Nonlocal continuum mechanics structures: the virtual powers method vs the extra fluxes topic

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Mauro Fabrizio*, Franca Franchi† and Roberta Nibbi‡

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Abstract

In this paper, once and for all, we want to strongly affirm that the thermodynamic framework for complex materials based on extra fluxes can lead to more severe restrictions with respect to the virtual powers scheme. Even if formally converging to the same final constitutive results, it seems only the consequence of an inappropriate approach to the problem itself. To achieve this goal, we proceed with a quick critical comparison between the different steps of the two methodologies: by way of example, we address novel constitutive settings for the third-order infinitesimal elasticity and viscoelasticity with an extension to thermo-elasticity, based on nonlocal revisited Green-Naghdi like thermal properties. Finally suitable thermodynamic restrictions are derived from the generalized Clausius-Duhem inequality.

1 Introduction

Nowadays there is an increasing interest in nonlocal thermo-elastic models that can predict the thermo-elastic behavior of micro- and nano-structures.

*Department of Mathematics, University of Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy - e-mail: mauro.fabrizio@unibo.it

†Department of Mathematics, University of Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy - e-mail: franca.franchi@unibo.it

‡Corresponding author: Department of Mathematics, University of Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy - e-mail: roberta.nibbi@unibo.it

These structures are widely employed in modern industry, in "extreme" environments, accounting for very short action times, very high temperature gradients requiring temperature-dependent conductivities, together with very small scale lengths, see e.g. [1] for nano-rods. Moreover, another challenge of higher-gradients theories is just represented by the constitutive boundary conditions naturally associated to them, within their thermodynamic construction. Higher-order boundary problems yield new and interesting insights in their applicability, since they are able to describe typical surface singularities of nano-technologies, see e.g. [2, 3]. Inspired by the evergreen great attention towards these smart materials, we aim in going back to the issue upstream, concerning the right thermodynamic approach to derive governing PDEs equations for nonlocal theories and related boundary conditions.

The nonlocal elastic behavior, herein addressed, is assumed to be consistent with third-order strain gradient elastic and viscoelastic theories, whereas the thermal properties are based on a suitable revisitation of Green-Naghdi like heat theories under nonlocal features. The choice of these modelings is just motivated by the increasing use of material bodies exhibiting micro- and nano-structures, at both mechanical and thermal point of view, in a wide range of applications. On the other hand, as already well known, the constitutive aspects of these nonlocal higher-gradient materials cannot be thermodynamically supported via the standard Thermodynamic Laws.

There are two very different approaches to implement, converging to apparently equivalent constitutive results: the variational principles and the so-called extra fluxes arguments. Following Germain's strategies (see e.g. [4]), the first format directly expresses the Laws of Thermodynamics in terms of appropriate internal and external mechanical/entropic powers, characteristics of the complex structure under study, as in Fabrizio et al. [5] and references therein. The second topic modifies the formulation of the two Laws, by introducing additional mechanical and entropic extra fluxes, in a divergence form, as suggested by Dunn and Serrin [6] and Müller [7].

In this way, it is not immediately clear which nonlocal constitutive setting is to be performed, with a posteriori choices for the extra fluxes, in order that the compatibility with the Laws of Thermodynamics is maintained, just in the spirit of the virtual powers developments.

It is worth to recall that in the literature, the two approaches are generally considered equivalent and often they are presented in parallel. However the first formulation is usually judged more general than the second one, as in Amendola et al. [8], with some theoretical advantage, as in Amendola et al.

[9]. Some constructive criticisms vs the second method are emphasized in Fabrizio et al. [5], within many nonlocal different research areas, whereas in Morro and Vianello [10] a "conciliation" between the two approaches is shown thoroughly, for second-gradient elasticity. Notably, Cardona et al. [11], working with a theory of second-gradient thermo-elasticity, point out the deficiencies of the extra fluxes topic vs the variational approach, within the isothermal frame.

In our opinion the extra fluxes topic does not hold up, without knowing the key constitutive developments, due to the variational approach. It is the primary goal of this paper to assert the universal validity of the virtual powers method in deriving thermodynamically consistent nonlocal continuum mechanics theories. To the best of our knowledge, this is the first time that, focusing on special nonlocal thermo-elastic/viscoelastic theories, and presenting both approaches in parallel, all the shortcomings of the extra fluxes topic are thoroughly highlighted to conclude, once and for all, that it is not the correct procedure to address such complex structures.

To this aim, hereafter, following the Lagrangian description we concentrate, by way of example, on third-gradient infinitesimal elasticity and thermo-elasticity, within analogous nonlocal thermal effects, via either Green-Naghdi or Fourier type heat theories. We choose to work with Green-Naghdi like thermal features, mainly motivated by the great past and recent research interest towards their revisitations, even in relation to the standard Fourier properties in their limiting cases, see e.g. [12, 13] and references therein.

In our opinion non standard Green-Naghdi heat conduction theories may be more accurate than the standard Fourier's theory, both theoretically and experimentally, since providing a general framework for a wide range of interacting thermal problems, see e.g. [14, 15].

Therefore our new idea to propose higher-order Green-Naghdi like thermal features, in order to also capture size dependencies related to nanoscale devices, may be worthwhile.

The outline of the paper is as follows. In section 2, after a brief overview of preliminary notations and definitions, novel non-isothermal third-order gradient elastic/viscoelastic theories, under analogous nonlocal thermal properties within a revisited Green-Naghdi heat theory, are set up via a non-standard version of the virtual powers method, which attributes a special role to the coldness displacement and its higher-gradients. The appropriate expressions for the virtual/real internal/external mechanical and entropic powers, incorporating also non standard surface effects, are deduced straightaway. As a

consequence, thermodynamic constraints, upon the general constitutive setting, are derived and the need of a limited number of additional internal lengths scale parameters is commented, within simplified gradient isotropic backgrounds. In section 3 we present the guidelines of the extra fluxes topic, by providing a step by step comparison with the previous procedure in order to highlight its irrefutable weaknesses. Moreover, we furnish two special constitutive examples in order to make a quick and incisive comparison also on the necessary smoothness requirements, driven by the two approaches.

