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THE FOSTER-GREER-THORBECKE INDEX AND THE INEQUALITY FACTORS: AN ANALYSIS THROUGH THE GINI INDEX DECOMPOSITION

Michele Costa *
Giuseppe Pignataro †

ABSTRACT

In this paper, we propose an alternative methodology to capture the impact of the inequality factors on poverty by decomposing the traditional Foster-Greer-Thorbecke index. We focus on the incidence and the intensity of poverty and on the inequality of the distribution of the poor. In particular, our proposal allows to evaluate the effect of each factor on the inequality part, which is further analyzed into the *within-*, *between-*, and the *overlapping-* components through Dagum's (1997) Gini index decomposition. We also introduce a subgroup decomposition able to detect the contribution of each subgroup to the poverty index. A case study on Italian income distribution highlights the usefulness of our proposal evaluating the effects of inequality factors as gender, education, and the area of residence on the Foster-Greer-Thorbecke index.

Keywords: *Foster-Greer-Thorbecke index, Gini index, inequality decomposition, inequality factors*

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1. INTRODUCTION

The current debate on poverty animates frequent and widespread connections to inequality measurement although there is still no universal consensus on the role that inequality plays in poverty analysis.

Since the '80s, different contributions have suggested how economic inequality and poverty can be deeply complementary; see, among others, Sen (1979); Yitzhaki (1994); Deutsch and Silber (2008). Despite an extensive literature on measuring poverty, the dimension of inequality and the advantage of understanding income differences among the poor still require further investigation.

Our analysis aims at measuring many dimensions of poverty offering a clear picture of the inequality among the poor. In this context, the interest in capturing the impact of different inequality sources stems from the fact that the correct identification of the causes of unfair distribution among the most deprived may justify some policy interventions, see Ravaillon (1994).

We investigate the Foster-Greer-Thorbecke (1984, 2010) poverty index of the second-order (FGT_2) which has many desirable properties and is widely adopted in the poverty measurement literature. Starting from the aggregate index, we focus on the role of inequality factors, e.g., gender, education and area of residence, and the related subgroups, i.e., female/male, highly educated/not educated, north/south.

Our first contribution is to decompose FGT_2 into the incidence and the intensity of poverty, plus the inequality among the poor. The inequality aspect is investigated by adopting the Dagum's decomposition of the Gini index.

Furthermore, we propose a subgroup decomposition able to detect the contribution of each subgroup to the poverty index. This novel result is generalized to some FGT_2 decompositions previously proposed in the literature, emphasizing the advantages that emerge in different approaches.

The interplay between poverty and inequality may help in determining the socio-economic policies. As a case study, we propose an empirical investigation on the Italian individual income distribution to illustrate the advantages of the new decomposition of the FGT_2 index. Our results are in line with other investigation on FGT_2 index, discussed in the literature as Celidoni (2015), Civardi and Chiappero-Martinetti (2008) and D'Alessio (2020).

Section 2 discusses the decompositions proposed in the literature for the FGT_2 index, while Section 3 introduces our decomposition, showing the advantage in disentangling the contribution of inequality factors. We also propose a new subgroup decomposition in Section 4, where the analysis is developed at the subgroup level. An empirical illustration in Section 5 shows a battery of results related to our proposal and to previous decompositions of the FGT_2 index. Concluding remarks follow in Section 6.

2. THE FGT INDICES

Consider a society of n persons with q poor individuals. Each individual i's income can be identified by $y_i \rightarrow R_+$, where $\mathbf{y}$ is a vector of incomes, $\mathbf{y} = (y_1, y_2, ..., y_n)$, with income mean μ.

Let us define a poverty line $z \rightarrow R_+$ on the basis of which the i-th individual is poor if $y_i \leq z$. Without loss of generality, we can arrange $\mathbf{y}$ in non decreasing order, with $y_1 \leq y_2 \leq ... \leq y_n$, and define the vector of poor incomes as $\mathbf{y_p} = (y_1, y_2, ..., y_q)$, with income mean μ_p.

Starting from the normalized poverty gap $g_i = (z - y_i)/z$, Foster *et al.* (1984) introduced a family of poverty indices as

$$FGT_\alpha = \frac{1}{n}\sum_{i=1}^{q} g_i^\alpha \tag{1}$$

where α can be interpreted as the inequality aversion parameter. For $\alpha = 0$,

$$FGT_0 = q/n = H$$

we get the frequency of poor on the population, that is the headcount ratio H; for $\alpha = 1$,

$$FGT_1 = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z - y_i}{z}\right) = I_n$$

is the mean of the normalized poverty gap over the community, that is the poverty gap ratio; and for $\alpha = 2$,

$$FGT_2 = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z - y_i}{z}\right)^2$$

refers to the squared normalized poverty gaps, where the normalized poverty gap is weighted not by one as in FGT_1, but by g_i. By increasing α, the weight attached to the income of the most unfortunate individuals increases.

FGT indices have an extensive set of desirable properties, which significantly contributed to their success and diffusion in many fields. Furthermore, this family of indices is straightforward, and their information is easily understandable even by not experts.

In particular, FGT_2 index, besides being intuitive and having optimal properties, has similarities with the Sen index (Sen, 1976) and its generalized version proposed by Shorrocks (1995).

