}}, \tag{4.10}$$

where

$$q:=p\,\frac{n+2}{n} \tag{4.11}$$

and C_2 is a positive constant depending only on n and p.

At this point, we introduce the sequence of dimensionless quantities

$$Y_j:=\fint\!\!\fint_{Q_j}(u-k_j)_+^p\,dx\,dt\,,\qquad j\in\mathbb{N}_0\,. \tag{4.12}$$

We shall derive an iterative inequality for Y_j by estimating the right-hand side of (4.10) by (4.9). Prior to this, by lengthy but elementary computations, we see that

$$\frac{|\widetilde{Q}_j|}{|Q_{j+1}|}<\left(\frac{3}{2}\right)^{n+1}\qquad\text{and}\qquad\frac{|Q_j|}{|\widetilde{Q}_j|}<4^{n+1}\,. \tag{4.13}$$

Therefore, using the properties of $\tilde{\zeta}_j$, (4.1), (4.13), Hölder's inequality, (4.6) with $s = p$ and (4.12), we obtain

$$\begin{aligned}
Y_{j+1} &= \fint\!\!\fint_{Q_{j+1}} (u-k_{j+1})_+^p \,\tilde{\zeta}_j^p \,dx\,dt \;\le\; \frac{|\widetilde{Q}_j|}{|Q_{j+1}|} \fint\!\!\fint_{\widetilde{Q}_j} (u-k_{j+1})_+^p \,\tilde{\zeta}_j^p \,dx\,dt \\
&\le \left(\frac{3}{2}\right)^{n+1} |\widetilde{Q}_j|^{\frac{p}{q}-1} \left(\fint\!\!\fint_{\widetilde{Q}_j} (u-k_{j+1})_+^q \,\tilde{\zeta}_j^q \,dx\,dt \right)^{\frac{p}{q}} |A_{j+1}|^{1-\frac{p}{q}} \\
&\le 6^{n+1} \left(\fint\!\!\fint_{\widetilde{Q}_j} (u-k_{j+1})_+^q \,\tilde{\zeta}_j^q \,dx\,dt \right)^{\frac{p}{q}} \left(\frac{|A_{j+1}|}{|Q_j|} \right)^{1-\frac{p}{q}} \\
&\le C_3 \left(\fint\!\!\fint_{\widetilde{Q}_j} (u-k_{j+1})_+^q \,\tilde{\zeta}_j^q \,dx\,dt \right)^{\frac{p}{q}} \left(\frac{2^{jp}}{k^p} Y_j \right)^{1-\frac{p}{q}},
\end{aligned}$$

where $C_3 \equiv C_3(n,p) > 0$. We now estimate the last integral via (4.10), and subsequently estimate the right-hand side of (4.10) using the inequality (4.9). Thus we obtain

$$\begin{aligned}
Y_{j+1} &\le \frac{C_3}{|\widetilde{Q}_j|^{\frac{p}{q}}} \left(\iint_{\widetilde{Q}_j} |D(u-k_{j+1})_+|^p \,dx\,dt + \iint_{\widetilde{Q}_j} (u-k_{j+1})_+^p \,|D\tilde{\zeta}_j|^p \,dx\,dt \right)^{\frac{p}{q}} \\
&\quad \times \left(\sup_{-\tilde{\theta}_j < \tau < 0} \int_{K_{\tilde{\rho}_j}} (u(x,\tau)-k_{j+1})_+^2 \,dx \right)^{\frac{p^2}{nq}} \left(\frac{2^{jp}}{k^p} Y_j \right)^{1-\frac{p}{q}} \\
&\le \frac{C_3\, 2^{\frac{p}{q}\left(\frac{p^2}{n}+q-p\right)j}}{(1-\sigma)^{\frac{p^3}{nq}}\, k^{\frac{p}{q}(q-p)}} \left(\iint_{\widetilde{Q}_j} |D(u-k_{j+1})_+|^p \,dx\,dt \right. \\
&\quad \left. + \frac{2^{(j+2)p}}{(1-\sigma)^p \rho^p} \iint_{Q_j} (u-k_j)_+^p \,dx\,dt \right)^{\frac{p}{q}} \\
&\quad \times \frac{|Q_j|^{\frac{p^2}{nq}}}{|\widetilde{Q}_j|^{\frac{p}{q}}} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right)^{\frac{p^2}{nq}} Y_j^{\frac{p^2}{nq}+1-\frac{p}{q}} \\
&\le \frac{C_3\, 2^{\frac{p}{q}\left(\frac{p^2}{n}+q\right)j}}{(1-\sigma)^{\frac{p^3}{nq}}\, k^{\frac{p}{q}(q-p)}} \, \frac{|Q_j|^{\frac{p^2}{nq}}}{|\widetilde{Q}_j|^{\frac{p}{q}}} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right)^{\frac{p^2}{nq}} Z_j \, Y_j^{\frac{p^2}{nq}+1-\frac{p}{q}},
\end{aligned} \tag{4.14}$$

where, in the last line, we have set

$$Z_j := \left(\iint_{\widetilde{Q}_j} |D(u-k_{j+1})_+|^p \,dx\,dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}}. \tag{4.15}$$