Notably, the variational integral formulation, related to the virtual powers format, lead to a weak methodology, allowing for an interpretation of all the quantities involved, in the sense of distributions, see e.g. [4]. These variational techniques allow to establish existence, uniqueness and stability results of weak solutions to initial-(modified) boundary value problems, related to the PDEs system governing higher-order gradient theories [16, 17, 18, 19, 20, 21]. Even in differential contexts, as it is herein emphasized, this method needs weaker smoothness assumptions.

To conclude, the virtual power format guarantees a universal and a priori tool since it is just based on the rheological aspects under study and furnishes all the balance and imbalance local equations together with the appropriate constitutive boundary conditions. It should be mentioned that the constitutive restrictions come out from the usual thermodynamic arguments; also, the basic regularity assumptions, herein used to justify the passage from integral formulations to local ones, are the classical ones within continuum thermo-mechanics frameworks, appropriately referred to nonlocal constitutive settings under study.

As a final comment, these novel third-order thermo-elastic and viscoelastic scenarios are herein derived on a theoretical point of view, and we might only conjecture their possible ability to capture the multidisciplinary size effects related to micro- and nano-structures or structural components, which are highlighted in a wide range of applications in modern industry [22, 23, 24]. More specifically, our nonlocal Green-Naghdi heat theories might better model nanoscale thermal devices which are proven to be important in a broad variety of research areas, including for example aerospace, optics, chemical and ceramics, see e.g [14, 15].

2 Third-gradient thermo-elastic models: the kinematical background, the guidelines of a non standard version of the virtual powers scheme and thermodynamic restrictions

As a briefly reviewing of kinematical aspects, let a typical simple/non simple body be pointwise identified with a reference/initial configuration $\mathcal{B}(0)$ of the ordinary 3D Euclidean space; $\mathcal{B}(t)$ denotes the configuration at the generic time t. Let $\mathcal{P}(0)$ be an arbitrary body part (sub-body) of $\mathcal{B}(0)$, with outward unit normal $\mathbf{n}$ on its boundary. So we work on the space-time cylinder $\mathcal{P}(0) \times (0,T)$, $(0,T)$ being the observational time interval. In what follows, we address the Lagrangian description of Continuum Mechanics, within the standard notations, granting us some slight nomenclature changes to avoid "writing distractions". For convenience, we refer to the small deformations of the continuum with respect to a fixed orthogonal Cartesian reference system. A compact notation is used, with capital bold face letters denoting vectors and tensors of any order. Also, a simple dot stands for the scalar product between vectors or tensors of any order, independently of contracted index pair. As usual, the material position $\mathbf{X}$ is related to the current position $\mathbf{x}(t)$, via the material displacement vector $\mathbf{u}$, as $\mathbf{u} = \mathbf{x}(t) - \mathbf{X}$, and, for brevity, henceforth the space dependence of the fields is generally omitted. Further un upper dot over a symbol denotes the time derivative, so that the velocity field is $\mathbf{v} = \dot{\mathbf{u}}$.

The tensor $\mathbf{E} = sym(\nabla \mathbf{u})$ represents the linearized form of the Green-St-Venant strain tensor to denote the infinitesimal strain tensor, and ρ_0 stands for the (positive and constant) reference mass density.

Without any misunderstanding the first Piola-Kirchhoff stress tensor will be still denoted by $\mathbf{T}$, likewise we use the notation θ for the material temperature and $\mathbf{q}$ represents the heat flux vector in its material form; e, η, $\mathbf{b}$ and r stand for the internal energy density, the entropy density, the body force and the heat supply respectively, for unit mass and in their material description.

In analogy with the displacement vector $\mathbf{u}$, we introduce the coldness displacement, denoted by $\hat{k}$, defined as a time primitive of the coldness $k = 1/\theta$, so that $\dot{\hat{k}} = k$. The "kinetics" of our thermo-elastic behavior is thus characterized by the pair $(\mathbf{u}, \hat{k})$, so that the pair $(\dot{\mathbf{u}}, \dot{\hat{k}})$ represents the real

"velocities", whereas the virtual (test) "velocities" will be given by $(\delta\mathbf{u}, \delta\hat{k})$. As usual, the virtual velocities are supposed sufficiently smooth with compact support on the space-time cylinder $\mathcal{P}(0) \times (0,T)$ in study, $\forall\, \mathcal{P}(0) \subset \mathcal{B}(0)$.

The Helmholtz free energy for unit mass, defined by

$$\psi = e - \theta\eta\,, \tag{1}$$

yields the modified free energy potential

$$\tilde{\psi} = k\psi = ke - \eta\,. \tag{2}$$

We believe that the coldness k just plays a universal role within thermoelasticity [25], so henceforth, dealing with thermal properties, we choose to work with k, rather than with the standard temperature $\theta = 1/k$.

In what follows, we regard simple and local as an equivalent description of materials. Likewise, non simple and nonlocal are interchangeable.

We conclude this preliminary section with some brief considerations over the strategic notion of the state $\sigma(t)$ and the associated process $P(t)$, within a local/nonlocal constitutive theory.

On the Lagrangian point of view, for simple isothermal linearly elastic materials, one has $\sigma(t) = \nabla\mathbf{u}(t)$ with process $P(t) = \nabla\dot{\mathbf{u}}(t)$. The first Piola Kirchhoff stress tensor $\mathbf{T}$ depends on the present value of $\nabla\mathbf{u}$, so that the internal mechanical power density has form $\mathbf{T}\cdot\nabla\dot{\mathbf{u}}$ in $\mathcal{P}(0)\times(0,T)$. We recall that in classical viscoelastic theories, the stress tensor $\mathbf{T}$ may also depend on the past history of $\mathbf{E}$, i.e. $\mathbf{E}^t(s) = \mathbf{E}(t-s)$, and hence the state and the process are dutifully modified to incorporate new memory effects [8].