The first decomposition of FGT_2 index (Foster *et al.*, 1984; Aristondo *et al.*, 2015) is,

$$FGT_2 = H(I_{2p} + (1 - I_p)^2 CV_{2p}) \tag{2}$$

where I_p is the mean over the poor of the normalized poverty gap,

$$I_p = \frac{1}{q} \sum_{i=1}^{q} \left(\frac{z - y_i}{z} \right)$$

I_{2p} is the mean over the poor of the squared normalized difference between z and μ_p,

$$I_{2p} = \frac{1}{q} \sum_{i=1}^{q} \left(\frac{z - \mu_p}{z} \right)^2$$

and CV_{2p} is the mean squared coefficient of variation among the poor,

$$CV_{2p} = \frac{1}{q} \sum_{i=1}^{q} \frac{(\mu_p - y_i)^2}{\mu_p^2}.$$

It exists a relation between I_p and I_{2p} which can be explained by the following corollary:

Corollary 1. *The square of the mean over the poor of the normalized poverty gap is equal to the mean over the poor of the squared normalized difference between z and μ_p: $I_p^2 = I_{2p}$.*

Proof. See Appendix A.1. □

Moreover, the next corollary explains a relation between $(1 - I_p)^2$, μ_p^2 and z^2 as follows:

Corollary 2. *The square of $(1 - I_p)$ is equal to the ratio between μ_p^2 and z^2.*

Proof. See Appendix A.2. □

Related to relation (2), H accounts for the incidence and I_p for the intensity, while CV_{2p} refers to poverty distribution. In this way, it is possible to understand how overall poverty depends on the number of poor, the depth of poverty, and the related distribution – aspects that represent the three poverty analysis pillars.*

Decomposition in relation (2) closely follows Sen Index (1976)

$$S = H(I_p + (1 - I_p)G_p)$$

where G_p is the Gini index of the poor. It is possible to note the strong similarities between relation (2) and Sens's intuition based on the three *Is*: the incidence, the intensity and the inequality of the distribution of the poor.

There are other different ways to decompose the FGT_2. Since the FGT_2 is a second-order index, its decomposition is compatible with the framework outlined by Yitzhaki and Schechtman (2013). They show the possibility of decomposing the poverty in its variance components.

Civardi and Chiappero-Martinetti (2008) propose a further decomposition of the FGT_2 index under the assumption of a population divided into M subgroups of size n_i. Their suggestion is to introduce, besides the poverty line for the community z, M subgroup-specific poverty lines $z_1, z_2, ..., z_M \to R_+$, and to derive, for each subgroup, the FGT_2 index for either z_i and z.

The resulting decomposition is

$$FGT_2 = \sum_{i=1}^{M} FGT_{2i}(z_i)\frac{n_i}{n} + \sum_{i=1}^{M}(FGT_{2i}(z) - FGT_{2i}(z_i))\frac{n_i}{n}$$

where the two terms represent the within and the between component, respectively.

Shorrocks (2013) instead adopts a different approach by proposing a solution based on the Shapley value.† Rather than simply decomposing the FGT index by population subgroups, he proposes to capture the marginal contribution of each factor to overall poverty.

Aristondo *et al.* (2010) introduce a further decomposition of FGT_2 by resorting to a generalized entropy index of income gaps of the poor such as

*Celidoni (2015) exploits the potential of relation (2) to measure the individual vulnerability to poverty.

†The Shapley value is a solution concept first employed in game theory to divide a given surplus among coalition members.

$$E_2 = \frac{1}{2q}\sum_{i=1}^{q}\left(\left(\frac{g_i}{\mu(g)}\right)^2 - 1\right).$$

Following their proposal, we can express FGT_2 as

$$FGT_2 = HI_p^2(1+2E_2).$$

As usually holds for multiplicative relations, this decomposition of the FGT_2 index, as well as relation (2), can be extremely useful when a dynamic is involved.

3. A NEW DECOMPOSITION OF THE FGT_2 INDEX

The vector of incomes, $\mathbf{y} = (y_1, y_2, ..., y_n)$ can be broadly partitioned into a vector Ω of circumstances C, e.g. different initial conditions, which belong to a finite set $\Omega = \{C_1, ..., C_m, ..., C_M\}$ for each type m, where $m \in \{1, 2, ..., M\}$.

Through the vector Ω, we can introduce one or more inequality factors and consider their effects on the FGT index. Each C_m corresponds to a population subgroup, e.g., considering gender as inequality factor, the related $M = 2$ subgroups are male and female.

Let us define the mean income of the poor μ_p as the parameter of the average economic affluence among poor people, i.e., $z > \mu_p$ (Dagum, 1980). Starting from the FGT index of the second order, we add and subtract μ_p, thus obtaining,

$$FGT_2 = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z-\mu_p+\mu_p-y_i}{z}\right)^2 = \frac{1}{n}\sum_{i=1}^{q}\frac{(z-\mu_p)^2}{z^2} + \frac{1}{n}\sum_{i=1}^{q}\frac{(\mu_p-y_i)^2}{z^2}$$

since $\sum_{i=1}^{q} 2(z-\mu_p)(\mu_p-y_i) = 0$.