Without loss of generality, we can now assume that $k \geq \rho$. Setting

$$\delta := \max\{\delta_i : i = 1, \dots, n\},$$

and using (4.1), (4.7), (4.6) with $s = p$, (4.9), (4.12) and the fact that $\frac{1}{k} \leq \frac{1}{\rho}$, we get

$$\begin{aligned}
Z_j &\leq C_4 \left(\iint_{\widetilde{Q}_j} \sum_{i=1}^{n} |[(u - k_{j+1})_+]_{x_i}|^p \, dx\, dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\
&= C_4 \left(\sum_{i=1}^{n} \iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} |u_{x_i}|^p \, dx\, dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\
&\leq C_4 \left(\sum_{i=1}^{n} \iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} [(|u_{x_i}| - \delta_i)_+ + \delta_i]^p \, dx\, dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\
&\leq 2^{\frac{p^2-p}{q}} C_4 \left(\sum_{i=1}^{n} \iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} (|u_{x_i}| - \delta_i)_+^p \, dx\, dt + n\,\delta^p\, |A_{j+1}| \right. \\
&\qquad \left. + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\
&\leq \frac{C_5}{(1-\sigma)^{\frac{p^2}{q}}} \left[2^{jp} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right) |Q_j|\, Y_j + |A_{j+1}| + \frac{1}{\rho^p} |Q_j|\, Y_j \right]^{\frac{p}{q}} \\
&\leq \frac{C_5\, 2^{(j+1)\frac{p^2}{q}}}{(1-\sigma)^{\frac{p^2}{q}}} \left[\left(\frac{1}{\theta\, k^{p-2}} + \frac{2}{\rho^p} + \frac{1}{k^p} \right) |Q_j|\, Y_j \right]^{\frac{p}{q}} \\
&\leq \frac{C_5\, 2^{(j+1)\frac{p^2}{q}}\, 3^{\frac{p}{q}}}{(1-\sigma)^{\frac{p^2}{q}}} \left[\left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right) |Q_j|\, Y_j \right]^{\frac{p}{q}},
\end{aligned} \tag{4.16}$$

where $C_4 \equiv C_4(n, p) > 1$ and $C_5 \equiv C_5(n, p, \delta) > 1$. Joining estimates (4.14) and (4.16), we deduce

$$Y_{j+1} \leq \frac{C_6\, 2^{\frac{p}{q}\left(\frac{p^2}{n} + q + p\right) j}}{(1-\sigma)^{\frac{p^2}{nq}(p+n)}\, k^{\frac{p}{q}(q-p)}} \, \frac{|Q_j|^{\frac{p}{nq}(p+n)}}{|\widetilde{Q}_j|^{\frac{p}{q}}} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right)^{\frac{p}{nq}(p+n)} Y_j^{1 + \frac{p^2}{nq}}$$

for some positive constant C_6 depending only on n, p and δ. From (4.13) and (4.1) again, we obtain

$$\frac{|Q_j|^{\frac{p}{nq}(p+n)}}{|\widetilde{Q}_j|^{\frac{p}{q}}} \leq \left(\frac{4^{n+1}}{|Q_j|} \right)^{\frac{p}{q}} |Q_j|^{\frac{p}{nq}(p+n)} \leq 4^{(n+1)\frac{p}{q}}\, |Q_0|^{\frac{p^2}{nq}} \leq C_7\, (\rho^n \theta)^{\frac{p^2}{nq}},$$

where $C_7 \equiv C_7(n, p) > 0$. Furthermore, we have

$$\left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p}\right)^{\frac{p}{nq}\,(p+n)} \le C_7 \left[\left(\frac{1}{\theta}\right)^{\frac{p+n}{p}} k^{(2-p)\,\frac{p+n}{p}} + \left(\frac{1}{\rho}\right)^{p+n}\right]^{\frac{p^2}{nq}}$$

