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Published Version:
Betti, G., Crescenzi, F., Mori, L. (2025). Parametric modelling of inequality aversion to reduce the computing time of distributional fuzzy poverty measures with application to EU countries. STATISTICAL METHODS & APPLICATIONS, 34(4 (September)), 687-705 [10.1007/s10260-025-00793-1].

Availability:
This version is available at: https://hdl.handle.net/11585/1018509 since: 2025-06-27

Published:
DOI: http://doi.org/10.1007/s10260-025-00793-1

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Abstract

This paper introduces a parametric approach for computing the Fuzzy Monetary (FM) index. We establish the relationship between this fuzzy index, which is frequently used to estimate inequality, and the Generalized Gini index through a parameter, α, that reflects the degree of aversion to inequality. While, for the Generalized Gini index α is chosen by the researcher for the FM index it is derived from data. To efficiently estimate FM, we first prove some technical results and then introduce a parametric method that accelerates the computation of α, incorporating the closed-form expression of the Generalized Gini index for selected distributions. To assess the variability of the index, we propose a parametric bootstrap method. The results are validated through simulations, where we compare the original FM index with its parametric counterpart. Finally, we apply our approach to estimate inequality aversion in European countries.

Keywords: EU-SILC, Fuzzy methods, Generalized Gini coefficient, Parametric Modelling, Statistical Computing

Data availability statement: Data available on request

1 Introduction

Economic sample surveys are often designed to estimate poverty based on monetary variables, typically related to income or consumption expenditure, offering a one-dimensional perspective on poverty. A common approach in official statistics classifies individuals or households into two groups: those considered poor and those considered not poor, with a predefined poverty line serving as the boundary between the two.

Since the seminal work of Cerioli and Zani [1], researchers have redefined poverty using fuzzy set theory [2] to develop a more nuanced model of the phenomenon. The introduction of fuzzy sets has transformed the perception of poverty from a binary classification to a matter of degree. This approach has garnered significant attention over the years, as evidenced by numerous studies proposing new fuzzy indices (see, for example, Betti and Verma [3], Belhadj [4], Zedini and Belhadj [5]) and books reviewing recent developments (Alkire et al. [6], Betti and Lemmi [7], Silber [8]). In essence, rather than assigning individuals to a rigid poor/non-poor classification, this methodology assigns each statistical unit a continuous score representing deprivation in terms of income or expenditure, determined by a so-called membership function.

The membership function is the core of the fuzzy approach to poverty analysis and to fuzzy statistics (Coppi et al. [9]). Broadly speaking, membership functions can be distance-based or distribution-based (for a concise but comprehensive overview see Crescenzi and Mori [10]). Here we focus on distribution-based membership functions, in particular the one defined in Betti and Verma [3], from which the Fuzzy Monetary (FM) index is calculated. The FM index is closely related to two important indices used in poverty analysis: the Head Count Ratio (HCR) – one of the Laeken indicators – which has made it attractive in the dissemination of fuzzy poverty statistics (see for example, Aassve et al. [11]), and the Generalised Gini Index (G_α), which is proven in the following. FM has also been applied in other research areas beyond poverty measurement (see, for example, Betti et al. [12], Bettio et al. [13], Betti et al. [14], Tavares and Betti [15]).

With respect to the HCR, the FM index is defined in such a way that while preserving fuzzy modelling of the poverty phenomenon, it is indistinguishable on average from the HCR when estimated over the whole sample. This is a great advantage over other fuzzy indices, as it is anchored to an official statistical estimate, mitigating a criticized feature of fuzzy statistics, namely the arbitrariness of the choice of membership function.

With respect to the G_α, which relies on an arbitrary parameter representing aversion to inequality, the FM index establishes an objective value for the aversion to inequality parameter (α), free of subjective influence, thereby mitigating inherent biases. A higher value of the parameter signifies greater aversion to inequality in a domain, resulting in a lower value of FM. Conversely, domains exhibiting lower aversion to inequality, as indicated by a lower value of α, can be expected to display higher levels of poverty according to the index. This relationship reflects the degree of anti-poverty efforts in a specific country.

Unfortunately, the FM index is computationally challenging, especially with methods like Monte Carlo simulations. Computing the FM and its variance for all countries by the techniques outlined in Betti et al. [16] can take days to weeks of computing time. In the field of small-area estimation (Rao and Molina [17]), Molina and Ferretti [18] proposed a Fast Empirical Best (Fast-EB) model to estimate the FM index for small areas. The computational burden of calculating α, however, forced the author to set it at an arbitrary value and to recognise the need for further study. Here we propose a computationally feasible solution to calculate parameter α that uses a parametric model over G_α. By leveraging its connection with FM, the model yields efficient computing times.

The remainder of this article is organized as follows. In Section 2, we introduce the FM index, while in Section 3 we prove the result that we use to introduce the new parametric version of FM. Section 4 offers a simulation comparing the results between the FM and its parametric version. In Section 5, an application of our parametric model is used to obtain estimates of poverty adversity for all European countries. In the end, Section 6 concludes the work and gives insights for further developments.