By multiplying and dividing by the number of poor q, we express the FGT index introducing the headcount ratio $H = q/n$ as

$$FGT_2 = H\frac{1}{q}\sum_{i=1}^{q}\frac{(z-\mu_p)^2}{z^2} + H\frac{\mu_p^2}{z^2}\frac{1}{q}\sum_{i=1}^{q}\frac{(\mu_p-y_i)^2}{\mu_p^2} \qquad (3)$$

Relation (3) is precisely equivalent to decomposition (2), since

$$I_{2p} = \frac{1}{q}\sum_{i=1}^{q}\left(\frac{z-\mu_p}{z}\right)^2 \quad \text{and} \quad (1-I_p)^2 = \frac{\mu_p^2}{z^2}.$$

We focus on the second term of (3), and our first result shows how CV_{2p} can be obtained as

$$CV_{2p} = \frac{1}{q}\sum_{i=1}^{q}\frac{(\mu_p - y_i)^2}{\mu_p^2} = \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i - y_j|^2 \tag{4}$$

where we can easily recognize a structure similar to the Gini index of the poor where the pairwise relative absolute differences between the incomes of individuals **i** and **j** are of the second order.

Therefore we express relation (4) as,

$$G_{2p} = \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i - y_j|^2 \tag{5}$$

Note that G_{2p} does not correspond to the square of the Gini coefficient among the poor.[‡] The proof of equation (4) is in turn illustrated in the next proposition.

Proposition 1. *The mean squared coefficient of variation among the poor CV_{2p} can be expressed as G_{2p}:*

$$CV_{2p} = \frac{1}{q}\sum_{i=1}^{q}\frac{(\mu_p - y_i)^2}{\mu_p^2} = \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i - y_j|^2 = G_{2p}$$

Proof. See Appendix B. □

Based on relations (4) and (5), we are finally able to express the FGT_2 index in (3) as proposed in the following corollary:

[‡]Our definition in relation (5) describes the square of the average absolute differences among individual incomes, while the square of the Gini coefficient would imply the square of the overall components.

Corollary 3. *The FGT_2 index of the second order can be decomposed as:*

$$\begin{aligned} FGT_2 &= HI_{2p} + H(1 - I_p)^2 G_{2p} = \\ &= H\frac{1}{q}\sum_{i=1}^{q}\frac{(z-\mu_p)^2}{z^2} + H\frac{\mu_p^2}{z^2}\left(\frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i - y_j|^2\right) \end{aligned} \tag{6}$$

Under Corollary 3, terms in equation (6) consider the incidence of poverty through the headcount ratio H, capturing the frequency of poverty in the income distribution. Moreover, H is combined with I_{2p} which measures the intensity of poverty, depicting how widespread poverty occurs. Furthermore, $(1 - I_p)^2$ captures an effect that decreases as the difference between μ_p and z increases. Recalling that $(1 - I_p)^2 = \mu_p^2/z^2$, it exactly claims that the higher the poverty line, the lower is, on average, the incidence of poverty. Finally, the interaction with G_{2p} reflects the effect that the level of inequality has on the distribution of the poor.

Our next step is to dig into relation (6) and, more specifically, its second term, by exploiting the advantages of the Gini index decomposition (see, e.g., Giorgi (2011) for a review), which allows us to evaluate the contribution of one or more inequality factors. In the following, we refer to the decomposition proposed by Dagum (1997), which is characterized by a high degree of simplicity and intuitiveness and explicitly considers overlapping components.

By dividing the poor units into M subgroups, we obtain

$$G_{2p} = \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i - y_j|^2 = \frac{1}{2q^2\mu_p^2}\sum_{m=1}^{M}\sum_{n=1}^{M}\sum_{i=1}^{q_m}\sum_{j=1}^{q_n}\left|y_{mi} - y_{nj}\right|^2 \tag{7}$$

where q_m and q_n identify the number of poor units, respectively, in subgroups m and n. Without loss of generality, we can order the M subgroups from the richest to the poorest, such that $\mu_{pm} \geq \mu_{pn}$, where μ_{pm} and μ_{pn} are the mean incomes of the poor, respectively, in subgroups m and n.

Substituting the relation (7) in (6), we have

$$FGT_2 = H\frac{1}{q}\sum_{i=1}^{q}\frac{(z-\mu_p)^2}{z^2} + H\frac{\mu_p^2}{z^2}\left(\frac{1}{2q^2\mu_p^2}\sum_{m=1}^{M}\sum_{n=1}^{M}\sum_{i=1}^{q_m}\sum_{j=1}^{q_n}\left|y_{mi} - y_{nj}\right|^2\right) \tag{8}$$

Following the intuition provided by Dagum's decomposition, we classify the differences $|y_{mi} - y_{nj}|^2$ into three categories, thus obtaining the three parts of the decomposition.

The first is related to inequality within the subgroups and refers to the case $m = n$, that is, it collects all differences between units belonging to the same subgroup,

$$G_{2pw} = \frac{1}{2q^2\mu_p^2} \sum_{m=1}^{M} \sum_{m=1}^{M} \sum_{i=1}^{q_m} \sum_{j=1}^{q_m} \left| y_{mi} - y_{mj} \right|^2$$

For the case $m \neq n$, let us define

$$(y_{mi} - y_{nj})^+ = max\left\{(y_{mi} - y_{nj}), 0\right\}$$

and

$$(y_{mi} - y_{nj})^- = max\left\{-(y_{mi} - y_{nj}), 0\right\}$$

such as

$$\left| y_{mi} - y_{nj} \right| = (y_{mi} - y_{nj})^+ + (y_{mi} - y_{nj})^-.$$

We then disaggregate the total sum of the differences between units belonging to two different subgroups m and n into two quantities:

$$d_{mn}^2 = \sum_{i=1}^{q_m} \sum_{j=1}^{q_n} \left[(y_{mi} - y_{nj})^+\right]^2 \text{ and } p_{mn}^2 = \sum_{i=1}^{q_m} \sum_{j=1}^{q_n} \left[(y_{mi} - y_{nj})^-\right]^2$$

where

- d_{mn}^2 refers to the inequality between the subgroups m and n, with $\mu_{pm} \geq \mu_{pn}$ and $y_{mi} \geq y_{nj}$,
- p_{mn}^2 evaluates the overlap between the subgroups m and n, with $\mu_{pm} \geq \mu_{pn}$ and $y_{mi} < y_{nj}$.