for a possibly different constant C_7. Finally, combining the three previous estimates and recalling that $q := p\,\frac{n+2}{n}$, we arrive at the recursive inequalities

$$Y_{j+1} \le \frac{\widetilde{C}\, b^j}{(1-\sigma)^{p\,\frac{n+p}{n+2}}\, k^{\frac{2p}{n+2}}}\, \mathcal{A}_k^{\frac{p}{n+2}}\, Y_j^{1+\frac{p}{n+2}}, \tag{4.17}$$

where $\widetilde{C}$ is a positive constant depending only on n, p and δ,

$$b := 2^{p\,\frac{p+2n+2}{n+2}}, \tag{4.18}$$

$$\mathcal{A}_k := \left(\frac{\rho^p}{\theta}\right)^{\frac{n}{p}} k^{(2-p)\,\frac{n+p}{p}} + \frac{\theta}{\rho^p}. \tag{4.19}$$

5 Proofs of the main results

We next turn to the proof of Theorem 1.1. The argument relies on the iterative inequalities (4.17), which, combined with Lemma 2.4, yield the desired local L^∞ bounds (1.3) and (1.4).

Proof of Theorem 1.1 Let us first consider the case $p > 2$. After a translation, we may assume that $(x_0, t_0) = (0, 0)$. Therefore, we can make use of the iterative inequalities (4.17), where Y_j, b and $\mathcal{A}_k$ are defined in (4.12), (4.18) and (4.19), respectively. Recalling that we assumed $k \ge \rho$ in obtaining (4.17), we now take k so large that, of the two terms composing $\mathcal{A}_k$, the second dominates the first, i.e.,

$$k \ge \max\left\{\rho, \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}}\right\}. \tag{5.1}$$

With this choice of k, we have

$$\mathcal{A}_k \le \frac{2\theta}{\rho^p}.$$

It follows from Lemma 2.4 that $Y_j \to 0$ as $j \to \infty$, provided we choose k from

$$Y_0 := \iint_{Q(\theta,\rho)} u_+^p\, dx\, dt = C\,\frac{\rho^p}{\theta}\,(1-\sigma)^{n+p}\, k^2,$$

where C is a positive constant depending only on n, p and $\max\{\delta_1, \dots, \delta_n\}$. For such a choice and (5.1), we obtain

$$\operatorname*{ess\,sup}_{Q(\sigma\theta,\sigma\rho)} u \le \max\left\{\rho, \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}}, \frac{C}{(1-\sigma)^{\frac{n+p}{2}}}\sqrt{\frac{\theta}{\rho^p}}\left(\fint\!\!\fint_{Q(\theta,\rho)} u_+^p\,dx\,dt\right)^{\frac{1}{2}}\right\}, \tag{5.2}$$

for a possibly different constant C. Now observe that $-u$ is also a local weak solution of (1.1). Then, replacing u with $-u$ in (5.2), we get

$$\operatorname*{ess\,inf}_{Q(\sigma\theta,\sigma\rho)} u \ge -\max\left\{\rho, \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}}, \frac{C}{(1-\sigma)^{\frac{n+p}{2}}}\sqrt{\frac{\theta}{\rho^p}}\left(\fint\!\!\fint_{Q(\theta,\rho)} (-u)_+^p\,dx\,dt\right)^{\frac{1}{2}}\right\}.$$

Combining the two previous estimates, we deduce the local L^∞ bound in (1.3).

Finally, if $p = 2$ we have

$$\mathcal{A}_k = \left(\frac{\rho^2}{\theta}\right)^{\frac{n}{2}} + \frac{\theta}{\rho^2},$$

and arguing as above we obtain estimate (1.4). This concludes the proof. □

For the sake of completeness, we now detail the modifications to the previous arguments that yield the

Proof of Corollary 1.2 Let u be a local weak solution of (1.5) and let ζ be a piecewise smooth cut-off function chosen as in Section 3. Now we set

$$A(\eta) := D_\xi G(\eta), \quad \eta \in \mathbb{R}^n,$$

where G is the second function defined in (2.1). Note that

$$|A(Du)| = (|Du| - \lambda)_+^{p-1} \le |Du|^{p-1}$$

and

$$\langle A(Du), Du\rangle = (|Du| - \lambda)_+^{p-1}\,|Du| \ge (|Du| - \lambda)_+^p.$$