2 The Fuzzy Monetary Index: preliminaries

In the fuzzy approach to poverty analysis, given a poverty predicate Y (*i.e.* a continuous non-negative variable such as the equivalised disposable income or consumption), its support $\mathcal{Y}$, and the (fuzzy) set $\mathcal{A}$, the membership function μ maps $\mathcal{Y}$ to $\mathcal{A} = [0,1] \subset \mathbb{R}$ ($\mu : \mathcal{Y} \to \mathcal{A}$). The higher the value of the membership function, the greater the belonging of a unit to the set of poor. Let $\mathcal{U} = \{1, \dots, N\}$ be a finite population of size N, and let Y_i be the value of the poverty predicate for unit i. Let us consider the finite population distribution function

$$F_Y(x) = \frac{1}{N}\sum_{j=1}^{N} I\{Y_j \leq x\}, \quad x \in \mathcal{R}^+, \tag{1}$$

where $I\{Y_j \leq x\} = 1$ if $Y_j \leq x$ and 0 otherwise. Consider also the finite population Lorenz Curve, given by:

$$L_Y(x) = \frac{\sum_{j=1}^{N} Y_j I\{Y_j \leq x\}}{\sum_{j=1}^{N} Y_j}, \quad x \in \mathcal{R}^+. \tag{2}$$

Following Betti et al. [16] and Molina and Ferretti [18] the population membership function of the i-th individual is:

$$\begin{aligned} \mu_i^\alpha =& \left\{\frac{N}{N-1}(1 - F_Y(Y_i))\right\}^{\alpha-1}\{1 - L_Y(Y_i)\} = \\ =& \left\{\frac{1}{N-1}\sum_{j=1}^{N} I\{Y_j > Y_i\}\right\}^{\alpha-1}\left\{\frac{\sum_{j=1}^{N} Y_j I\{Y_j > Y_i\}}{\sum_{j=1}^{N} Y_j}\right\}, \quad i \in \mathcal{U}, \end{aligned} \tag{3}$$

where $\alpha \geq 1$ is a parameter to be interpreted as the aversion to inequality. In a traditional sample survey framework, let $\mathcal{S}$ be a sample of n units from a population of size N, ($n < N$), and let y_i, be the poverty predicate observed for unit i in the sample. The value of the membership function in Equation (3) – with a slight abuse in notation – is calculated using the information available in the sample as follows

$$\hat{\mu}_i^\alpha = \left(\frac{\sum_{j\in\mathcal{S}} w_j I\{y_j > y_i\}}{\sum_{j\in\mathcal{S}} w_j}\right)^{\alpha-1}\left(\frac{\sum_{j\in\mathcal{S}} w_j y_j I\{y_j > y_i\}}{\sum_{j\in\mathcal{S}} w_j y_j}\right), \tag{4}$$

where w_j denotes the sample weight of unit j, $F(y_i)$ is the empirical distribution function of the predicate, and $L(y_i)$ represents the value of the Lorenz curve for the individual i.

In the same population, let also N_p be the set of units that are traditionally regarded to as poor with respect to Y, *i.e.* those units for whom Y is less than the value taken by the poverty line (τ). The proportion of poor units in the population (N_p/N)

– *i.e.* the Head Count Ratio (HCR) – is estimated using the sample information as follows

$$\widehat{HCR} = \frac{\sum_{j\in\mathcal{S}} y_j I_{\{y_j<\tau\}}}{\sum_{j\in\mathcal{S}} w_j}. \tag{5}$$

Following Betti and Verma [3] the parameter α is chosen so that the mean of the membership function in Equation (4) – *i.e.* the FM index – equals the HCR of the whole population, if known, or its sample estimate given in Equation (5).

Using the value thus obtained, the FM index (formalized in Section 3) can be computed not only at the national level but also for arbitrary sub-domains. In practice, higher values of α will give more weight to the left tail of the income distribution.

3 The parametric proposal for the Fuzzy Monetary Index

The FM index is computationally demanding due to the calculation of the parameter α. To the best of our knowledge, the only method of finding this parameter (Crescenzi and Mori [10]) is to use a recursive algorithm that iteratively computes the mean of the membership function in Equation (4) for different α values until it equals the HCR or its sample estimate. Unfortunately, this process involves arranging the sample and solving a non-linear equation to compute it, resulting in long computation times, especially for official sample surveys, which typically consist of several thousand observations. This issue is particularly relevant for variance estimation, as the variance of the FM index is computed using jack-knife repeated replications (see Betti et al. [16]).

The issue of computational burden associated with the FM index was highlighted by Molina and Ferretti [18]. In the authors' words, *"Regarding the parameter α which is involved in the fuzzy measures, we have considered $\alpha = 2$ in our analysis. Determining the value of α to make the FM computationally feasible deserves further study"*.

To this end, we propose a fast and reliable parametric method to compute α. First, we redefine the FM index as

$$FM = \int_0^1 \mu(p)^\alpha dp, \quad p \in [0,1], \tag{6}$$

we then prove the following simple result.

Proposition 1 *The value of the FM index expressed in Equation (6) is equal to*

$$FM = \frac{\alpha + G_\alpha}{\alpha(\alpha+1)},$$

where $G_\alpha = 1 - \frac{1}{\mu_y}\int_0^{+\infty}(1-F(y))^\alpha dy$, where μ_y represents the expected value of y

Proof The proof is straightforward.

$$\int_0^1 \mu(p)^\alpha dp = \int_0^1 (1-p)^{\alpha-1}(1-L(p))dp =$$

$$\int_0^1 (1-p)^{\alpha-1}(1-p) + \int_0^1 (1-p)^{\alpha-1}(p-L(p))dp,$$

and since the Generalized Gini index can also be written $G_\alpha = \alpha(\alpha+1)\int_0^1 (1-p)^{\alpha-1}(p-L(p))dp$, so that

$$\frac{G_\alpha}{\alpha(\alpha+1)} = \int_0^1 (1-p)^{\alpha-1}(p-L(p))dp,$$

we have

$$\int_0^1 (1-p)^\alpha dp + \frac{G_\alpha}{\alpha(\alpha+1)} = \frac{\alpha+G_\alpha}{\alpha(\alpha+1)},$$

where the last equality follows from the fact that $\int_0^1 (1-p)^\alpha dp = 1/(\alpha+1)$. □

As already said, in practice the value of α is unknown and is selected so that the numerical solution of the integral in Equation (6) equals the HCR (if known) or its estimate. However, as suggested by Molina and Ferretti [18], since this requires long computation time, one may opt for setting the value of the parameter at a known value (*e.g.* $\alpha = 2$). Although this option is not discouraged – for example the R package FuzzyPovertyR (Crescenzi and Mori [19]) specifically envisages this option – anchoring the value of the index to an official statistical measure somehow limits the arbitrariness of the choice of the membership function.