The second component derived from the decomposition *à la* Dagum is the sum of all d_{mn}^2 and therefore refers to the inequality between M subgroups:

$$G_{2pb} = \frac{1}{2q^2\mu_p^2} \sum_{m=1}^{M} \sum_{n=1}^{M} d_{mn}^2.$$

Analogously, the overlap between M subgroups is evaluated through the sum of all p_{mn}^2 which provides the following third component:

$$G_{2po} = \frac{1}{2q^2\mu_p^2} \sum_{m=1}^{M} \sum_{n=1}^{M} p_{mn}^2.$$

Overall we have

$$G_{2p} = G_{2pw} + G_{2pb} + G_{2po}.$$

In this way, we decompose the second term in relation (8) into the *within-*, *between-* and *overlapping-* components, thus obtaining a decomposition of FGT_2 index as in the following corollary, which represents our second result.

Corollary 4. *The FGT_2 index proposed in equation (6) can be decomposed as follows:*

$$\begin{aligned} FGT_2 &= H\frac{1}{q}\sum_{i=1}^{q}\frac{(z-\mu_p)^2}{z^2} + H\frac{\mu_p^2}{z^2}G_{2pw} + H\frac{\mu_p^2}{z^2}G_{2pb} + H\frac{\mu_p^2}{z^2}G_{2po} = \\ &= HI_{2p} + H(1-I_p)^2G_{2pw} + H(1-I_p)^2G_{2pb} + H(1-I_p)^2G_{2po} = \\ &= H(I_{2p} + (1-I_p)^2(G_{2pw} + G_{2pb} + G_{2po})) \end{aligned} \tag{9}$$

Under Corollary 4, equation (9) extends the analysis proposed in Corollary 3, including the contribution of each inequality factor.

This is possible by investigating the second term of equation (6) through the Dagum's decomposition. More specifically, G_{2pw} reflects the income dispersion within the subgroups and measures how poverty can be unfair in the same community.

Furthermore, G_{2pb} considers the influence of the inequality factor between subgroups. More in detail, it is now gauging the extent of the unfair poverty by looking at the poor with different characteristics.

Finally, G_{2po} measures the effects of the overlapping which can be interpreted as the stratification in the society. Stratification plays a vital role in relative deprivation theory, as suggested by Yitzhaki and Lerman (1991). They argue that the larger the stratification of society, the more the society can tolerate a significant level of inequality.

Overall, G_{2pw}, G_{2pb} and G_{2po} in equation (9) provide a powerful insight about the role of inequality in the decomposition of the FGT_2 index. In particular, an increase in G_{2pb} is directly interpretable as a rise in the importance of the analyzed inequality factor, while an increase in G_{2po} suggests minor importance of such a factor.

Our proposal is even complementary to a strand of literature on regression analysis involving the estimation of the Gini index through a stochastic approach, see Ogwang (2014).[§] In particular, the possibility of computing the poverty index from estimated regression model parameters is perfectly compatible as we get the *within-*, *between-* and *overlapping* components through the Dagum's method, see Maasoumi (1994).

Starting from relation (9), we can extend the analysis of the decomposition of the FGT_2 index by looking at the contribution of each subgroup to FGT_2, which we discuss in the next section.

4. FGT_2 SUBGROUPS DECOMPOSITION

To detect the contribution of each subgroup to FGT_2, we can easily exploit its additive structure and express the overall index as

$$FGT_2 = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z-y_i}{z}\right)^2 = \frac{1}{n}\sum_{m=1}^{M}\sum_{i=1}^{q_m}\left(\frac{z-y_{mi}}{z}\right)^2 = \sum_{m=1}^{M}\frac{n_m}{n}FGT_{2m} \qquad (10)$$

where n_m represents the number of people belonging to the m-th subgroup. In turn, the Foster-Greer-Thorbecke index FGT_{2m} is traditionally identified as

$$FGT_{2m} = \frac{1}{n_m}\sum_{i=1}^{q_m}\left(\frac{z-y_{mi}}{z}\right)^2$$

which captures the poverty among people within the $m-$th subgroup.

A relevant result that can be derived from relation (9) is the possibility of evaluating the contribution of each subgroup to the FGT_2 index.

Starting from the expressions of μ_p and μ_p^2,

[§]A weight-least square estimator can then be adopted to estimate the Pseudo-Gini index as in Ogwang (2007) similarly for the poverty gaps.

$$\mu_p = \sum_{m=1}^{M} \frac{q_m}{q} \mu_{pm}$$

$$\mu_p^2 = \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{q_m}{q} \frac{q_n}{q} \mu_m \mu_n = \sum_{m=1}^{M} \frac{q_m}{q} \mu_m \sum_{n=1}^{M} \frac{q_n}{q} \mu_n$$

we can indicate I_{2p} as,

$$I_{2p} = \frac{1}{q} \sum_{i=1}^{q} \frac{(z-\mu_p)^2}{z^2} = 1 - \frac{2\mu_p}{z} + \frac{\mu_p^2}{z^2} = \sum_{m=1}^{M} \frac{q_m}{q} \left(1 - \frac{2\mu_{pm}}{z} + \frac{\mu_{pm}}{z^2} \sum_{n=1}^{M} \frac{q_n}{q} \mu_{pn} \right) \tag{11}$$

that is the sum of the contributions of the m subgroups.

