

THE DUAL APPROACH TO THE $K(\pi, 1)$ CONJECTURE

by

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Abstract. — Dual presentations of Coxeter groups have recently led to breakthroughs in our understanding of affine Artin groups. In particular, they led to the proof of the $K(\pi, 1)$ conjecture and to the solution of the word problem. Will the “dual approach” extend to more general classes of Coxeter and Artin groups? In this paper, we describe the techniques used to prove the $K(\pi, 1)$ conjecture for affine Artin groups and we ask a series of questions that are mostly open beyond the spherical and affine cases.

Résumé (Approche duale à la conjecture du $K(\pi, 1)$). — Les présentations duales de groupes de Coxeter ont récemment permis des avancées spectaculaires dans notre compréhension des groupes d’Artin affines. Elles ont en particulier permis une démonstration de la conjecture du $K(\pi, 1)$ et une solution du problème des mots. Est-ce que cette « approche duale » s’étend à des classes plus générales de groupes d’Artin et de Coxeter? Dans ce papier nous décrivons les techniques utilisées pour prouver la conjecture du $K(\pi, 1)$ pour les groupes d’Artin affines et posons une série de questions, souvent ouvertes au-delà des cas sphériques et affines.

The $K(\pi, 1)$ conjecture is one of the most important open problems on Artin groups, dating back to Brieskorn, Arnol’d, Pham, and Thom in the 60s [20, 71]. Due to its numerous consequences and connections, over the years it attracted the attention of mathematicians from several areas. It was solved for certain classes of Artin groups with approaches from algebraic topology, geometric group theory, and combinatorics.

The $K(\pi, 1)$ conjecture states that a certain topological space Y , constructed from the geometric action of a Coxeter group W on the Tits cone, is a classifying space (or “ $K(\pi, 1)$ ”) for the corresponding Artin group G_W . If W is a finite or affine reflection group acting on \mathbb{R}^n , then Y is the complement in \mathbb{C}^n of the complexification of all reflection hyperplanes (see Figure 1, left, and Figure 2, center).

In this paper, we focus on a combinatorial approach which has been very fruitful when studying the class of spherical Artin groups and, more recently, affine Artin groups. The fundamental idea, which goes back to Garside [61], is the following: given

a group G (in our case, an Artin group), fix a generating set and find a special element whose divisors generate the whole group and form a lattice under the divisibility relation. For a spherical Artin group with its standard generating set, such a special element can be obtained by lifting the longest element of the corresponding finite Coxeter group W . In this case, the lattice of divisors is isomorphic to W with the weak Bruhat order. When a group admits a special element, then it is called a *Garside group*. Together, the data of a Garside group, its generating set, and its special element form a *Garside structure*. Garside structures are very useful, thanks to an elegant solution to the word problem and an explicit combinatorial construction of a classifying space. Deligne’s proof of the $K(\pi, 1)$ conjecture for spherical Artin groups [48] can be reinterpreted with the language of Garside structures (although Garside groups were introduced later [44, 45, 47]).

The aforementioned Garside structure on spherical Artin groups is arguably one of the greatest milestones in the study of Artin groups. However, it does not generalize beyond the spherical case, since infinite Coxeter groups have no longest element. Birman-Ko-Lee [9] and Bessis [6] introduced and studied alternative “dual” realizations of spherical Artin groups as Garside groups, using a different generating set (a lift of all reflections in W) and different special elements (Coxeter elements). Besides providing new interesting insights into the spherical case, this dual approach has the advantage that Coxeter elements exist also in infinite Coxeter groups. For some families of affine Artin groups, Digne showed that this larger generating set together with a Coxeter element form a dual Garside structure [53, 54], but McCammond proved that the divisors of a Coxeter element do not form a lattice in almost all of the remaining affine Artin groups [74]. On the positive side, McCammond and Sulway [76] exhibited a way to embed any affine Artin group into a Garside group, thus recovering some of the benefits of a Garside structure such as a solution to the word problem. More recently, Salvetti and the author succeeded in proving the $K(\pi, 1)$ conjecture for all affine Artin groups [87] with an approach that is based on the dual structure even though the lattice property does not necessarily hold.

The purpose of this paper is to outline this dual approach to the $K(\pi, 1)$ conjecture. We are going to discuss the various combinatorial, topological, and geometric ingredients that went into the proof of the affine case given in [87], how they might possibly generalize to other Artin groups, and the multitude of questions that naturally arise along the way. We do not know to what extent the dual approach is viable to obtain a full proof of the $K(\pi, 1)$ conjecture. Regardless, we want to lay the foundations for exploring new exciting directions.

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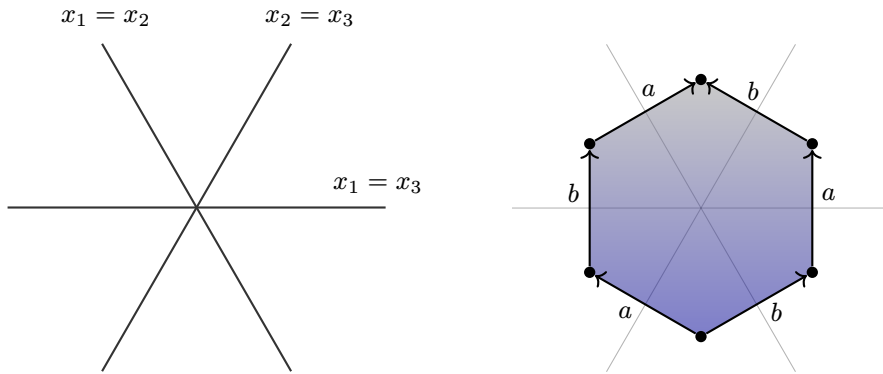


FIGURE 1. The symmetric group \mathfrak{S}_3 on a three-element set is a spherical Coxeter group with presentation $\mathfrak{S}_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$. It can be represented as the group of linear isometries of \mathbb{R}^3 that permutes the three coordinates. This representation can be made essential by restricting to the plane $\{x_1 + x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$. LEFT: the reflection lines (corresponding to the transpositions of \mathfrak{S}_3) in the plane $\{x_1 + x_2 + x_3 = 0\}$. RIGHT: the Salvetti complex $X_{\mathfrak{S}_3}$, which has one 0-cell, two 1-cells (labeled a and b), and one hexagonal 2-cell. The fundamental group of $X_{\mathfrak{S}_3}$ is the braid group $G_{\mathfrak{S}_3} = \langle a, b \mid aba = bab \rangle$.

1. The general picture

1.1. Coxeter groups. — A *Coxeter group* is a group with a presentation of the following form:

$$(1) \quad W = \langle S \mid (st)^{m(s,t)} = 1 \ \forall s, t \in S \text{ such that } m(s, t) \neq \infty \rangle,$$

where S is a finite set and $m: S \times S \rightarrow \mathbb{N}$ is any symmetric integer matrix with 1's on the diagonal and all other entries ≥ 2 . The off-diagonal entries are allowed to take the value $m(s, t) = \infty$, in which case there is no relation involving s and t . The 1's on the diagonal ensure that all generators have order 2. In this section, we recall some facts about Coxeter groups that are particularly relevant for the rest of this paper. We point to [14, 68] for a detailed introduction.