The parametric calculation of alpha described above helps to reduce the calculation time of the index by solving the following equation

$$\frac{\alpha+G_\alpha}{\alpha(\alpha+1)} = HCR, \tag{7}$$

instead of finding α by numerical evaluation of Equation (6). In other words, the real advantage of the proposed parametric calculation is that we can calculate the parameter without having to calculate the membership function because G_α is known for some distributions. Furthermore, by looking at Equation (7) we can find conditions for the existence of a real solution.

Proposition 2 *The equation*

$$\frac{\alpha+G_\alpha}{\alpha(\alpha+1)} = HCR,$$

admits at least a real solution for $\alpha \in [1,\infty)$ as long as HCR is less than $(G_1+1)/2$.

Proof the proof is a straightforward application of the intermediate value theorem. For the sake of brevity, let us denote the head count ratio as $h, h \in [0,1]$. If we re-arrange the equation as follows

$$G_\alpha = h\alpha^2 + h\alpha - \alpha, \tag{8}$$

and define the function

$$f(\alpha) = G_\alpha - \alpha(h\alpha + h - 1),$$

we need to show that $f(\alpha) = 0$ has at least one solution for $\alpha \in [1, \infty)$. At $\alpha = 1$ we have

$$f(1) = G_1 - 2h - 1,$$

which can be positive or negative depending on the value of h, *i.e.* $f(1) > 0$ if $G_1 > 2h + 1$ and $f(1) < 0$ if $G_1 < 2h + 1$. If α is large enough, given that G_α is bounded and that $\alpha(h\alpha + h - 1)$ is quadratic, we have that $f(\alpha)$ will tend to assume large negative values. $\square$

An additional result emerging from above is that by assigning a certain value to α (as in Molina and Ferretti [18]), we implicitly assume that the proportion of individuals regarded as poor has to lie in a given range. We can see this by noting that since

$$0 < G_\alpha < 1,$$

by substituting Equation (8), we obtain

$$0 < h\alpha^2 + h\alpha - \alpha < 1,$$

therefore we have that

$$h \in \left(\frac{1}{\alpha+1}, \frac{1}{\alpha}\right).$$

For example, if we specified $\alpha = 2$ to speed up the computation of the index, we implicitly assume that the proportion of poor people is between 1/3 and 1/2, which is very high in practice.

Although the FM index was originally designed as a non-parametric index, the result above suggests that if we could find a suitable parametric model of the poverty predicate, then we would simplify the calculation of α.

Table 1 presents the Generalized Gini index formulas for some distributions[1], we use the notation G_{α^P} to underline the parametric definition of the index. Most of these results are well-known in the literature (Kleiber and Kotz [21, 22], Chotikapanich [23], Krause [24]). Unfortunately, according to Kleiber and Kotz [21], G_{α^P} does not exist in a closed-form expression for the Log-Normal distribution, which is often used to model monetary variables.

Furthermore, using the international definition of the HCR as the percentage of a population that falls below the poverty line, set at 60% of the median of the equivalized disposable income distribution, Equation (7) becomes:

$$\frac{\alpha^P + G_{\alpha^P}}{\alpha^P(\alpha^P + 1)} = F^{-1}(0.6\tilde{y}), \tag{9}$$

where $\tilde{y}$ is the median of y and the superscript P is used to denotes the parametric version of α.

By substituting one of the closed-form expressions from Table 1 for G_{α^P} in Equation (7), the computational effort is significantly reduced. since it does not involve the calculation of G_α, *i.e.*, the arrangement of the sample.

[1] In the table the distributions are defined in terms of a location parameter (b), a scale parameter (c) and a shape parameter (q). For the parametrization of the distributions see: Rigby et al. [20]. $\Gamma()$ denotes the Gamma function.

Table 1 Generalized Gini indices of classical income distributions

Distribution	G_{α^P}
Exponential	$1-\frac{1}{\alpha^P}$
Pareto	$\frac{\alpha^P-1}{c\alpha^P-1}$
Singh-Maddala	$1-\frac{\Gamma(q\alpha^P-\frac{1}{b})\Gamma(q)}{\Gamma(q\alpha^P)\Gamma(q-\frac{1}{b})}$
Uniform	$1-\frac{2c+b\alpha^P}{(\alpha^P+1)(c+b)}$
Weibull	$1-(\alpha^P)^{-\frac{1}{b}}$

3.1 Mean Squared Error estimation

Estimation of the Mean Square Error (MSE) of the FM index has been discussed in the academic literature. There are two methods for estimating it: the main one is the simplified Jack-knife repeated replications method (Betti et al. [16]), implemented in the R package FuzzyPovertyR which also offers a non-parametric bootstrap approach (Crescenzi and Mori [19]). While the bootstrap estimator is a broad and general method for estimating the MSE (Field and Welsh [25]), the simplified Jack-knife estimator has been modified ad-hoc for the FM index. A drawback of the simplified Jack-knife method is that it requires full information on the sampling design, *e.g.* strata and primary selection units, which are not always available to researchers. In any case, the two methods are revised and compared in Crescenzi and Mori [10]. For the sake of clarity, we use the notation FM^{α} and FM^{α^P} to denote the FM index with α obtained as usual, and the FM index with α obtained using a parametric approach, respectively.