Analogously, we can also express G_{2p} as the sum of the contributions of the m subgroups such that,

$$G_{2p} = \frac{1}{2q^2\mu_p^2} \sum_{m=1}^{M} (G_{2pwm} + G_{2pbm} + G_{2pom}) \tag{12}$$

where

$$G_{2pwm} = \sum_{i=1}^{q_m} \sum_{j=1}^{q_m} \left| y_{mi} - y_{mj} \right|^2, \quad G_{2pbm} = \sum_{n=1}^{M} d_{mn}^2, \quad G_{2pom} = \sum_{n=1}^{M} p_{mn}^2.$$

The terms G_{2pwm} G_{2pbm} and G_{2pom} explain the inequality in the distribution by pointing out the *within-*, *between-* and *overlapping-* differences that emerge in the subgroups.

Overall, starting from (9), we can define $FGT_{2m}^{\star}$ as the measure of poverty in the $m-$th subgroup in the framework of our new decomposition, as expressed in the following proposition, which represents our third result.

Proposition 2. *The FGT_2 index is the weighted average of the contribution of each subgroup $FGT_{2m}^{\star}$*

$$FGT_2 = \sum_{m=1}^{M} \frac{n_m}{n} FGT_{2m}^{\star}$$

where

$$FGT^{\star}_{2m} = \frac{q_m}{n_m}\left(1 - \frac{2\mu_{pm}}{z} + \frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}\right) + \frac{1}{2qn_m z^2}(G_{2pwm} + G_{2pbm} + G_{2pom}). \tag{13}$$

Proof. See Appendix C. □

Proposition 2 and equation (13) show that it is possible to evaluate the contribution of each subgroup to the FGT_2 index. We extend the analysis for each subgroup by disentangling the role of the incidence and the intensity of poverty together with the inequality that emerges through a decomposition *à la* Dagum (1997).

With respect to the decompositions of the FGT_2 index mentioned in Section 2, the proposal of Civardi and Chiappero Martinetti (2008) explicitly takes into account the contribution of the subgroups such that,

$$FGT_{2m} = FGT_{2m}(z_m) + (FGT_{2m}(z) - FGT_{2m}(z_m))$$

where the first and second terms measure, respectively, the inequality within- and between- subgroups. This is complementary to our analysis as it provides a decomposition by *population subgroups* looking at different poverty lines. In our analysis, instead, we focus on the information on poverty measurement that each subgroup may provide in a society with a unique poverty line.

The subgroup decomposition can also be extended in the framework of Aristondo *et al.* (2010). In this case, it is possible to derive the contribution of the subgroups by disaggregating the term E_2:

$$E_2 = \frac{1}{2q}\sum_{i=1}^{q}\left(\left(\frac{g_i}{\mu(g)}\right)^2 - 1\right) = \frac{1}{2q}\sum_{m=1}^{M}\sum_{i=1}^{q_m}\left(\left(\frac{g_{mi}}{\mu(g)}\right)^2 - 1\right) = \frac{1}{2q}\sum_{m=1}^{M}E_{2m}.$$

Finally, by recalling that $I_p^2 = I_{2p}$ and relation (11), we get

$$FGT^{A}_{2m} = \frac{q_m}{n_m}(1 - \frac{2\mu_{pm}}{z} + \frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}) + \frac{I_p^2}{n_m}E_{2m} \tag{14}$$

with

$$FGT_2 = \sum_{m=1}^{M} \frac{n_m}{n} FGT_{2m}^{A}.$$

The advantage of FGT_{2m}^{A} is to capture the relative differences between the poverty gaps for each subgroup through a structure based on generalized entropy indices, while FGT_{2m} focuses on the internal dynamics of each subgroup underlining the differences in the poverty lines.

By comparing relations (13) and (14), differences emerge in the second term only. Our proposal in $FGT_{2m}^{\star}$ allows to expand the set of analysis by disentangling the impact of inequality within and among subgroups by means of the Dagum's methodology.

To sum up, the analysis and the comparison of FGT_{2m}, $FGT_{2m}^{\star}$ and FGT_{2m}^{A} highlight the advantages of different decompositions, allowing a comprehensive evaluation of the contribution of each subgroup to the FGT_2 index.

5. EMPIRICAL EVIDENCE ON ITALIAN DATA

Based on the results illustrated in the previous sections, we analyze the Italian individual income distribution for 2006 and 2016. We use the Survey on Households Income and Wealth run by the Bank of Italy and we evaluate the relevance of education, gender, and the area of residence as inequality factors, measuring their effects on FGT_2 Index

As the number of subgroups influences the FGT_2 decomposition, we choose to compare the inequality factors by always referring to $M = 2$ subgroups. However, we can easily extend the analysis to the case of $M > 2$ enriching the results of the empirical design. We can even consider potential interactions among factors like gender conditional on the level of education or located in the South/North of Italy.