Therefore, we can proceed as in the proof of Proposition 3.1, thus obtaining, for every $k > 0$, the following local energy estimate:

$$\begin{aligned}
&\sup_{t_0-\theta<\tau<t_0} \int_{[x_0+K_\rho]} (u-k)_+^2\,\zeta^p(x,\tau)\,dx \\
&\quad + \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (|Du|-\lambda)_+^p\,\zeta^p\,\mathbb{1}_{\{u>k\}}\,dx\,dt \\
&\quad \le p \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (u-k)_+^2\,\zeta^{p-1}\,\partial_t\zeta\,dx\,dt \\
&\quad + C(p) \iint_{[(x_0,t_0)+Q(\theta,\rho)]} (u-k)_+^p\,|D\zeta|^p\,dx\,dt.
\end{aligned}$$

Starting from this estimate, assuming again that $(x_0, t_0) = (0, 0)$, and using the same notations and arguments as in Section 4, we find the iterative inequality

$$\sup_{-\tilde{\theta}_j < \tau < 0} \int_{K_{\tilde{\rho}_j}} (u(x,\tau) - k_{j+1})_+^2 \, dx + \iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} (|Du| - \lambda)_+^p \, dx \, dt, \leq \frac{C\, 2^{jp}}{(1-\sigma)^p} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right) \iint_{Q_j} (u - k_j)_+^p \, dx \, dt \,, \tag{5.3}$$

for every $j \in \mathbb{N}_0$. Moreover, we again arrive at estimate (4.14), where q, Y_j and Z_j are defined respectively in (4.11), (4.12) and (4.15). Without loss of generality, we can now assume that $k \geq \rho$. Thus, using (4.1), (4.7), (4.6) with $s = p$, (5.3), (4.12) and the fact that $\frac{1}{k} \leq \frac{1}{\rho}$, we get

$$\begin{aligned} Z_j &= \left(\iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} |Du|^p \, dx \, dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\ &\leq \left(\iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} [(|Du| - \lambda)_+ + \lambda]^p \, dx \, dt + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\ &\leq 2^{\frac{p^2-p}{q}} \left(\iint_{\widetilde{Q}_j \cap \{u > k_{j+1}\}} (|Du| - \lambda)_+^p \, dx \, dt + \lambda^p \, |A_{j+1}| + \frac{|Q_j|}{(1-\sigma)^p \rho^p} Y_j \right)^{\frac{p}{q}} \\ &\leq \frac{C_1}{(1-\sigma)^{\frac{p^2}{q}}} \left[2^{jp} \left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right) |Q_j| \, Y_j + |A_{j+1}| + \frac{1}{\rho^p} |Q_j| \, Y_j \right]^{\frac{p}{q}} \\ &\leq \frac{C_1\, 2^{(j+1)\frac{p^2}{q}}}{(1-\sigma)^{\frac{p^2}{q}}} \left[\left(\frac{1}{\theta\, k^{p-2}} + \frac{2}{\rho^p} + \frac{1}{k^p} \right) |Q_j| \, Y_j \right]^{\frac{p}{q}} \\ &\leq \frac{C_1\, 2^{(j+1)\frac{p^2}{q}} \, 3^{\frac{p}{q}}}{(1-\sigma)^{\frac{p^2}{q}}} \left[\left(\frac{1}{\theta\, k^{p-2}} + \frac{1}{\rho^p} \right) |Q_j| \, Y_j \right]^{\frac{p}{q}}, \end{aligned} \tag{5.4}$$

where $C_1 \equiv C_1(n, p, \lambda) > 1$. Joining estimates (4.14) and (5.4), and arguing as in the final part of Section 4, we obtain the recursive inequalities (4.17), where $\widetilde{C}$ is now a positive constant depending only on n, p and λ, while b and $\mathcal{A}_k$ are defined in (4.18) and (4.19), respectively. The desired conclusion then follows by proceeding exactly as in the proof of Theorem 1.1. □

Acknowledgements This work has been partially supported by the INdAM–GNAMPA 2025 Project "Regolarità ed esistenza per operatori anisotropi" (CUP E5324001950001). The authors wish to express their gratitude to the Department of Mathematics of the University of Bologna. In addition, P. Ambrosio acknowledges financial support under the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.1, Call for tender No. 104 published on 2.2.2022 by the Italian Ministry of University and Research (MUR), funded by the European Union - NextGenerationEU - Project PRIN_CITTI 2022 - Title "Regularity problems in sub-Riemannian structures" - CUP J53D23003760006 - Bando 2022 - Prot. 2022F4F2LH.

Funding Open access funding provided by Alma Mater Studiorum - Università di Bologna within the CRUI-CARE Agreement.

Declarations

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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