3.1.1 A parametric bootstrap

As noted before, the Jack-knife method requires appropriate sampling information in order to be correctly carried out, otherwise, to the best of our knowledge, the only alternative is a (naive) non-parametric bootstrap. Even if $Var(FM) = \frac{Var(G_\alpha)}{\alpha^2(\alpha+1)^2}$, for a given value of α, to the best of our knowledge, there are no closed formulae for the variance of the Generalized Gini index in the literature.

We now propose a parametric bootstrap to estimate the MSE of the FM^{α^P} index. Usually, the parametric bootstrap assumes a parametric model for the poverty predicate and repeatedly samples from the model, say B times, to obtain an estimate $\widehat{FM}^{\alpha^P}(b)$ at each loop. At the end of the procedure the estimated MSE is

$$\widehat{MSE}_B(\widehat{FM}^{\alpha^P}) = B^{-1}\sum_{b=1}^{B}\left[\widehat{FM}^{\alpha^P}(b) - FM^{\alpha^P}(b)\right]^2. \tag{10}$$

The parametric bootstrap should yield more accurate MSE estimates, even when the sample size is small.

4 Simulation results

In this first attempt to accelerate the estimation of the α parameter in the FM^α index, we compare results from three public available different datasets: one that collect data from a real sample survey – the 2019 Italian Household Budget Survey[2] (HBS) – and other two synthetic datasets, namely, the "Income-data" dataset (Molina and Marhuenda [26]) and the "Pareto-data" dataset (Millard et al. [27]). In the case of "Pareto Data" we increased the sample size to 30,000 to obtain a more realistic scenario.

As first step, for all datasets, we analyzed the monetary variable to find the best parametric model to describe its distribution using the Generalized Akaike Criterion (GAIC). The distribution that minimizes the GAIC, as shown in Table 2, is the Singh-Maddala for both HBS and "Income-data", and the Pareto distribution for "Pareto-data". Table 2 shows estimated values of α and α^P, and the percentages of time saved, denoted as Δ-time %, using the selected distributions. This percentage is the time saved using the parametric version of FM^{α^P}, *i.e.* calculating α^P instead of α. More precisely, the computational times compared include: (i) non-parametric - the time needed to compute the membership function and α using the FuzzyPovertyR package, (ii) parametric - the time needed to estimate the parameters of the chosen model, to calculate α^P using Equation (9), and to compute the membership function given the value of α^P.

The estimates of α^P show negligible errors when compared with the values of α. More importantly, the reduction in time appears to be extremely significant across all datasets. Figure 1 shows the membership functions calculated for the HBS dataset. For the sake of clarity, we only show values of the membership function between 0.2 and 0.7. The choice of the most suitable model is fundamental because if we use a distribution that does not minimize the GAIC, we obtain progressively larger errors in the estimation of α^P.

Table 2 Estimates of α and α^P and computational time reduction

Data	Distribution	α^P	α	Δ-time%
HBS (2019)	Singh-Maddala	4.613	4.637	$\sim -87\%$
Income-data	Singh-Maddala	4.236	4.235	$\sim -75\%$
Pareto-data	Pareto	5.354	5.356	$\sim -90\%$

[2] In the simulation we prefer to use HBS instead of EU-SILC for his public availability.

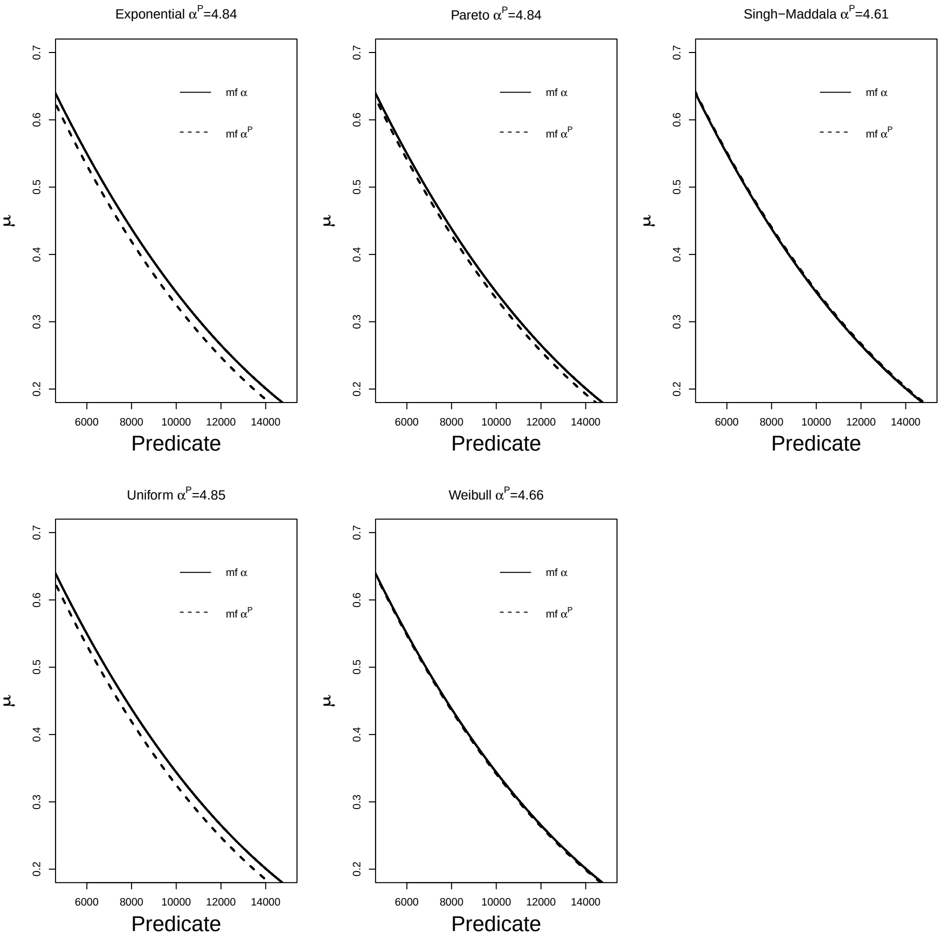

Fig. 1 Comparison of membership functions obtained with parametric and non-parametric method based on HBS data.