More generally, the deformation can be better approximated considering deformation gradients of a higher-order $N > 1$, thus leading to non simple elastic materials of grade $N > 1$. Due to this typology of complexity, the concept of *stress* has to be revisited, by introducing *stresses* of higher-order with the role of hyper-stresses. For example, for a second-gradient mechanical structure the effective stress tensor splits as follows:

$$\mathbf{T} = \mathbf{T}_2 - \nabla\cdot\mathbf{T}_3\,, \tag{3}$$

where the hyper-stress $\mathbf{T}_3$ is a third-order tensor, symmetric with respect to the last two indices (see e.g.[10]), which, in a simplified constitutive setting [26], may be chosen proportional to the gradient of $\mathbf{T}_2$; the proportionality parameter represents an internal length modulo able to capture different size-dependent effects, see e.g. [27]. In this way the constitutive choice for $\mathbf{T}_2$

would affect the constitutive form of $\mathbf{T}_3$; as a consequence the state and the process broaden to include the second gradients of $\mathbf{E}$ and $\dot{\mathbf{E}}$ respectively.

Raising the order of complexity to 3, we propose for the effective stress tensor the form

$$\mathbf{T} = \mathbf{T}_2 - \nabla \cdot \mathbf{T}_3 + \nabla \cdot (\nabla \cdot \mathbf{T}_4)\,, \tag{4}$$

accounting for a further fourth-order hyper-stress $\mathbf{T}_4$.

Now dimensional arguments, confirmed by experimental evidences, suggest that, if $\mathbf{T}_3$ may be proportional to the gradient of the standard stress tensor $\mathbf{T}_2$, for $\mathbf{T}_4$ can be predicted a proportionality with the double gradient of $\mathbf{T}_2$. By all means it is the constitutive relation for the symmetric $\mathbf{T}_2$ to be decisive for constitutive features of higher-gradient theories. In this Lagrangian context the mechanical state σ is characterized by $(\nabla\mathbf{u}, \nabla\nabla\mathbf{u}, \nabla\nabla\nabla\mathbf{u})$, whereas the process P is given by their temporal derivatives.

For a non isothermal third-gradient elastic theory, we may propose simple thermal properties of either Cattaneo or Green-Naghdi type: the state σ is then redefined to incorporate not only the coldness, as in the Fourier frame, but also other thermal variables, like the heat flux vector or the coldness displacement gradient, respectively. However, it is interesting to argue on a complex heat theory, by introducing nonlocal thermal effects of grade $M > 1$, via additional hyper-heat flux tensors. Just for a better understanding, within a rigid heat conduction of grade 3, the effective heat flux vector has form

$$\mathbf{q} = \mathbf{q}_1 - \nabla \cdot \mathbf{Q}_2 + \nabla \cdot (\nabla \cdot \mathbf{Q}_3)\,, \tag{5}$$

where the second-order symmetric tensor $\mathbf{Q}_2$ and the third-order tensor $\mathbf{Q}_3$, symmetric with respect to its two last indices, play the role of hyper-heat fluxes. Indeed the divergence and the double divergence account for the flux exchanging between bulk and surface points. However the knowledge of inflowing/outflowing hyper-fluxes is still not well understood: hence the signs are random, and again it will be up to the thermodynamics restrictions to settle them.

Again it is the constitutive relation for $\mathbf{q}_1$ to play a key role on the third-gradient theory; for Fourier type thermal features, $\mathbf{q}_1$ is proportional to the coldness gradient, via a generally non constant conductivity, so as for the nonlocal mechanical structure we address a state σ dependent only on the coldness k. Instead the thermal process P includes first-, second- and third-coldness gradients. Obviously, in analogy with nonlocal mechanical

structures, one may generalize nonlocal thermal properties to include memory effects.

Let us now briefly present the main steps of a nonstandard version of the virtual powers method, based on the notion of the coldness displacement, in order to deduce all the balance and imbalance laws, together with the appropriate boundary conditions, for nonlocal third-gradient elasticity and thermo-elasticity.

Taking into account the relation (4), the equation of motion for third-gradient elasticity has the form

$$\rho_0 \ddot{\mathbf{u}} = \nabla \cdot [\mathbf{T}_2 - \nabla \cdot \mathbf{T}_3 + \nabla \cdot (\nabla \cdot \mathbf{T}_4)] + \rho_0 \mathbf{b} \, . \tag{6}$$

Considering the inner product of (6) with $\delta \mathbf{u}$ and following the standard procedure, we recover its weak formulation in the d'Alembert form

$$\int_{\mathcal{P}(0)} \rho_0 \ddot{\mathbf{u}} \cdot \delta \mathbf{u} \, dV = \mathcal{P}_m^{(e)}(\mathcal{P}(0); \delta \mathbf{u}) - \mathcal{P}_m^{(i)}(\mathcal{P}(0); \delta \mathbf{u}) \tag{7}$$

for any reference sub-bodies $\mathcal{P}(0) \subset \mathcal{B}(0)$ and any sufficiently smooth virtual velocity field $\delta \mathbf{u}$. The internal and external virtual mechanical powers are expressed as

$$\begin{aligned} \mathcal{P}_m^{(i)}(\mathcal{P}(0); \delta \mathbf{u}) &= \int_{\mathcal{P}(0)} \tilde{p}_m^{(i)} \, dV \\ &= \int_{\mathcal{P}(0)} (\mathbf{T}_2 \cdot \nabla \delta \mathbf{u} + \mathbf{T}_3 \cdot \nabla \nabla \delta \mathbf{u} + \mathbf{T}_4 \cdot \nabla \nabla \nabla \delta \mathbf{u}) \, dV \, , \end{aligned} \tag{8}$$

$$\begin{aligned} \mathcal{P}_m^{(e)}(\mathcal{P}(0); \delta \mathbf{u}) &= \int_{\mathcal{P}(0)} \rho_0 \mathbf{b} \cdot \delta \mathbf{u} \, dV \\ &+ \int_{\partial \mathcal{P}(0)} [\mathbf{T} \delta \mathbf{u} + (\mathbf{T}_3 - \nabla \cdot \mathbf{T}_4) \nabla \delta \mathbf{u} + \mathbf{T}_4 \nabla \nabla \delta \mathbf{u}] \cdot \mathbf{n} \, da \, . \end{aligned} \tag{9}$$