The presentation (1) can be encoded into a graph called the *Coxeter graph*: the vertices are indexed by S , and there is an edge connecting s and t whenever $m(s, t) \geq 3$; this edge is labeled by $m(s, t)$ if $m(s, t) \geq 4$. We always assume that W is *irreducible*, i.e., its Coxeter graph is connected. The size of the generating set S is called the *rank* of W . Any subset $T \subseteq S$ generates a subgroup of W called a *standard parabolic subgroup*, which is itself a Coxeter group, with a presentation obtained by restricting (1) to the generators in T and the relations between them. Any conjugate of S can be used in place of S to write a presentation for W of the form (1). The conjugates of S are called *sets of simple reflections*. Usually, a set S of simple reflections is fixed, in which case the elements of S are called *simple reflections*. However, when taking

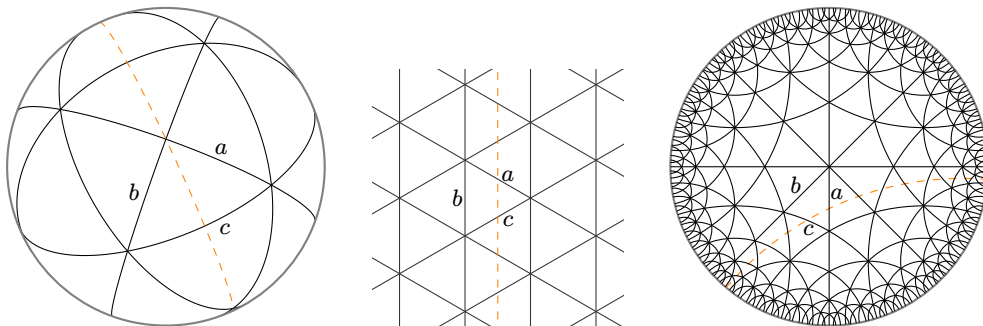


FIGURE 2. Reflection arrangements of some Coxeter groups of rank 3 (also known as *triangle groups*). The dashed line is the axis of the Coxeter element $w = abc$ (see Section 2). LEFT: the spherical $(2, 3, 3)$ triangle group, a.k.a. the symmetric group \mathfrak{S}_4 . CENTER: the affine $(3, 3, 3)$ triangle group, a.k.a. the affine symmetric group of type \tilde{A}_3 . RIGHT: the hyperbolic $(4, 3, 3)$ triangle group. In all three cases, the triple (p, q, r) consists of the upper-triangular entries of the Coxeter matrix. The sphere \mathbb{S}^2 (left), Euclidean plane \mathbb{R}^2 (center), and the hyperbolic plane \mathbb{H}^2 (right) are tiled by triangles with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

the dual point of view (Section 1.4), it is useful to think of a Coxeter group without any preferred set of simple reflections.

For us, a Coxeter group W always carries with it a fixed set S of simple reflections (the pair (W, S) is usually called a *Coxeter system*) or the set of all sets of simple reflections (when the particular choice of S is not relevant). Indeed, two Coxeter groups may be isomorphic as groups (if we forget about simple reflections) but not as Coxeter groups. For example, the dihedral group $W = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$ is irreducible when using s, t as simple reflections, but it is the direct product of two Coxeter subgroups: the symmetric group \mathfrak{S}_3 (generated by s and tst), and \mathbb{Z}_2 (generated by $(st)^3$). See for example [78].

The family of Coxeter groups encompasses all discrete groups generated by (linear or affine) Euclidean reflections in \mathbb{R}^k (we call them *real reflection groups*). The linear reflection groups are exactly the finite Coxeter groups, and they are also called *spherical* because they act on the unit sphere $\mathbb{S}^{k-1} \subseteq \mathbb{R}^k$. The action on \mathbb{S}^{k-1} is cocompact, provided that $k = |S|$ (the representation is *essential*). For example, the symmetric groups are spherical Coxeter groups (Figure 1). The infinite Euclidean reflection groups are called *affine* Coxeter groups. They act cocompactly on \mathbb{R}^k , provided that $k = |S| - 1$. See Figure 2 for more examples.

The representations of finite and affine Coxeter groups as Euclidean reflection groups inspired a more general *geometric representation* for an arbitrary Coxeter group W as a group generated by linear reflections with respect to a suitable bilinear form B in \mathbb{R}^n with $n = |S|$ [68, Section 5.3]. The bilinear form is defined using the

data of the presentation (1). It is positive definite if W is finite, positive semi-definite if W is affine, and it otherwise admits both positive and negative vectors.

By switching to the contragradient representation [68, Section 5.13], one gains a geometric picture that resembles the Euclidean case: W is generated by reflections with respect to hyperplanes of \mathbb{R}^n , these hyperplanes divide \mathbb{R}^n into simplicial cones called *chambers*, and the generating set S consists of the reflections with respect to the walls of a *fundamental chamber* $C_0 \subseteq \mathbb{R}^n$. The union of all W -translates $w(\bar{C}_0)$ forms a convex cone I called the *Tits cone*. The closure \bar{C}_0 of the fundamental chamber is a fundamental domain for the action of W on the Tits cone I . In addition, W acts simply transitively on the chambers contained in the Tits cone I , so the choice of a fundamental chamber induces a bijection between the elements of W and the chambers in I . The Tits cone is the whole space \mathbb{R}^n if W is finite, an open half-space (with the origin added) if W is affine, and otherwise it does not contain any line.

The Coxeter groups that are neither finite nor affine are coarsely classified as: *Lorentzian*, if the bilinear form B has exactly one negative eigenvalue; *higher-rank*,⁽¹⁾ if B has at least two negative eigenvalues. A Lorentzian Coxeter group is called *hyperbolic* if (i) the bilinear form B is non-degenerate and (ii) every vector in the interior of the Tits cone is negative. Hyperbolic Coxeter groups act by isometries on the hyperbolic space \mathbb{H}^{n-1} (realized as the hyperboloid model inside the Tits cone) and the chambers are simplicial (possibly with ideal vertices). Higher-rank Coxeter groups act by isometries on the projectivization of the Tits cone with the Hilbert metric [77].

1.2. The $K(\pi, 1)$ conjecture and Artin groups. — Let \mathcal{A} be the set of all fixed hyperplanes of reflections of W , where W acts on the Tits cone $I \subseteq \mathbb{R}^n$ via the representation introduced in the previous section.

Conjecture 1.1 ($K(\pi, 1)$ conjecture). — The space

$$Y = (I \times I) \setminus \bigcup_{H \in \mathcal{A}} (H \times H)$$

is a $K(\pi, 1)$ space.

This conjecture dates back to the '60s when it was proved for the symmetric group by Fox and Neuwirth [59]. It was then proved for most finite Coxeter groups by Brieskorn [20] and for all finite Coxeter groups by Deligne [48]. The above formulation for general Coxeter groups is attributed to Arnol'd, Pham, and Thom [71]. If W is finite or affine, then the $K(\pi, 1)$ conjecture says that the complement of the complexification of a (locally finite) Euclidean reflection arrangement is a $K(\pi, 1)$ space. See [89] for a survey on this problem.

⁽¹⁾ Here *rank* does not refer to the rank of the Coxeter group as defined earlier. Rather, it refers to the rank of the Lie group $SO(p, q)$ where (p, q) is the signature of the bilinear form B .

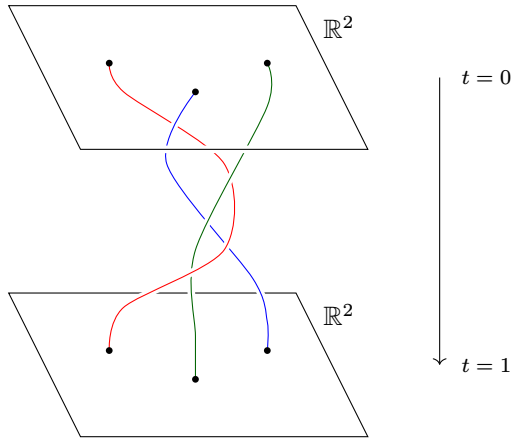


FIGURE 3. A loop in the orbit configuration space Y_W for $W = \mathfrak{S}_3$. In this case, Y_W is the space of configurations of 3 points in \mathbb{R}^2 . Elements of the fundamental group $G_{\mathfrak{S}_3} = \pi_1(Y_W)$ are homotopy classes of loops, also known as *braids*.