4.1 A design-based simulation

Earlier results support the idea that using a parametric specification for G_{α^P} accelerates the computation of the index without introducing any sources of bias. We further investigate these results by comparing: (i) the computed values of α and α^P, (ii) the bias of the index for sub-domains using α and α^P, and (iii) the performance of the non-parametric bootstrap to estimate the MSE of FM^{α} versus the performance of the parametric bootstrap to estimate the MSE of FM^{α^P}. To do this, we use the HBS dataset consisting in 42,760 individuals, and re-sampling $T = 500$ subsets with a 0.01 sample fraction mimicking the original sampling scheme. As sub-domains, we use the twenty Italian Regions. Again, the best parametric model for our monetary variable turns out to be the Singh-Maddala distribution. Denoting the sub-domains as j with $j = 1, \ldots, J = 20$ and the replicates as t with $t = 1, \ldots, T$, we assess the performance of the consider FM indices using three different indicators[3] [28]. The first one is the Average Relative Bias (ARB)

$$ARB = \frac{1}{J}\sum_{j=1}^{J}\left\{\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{FM}_{jt}}{FM_j} - 1\right)\right\} \times 100.$$

We also report the ARB relative of the parameter α, ($ARB(\alpha)$). To evaluate the reliability of the estimates, we use the Average Coefficient of Variation (ACV)

$$ACV = \frac{1}{J}\sum_{j=1}^{J}\left\{\frac{1}{T}\sum_{t=1}^{T}\frac{\sqrt{\widehat{MSE}(\widehat{FM}_{jt})}}{FM_{jt}}\right\} \times 100,$$

where $\widehat{MSE}(\widehat{FM}_{jt})$ is the estimated MSE at the replicate t for the area j. To evaluate the two MSE procedures we also compare the Average Relative Bias of the MSE (ARBMSE)

$$ARBMSE = \frac{1}{J}\sum_{j=1}^{J}\left\{\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{MSE}(\widehat{FM}_{jt})}{MSE(FM_{jt})} - 1\right)\right\} \times 100,$$

where $MSE(FM_{jt})$ is the true MSE for area j at the replicate t.

Table 3 shows the performance measure results. The non-parametric approach shows a negative ARB, and the parametric one a slightly positive ARB, both of which are almost negligible. The ARB related to parameter α reveals that for a small sample fraction, the parametric computation can even outperform the non-parametric one. Examination of the ARBMSE suggests that the parametric bootstrap employing α^P outperforms the non-parametric bootstrap and that the former has a negligible bias, while the latter's is negative. These results are evident in Figure 2, where the real value of the MSE and the two estimated one, with the different bootstraps, are reported for each sub-domain.

The underestimation of the MSE also influences the values of the ACV. In fact, the non-parametric approach results in a lower ACV than the parametric one. This is mainly due to the underestimation of the numerator of the CV.

Regarding the computational effort, we have that the average Δ-time% to estimate the FM values is -81%, while the average Δ-time% to estimate the MSE is -12%.

5 An application to European countries

As an application of our findings, we estimate the FM index for European countries. Poverty in Europe has been extensively researched. Notably, Clark [29] conducted a comprehensive review of poverty measurement spanning the past four decades. Empirical evidence consistently highlights a discernible regional divergence, with southern European countries exhibiting significantly higher levels of poverty, insecurity, and

[3] The performance indices are defined for a generic FM index. When the non-parametric (parametric) version is used, it is to be read as FM^{α} (FM^{α^P}).

Table 3 Simulation results: performance indices

	$\widehat{FM}^{\alpha^P}$	$\widehat{FM}^{\alpha}$
ARB	0.27	-0.51
$ARB(\alpha)$	1.11	1.94
ACV	32.0	30.1
ARBMSE	0.78	-8.17

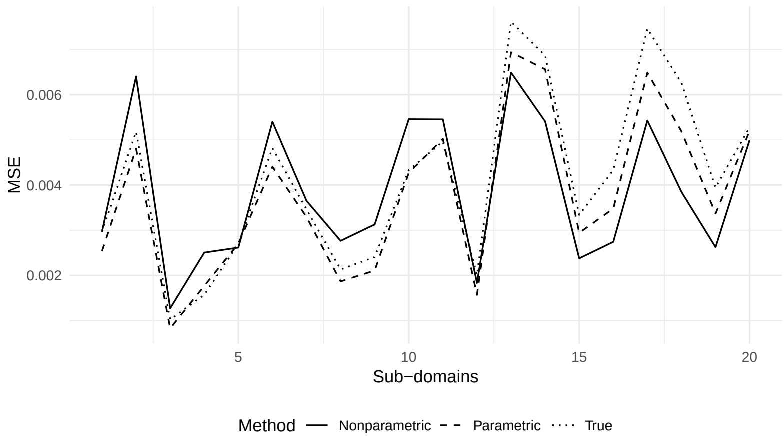

Fig. 2 Comparison of estimated MSE using parametric and non-parametric bootstrap across areas

instability than their northern counterparts.This regional dichotomy is repeatedly documented in the academic literature, as demonstrated by the work of scholars such as Heidenreich and Wunder [30], Acemoglu [31], Castells-Quintana et al. [32]. Moreover, an examination of material deprivation, rather than solely economic variables, reveals that while the ranking of countries may vary, overarching trends remain consistent, as noted by Bossert et al. [33].