Whence the real form of the Kinetic Energy Theorem can be written as

$$\frac{d}{dt} \int_{\mathcal{P}(0)} \rho_0 \frac{\dot{\mathbf{u}}^2}{2} \, dV = \mathcal{P}_m^{(e)}(\mathcal{P}(0)) - \mathcal{P}_m^{(i)}(\mathcal{P}(0)) \tag{10}$$

where the external and internal mechanical powers are defined as

$$\begin{aligned} \mathcal{P}_m^{(e)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} \rho_0 \mathbf{b} \cdot \dot{\mathbf{u}} \, dV \\ &+ \int_{\partial \mathcal{P}(0)} [\mathbf{T} \dot{\mathbf{u}} + (\mathbf{T}_3 - \nabla \cdot \mathbf{T}_4) \nabla \dot{\mathbf{u}} + \mathbf{T}_4 \nabla \nabla \dot{\mathbf{u}}] \cdot \mathbf{n} \, da \, , \end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{P}_m^{(i)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} p_m^{(i)}\, dV \\ &= \int_{\mathcal{P}(0)} \left(\mathbf{T}_2 \cdot \nabla\dot{\mathbf{u}} + \mathbf{T}_3 \cdot \nabla\nabla\dot{\mathbf{u}} + \mathbf{T}_4 \cdot \nabla\nabla\nabla\dot{\mathbf{u}}\right)\, dV\,,\end{aligned} \tag{12}$$

$p_m^{(i)}$ representing the density of the global internal mechanical power.

It is worth to note that the nonlocal mechanical theory under study is responsible of additional constitutive boundary conditions driven by the presence of hyper-stress tensors via the superficial integral in (11).

The interaction between the First Law of Thermodynamics and the Kinetic Energy Theorem (10), under suitable smoothness assumptions, locally yields

$$\rho_0\dot{e} - p_m^{(i)} = \rho_0 h \tag{13}$$

where $p_m^{(i)}$ is defined in (12) and, from the heat balance equation, the right-hand side satisfies

$$\rho_0 h = -\nabla \cdot \mathbf{q} + \rho_0 r\,, \tag{14}$$

$(\mathbf{q}, \rho_0 r)$ being the internal energy thermal inflow.

We highlight that the integral formulation of the internal energy mechanical inflow is just represented by the right-hand side of (11).

Let us multiply (14) by $\delta\hat{k}$ and then integrate over $\mathcal{P}(0)$ to yield the balance equation for the virtual entropy powers

$$\mathcal{P}_{en}^{(i)}(\mathcal{P}(0); \delta\hat{k}) = \mathcal{P}_{en}^{(e)}(\mathcal{P}(0); \delta\hat{k})\,. \tag{15}$$

Herein, thinking of the third-gradient heat theory, the virtual internal and external entropy powers are expressed as

$$\begin{aligned}\mathcal{P}_{en}^{(i)}(\mathcal{P}(0); \delta\hat{k}) &= \int_{\mathcal{P}(0)} \left(\rho_0 h\delta\hat{k} - \mathbf{q}_1 \cdot \nabla\delta\hat{k} - \mathbf{Q}_2 \cdot \nabla\nabla\delta\hat{k} - \mathbf{Q}_3 \cdot \nabla\nabla\nabla\delta\hat{k}\right)\, dV \\ \mathcal{P}_{en}^{(e)}(\mathcal{P}(0); \delta\hat{k}) &= \int_{\mathcal{P}(0)} \rho_0 r\delta\hat{k}\, dV \\ &\quad - \int_{\partial\mathcal{P}(0)} \left\{\mathbf{q}\delta\hat{k} + \left[\left(\mathbf{Q}_2 - (\nabla \cdot \mathbf{Q}_3)\right)\nabla\delta\hat{k} + \mathbf{Q}_3\nabla\nabla\delta\hat{k}\right]\right\} \cdot \mathbf{n}\, da\,,\end{aligned} \tag{16}$$

which, in a complete analogy with the mechanical background, lead to the

definitions

$$\begin{aligned}\mathcal{P}_{en}^{(i)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} p_{en}^{(i)}\, dV \\ &= \int_{\mathcal{P}(0)} \left(\rho_0 hk - \mathbf{q}_1 \cdot \nabla k - \mathbf{Q}_2 \cdot \nabla\nabla k - \mathbf{Q}_3 \cdot \nabla\nabla\nabla k\right)\, dV \\ \mathcal{P}_{en}^{(e)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} \rho_0 rk\, dV \\ &\quad - \int_{\partial\mathcal{P}(0)} \left\{k\mathbf{q} + \left[\left(\mathbf{Q}_2 - (\nabla \cdot \mathbf{Q}_3)\right)\nabla k + \mathbf{Q}_3 \nabla\nabla k\right]\right\} \cdot \mathbf{n}\, da\,,\end{aligned} \tag{17}$$

$p_{en}^{(i)}$ being the density of the real internal entropy power.