In the following, the partitioning vectors are: *i)* for the case of gender, $\Omega = \{C_f, C_m\}$, where the conditions C_f and C_m identify female and male subgroups, *ii)* for education, $\Omega = \{C_{wh}, C_h\}$, where C_{wh} and C_h indicate *without* and *with* high school, *iii)* for the area of residence $\Omega = \{C_s, C_n\}$, where C_s and C_n correspond to *south-and-islands* and *north-central* Italy.

Furthermore, to highlight the effects of the inequality factors, we analyze and compare different subsets of the poor, detected through a set of poverty lines, starting from the median income $z = \mu_{me}$ up to $z = 0.4 * \mu_{me}$.

Table 1 reports the Sen and the FGT_2 indices and their components by looking at different poverty lines for 2006 and 2016, and illustrates the results related to Proposition 1 and Corollary 3.

In 2006, the headcount ratio H naturally decreases as the poverty line reduces, while this effect is less relevant for the poverty gap ratio I_p and also for I_{2p}. However, we detect a different behaviour of CV_{2p} showing that inequality increases when the poverty line varies, as also confirmed by G_p.

Table 1: *Poverty line z, Sen index S, FGT_2 index and their components: head count ratio H, poverty gap ratio I_p, squared poverty gap ratio I_p^2, Gini index of the poor G_p, squared coefficient of variation among the poor CV_{2p}, Italian individual income 2006 and 2016.*

z	S	FGT_2	H	I_p	I_{2p}	G_p	CV_{2p}
	2006						
μ_{me}	0.266	0.108	0.500	0.384	0.147	0.241	0.179
$0.8\mu_{me}$	0.187	0.076	0.349	0.387	0.149	0.243	0.183
$0.6\mu_{me}$	0.119	0.049	0.222	0.389	0.151	0.242	0.185
$0.4\mu_{me}$	0.065	0.027	0.122	0.363	0.132	0.261	0.224
	2016						
μ_{me}	0.285	0.123	0.500	0.415	0.172	0.267	0.218
$0.8\mu_{me}$	0.208	0.091	0.357	0.425	0.180	0.271	0.226
$0.6\mu_{me}$	0.141	0.063	0.241	0.424	0.180	0.280	0.243
$0.4\mu_{me}$	0.084	0.039	0.140	0.425	0.181	0.304	0.293

Looking at the differences between 2006 and 2016, Sen and FGT_2 indices increase. This result is due to the increase of the incidence of poverty measured by I_p and I_{2p} as well as to the increase of inequality in the distribution through G_p and even more CV_{2p}. We also note that, analyzing poorer subsets of the population, the importance of inequality in the distribution of the poor increases considerably.

Moving to our second result proposed in Corollary 4, we look at the inequality in the distribution of the poor under the three components of the Dagum's decomposition.

Table 2 summarizes the results. For instance, in education, the increase in both G_{2pw} and G_{2pb} shows that the inequality within- and between- subgroups, i.e., *without* and *with* high school, plays a larger role when the poverty line reduces. This is associated to a reduction in G_{2po} suggesting a change in the stratification among subgroups. These results are mostly confirmed moving from 2006 to 2016. Taking into account gender, G_{2pb} shows a decline in the difference between male and female in 2006, a pattern which is not confirmed in 2016. Analogous results can be observed for the area of residence.

Interestingly, from the fifth column of Table 2, it is possible to state that, for 2006, as the poverty line decreases, education has greater relevance, and there is a reduction of the gender-related effects, while the area of residence seems relatively stable. Looking for instance to gender, we can observe how, for $z = \mu_{me}$, the component $H(1 - I_p)^2 G_{2pb}$ represents 11.1% of FGT_2, while,

Table 2: *Poverty line z, components of the FGT_2 index new decomposition, Italian individual income 2006 and 2016.*

z	G_{2pw}	G_{2pb}	G_{2po}	$\frac{H(1-I_p)^2 G_{2pb}}{FGT_2}$	G_{2pw}	G_{2pb}	G_{2po}	$\frac{H(1-I_p)^2 G_{2pb}}{FGT_2}$
	2006				2016			
	education							
μ_{me}	0.111	0.029	0.040	0.051	0.126	0.044	0.048	0.061
$0.8\mu_{me}$	0.116	0.032	0.036	0.055	0.132	0.046	0.047	0.060
$0.6\mu_{me}$	0.120	0.038	0.027	0.064	0.141	0.047	0.054	0.060
$0.4\mu_{me}$	0.145	0.050	0.028	0.091	0.176	0.056	0.061	0.066
	gender							
μ_{me}	0.090	0.063	0.026	0.111	0.111	0.064	0.042	0.089
$0.8\mu_{me}$	0.098	0.054	0.031	0.093	0.119	0.060	0.048	0.077
$0.6\mu_{me}$	0.105	0.041	0.039	0.069	0.127	0.065	0.051	0.083
$0.4\mu_{me}$	0.132	0.028	0.064	0.051	0.160	0.063	0.070	0.075
	area of residence							
μ_{me}	0.092	0.056	0.031	0.100	0.109	0.073	0.036	0.101
$0.8\mu_{me}$	0.093	0.053	0.038	0.091	0.113	0.067	0.047	0.086
$0.6\mu_{me}$	0.093	0.046	0.046	0.078	0.121	0.068	0.053	0.087
$0.4\mu_{me}$	0.113	0.053	0.058	0.096	0.146	0.086	0.062	0.102

for $z = 0.4\mu_{me}$, its relevance is only 5.1%, thus suggesting a weaker influence of gender on the FGT_2 index.