With the FM index, we expect to identify the same pattern while also highlighting a new socio-economic trend: "aversion to inequality". Given the differences among European countries, interpreting the values of α (aversion to inequality) in the FM index should reveal which countries exhibit greater aversion to inequality.

Our analysis encompasses a total of 32 countries using the 2022 EU-SILC data.[4]. Equivalized income, or simply income for brevity, serves as the foundation of our analysis. Figure 3 illustrates the estimated income distribution across 32 countries, categorized by geographical regions (North, Center, East, South). We employ the Singh-Maddala distribution for each country.

Focusing on Central Europe, income distributions appear similar across countries, with the exception of the Czech Republic (CZ), which exhibits a notably lower mean

[4]Note as for five countries we do not have the survey for the 2022, so we consider the latest available: Switzerland (2021), Iceland (2018), Norway (2020), Republic of Serbia (2021) and United Kingdom (2018).

income level. Luxembourg (LU) emerges as the wealthiest nation in this region. Moving eastward, income distributions also show similarities, with Estonia (EE) being marginally wealthier than its counterparts. A comparable pattern is observed in Southern Europe, though Croatia (HR) appears slightly less affluent. In Northern Europe, Iceland (IS) stands out as the wealthiest nation.

Figure 4 shows the estimated values of the FM^{α^P} index for different countries (see Table 4 for numerical values). Two distinct yet similar trends emerge. The index follows a decreasing pattern from south to north and from east to west. The poorest countries are located in the Balkans and the Baltic region, while the wealthiest are in northwestern Europe. As highlighted in the introduction, this trend is well documented among European countries and is also observed with other indicators, such as the HCR (e.g., Iammarino et al. [34], García-Pardo et al. [35], D'agostino et al. [36]).

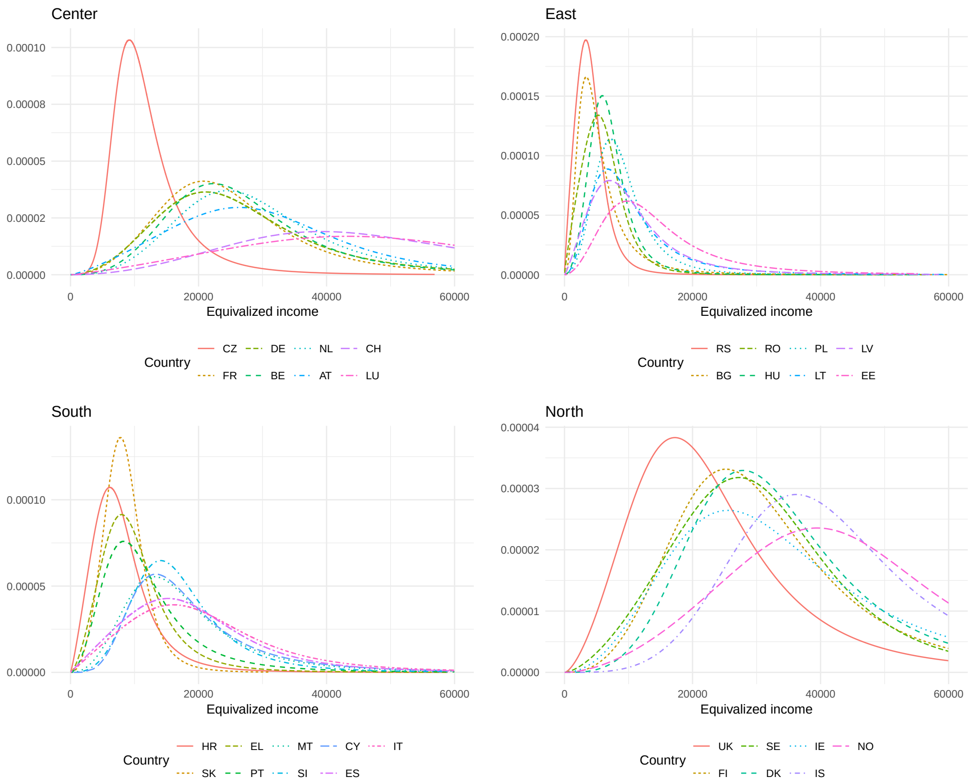

Fig. 3 Estimated distribution of income (European Countries). Legends are arranged in ascending order of median income.

Note that the FM^{α^P} index automatically determines the level of α^P, unlike the Generalized Gini index, where the value is chosen by the researcher. As this selection is data-driven, we expect countries with lower FM^{α^P} values to have higher values of α^P.

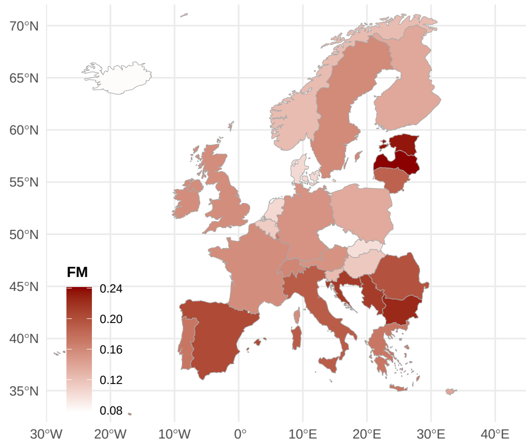

Fig. 4 Estimated values of FM^{α^P} *index*

Theoretically, this parameter reflects the system's aversion to inequality. The FM^{α^P} becomes more sensitive to changes at the lower end of the income distribution as α^P increases. Conversely, as the level of inequality aversion falls (i.e., as it approaches 1), the FM^{α^P} becomes less sensitive to changes at the lower end of the distribution. A system that generates less poverty is expected to exhibit a greater aversion to this phenomenon, hence a higher value of α^P. This intuition is corroborated by the findings presented in Figure 5 and Table 4, which illustrate the calculated values of α^P for all European countries.