The primary motivation for the $K(\pi, 1)$ conjecture, besides its elegance, comes from the study of *Artin groups*. To every Coxeter group W presented as in (1), there is an associated Artin group defined as follows:

$$(2) \quad G_W = \langle S \mid \underbrace{stst \cdots}_{m(s,t) \text{ terms}} = \underbrace{tsts \cdots}_{m(s,t) \text{ terms}} \quad \forall s, t \in S \text{ such that } m(s, t) \neq \infty \rangle.$$

The Artin group G_W also arises as the fundamental group of the quotient space $Y_W = Y/W$ [71, 92]. The quotient $Y \rightarrow Y_W$ is a covering map, so the $K(\pi, 1)$ conjecture can be equivalently formulated by asking that Y_W be a *classifying space* for the Artin group G_W . The space Y_W is called the *orbit configuration space* associated with W . This name comes from the case of the symmetric group $W = \mathfrak{S}_n$, for which Y_W is the space of configurations of n (indistinguishable) points in \mathbb{R}^2 and G_W is the *braid group* on n strands (see Figure 3).

As shown by Salvetti [91, 92], the orbit configuration space Y_W has the homotopy type of a CW complex X_W with k -cells indexed by the finite standard parabolic subgroups of rank k . The CW complex X_W is known as the *Salvetti complex* of W (see Figure 1). The presentation (2) can be read off the 2-skeleton of the Salvetti complex, thus providing a simple proof that $G_W \cong \pi_1(X_W)$. Since X_W is finite-dimensional, the $K(\pi, 1)$ conjecture for W implies that the Artin group G_W is torsion-free (a property that is not known in general). In addition, the $K(\pi, 1)$ conjecture makes it possible to compute the homology and cohomology of an Artin group G_W using the configuration space Y_W or, equivalently, the Salvetti complex X_W , as done in several works already [2, 22, 23, 24, 25, 26, 27, 31, 33, 34, 35, 39, 40, 41, 42, 43, 60, 62, 63, 84, 86, 90, 92, 93, 94].

Of course, having a CW model for Y_W can be also useful to prove the $K(\pi, 1)$ conjecture. This is particularly true if we want to approach the conjecture with combinatorial techniques, as in this case, we would rather work with CW complexes (indexed by combinatorial objects associated with W) than with “raw” topological spaces. This combinatorial spirit is at the heart of the approach we discuss in this paper.

To date, the $K(\pi, 1)$ conjecture has been proved in the following cases: spherical Artin groups [48] (see Section 1.3); affine Artin groups [87] (see Section 1.4 and the rest of this paper); 2-dimensional and FC-type Artin groups [30] (the proof is based on finding a CAT(0) metric on the *Deligne complex*, see also [29]). It was previously proved for some subclasses of these Artin groups, with different methods: braid groups [59] and spherical Artin groups of type C_n , D_n , G_2 , $I_2(m)$ [21]; ⁽²⁾ affine Artin groups of type \tilde{A}_n , \tilde{C}_n [79] and of type \tilde{B}_n [26]; Artin groups of large type [66].

1.3. The “standard” approach. — The (right) Cayley graph of W with respect to the generating set S is the Hasse diagram of a partial order on W known as the (right) *weak Bruhat order* \leq_S :

$$u \leq_S v \quad \text{if and only if } l_S(v) = l_S(u) + l_S(u^{-1}v),$$

where $l_S(u)$ is the length of u with respect to S . Choosing “left” instead of “right” (and replacing $u^{-1}v$ with vu^{-1} in the definition of \leq_S) does not have an impact, since the resulting partial order \leq'_S is isomorphic to \leq_S via the map $u \mapsto u^{-1}$. Notice that the edges of the Cayley graph are labeled by elements of S , so (W, \leq_S) is an edge-labeled poset.

If W is finite, then it has a unique longest element δ (where the length is measured by l_S). Geometrically, δ is the element that sends the fundamental chamber C_0 to its opposite. It is useful to think of the poset (W, \leq_S) as the interval between 1 and δ in the Cayley graph of W : every element $u \in W$ lies on at least one geodesic from 1 to δ , and the relation $u \leq_S v$ holds if and only if there is a geodesic from 1 to δ which passes through u and v (in this order). To emphasize this interval structure, we denote the poset (W, \leq_S) by $[1, \delta]_S$. For the symmetric group \mathfrak{S}_3 , this is shown in Figure 4.

Crucially, the interval $[1, \delta]_S$ is a *lattice*: every pair of elements has a unique minimal upper bound (a least common multiple) and a unique maximal lower bound (a greatest common divisor). This was first shown by Deligne [48] ⁽³⁾ and used to prove the $K(\pi, 1)$ conjecture in the spherical case (actually, Deligne’s proof works more generally for finite simplicial arrangements of linear hyperplanes in \mathbb{R}^n). To prove the $K(\pi, 1)$ conjecture, Deligne showed that the universal cover of Y_W is an increasing union of copies of the subspace determined by the *positive paths*, which can be proved to be

⁽²⁾ I was pointed out by Georges Neaime (private communication) that Brieskorn’s construction is incorrect in the case F_4 .

⁽³⁾ Another proof of the lattice property was given later by Björner-Edelman-Ziegler [12]. See also [11, Section 3.2].

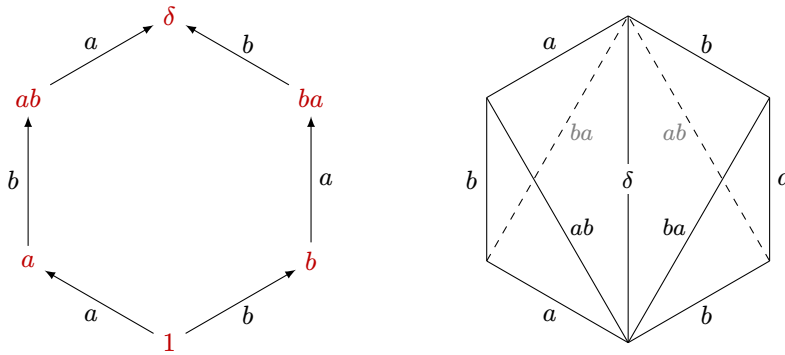


FIGURE 4. LEFT: the interval $[1, \delta]_S$ for the symmetric group \mathfrak{S}_3 . The edges are labeled by the simple reflections a and b . The longest element is given by $\delta = aba = bab$. RIGHT: the labeled order complex of $[1, \delta]_S$. Its quotient K is a classifying space for the braid group $G_{\mathfrak{S}_3}$ and consists of the following simplices: the 0-simplex $[\]$; the five 1-simplices $[a]$, $[b]$, $[ab]$, $[ba]$, and $[\delta]$; the six 2-simplices $[a|b]$, $[b|a]$, $[a|ba]$, $[ab|a]$, $[ba|b]$, and $[b|ab]$; the two 3-simplices $[a|b|a]$ and $[b|a|b]$.

contractible using the lattice property. It follows that the whole universal cover is itself contractible. This idea goes back to Garside [61]. See also [49, 50, 82, 89] for different reformulations of Deligne’s proof.