So, under suitable hypotheses of regularity, the Second Law of Thermodynamics can be locally written as

$$\rho_0 \dot{\eta} \geq \rho_0 hk - \mathbf{q}_1 \cdot \nabla k - \mathbf{Q}_2 \cdot \nabla\nabla k - \mathbf{Q}_3 \cdot \nabla\nabla\nabla k = p_{en}^{(i)}\,. \tag{18}$$

The interlacement between (13), multiplied by k, and (18) yields

$$\begin{aligned}\rho_0 \dot{\eta} \geq \rho_0 \dot{e} k &- \left(\mathbf{T}_2 \cdot \nabla\dot{\mathbf{u}} + \mathbf{T}_3 \cdot \nabla\nabla\dot{\mathbf{u}} + \mathbf{T}_4 \cdot \nabla\nabla\nabla\dot{\mathbf{u}}\right) k \\ &- \mathbf{q}_1 \cdot \nabla k - \mathbf{Q}_2 \cdot \nabla\nabla k - \mathbf{Q}_3 \cdot \nabla\nabla\nabla k\end{aligned} \tag{19}$$

whereas, via the definition of the modified free energy $\tilde{\psi}$, defined in (2), we recover the general Clausius-Duhem inequality for our theory

$$\begin{aligned}&- \rho_0 \dot{\tilde{\psi}} + \rho_0 e \dot{k} + \mathbf{q_1} \cdot \nabla k + \mathbf{Q}_2 \cdot \nabla\nabla k + \mathbf{Q}_3 \cdot \nabla\nabla\nabla k \\ &+ \left(\mathbf{T}_2 \cdot \nabla\dot{\mathbf{u}} + \mathbf{T}_3 \cdot \nabla\nabla\dot{\mathbf{u}} + \mathbf{T}_4 \cdot \nabla\nabla\nabla\dot{\mathbf{u}}\right) k \geq 0\,.\end{aligned} \tag{20}$$

It is worth to observe that this inequality holds also for nonlocal higher-order thermo-elastic structures, by suitably updating the internal mechanical and entropic densities. The universality of this thermodynamic approach just stands on different expressions of the internal virtual/real powers, according to the nonlocal properties of the material under study.

Clearly we reduce to the classical Clausius-Duhem inequality in the absence of both nonlocal effects, while, in an isothermal context, from (20) we provide the Dissipation Principle in the form $\rho_0 \dot{\psi} \leq p_m^{(i)}$. On the other hand, within a nonlocal rigid heat conduction of grade 3, in (20) all the mechanical contributions are neglected. It is perhaps superfluous to note that for

non-isothermal third-strain gradient elasticity, under simple (Fourier, Cattaneo or Green-Naghdi type) thermal properties, in (20) the hyper-heat flux tensors disappear so that the effective heat flux vector is just $\mathbf{q_1}$.

The final step of this section is the application of the classical Coleman-Noll arguments towards the thermodynamic restrictions on our constitutive settings, taking into account the dependences of the modified free energy potential on the state functions.

In order to describe non-isothermal anisotropic/isotropic third-strain gradient elasticity, under nonlocal anisotropic/isotropic Green-Naghdi type thermal effects of grade 3, we take $\tilde{\psi}$ as a continuously differentiable function of all the independent variables at the current time t, representing the state $\sigma = (k, \nabla\mathbf{u}, \nabla\nabla\mathbf{u}, \nabla\nabla\nabla\mathbf{u}, \nabla\hat{k}, \nabla\nabla\hat{k}, \nabla\nabla\nabla\hat{k})$. Upon evaluation of $\dot{\tilde{\psi}}$, by using the chain rule, and substitution in (20), with the usual line of arguments, besides the classical relations

$$e = \frac{\partial\tilde{\psi}}{\partial k}\,, \qquad \eta = k^2\frac{\partial\psi}{\partial k}\,, \tag{21}$$

we provide the following "split" thermo-elastic restrictions

$$\begin{aligned} &k\left(\mathbf{T_2} - \rho_0\frac{\partial\psi}{\partial\nabla\mathbf{u}}\right)\cdot\nabla\dot{\mathbf{u}} \geq 0\,, \quad k\left(\mathbf{T_3} - \rho_0\frac{\partial\psi}{\partial\nabla\nabla\mathbf{u}}\right)\cdot\nabla\nabla\dot{\mathbf{u}}\,, \\ &k\left(\mathbf{T_4} - \rho_0\frac{\partial\psi}{\partial\nabla\nabla\nabla\mathbf{u}}\right)\cdot\nabla\nabla\nabla\dot{\mathbf{u}} \geq 0\,, \end{aligned} \tag{22}$$

together with

$$\begin{aligned} &\left(\mathbf{q_1} - \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\hat{k}}\right)\cdot\nabla k \geq 0\,, \quad \left(\mathbf{Q_2} - \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\nabla\hat{k}}\right)\cdot\nabla\nabla k \geq 0\,, \\ &\left(\mathbf{Q_3} - \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\nabla\nabla\hat{k}}\right)\cdot\nabla\nabla\nabla k \geq 0\,. \end{aligned} \tag{23}$$

These inequalities hold if we choose

$$\begin{aligned} &\mathbf{T_2} = \rho_0\frac{\partial\psi}{\partial\nabla\mathbf{u}} + \mathbf{D_1}\nabla\dot{\mathbf{u}}\,, \quad \mathbf{T_3} = \rho_0\frac{\partial\psi}{\partial\nabla\nabla\mathbf{u}} + \mathbf{D_2}\nabla\nabla\dot{\mathbf{u}}\,, \\ &\mathbf{T_4} = \rho_0\frac{\partial\psi}{\partial\nabla\nabla\nabla\mathbf{u}} + \mathbf{D_3}\nabla\nabla\nabla\dot{\mathbf{u}} \end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\mathbf{q_1} = \rho_0 \frac{\partial \tilde{\psi}}{\partial \nabla \hat{k}} + \mathbf{K_1} \nabla k \,, \quad \mathbf{Q_2} = \rho_0 \frac{\partial \tilde{\psi}}{\partial \nabla \nabla \hat{k}} + \mathbf{K_2} \nabla \nabla k \,, \\
&\mathbf{Q_3} = \rho_0 \frac{\partial \tilde{\psi}}{\partial \nabla \nabla \nabla \hat{k}} + \mathbf{K_3} \nabla \nabla \nabla k \,,
\end{aligned} \tag{25}$$

$\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ and $\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3}$ being positive tensorial valued functions, depending on at most $(\mathbf{X}, t)$ and the coldness k. It is worth to comment that relations (24) generalize our third-strain gradient constitutive setting to account of third-strain rate gradient viscoelastic features. The coefficients $\mathbf{D_1}, \mathbf{D_2}$ and $\mathbf{D_3}$ may be interpreted as viscoelastic and hyper-viscoelastic tensorial moduli, likewise the additional $\mathbf{K_1}, \mathbf{K_2}$ and $\mathbf{K_3}$ play the role of conductivity and hyper-conductivity tensors, respectively.