Overall, Iceland (IS) and the Czech Republic (CZ) are the two countries with the lowest FM^{α^P} value ($\simeq 0.08$) and the highest α^P value (≥ 12). On the other hand, Latvia (LV) has the highest FM^{α^P} value ($\simeq 0.24$) and the lowest α^P value ($\simeq 3.79$). Once more, when observing from southern Europe toward the north, there is an increase in the parameter α^P.

Linking these findings with those depicted in Figure 3, it is crucial to note that the parametric FM^{α^P} is clearly associated with the shape of the distribution. For instance, in the case of the Czech Republic (CZ), the distribution exhibits a slight right tail, resulting in a low FM^{α^P} and a high α^P. Similar conclusions can be drawn for Slovakia (SK) in Southern Europe. Conversely, Switzerland (CH) and Luxembourg (LU), both recognized as among the wealthiest nations globally, display distributions with heavy tails, which are associated with high values of FM^{α^P} and low values of α^P.

Figure 6 shows the membership function, the Weighted Empirical Cumulative Distribution Function (WECDF), and the Lorenz Curve for various countries. As expected, countries with a higher FM^{α^P} have a long right tail. Denmark (DK), for example, exhibits a relatively short right tail, reflected by $FM^{\alpha^P} \simeq 0.10$, while Estonia (EE) has a long right tail and $FM^{\alpha^P} \simeq 0.23$. Once again, this figure confirms the trend of greater poverty in Southern and Eastern European countries. Similarly, Northern countries appear to have a higher aversion to poverty.

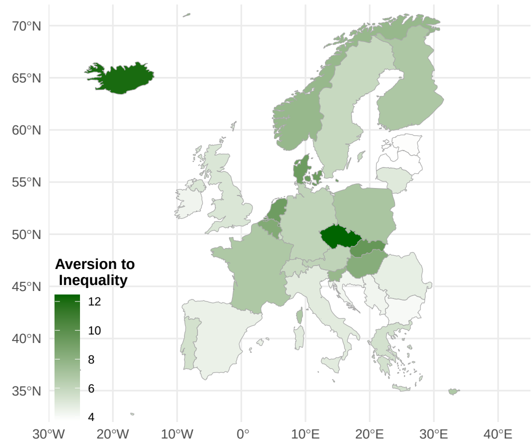

Fig. 5 Computed values of aversion to inequality (α^P) parameter

To conclude, Figure 7 reports the CV of the estimates. As expected the values of the CV are very low confirming the reliability of our estimates.

6 Conclusion

The approach of the innovative method for accelerating the computation of parameter α^P in the FM^{α^P} index proposed here is based on an equation that allows us to express the FM index in terms of the Generalized Gini index. Being a parametric approach, it leverages its closed-form expressions to significantly reduce the computation time of the parameter, especially for large datasets. The real advantage of our method is that we can compute α^P without calculating the entire membership function. The method's effectiveness is demonstrated by comparisons across real and synthetic datasets, which show a substantial saving of time. In addition, the parametric bootstrap introduced for estimating the MSE of FM^{α^P} outperforms the traditional non-parametric bootstrap.

Our findings emphasize the importance of selecting an appropriate distribution for the parametric version, as this significantly influences computational efficiency and the accuracy of the results. Among the distributions tested, the Singh-Maddala distribution emerged as a robust choice in all our simulations.

Overall, this study contributes to the ongoing discourse on the computational efficiency of poverty indices, shedding light on a promising avenue for accelerating the estimation of parameter α^P in the FM^{α^P} index.

From an applied standpoint, when we estimated the Fuzzy Monetary Index for all European countries, a clear pattern of poverty across Europe emerged, highlighting higher poverty values in southern countries linked to lower aversion to inequality. A further development would be to apply these results to the Fast-EB proposed by Molina and Ferretti [18] to determine whether the computation time remains prohibitive or if the parametric solution makes the problem tractable.