One can also *define* spherical Artin groups in terms of the interval $[1, \delta]_S$: G_W is the group generated by the set S and subject to all relations that identify any two words that can be read along maximal chains with the same initial and final point. For example, in the symmetric group \mathfrak{S}_3 (Figure 4), the only relation is $aba = bab$ (obtained by reading the labels along the two geodesics from 1 to δ), so the Artin group is presented as $G_{\mathfrak{S}_3} = \langle a, b \mid aba = bab \rangle$. We say that G_W is the *interval group* associated with the labeled poset $[1, \delta]_S$. This construction was generalized by Dehornoy and Paris, replacing $[1, \delta]_S$ with any labeled lattice P satisfying some additional conditions. They called *Garside groups* the interval groups arising in this way [32, 46, 47, 73]. The case $P = [1, \delta]_S$ is known as the “standard” Garside structure on spherical Artin groups.

Elements in a Garside group admit a normal form $\delta^m y$, where δ is the top element of the defining lattice P , the exponent m is an integer, and y is a positive word in the generators. In addition, a suitable quotient K of the order complex of P is a classifying space for the corresponding Garside group (see Figure 4). More specifically, K is obtained by identifying any two simplices $\{y_0 < y_1 < \dots < y_k\}$ and $\{z_0 < z_1 < \dots < z_k\}$ such that $y_i^{-1} y_{i+1} = z_i^{-1} z_{i+1}$ for all $i = 0, \dots, k - 1$. Then a simplex of K is uniquely determined by the sequence $(x_1, \dots, x_k) = (y_0^{-1} y_1, \dots, y_{k-1}^{-1} y_k)$ and we denote this simplex by $[x_1|x_2|\dots|x_k]$. The complex K was first introduced by Brady for braid groups [16] and then extended to spherical Artin groups by Brady-Watt [18] and

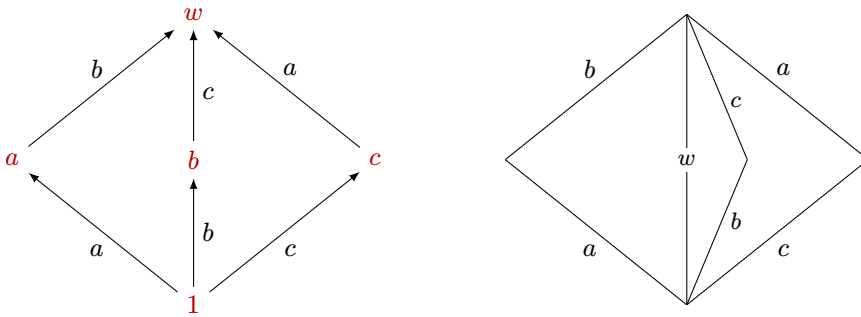


FIGURE 5. LEFT: the “dual” interval $[1, w]_R$ for the symmetric group \mathfrak{S}_3 with $w = ab$ as the chosen Coxeter element. The edges are labeled by the three reflections a, b, c (note that the third reflection c is equal to the longest element $\delta = aba = bab$, but this is just a coincidence). RIGHT: the labeled order complex of $[1, w]_R$. Its quotient K is a classifying space for the (dual) braid group $G_{\mathfrak{S}_3}$ and consists of the following simplices: the 0-simplex $[\]$; the four 1-simplices $[a], [b], [c]$, and $[w]$; the three 2-simplices $[a|b], [b|c]$, and $[c|a]$.

Bestvina [8], and to general Garside groups by Charney-Meier-Whittlesey [32]. We call K the *interval complex* associated with P . The lattice property of P is crucial to obtain the normal form mentioned above and to prove that K is a classifying space.

For spherical Artin groups G_W , one can show that K is homotopy equivalent to the Salvetti complex X_W and this is another way to prove the $K(\pi, 1)$ conjecture in the spherical case. This homotopy equivalence is shown in [50]. Another complex that is homotopy equivalent to both K and X_W is the classifying space of the Artin monoid G_W^+ , appearing in [55, 80, 83].

1.4. The “dual” approach. — The absence of the longest element in infinite Coxeter groups makes it impossible to extend the standard Garside structure to non-spherical Artin groups. An alternative and promising direction to study Artin groups is based on a “dual” presentation of Coxeter groups, where the standard generating set S is replaced by the set R of all reflections (i.e., all conjugates of elements of S). The associated length function l_R is called the *absolute length* and the induced partial order \leq_R on W is called the *absolute order*. Note that R is infinite if W itself is infinite, whereas S is always finite. The absolute order does not depend on the choice of left or right, thanks to the generating set R being closed under conjugation.

Let us restrict for now to finite Coxeter groups, where the absolute length of all \leq_R -maximal elements is equal to the rank of W .⁽⁴⁾ Among the maximal elements, a special role is played by *Coxeter elements*, defined as $w = s_1 s_2 \cdots s_n$ where $\{s_1, s_2, \dots, s_n\}$ is any set of simple reflections and the product is taken in any order.

⁽⁴⁾ This is not the case for infinite Coxeter groups, see for example [72].

If W is finite, then all Coxeter elements $w \in W$ are conjugate and the intervals $[1, w]_R$ (inside the Cayley graph of W with respect to the generating set R) are lattices (see Figure 5). This gives rise to several “dual” Garside structures, all isomorphic to each other. Perhaps surprisingly, the interval group associated with any of these intervals $[1, w]_R$ is naturally isomorphic to the Artin group G_W . Therefore, it is really the usual Artin group that we are studying and not some new Garside group.⁽⁵⁾

The duality between standard and dual presentations manifests itself in multiple numerical “coincidences,” the most apparent being that $l_S(\delta) = |R|$ and $l_R(w) = |S|$. In other words, the length of the standard interval $[1, \delta]_S$ is equal to the cardinality of R (which is the set of atoms of $[1, w]_R$) and conversely the length of any dual interval $[1, w]_R$ is equal to the cardinality of S (which is the set of atoms of $[1, \delta]_S$). The dual presentation was first introduced by Birman-Ko-Lee [9] for the braid group and then by Bessis [6] for all finite Coxeter groups.

The intervals $[1, w]_R$ also exist in infinite Coxeter groups. So it is natural to ask: Are they lattices, thus giving rise to Garside structures?⁽⁶⁾ Are the corresponding interval groups (called *dual Artin groups*) isomorphic to the usual Artin groups? Can they help us find a solution to the word problem and the $K(\pi, 1)$ conjecture?

These questions are motivated by the success in understanding affine Artin groups by means of the dual approach. McCammond showed that $[1, w]_R$ fails to be a lattice in most affine cases [74]. However, McCammond and Sulway were able to prove that affine dual Artin groups are always isomorphic to the corresponding standard Artin groups, and can be included in larger Garside groups [76]. In particular, this solves the word problem and shows that affine Artin groups are torsion-free. More recently, Salvetti and the author proved the $K(\pi, 1)$ conjecture for affine Artin groups [87]. At a very high level, the proof consists of the following three components.

1. Show that the interval complex K_W associated with $[1, w]_R$ is a classifying space, despite the failure of the lattice property.
2. Introduce a new subcomplex $X'_W \subseteq K_W$ with the same homotopy type as the orbit configuration space Y_W .
3. Find a deformation retraction of K_W onto X'_W .

Together, these three steps imply that Y_W is a classifying space and that the dual Artin group $\pi_1(K_W)$ is isomorphic to the standard Artin group $G_W = \pi_1(Y_W)$. Therefore, they prove the $K(\pi, 1)$ conjecture and re-prove the isomorphism between standard and dual affine Artin groups. In the rest of the present paper, we dive deeper into the different geometric, combinatorial, and topological aspects of this proof, with the hope that some of the key ideas can be generalized beyond the affine case.

⁽⁵⁾ This is not necessarily the case if w is not a Coxeter element, see [5].

⁽⁶⁾ When the defining interval is infinite, the term *quasi-Garside* is often used in place of *Garside* (see for example [45]). In the present paper, we will not make this distinction.