Our final set of results regards the contribution of each subgroup to FGT_2 reported in Table 3 and based on Proposition 2 as well as equations (10) and (14). We refer to the contribution of the most deprived subgroup for each inequality factor: *i)* without high school for education, *ii)* female for gender, and *iii)* located in the South for the area of residence.

In particular, for 2006, in the third and fourth columns, we can observe how FGT_{12}^{*} and FGT_{12}^{A} point out a relevant increase of the contribution of the female subgroup to the poverty index as the poverty line decreases. This dynamic is absent for the most deprived subgroup in the other two inequality factors.

An interesting comparison is between 2006 and 2016, where we can observe, for $z = 0.4\mu_{me}$, a lower contribution of the most deprived subgroup for education and gender, and a higher contribution for the area of residence.

Finally, in order to assess the significance of our results, building on the statistical inference procedures developed for poverty measurement (see, e.g., Bishop *et al.* (1997)), we implement a bootstrap procedure able to derive confidence intervals for all relevant indicators. In Table 4 we show the 95% bootstrap confidence intervals for the main object of our work, the FGT_2 index, and also for the component G_{2pb}, to which we paid a specific attention and which plays a major role throughout the paper. Bootstrap confidence intervals are

Table 3: *Poverty line z, subgroups contribution to FGT_2 index, Italian individual income 2006 and 2016.*

z	$\frac{FGT_{12}}{FGT_2}$	$\frac{FGT^*_{12}}{FGT_2}$	$\frac{FGT^A_{12}}{FGT_2}$	$\frac{FGT_{12}}{FGT_2}$	$\frac{FGT^*_{12}}{FGT_2}$	$\frac{FGT^A_{12}}{FGT_2}$
	2006			2016		
	education					
μ_{me}	0.781	0.656	0.628	0.705	0.590	0.584
$0.8\mu_{me}$	0.777	0.632	0.625	0.706	0.584	0.582
$0.6\mu_{me}$	0.761	0.751	0.721	0.708	0.589	0.580
$0.4\mu_{me}$	0.730	0.692	0.640	0.721	0.556	0.543
	gender					
μ_{me}	0.295	0.218	0.122	0.369	0.337	0.292
$0.8\mu_{me}$	0.282	0.227	0.169	0.362	0.346	0.323
$0.6\mu_{me}$	0.287	0.264	0.261	0.358	0.336	0.312
$0.4\mu_{me}$	0.334	0.560	0.478	0.359	0.473	0.463
	area of residence					
μ_{me}	0.458	0.512	0.578	0.485	0.536	0.611
$0.8\mu_{me}$	0.466	0.501	0.538	0.499	0.522	0.560
$0.6\mu_{me}$	0.463	0.470	0.469	0.513	0.527	0.559
$0.4\mu_{me}$	0.446	0.510	0.494	0.529	0.556	0.592

obtained using the percentile method and are based on 1,000 bootstrap repetitions.

From Table 4, we can observe how the different poverty lines do not influence the confidence intervals, which are quite stable across them. Furthermore, the intervals are mostly negatively skewed, with the lower bound more distant from the mean than the upper bound. Almost all the intervals are centered on the value of the parameter reported in Tables 1 and 2, thus lending a strong support to our results.[¶]

6. CONCLUSIONS

This paper aims at generalizing the well-known standard decomposition of the FGT_2 index by stressing the role of unfair income distribution among the poor and the effects of the inequality factors, also evaluating their influence on overall poverty.

We start from the literature on FGT_2 decomposition, and we investigate different aspects of inequality in the distribution of the poor by means of the Dagum's method.

The analysis is extended by looking at the contribution of each subgroup to the overall index. This approach is so general that it can even be included

[¶]Bootstrap confidence intervals for the remaining indicators are available from the Authors upon request and also support our previous conclusions.

Table 4: *95% bootstrap confidence intervals for FGT_2 index and G_{2pb} component, Italian individual income 2006, 2016.*

	FGT_2				G_{2pb} - education			
	2006		2016		2006		2016	
	low	high	low	high	low	high	low	high
μ_{me}	0.104	0.109	0.119	0.126	0.027	0.031	0.040	0.047
$0.8\mu_{me}$	0.072	0.078	0.088	0.093	0.028	0.035	0.042	0.050
$0.6\mu_{me}$	0.047	0.051	0.060	0.065	0.033	0.041	0.042	0.053
$0.4\mu_{me}$	0.025	0.029	0.036	0.041	0.043	0.056	0.049	0.063
	G_{2pb} - gender				G_{2pb} - area			
	2006		2016		2006		2016	
	low	high	low	high	low	high	low	high
μ_{me}	0.061	0.066	0.061	0.068	0.053	0.059	0.068	0.076
$0.8\mu_{me}$	0.050	0.056	0.054	0.064	0.048	0.056	0.062	0.071
$0.6\mu_{me}$	0.038	0.044	0.059	0.071	0.041	0.050	0.059	0.072
$0.4\mu_{me}$	0.025	0.032	0.054	0.071	0.046	0.058	0.075	0.095

in some previous decompositions, leading to a further perspective on poverty assessment.

We develop a case study on Italian data for empirical illustration. Our decomposition of the standard FGT_2 poverty index highlights the role that inequality factors such as education, gender and area of residence play in both inequality and poverty measurement.

Overall we are confident that the information provided by the decomposition of the FGT Index introduced in the paper allows for a more exhaustive knowledge of the inequality structure and its effects on poverty evaluation.

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Compliance with Ethical Standards: The authors declare that there is no conflict of interest and that they do not have no competing interests to declare.