By summarizing, different nonlocal Green-Naghdi thermo-elastic/viscoelastic theories depend on different expressions for the free energy potentials. Notably, a significant number of additional constitutive parameters enter due to the complex structure under study.

In special linearly isotropic backgrounds, we recall that a positive-definite fourth-order tensor can be written in terms of only two positive phenomenological constants, which become characteristics of the structure under study; obviously the picture changes a lot for higher-order isotropic tensors, see e.g. [28].

When ψ does not depend on (simple, double and triple) coldness displacement gradients, so that k represents its only thermal dependence, we reduce to a nonlocal Fourier type heat theory of grade 3. In this case, we may confine our attention to the following free energy potential, accounting for a quadratic dependence on the first-, second- and third-strain gradients, together with the mentioned dependence on the coldness,

$$\begin{aligned}
\psi = \psi_E(k) &+ \frac{1}{2\rho_0} \mathbf{C_1}(k) \nabla \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2\rho_0} \mathbf{C_2}(k) \nabla \nabla \mathbf{u} \cdot \nabla \nabla \mathbf{u} \\
&+ \frac{1}{2\rho_0} \mathbf{C_3}(k) \nabla \nabla \nabla \mathbf{u} \cdot \nabla \nabla \nabla \mathbf{u} \,.
\end{aligned} \tag{26}$$

Herein $\mathbf{C_1}$ is the standard fourth-order elasticity tensor, $\mathbf{C_2}$ and $\mathbf{C_3}$ represent the sixth- and eighth-order hyper-elasticity tensors respectively, obeying suitable symmetries strictly related to the symmetry requirements on $\mathbf{T_2}$ and $\mathbf{T_3}$. Since it is reasonable to assume for $\mathbf{T_2}$ a symmetry in the last pair of

indices, as in [10, 29], we propose for $\mathbf{T_3}$ a symmetry with respect to its last three indices.

Thus, by restricting to (24) with vanishing $\mathbf{D_i}$, $i = 1, 2, 3$, we find the constitutive relations

$$\mathbf{T_2} = \mathbf{C_1}\nabla\mathbf{u}\,, \quad \mathbf{T_3} = \mathbf{C_2}\nabla\nabla\mathbf{u}\,, \quad \mathbf{T_4} = \mathbf{C_3}\nabla\nabla\nabla\mathbf{u}\,. \tag{27}$$

Obviously, from (25), beyond the Fourier Law $\mathbf{q_1} = \mathbf{K_1}(k)\nabla k$ ($\mathbf{K_1}$ being the non-negative conductivity tensor), we recover the constitutive relations for the additional hyper-heat fluxes $\mathbf{Q_2}$ and $\mathbf{Q_3}$, interpreting $\mathbf{K_2}$ and $\mathbf{K_3}$ as the non-negative fourth- and sixth-order hyper-conductivity tensors, satisfying the same symmetries as $\mathbf{C_1}$ and $\mathbf{C_2}$, respectively.

By making recourse to the simplified isotropic Aifantis gradient theory, within the so called internal lengths gradient (in short ILG) framework, see e.g.[27], the effective heat flux vector for third-coldness gradient thermal behaviors has form

$$\mathbf{q} = \left(1 - L_{c_1}^2\Delta + L_{c_2}^2\Delta^2\right)\mathbf{K_1}(k)\nabla k\,, \tag{28}$$

where L_{c_1} and L_{c_2} denote two thermal internal lengths (ILs), entering through the additional nonlocal terms.

In an analogous manner, starting from the Hookean form for the standard stress tensor $\mathbf{T_2}$, in terms of $\mathbf{E}$, the effective stress tensor may be proposed as

$$\begin{aligned}\mathbf{T} = \mathbf{T}_2 - \nabla\cdot\mathbf{T}_3 + \nabla\cdot(\nabla\cdot\mathbf{T}_4) = 2\mu\left(1 - L_{e_1}^2\Delta + L_{e_2}^2\Delta^2\right)\mathbf{E} \\ + \lambda\left(1 - L_{e_1}^2\Delta + L_{e_2}^2\Delta^2\right)(tr\mathbf{E})\mathbf{I}\,.\end{aligned} \tag{29}$$

Here μ and λ denote the (positive) Lamè constants, whereas L_{e_1} and L_{e_2} represent the elastic ILs, due to the additional hyper-stresses, with an important role in detecting different size-effects.

Also, it is worth to highlight that, within the Aifantis scheme, only a minimum number of dominant ILs gradient parameters needs to be introduced to successfully account for size dependent characteristics of nonlocal multidisciplinary interacting phenomena, at very different scales, as in [30].

Finally we observe that, for a better description of memory behaviors, a generalization of these nonlocal gradient theories to involve the Caputo-Fabrizio fractional calculus [31] is also desirable, see e.g. [32, 33].

3 The virtual powers method vs the extra fluxes format

We now present the key steps of the (mechanical/entropic) extra fluxes type thermodynamic framework to address nonlocal thermo-mechanical structures, like third-gradient elasticity and thermo-elasticity.