2. Coxeter elements

In this section, we introduce the main characters of the dual approach outlined in Section 1.4: Coxeter elements. Let W be a Coxeter group. For any set $S = \{s_1, s_2, \dots, s_n\}$ of simple reflections (not necessarily the one used to define W), we say that the product $w = s_1 s_2 \cdots s_n$ is a *Coxeter element* of W [36, 37, 68]. Any order of the simple reflections s_1, \dots, s_n can be used and different orders can give rise to different Coxeter elements. It is noted in [70, Lemma 3.8] and [87, Lemma 5.1] that the reflection length of any Coxeter element is equal to $n = |S|$. In other words, it is not possible to write a Coxeter element as a product of less than n reflections.

If W is finite, then all Coxeter elements form a single conjugacy class and the order of any Coxeter element is $h = 2|R|/n$ (h is called the *Coxeter number* of W). The eigenvalues of w are of the form ζ^e , where $\zeta = e^{2\pi i/h}$ and e runs through the exponents of W (listed in Table 1). \mathbb{R}^n is the orthogonal sum of w -invariant subspaces V_1, \dots, V_k of dimension 1 or 2: for each exponent $e < h/2$, there is a plane V_i where w acts as a rotation of $2\pi e/h$; for each exponent $e = h/2$ (there are at most 2 of them), there is a line V_i where w acts as a reflection with respect to the origin. There is a unique w -invariant plane P corresponding to the exponent $e = 1$, called the *Coxeter plane*. The Coxeter element w acts on P as a rotation of $2\pi/h$. The Coxeter plane is also interesting thanks to the following simple observation.

Lemma 2.1. — Let w be a Coxeter element in an irreducible finite Coxeter group W acting on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ with $n = |S|$. Let $P \subseteq \mathbb{R}^n$ be the Coxeter plane. Then the circle $P \cap \mathbb{S}^{n-1}$ is the set of points $x \in \mathbb{S}^{n-1}$ for which the spherical distance $d(x, w(x))$ is minimized.

Proof. — Given $x \in \mathbb{S}^{n-1}$, write $x = v_1 + \cdots + v_k$ with $v_i \in V_i$. Then

$$\begin{aligned} \cos(d(x, w(x))) &= \langle x, w(x) \rangle \\ &= \langle v_1, w(v_1) \rangle + \cdots + \langle v_k, w(v_k) \rangle \\ &= \|v_1\|^2 \cos \theta_1 + \cdots + \|v_k\|^2 \cos \theta_k, \end{aligned}$$

where θ_i is equal to the rotation angle of $w|_{V_i}$ if V_i is a plane and to π if V_i is a line. Since $\|v_1\|^2 + \cdots + \|v_k\|^2 = \|x\|^2 = 1$, the previous expression is maximized if and only if all v_i are 0 except for the one corresponding to the largest $\cos \theta_i$. Such an i is the one for which $\theta_i = 2\pi/h$ and $V_i = P$ is the Coxeter plane. \square

The fact that all Coxeter elements form a single conjugacy class holds more generally whenever the Coxeter graph is a tree. This applies to all finite Coxeter groups (discussed above), but also to several infinite Coxeter groups, including all irreducible affine Coxeter groups except for the infinite family \tilde{A}_n . In an arbitrary Coxeter group, however, Coxeter elements can form more than one conjugacy class and exhibit substantially different geometric properties and different spectra.

In affine Coxeter groups, Coxeter elements act as hyperbolic isometries on the Euclidean space [74]. The set of points $x \in \mathbb{R}^n$ minimizing the Euclidean distance

Type	Exponents	h	$ W $
A_n	$1, 2, 3, \dots, n$	$n + 1$	$(n + 1)!$
B_n	$1, 3, 5, \dots, 2n - 1$	$2n$	$2^n n!$
D_n	$1, 3, 5, \dots, 2n - 3, n - 1$	$2n - 2$	$2^{n-1} n!$
E_6	$1, 4, 5, 7, 8, 11$	12	$2^7 3^4 5$
E_7	$1, 5, 7, 9, 11, 13, 17$	18	$2^{10} 3^4 5 7$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	30	$2^{14} 3^5 5^2 7$
F_4	$1, 5, 7, 11$	12	48
G_2	$1, 5$	6	12
H_3	$1, 5, 9$	10	120
H_4	$1, 11, 19, 29$	30	14400
$I_2(m)$	$1, m - 1$	m	$2m$

TABLE 1. Exponents, Coxeter number, and cardinality of irreducible finite Coxeter groups (see [68, Sections 2.11, 2.13, and 3.7] and [11, Appendix A1]).

$d(x, w(x))$ is a line called the *Coxeter axis*. Analogously, in hyperbolic Coxeter groups, Coxeter elements act as hyperbolic isometries on the hyperbolic space \mathbb{H}^n by [75, Lemma 5.5] and thus possess an axis (a hyperbolic line consisting of the points that are minimally moved). In view of Lemma 2.1, the circle $P \cap \mathbb{S}^{n-1}$ can be regarded as the axis of the Coxeter element w in a finite Coxeter group. Therefore, in all three geometries (spherical, Euclidean, and hyperbolic), Coxeter elements w have a well-defined axis, and the axis is a w -invariant geodesic. Coxeter axes in triangle groups are shown in Figure 2.

Question 2.2. — Do all Coxeter elements (in arbitrary Coxeter groups) have an axis? Is the axis not contained in any reflection hyperplane?

If the previous question has a positive answer, then the axis ℓ of a Coxeter element w passes through the interior of several chambers that we call *axial chambers* (as in [74]). The following question was positively answered for affine Coxeter groups in [74, Theorem 8.10] and [87, Theorem 3.8].

Question 2.3. — Is it true that a Coxeter element can be written as the product of the reflections with respect to the walls of any axial chamber (in some order)?

To answer this question, it is enough to show that a Coxeter element has length $\leq n$ with respect to the generating set S consisting of the reflections with respect to some axial chamber. Indeed, the length then needs to be exactly n (because the reflection length is n) and a factorization into reflections in S needs to use all of them because Coxeter elements are *essential* (not contained in any proper parabolic subgroup)

[88]. McMullen's interpretation of Coxeter axes as billiard trajectories [77] could help answering Question 2.3.

When the Coxeter graph is bipartite, one can construct so-called *bipartite Coxeter elements*: these are obtained as $w = s_1 s_2 \cdots s_n$, where $S = \{s_1, \dots, s_k\} \cup \{s_{k+1}, \dots, s_n\}$ is a bipartition of any set S of simple reflections. In other words, s_i and s_j commute whenever $i, j \leq k$ or $i, j \geq k + 1$. If the Coxeter graph is a tree, then every Coxeter element is a bipartite Coxeter element with respect to some set of simple reflections. This is especially useful to study Coxeter elements in finite [6, 68] and affine [74] Coxeter groups. In fact, [87, Section 3] shows a clear dichotomy between bipartite and non-bipartite affine Coxeter elements, the former having a significantly simpler geometrical behavior. Bipartite Coxeter elements also minimize the spectral radius among all Coxeter elements in a fixed (hyperbolic or higher-rank) Coxeter group [77]. In Figure 2, the Coxeter element on the left is bipartite, whereas the other two are not.

3. Factoring Coxeter elements: the noncrossing partition poset $[1, w]$

As explained in Section 1.4, the dual approach is based on understanding the poset $[1, w] = [1, w]_R$ which encodes the combinatorial data of all minimal factorizations of a Coxeter element w into reflections. For this, it can be useful to understand the minimal factorizations of w into arbitrary reflections of the ambient bilinear form, not necessarily belonging to W . These factorization posets are studied in [17] for spherical isometries, in [15] for Euclidean isometries, and in [75] for arbitrary non-degenerate quadratic spaces. In this general setting, a recurring theme is that (under certain hypotheses) an isometry u below w is uniquely determined by w and by its moved space $\text{MOV}(u) := \text{im}(u - \text{id})$. In the spherical case, intervals are easy to describe: for any isometry w in the orthogonal group $O(n)$, the interval $[1, w]$ in $O(n)$ is isomorphic to the poset of all subspaces of $\text{MOV}(w)$ ordered by inclusion.