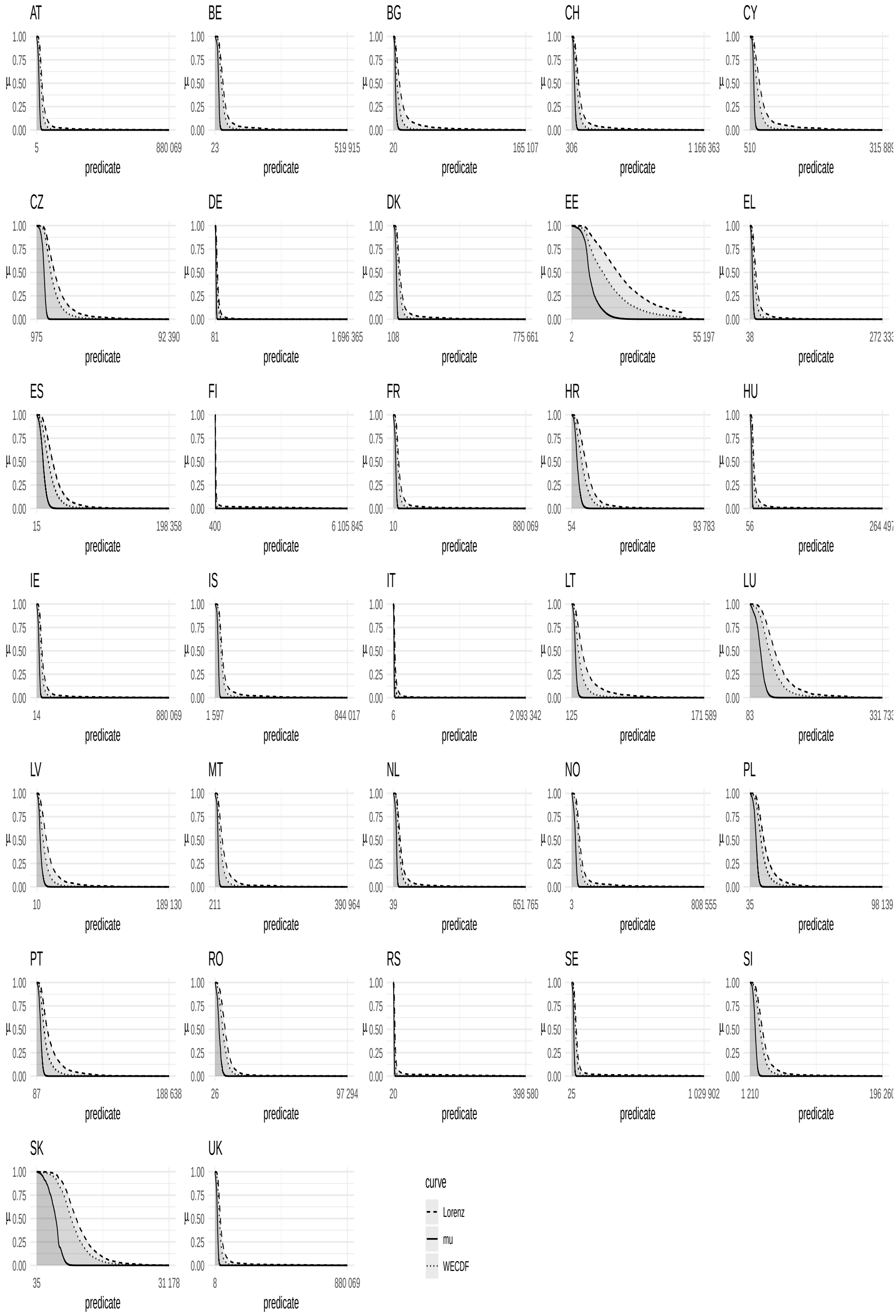

Fig. 6 Membership Function, Lorenz curve and weighted empirical Cumulative Distribution Function for European countries

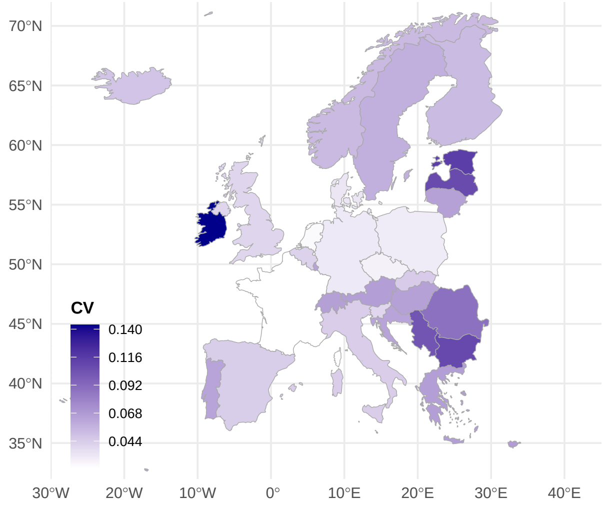

Fig. 7 Estimated Coefficient of Variation of FM^{α^P} *index*

Table 4 Estimated Fuzzy Monetary index (FM^{α^P}), aversion to inequality (α^P) and Coefficient of Variation (CV) for each country

	FM^{α^P}	α^P	Country
1	0.15	6.15	AT – Austria
2	0.11	8.48	BE – Belgium
3	0.23	4.09	BG – Bulgaria
4	0.16	5.80	CH – Switzerland
5	0.14	6.75	CY – Cyprus
6	0.08	12.44	CZ – Czech Republic
7	0.15	6.12	DE – Germany
8	0.10	9.25	DK – Denmark
9	0.23	3.88	EE – Estonia
10	0.17	5.41	EL – Greece
11	0.20	4.52	ES – Spain
12	0.14	6.84	FI – Finland
13	0.16	6.79	FR – France
14	0.21	4.28	HR – Croatia
15	0.12	8.28	HU – Hungary
16	0.16	4.36	IE – Ireland
17	0.08	12.05	IS – Iceland
18	0.19	4.87	IT – Italy
19	0.19	4.98	LT – Lithuania
20	0.17	5.52	LU – Luxembourg
21	0.24	3.79	LV – Latvia
22	0.14	6.95	MT – Malta
23	0.10	9.23	NL – Netherlands
24	0.12	7.65	NO – Norway
25	0.14	6.95	PL – Poland
26	0.17	5.43	PT – Portugal
27	0.20	4.66	RO – Romania
28	0.21	4.32	RS – Republic of Serbia
29	0.16	5.87	SE – Sweden
30	0.12	7.66	SI – Slovenia
31	0.10	9.54	SK – Slovakia
32	0.16	5.15	UK – United Kingdom

Acknowledgment

This paper is dedicated to the memory of Professor Vijay Verma. Vijay contributed in the early stages of the proof of Proposition 1.

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