Following the same notation as in [8], for the mechanical frame it is only the First Law of Thermodynamics to be modified by supposing that the internal energy mechanical influx is given by the couple $(-\mathbf{T}\dot{\mathbf{u}}, \mathbf{\Phi_e})$, where $\mathbf{\Phi_e}$ represents an extra influx vector, playing the role of the interstitial working conceived by Dunn and Serrin [6]. In this way the local form of the First Law of Thermodynamics reads

$$\rho_0 \dot{e} - \mathbf{T} \cdot \nabla \dot{\mathbf{u}} = -\nabla \cdot \mathbf{\Phi_e} + \rho_0 h \tag{30}$$

so that, compared with (13), by a quick inspection the density of the internal mechanical power has now the following form

$$p_m^{(i)} = \mathbf{T} \cdot \nabla \dot{\mathbf{u}} - \nabla \cdot \mathbf{\Phi_e}\,, \tag{31}$$

which exhibits different expressions according to the different constitutive setting to face up.

Again, we focus on a third-order strain gradient elasticity theory, with the primary goal to state the thermodynamic legitimation of the virtual power format vs the extra fluxes topic and, at the same time, carrying on a critical comparison of the two methods.

If we consider the effective stress tensor of the form (4), by using standard identities, we easily find

$$\begin{aligned} p_m^{(i)} &= \mathbf{T}_2 \cdot \nabla \dot{\mathbf{u}} + \mathbf{T}_3 \cdot \nabla \nabla \dot{\mathbf{u}} + \mathbf{T}_4 \cdot \nabla \nabla \nabla \dot{\mathbf{u}} \\ &\quad - \nabla \cdot \left[\mathbf{\Phi_e} + (\mathbf{T_3} - \nabla \cdot \mathbf{T_4}) \nabla \dot{\mathbf{u}} + \mathbf{T_4} \nabla \nabla \dot{\mathbf{u}} \right] , \end{aligned} \tag{32}$$

which coincides with the density defined in (12) whenever we choose

$$\mathbf{\Phi_e} = -\left(\mathbf{T_3} - \nabla \cdot \mathbf{T_4}\right) \nabla \dot{\mathbf{u}} - \mathbf{T_4} \nabla \nabla \dot{\mathbf{u}}\,, \tag{33}$$

within an additive divergence-free term.

It is worth to enhance that this a posteriori choice does not change however the standard boundary value problem. This becomes a strong drawback

towards the challenge concerning the role of additional *constitutive* boundary conditions due to the non locality, via the presence of hyper-stress tensors. Therefore this approach therefore just pierces the target of the important hyper-boundary conditions topic, strictly related to the applicability of complex mechanical structures in a wide range of phenomena, see e.g. [2, 3].

Likewise, in order to model nonlocal thermal properties as suggested by Müller [7], an extra entropic flux appears to modify the Second Law of Thermodynamics, by assuming that the entropy inflow is given by the couple $(\mathbf{q}k + \mathbf{\Phi}_\eta, \rho_0 rk)$. In this way inequality (18) has form

$$\rho_0 \dot{\eta} \geq \rho_0 hk - \nabla \dot{\hat{k}} \cdot \mathbf{q} - \nabla \cdot \mathbf{\Phi}_\eta \,, \tag{34}$$

so, when the heat flux vector splits as in (5), we arrive at

$$\begin{aligned} \rho_0 \dot{\eta} \geq \rho_0 hk - \nabla \dot{\hat{k}} \cdot \mathbf{q_1} - \nabla \nabla \dot{\hat{k}} \cdot \mathbf{Q_2} - \nabla \nabla \nabla \dot{\hat{k}} \cdot \mathbf{Q_3} \\ + \nabla \cdot \left[(\mathbf{Q_2} - \nabla \cdot \mathbf{Q_3}) \nabla \dot{\hat{k}} + (\nabla \cdot \mathbf{Q_3}) \nabla \nabla \dot{\hat{k}} - \mathbf{\Phi}_\eta \right] , \end{aligned} \tag{35}$$

which coincides with the entropic density in (18), whenever we choose

$$\mathbf{\Phi}_\eta = (\mathbf{Q_2} - \nabla \cdot \mathbf{Q_3}) \nabla \dot{\hat{k}} + (\nabla \cdot \mathbf{Q_3}) \nabla \nabla \dot{\hat{k}} \,, \tag{36}$$

within an additive divergence-free term.

Also, we stress an intrinsic non-uniqueness of the extra fluxes in that a divergence-free term may be added, without affecting the two Laws of Thermodynamics. Moreover it is usual to require that $\mathbf{\Phi}_\eta \cdot \mathbf{n} = 0$ on the boundary $\partial \mathcal{B}(0)$, so that the global formulation of the entropy inequality for the whole region $\mathcal{B}(0)$ is free from $\mathbf{\Phi}_\eta$.

As an overall comment, we note that the presence of $\mathbf{\Phi_e}$ and $\mathbf{\Phi}_\eta$ is formally inhibited by the above a posteriori choices, totally constrained by the nonlocal thermo-elastic theory of grade 3 under study. This, in turn, precludes the universal character of the two Laws of Thermodynamics.

It should also be stressed that the internal powers are not given by a differential form, but exhibit a hybrid representation, containing also divergence terms.

Finally, beyond all the above shareable criticisms of this approach versus the other one, we are now interested in arguing also on the need of more severe smoothness requirements on the constitutive setting in study.

To this aim let us consider the following linearly anisotropic constitutive relations for third-order strain gradient elasticity in an isothermal background:

$$\mathbf{T_2} = \mathbf{C_1}\nabla\mathbf{u}\,, \quad \mathbf{T_3} = \mathbf{C_2}\nabla\nabla\mathbf{u}\,, \quad \mathbf{T_4} = \mathbf{C_3}\nabla\nabla\nabla\mathbf{u}\,, \tag{37}$$

where $\mathbf{C_i}$, $i = 1, 2, 3$, are chosen as in the previous section.

The standard combination between (30), multiplied by k, and (34) leads to the following Clausius-Duhem inequality:

$$-\rho_0\dot{\tilde{\psi}} + \rho_0 e\dot{k} + p_m^{(i)}k + \nabla k \cdot \mathbf{q} + \nabla \cdot \mathbf{\Phi}_\eta \geq 0 \tag{38}$$

where $p_m^{(i)}$ is now defined in (31).