Let us go back to the setting of Coxeter groups. If $w \in W$ is a Coxeter element, the interval $[1, w] = [1, w]_R$ is called a (generalized) *noncrossing partition poset*. The terminology comes from the case where W is the symmetric group \mathfrak{S}_{n+1} , whose reflections are all transpositions $(i j)$. Here Coxeter elements are the $(n + 1)$ -cycles, such as $(1 2 \cdots n + 1)$. Then, the poset $[1, w]$ is naturally isomorphic to the classical lattice of noncrossing partitions of an $(n + 1)$ -gon (see for instance [1]).

The maximal chains in $[1, w]$ correspond to the minimal factorizations of w as a product of reflections, $w = r_1 r_2 \cdots r_n$. For example, in \mathfrak{S}_3 we have three reflections: $a = (1 2)$, $b = (2 3)$, and $c = (1 3)$. The Coxeter element $w = ab = (1 2 3)$ has three minimal factorizations: $w = ab = bc = ca$. For this case, the interval $[1, w]$ is depicted in Figure 5. There is a natural action of the braid group on the set of all minimal factorizations of a Coxeter element. This is called the *Hurwitz action* and is defined as follows: the i -th generator σ_i of the braid group sends the factorization $r_1 r_2 \cdots r_n$ to the factorization $r_1 r_2 \cdots r_{i-1} r_{i+1} \tilde{r}_i r_{i+2} \cdots r_n$ where $\tilde{r}_i = r_{i+1} r_i r_{i+1}$. In words, σ_i swaps the reflections r_i and r_{i+1} while conjugating r_i by r_{i+1} . Similarly, σ_i^{-1} swaps r_i

and r_{i+1} while conjugating r_{i+1} by r_i . It is known that the Hurwitz action is transitive on the set of all minimal factorizations of w [4, 6, 69].

In the spherical case, the interval $[1, w]$ in W (let us temporarily denote it by $[1, w]^W$) is an induced subposet of the interval $[1, w]$ in the whole orthogonal group $O(n)$. However, [87, Example 3.31] shows that this is not true in the affine case, where there can be elements $u, v \in [1, w]^W$ such that $u \leq v$ in $[1, w]^L$ but $u \not\leq v$ in $[1, w]^W$. Here $[1, w]^{\text{ISOM}(\mathbb{R}^n)}$ is the interval inside the group $\text{ISOM}(\mathbb{R}^n)$ of all Euclidean isometries of \mathbb{R}^n .

If W is finite, then Bessis proved that every element $u \in [1, w]$ is a Coxeter element for the parabolic subgroup generated by the reflections $\leq u$ [6, Lemma 1.4.3 and Proposition 1.6.1]. This result was extended to *crystallographic* Coxeter groups⁽⁷⁾ by Hubery-Krause [67, Corollary 5.8]. In the affine case, it was also proved in [87, Theorem 3.22]. Note that, if W is infinite, the subgroup generated by the reflections below u is not necessarily a parabolic subgroup.

Question 3.1. — Is any element $u \in [1, w]$ a Coxeter element for the subgroup of W generated by the reflections $\leq u$?

The previous question is closely related to the following ones.

Question 3.2. — Is the Hurwitz action transitive on the minimal reflection factorizations of any element $u \in [1, w]$?

Question 3.3. — Let $u = r_1 r_2 \cdots r_k$ be a minimal reflection factorization of an element $u \in [1, w]$. Does the subgroup $\langle r_1, r_2, \dots, r_k \rangle$ only depend on u (and not on the chosen factorization)?

Lemma 3.4. — The following implications hold: Question 3.1 \Rightarrow Question 3.2 \Rightarrow Question 3.3.

Proof. — If Question 3.1 has a positive answer, then Question 3.2 also does because of the transitivity of the Hurwitz action on Coxeter elements. After applying a Hurwitz move to a minimal reflection factorization $u = r_1 r_2 \cdots r_k$, the subgroup $\langle r_1, r_2, \dots, r_k \rangle$ does not change. Therefore Question 3.2 implies Question 3.3. \square

⁽⁷⁾ In this context, W is crystallographic if it is the Weyl group of a symmetrisable Kac-Moody Lie algebra. This happens if and only if the following two conditions are satisfied: (1) $m(s, t) \in \{2, 3, 4, 6, \infty\}$ for all $s \neq t$; (2) in each circuit of the Coxeter graph not containing the edge label ∞ , the number of edges labeled 4 (resp. 6) is even [67, Theorem B.2].

4. Combinatorics of $[1, w]$

Both the standard and the dual structure are particularly powerful to study spherical Artin groups, because the corresponding intervals $[1, \delta]_S$ and $[1, w] = [1, w]_R$ are lattices (and therefore give rise to Garside structures). In the spherical case, $[1, w]$ was shown to be a lattice by Bessis with a case-by-case proof [6] and then later by Brady and Watt with a case-free proof [19]. Digne showed that $[1, w]$ is a lattice in the affine cases \tilde{A}_n (for certain choices of the Coxeter element w) and \tilde{C}_n [53, 54]. It turns out that these and \tilde{G}_2 are the only affine cases where the lattice property holds, as proved by McCammond [74]. It seems reasonable to expect that most noncrossing partition posets are not lattices, but a general characterization is not known.

Question 4.1. — For which Coxeter groups W (and Coxeter elements w) the noncrossing partition poset $[1, w]$ is a lattice?

There is another combinatorial property of $[1, w]$ which emerged as part of the proof of the $K(\pi, 1)$ conjecture in the affine case: lexicographic shellability [13]. Shellability is ubiquitous in the theories of Coxeter groups and of subspace arrangements, where several naturally arising posets turn out to be shellable [3, 10, 38, 51, 56, 64, 81, 85].

EL-shellability of $[1, w]$ was an essential ingredient in [87] towards showing that the interval complex K_W deformation retracts onto the subcomplex X'_W which, in turn, is homotopy equivalent to the Salvetti complex X_W . Technically, the proof did not just use the existence of *any* EL-labeling of $[1, w]$. Rather, it used the fact that a certain family of total orderings \prec of the reflections $R_0 = R \cap [1, w]$ makes the natural labeling of $[1, w]$ an EL-labeling. In this setting, the EL-labeling property can be phrased as follows: for every $u \in [1, w]$, there is exactly one minimal reflection factorization $u = r_1 r_2 \cdots r_k$ that is \prec -increasing (i.e., $r_1 \prec r_2 \prec \cdots \prec r_k$); furthermore, this factorization is lexicographically smallest among all factorizations of $[1, u]$. Shellability of $[1, w]$ for finite Coxeter groups was proved by Athanasiadis-Brady-Watt [3].

The idea for the construction of suitable orderings \prec of R_0 is geometric. Suppose that Questions 2.2 and 2.3 have a positive answer. In particular, the Coxeter element w has an axis ℓ . Fix an axial chamber C_0 and a point $p \in C_0 \cap \ell$. When working in the contragradient representation of W , the axis becomes a two-dimensional plane P (for finite W , this is the Coxeter plane), and the point p becomes a line $\bar{p} \subseteq P$ through the origin. Any reflection hyperplane H intersects P in a line through the origin (Question 2.2 asks that no reflection hyperplane H contains P). The lines through the origin in P have a natural cyclic ordering \prec_c based on the orientation of the axis ℓ . The line \bar{p} provides a way to make this cyclic ordering into a total ordering: given two lines through the origin λ, λ' , we say that $\lambda \prec \lambda'$ if $\bar{p}, \lambda, \lambda'$ appear in this order in \prec_c . Then we can order reflections $r \in R_0$ based on the intersection between the reflection hyperplane H and the Coxeter plane P . See Figure 6.