Data availability: The data that support the findings of this study are publicly available in the *Bank of Italy* repository at the following link `https://www.bancaditalia.it/statistiche`

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APPENDIX

A Proofs of Corollaries 1 and 2

A.1 Proof of Corollary 1

$$I_p = \frac{1}{q}\sum_{i=1}^{q}\left(\frac{z-y_i}{z}\right) = \frac{1}{q}\left(\sum_{i=1}^{q}\frac{z}{z} - \sum_{i=1}^{q}\frac{y_i}{z}\right) = \frac{1}{q}\left(q - \frac{1}{z}\sum_{i=1}^{q}y_i\right) = \frac{1}{q}\left(q - \frac{q\mu_p}{z}\right) =$$

$$= 1 - \frac{\mu_p}{z}$$

From which,

$$I_p^2 = \left(1 - \frac{\mu_p}{z}\right)^2 = 1 - \frac{2\mu_p}{z} + \frac{\mu_p^2}{z^2}$$

Moreover we derive,

$$I_{2p} = \frac{1}{q}\sum_{i=1}^{q}\left(\frac{z-\mu_p}{z}\right)^2 = \frac{1}{q}\sum_{i=1}^{q}\left(1 - \frac{\mu_p}{z}\right)^2 = \frac{1}{q}\sum_{i=1}^{q}\left(1 - \frac{2\mu_p}{z} + \frac{\mu_p^2}{z^2}\right) =$$

$$= 1 - \frac{2\mu_p}{z} + \frac{\mu_p^2}{z^2} = I_p^2$$

A.2 Proof of Corollary 2

From Corollary 1, we have

$$I_p = 1 - \frac{\mu_p}{z}, \text{ from which } (1 - I_p) = 1 - 1 + \frac{\mu_p}{z}$$

$$\text{and } (1 - I_p)^2 = \frac{\mu_p^2}{z^2}$$

B Proof of Proposition 1

Starting from relation (4), it follows that:

$$
\begin{aligned}
\frac{1}{q}\sum_{i=1}^{q}\frac{(\mu_p-y_i)^2}{\mu_p^2} &= \frac{\frac{1}{q}(q\mu_p^2+\sum_{i=1}^{q}y_i^2-2q\mu_p^2)}{\mu_p^2}=\frac{\frac{1}{q}\sum\limits_{i=1}^{q}y_i^2-\mu_p^2}{\mu_p^2}= \\
&= \frac{1}{2q\mu_p^2}\sum_{i=1}^{q}y_i^2+\frac{1}{2q\mu_p^2}\sum_{j=1}^{q}y_j^2-\frac{1}{\mu_p^2}\left(\frac{1}{q}\sum_{i=1}^{q}y_i\right)\left(\frac{1}{q}\sum_{j=1}^{q}y_j\right)= \\
&= \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_i^2+\frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_j^2-\frac{1}{q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_iy_j= \\
&= \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_i^2+\frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_j^2-\frac{2}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}y_iy_j= \\
&= \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}(y_i^2+y_j^2-2y_iy_j)= \\
&= \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}(y_i-y_j)^2= \\
&= \frac{1}{2q^2\mu_p^2}\sum_{i=1}^{q}\sum_{j=1}^{q}|y_i-y_j|^2
\end{aligned}
$$

C Proof of Proposition 2

Starting from (9), we first substitute $H=\frac{q}{n}$ and $(1-I_p)^2=\frac{\mu_p^2}{z^2}$,

$$
\begin{aligned}
FGT_2 &= H(I_{2p}+(1-I_p)^2(G_{2pw}+G_{2pb}+G_{2po}))= \\
&= \frac{q}{n}(I_{2p}+\frac{\mu_p^2}{z^2}(G_{2pw}+G_{2pb}+G_{2po}),
\end{aligned}
$$

then we resort to equation (11) to express I_{2p},

$$
FGT_2=\frac{q}{n}\sum_{m=1}^{M}\frac{q_m}{q}\left(1-\frac{2\mu_{pm}}{z}+\frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}\right)+\frac{q}{n}\frac{\mu_p^2}{z^2}(G_{2pw}+G_{2pb}+G_{2po})
$$

and finally we apply (12) to $(G_{2pw} + G_{2pb} + G_{2po})$ such that:

$$\begin{aligned}
FGT_2 &= \frac{q}{n}\sum_{m=1}^{M}\frac{q_m}{q}\left(1-\frac{2\mu_{pm}}{z}+\frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}\right)+\frac{q}{n}\frac{\mu_p^2}{z^2}\frac{1}{2q^2\mu_p^2}\sum_{m=1}^{M}(G_{2pwm}+G_{2pbm}+G_{2pom}) = \\
&= \sum_{m=1}^{M}\frac{q_m}{n}\left(1-\frac{2\mu_{pm}}{z}+\frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}\right)+\frac{1}{2nqz^2}\sum_{m=1}^{M}(G_{2pwm}+G_{2pbm}+G_{2pom}) = \\
&= \sum_{m=1}^{M}\frac{n_m}{n}\left(\frac{q_m}{n_m}\left(1-\frac{2\mu_{pm}}{z}+\frac{\mu_{pm}}{z^2}\sum_{n=1}^{M}\frac{q_n}{q}\mu_{pn}\right)+\frac{1}{2n_mqz^2}(G_{2pwm}+G_{2pbm}+G_{2pom})\right).
\end{aligned}$$