On the other hand from (38), within this isothermal constitutive setting, the appropriate Dissipation Principle reads

$$\rho_0\dot{\psi} \leq [\mathbf{C_1}\nabla\mathbf{u} - \nabla\cdot(\mathbf{C_2}\nabla\nabla\mathbf{u}) + \nabla\cdot(\nabla\cdot(\mathbf{C_3}\nabla\nabla\nabla\mathbf{u}))]\cdot\nabla\dot{\mathbf{u}} - \nabla\cdot\mathbf{\Phi_e}\,, \tag{39}$$

which, via standard identities, may be compared with the analogous Principle as due to the variational approach:

$$\rho_0\dot{\psi} \leq \mathbf{C_1}\nabla\mathbf{u}\cdot\nabla\dot{\mathbf{u}} + \mathbf{C_2}\nabla\nabla\mathbf{u}\cdot\nabla\nabla\dot{\mathbf{u}} + \mathbf{C_3}\nabla\nabla\nabla\mathbf{u}\cdot\nabla\nabla\nabla\dot{\mathbf{u}}\,. \tag{40}$$

Obviously, if $\mathbf{\Phi_e} = [-(\mathbf{C_2}\nabla\nabla\mathbf{u}) + \nabla\cdot(\mathbf{C_3}\nabla\nabla\nabla\mathbf{u})]\,\nabla\dot{\mathbf{u}} - (\mathbf{C_3}\nabla\nabla\nabla\mathbf{u})\,\nabla\nabla\dot{\mathbf{u}}$, these two principles eventually come in the same form, but it is immediately evident that the right-hand side of (40) is well defined for continuous hyper-elasticities $\mathbf{C_2}$ and $\mathbf{C_3}$, whereas the right-hand side of (39) requires the differentiability of $\mathbf{C_2}$ and the double differentiability of $\mathbf{C_3}$. Therefore we may conclude that the two approaches are by no means formally equivalent and, in our opinion, the correct procedure to address nonlocal structures turns out the virtual powers format [11].

Also, within a purely thermal structure, let us now consider the following constitutive relations for a third-coldness gradient Fourier type rigid heat theory:

$$\mathbf{q_1} = \mathbf{K_1}\nabla k\,, \quad \mathbf{Q_2} = \mathbf{K_2}\nabla\nabla k\,, \quad \mathbf{Q_3} = \mathbf{K_3}\nabla\nabla\nabla k \tag{41}$$

where, as before, $\mathbf{K_1}$ is the (non-negative) second-order conductivity tensor, while $\mathbf{K_2}$ and $\mathbf{K_3}$ are the (non-negative) fourth- and sixth-order hyperconductivity tensors, respectively. In this case the right-hand side of (34) reads

$$\rho_0 hk - \nabla k\cdot[\mathbf{K_1}\nabla k - \nabla\cdot(\mathbf{K_2}\nabla\nabla k) + \nabla\cdot(\nabla\cdot(\mathbf{K_3}\nabla\nabla\nabla k))] - \nabla\cdot\mathbf{\Phi}_\eta\,. \tag{42}$$

With the help of standard identities, under the final choice

$$\mathbf{\Phi}_\eta = [\mathbf{K_2}\nabla\nabla k - \nabla\cdot(\mathbf{K_3}\nabla\nabla\nabla k)]\,\nabla k + (\mathbf{K_3}\nabla\nabla\nabla k)\nabla\nabla k\,, \qquad (43)$$

we recover the right expression of the density of the (real) internal entropy power, given in (18). Again, concerning the regularity assumptions on the constitutive setting, we point out that this approach needs the differentiability of $\mathbf{K_2}$ together with a twice differentiability for $\mathbf{K_3}$ On the contrary, it is enough the continuity requirement along the variational formulation, thus confirming our point of view.

In conclusion, we believe that these two thermodynamics arguments are by no means comparable. Indeed we want to underline that the virtual powers format just represents the most efficient and right approach for the constitutive description of nonlocal continuum mechanics structures, like higher-order gradients thermo-elasticity. Even if, for sake of simplicity, we focused on purely mechanical and thermal structures, our theories may be easily extended to the presence of interaction terms accounting for temperature driven motions, as in [12], also generalizable to coupled nonlocal hereditary effects, as in [8, 9].

By summarizing, along the variational approach, we may develop general balance equations together with appropriate boundary conditions for third-/higher-order thermo-elasticity, undergoing infinitesimal/finite deformations. Also, the theory provides thermodynamically consistent general constitutive equations and furnishes a weak formulation of related initial-boundary value problems towards the well-posedness issues.

4 Concluding remarks

The primary idea of this brief paper is to assert the universal validity of the virtual powers method in deriving thermodynamically consistent nonlocal continuum mechanics theories vs the extra fluxes topic.

We emphasize that all strengths of the first method are quite overlooked by the second one. The key support points are

- the a priori role played by the concepts of state and process to assess the typology of nonlocal behaviors to face;
- internal mechanical and/or entropic powers formulated through linear

dependences on the quantities that define the process and hence, in a predictable way, generalizable to higher-order non localities;

- the presence of additional higher-order constitutive boundary conditions, justified through the procedural developments, with a challenge role within micro- to nano-applications;

- last, but not least, the variational arguments allow for the weak formulation of initial-(modified) boundary value problems within the distribution theory. Whence less smoothness needs to be required on the solutions of the governing equations, in order to test their well-posedness problem via existence, uniqueness and stability results.

We achieve these conclusions from a procedural point of view, by comparing, step by step, the different thermodynamic developments leading to nonlocal constitutive structures. We address, by way of example, third-strain gradient linearized elasticity, under analogous nonlocal thermal properties, due to a revisitation of the Green-Naghdi heat conduction theory.

For completeness, we have emphasized the need of taking a minimum number of additional constitutive/internal length gradient parameters in triggering the size dependent characteristics of nonlocal higher-gradient multidisciplinary phenomena, like the third-gradient elastic and thermo-elastic behaviors, herein considered.

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