The procedure we have described is ambiguous whenever there are two reflection hyperplanes that intersect P in the same line. For finite Coxeter groups, any resolution of “ties” works, and we get the reflection orderings used in [3] to prove that $[1, w]$ is

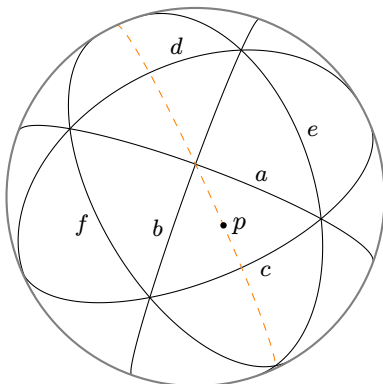


FIGURE 6. Procedure to define an axial ordering of the reflections in \mathfrak{S}_4 . The Coxeter element is $w = abc$ (as in Figure 2) and its axis is the dashed line, oriented upwards. Starting from the point p and moving along the axis, we encounter the reflections in the following order: $a \prec b \prec d \prec e \prec f \prec c$. Note that it is possible to swap a and b (they commute and fix the same point of the axis), as well as e and f .

EL-shellable. For affine Coxeter groups, a more sophisticated definition is needed to handle ties between so-called horizontal reflections. Nevertheless, we hope that the general idea of *axial orderings* can be extended to more general Coxeter groups.

Question 4.2. — Can axial orderings be defined for general Coxeter groups in such a way that the natural labeling of $[1, w]$ is an EL-labeling?

A proof of EL-shellability using axial orderings is likely going to need the following compatibility property as an intermediate step.

Question 4.3. — Does an axial ordering for W restrict to axial orderings for all subgroups generated by the reflections below $u \in [1, w]$?

5. Dual Artin groups and classifying spaces

As usual, let W be a Coxeter group and w one of its Coxeter elements. Recall from Section 1 that the interval complex K_W of $[1, w]$ is a Δ -complex (as in Hatcher’s book [65]) with one d -dimensional simplex $[x_1|x_2|\cdots|x_d]$ for every factorization $u = x_1x_2\cdots x_d$ such that $l_R(u) = l_R(x_1) + l_R(x_2) + \cdots + l_R(x_d)$ and $u \in [1, w]$. In other words, simplices correspond to partial minimal factorizations of w . The faces of $[x_1|x_2|\cdots|x_d]$ are given by:

- $[x_2|\cdots|x_d]$;
- $[x_1|x_2|\cdots|x_i x_{i+1}|\cdots|x_d]$ for $i = 1, \dots, d - 1$;
- $[x_1|x_2|\cdots|x_{d-1}]$.

The fundamental group $W_w = \pi_1(K_W)$ is the dual Artin group associated with the Coxeter group W and the Coxeter element w .

If $[1, w]$ is a lattice, then W_w is a Garside group and K_W is a classifying space for W_w . Interestingly, it was shown in [87, Theorem 6.6] that K_W is a classifying space for every affine Coxeter group W , even when the lattice property does not hold. The proof makes use of a construction by McCammond-Sulway [76] of a crystallographic group $C \supseteq W$ where the interval $[1, w]_{R'}$ (with respect to an extended generating set $R' \supseteq R$) is a lattice. Therefore, we naturally pose the following two questions.

Question 5.1. — For a general Coxeter group W with a Coxeter element w , is it possible to extend the generating set R to a larger set R' of isometries which generate a discrete group $C \supseteq W$ so that (1) the interval $[1, w]_{R'}$ is a lattice; (2) the dual Artin group W_w embeds into the interval group C_w ?

Question 5.2. — Is the interval complex K_W a classifying space?

If K_W is a classifying space for the dual Artin group W_w , then one can hope to prove the $K(\pi, 1)$ conjecture as in the affine case, by constructing a homotopy equivalence between K_W and the orbit configuration space Y_W (or equivalently, the Salvetti complex X_W). Regardless of whether K_W is a classifying space, a homotopy equivalence $K_W \simeq Y_W$ implies that the dual Artin group W_w is isomorphic to the standard Artin group G_W . Note that there is a natural map $G_W \rightarrow W_w$ which sends the standard generators of G_W to the same generators inside W_w , and it makes sense to expect this map to be an isomorphism.

Question 5.3. — Is the interval complex K_W homotopy equivalent to the orbit configuration space Y_W (or equivalently, to the Salvetti complex X_W)?

Question 5.4. — Is any dual Artin group W_w (naturally) isomorphic to the corresponding standard Artin group G_W ?

In the spherical case, Bessis proved that $[1, w]$ is always a lattice (Question 4.1) and that W_w is isomorphic to G_W (Question 5.4).⁽⁸⁾ Then Question 5.3 was a consequence of the fact that K_W and Y_W are classifying spaces of isomorphic groups. In the affine case, Question 5.4 was first settled by McCammond-Sulway [76] and then re-proved in [87] by answering Question 5.3.

In order to attempt a proof that $K_W \simeq X_W$, it is convenient to fix a totally ordered set of simple reflections $S = \{s_1, s_2, \dots, s_n\}$ such that $w = s_1 s_2 \cdots s_n$. Then, the cells of X_W are indexed by the subsets $T \subseteq S$ such that the (standard) parabolic

⁽⁸⁾ There is in fact a uniform proof of the isomorphism $W_w \cong G_W$ in the spherical case, as discussed in [28].

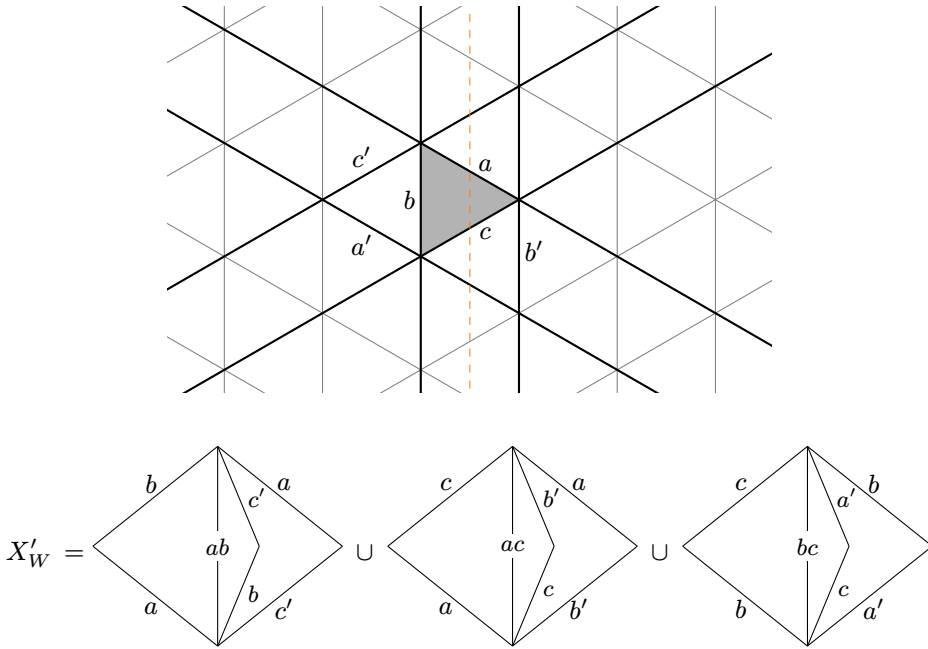


FIGURE 7. TOP: the affine reflection arrangement of type \tilde{A}_2 . The chamber corresponding to the set of simple reflections $S = \{a, b, c\}$ is highlighted. The dashed line is the axis of the Coxeter element $w = abc$. BOTTOM: the complex X'_W in the case \tilde{A}_2 . It is the union of three copies of the dual interval complex of type A_2 (Figure 5), one for each standard parabolic subgroup. Overall, X'_W consists of the following simplices: the 0-simplex $[\]$; the nine 1-simplices $[a], [b], [c], [a'], [b'], [c'], [ab], [ac]$, and $[bc]$; the nine 2-simplices $[a|b], [b|c'], [c'|a], [a|c], [c|b'], [b'|a], [b|c], [c|a']$, and $[a'|b]$.

subgroup $W_T \subseteq W$ is finite. With this setup, [87, Section 5] introduces a new CW complex $X'_W \simeq Y_W$ which is naturally included in K_W , defined as

$$X'_W = \bigcup_{\substack{T \subseteq S \\ W_T \text{ finite}}} K_{W_T}.$$

If W is finite, then $X'_W = K_W$ and nothing interesting happens. In all other cases, K_W has infinitely many cells, whereas X'_W is always a finite subcomplex. The affine case \tilde{A}_2 is described in Figure 7.

Note that the definition of X'_W requires to fix a Coxeter element $w_T \in W_T \cap [1, w]$ for every T , and this is done by multiplying the elements of T in the same order as they appear in S . Roughly speaking, X'_W locally looks like K_W but globally it has the structure of the Salvetti complex X_W . This, together with the known fact that $K_{W_T} \simeq X_{W_T}$ for finite W_T , implies that the newly defined X'_W is indeed homotopy

equivalent to the Salvetti complex X_W . Note that X'_W is defined and proved to be homotopy equivalent to X_W (and thus to Y_W) for general Coxeter groups. Note also that X'_W depends on the choice of S and not only on the Coxeter element w (as opposed to K_W , which depends on w but not on S).

The complexes X'_W provide a more direct link between K_W and Y_W . With them, we can strengthen Question 5.3 while possibly coming closer to its proof.

Question 5.5. — Does the interval complex K_W deformation retract onto any (or all) of its subcomplexes X'_W ?

In the affine case, this is proved within the framework of discrete Morse theory [57, 58]. The set of simple reflections S used to construct X'_W consists of the reflections with respect to the walls of an axial chamber (see Question 2.3). The “discrete Morse vector field” (a.k.a. the Morse matching) is constructed in two stages. First, all but a finite number of the simplices in $K_W \setminus X'_W$ of the form $[x_1|x_2|\cdots|x_d]$ with $x_1x_2\cdots x_d = w$ are matched with either $[x_1|x_2|\cdots|x_{d-1}]$ or $[x_2|x_3|\cdots|x_d]$. This is a natural way to collapse a large number of simplices of K_W and could potentially be useful in other cases. Second, the (finitely many) remaining simplices of $K_W \setminus X'_W$ are collapsed by using an axial ordering \prec of R_0 and the EL-shellability property (see Question 4.2). Roughly speaking, a simplex $\sigma = [x_1|x_2|\cdots|x_d]$ is matched with the simplex τ computed through the following procedure:

1. let $i = 1$;
2. if $l(x_i) > 1$, then let τ be the simplex obtained from σ by replacing x_i with $r|rx_i$ where r is the \prec -smallest reflection below x_i ;
3. if x_i is a reflection \prec -smaller than all reflections below x_{i+1} , then let τ be the simplex obtained from σ by replacing $x_i|x_{i+1}$ with x_ix_{i+1} ;
4. otherwise, increase i by 1 and repeat from step 2.

The uniqueness of \prec -increasing factorizations, granted by the EL-shellability property, ensures that the previous procedure actually defines an involution (if σ is matched with τ , then τ is matched with σ). For example, in the case \tilde{A}_2 (with the notation of Figure 7), the simplex $[w] = [abc]$ is matched with $[a|bc]$ because a is the \prec -smallest reflection below w . Conversely, $[a|bc]$ is matched with $[abc] = [w]$ because a is \prec -smaller than all reflections below bc (these reflections are b, c, a'). We also have that $[ab|c]$ is matched with $[a|b|c]$. This procedure could be used more generally, provided that the EL-shellability property holds.

The lattice property for $[1, w]$ (Question 4.1) implies that the dual Artin group W_w is a Garside group and thus it has a solvable word problem, provided that one can effectively compute meets and joins in $[1, w]$. Together with a constructive proof of the isomorphism $W_w \cong G_W$ (Question 5.4), this implies that the word problem for the Artin group G_W is also solvable. When the lattice property does not hold, then a “completion” $C_w \supseteq W_w$ as in [76] (Question 5.1) could be used to solve the

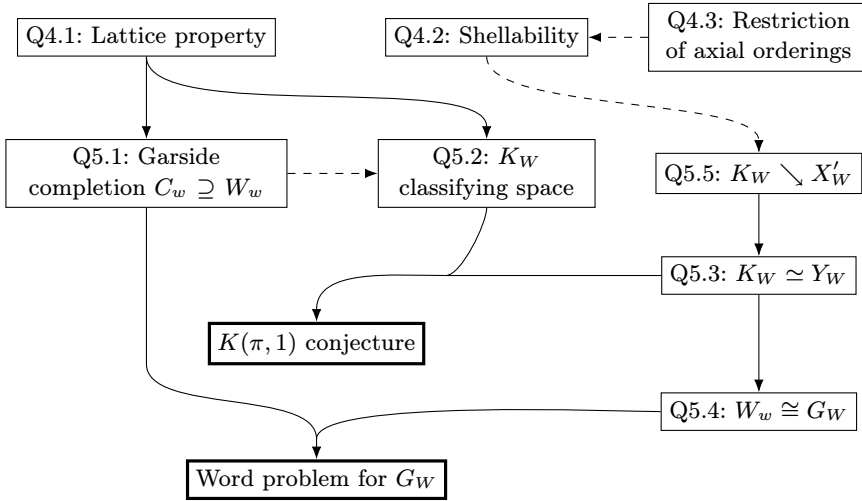


FIGURE 8. Diagram of the questions from Sections 4 and 5. Solid arrows mean “implies,” whereas dashed arrows mean “could be helpful for”.

word problem for W_w by leveraging a solution to the word problem for the Garside group C_w .

Figure 8 summarizes all questions from this and the previous section, as well as the implications between them and with the $K(\pi, 1)$ conjecture and the word problem.

6. Beyond spherical and affine cases

The questions posed in this paper are answered for spherical and affine Coxeter/Artin groups but mostly remain mysterious beyond those cases. Bessis proved that $[1, w]$ is a lattice if W is a *universal Coxeter group* (all the labels in the Coxeter graph are ∞), thus positively answering Question 4.1 in this case [7]. In an upcoming work with Emanuele Delucchi and Mario Salvetti, we are going to completely address Coxeter groups of rank 3 (i.e., with 3 generators):

Theorem 6.1 (Delucchi-Paolini-Salvetti [52]). — All questions in this paper have a positive answer for Coxeter groups of rank 3.

Examples of such groups are given in Figure 2. Note that the $K(\pi, 1)$ conjecture was already known if W has rank 3 because W is either spherical or 2-dimensional. However, most of the other questions are not trivial and require a study of the geometry and combinatorics of the dual structure. Hopefully, this is going to be a useful step towards a better understanding of the dual approach for general Coxeter groups.

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