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ON A BREZIS-OSWALD-TYPE RESULT FOR DEGENERATE KIRCHHOFF PROBLEMS

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ABSTRACT. In the present note we establish an *almost-optimal* solvability result for Kirchhoff-type problems of the following form

$$\begin{cases} -M\big(\|\nabla u\|^2_{L^2(\Omega)}\big)\Delta u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where f has sublinear growth and M is a non-decreasing map with $M(0) \geq 0$. Our approach is purely variational, and the result we obtain is resemblant to the one established by Brezis and Oswald (Nonlinear Anal., 1986) for sublinear elliptic equations.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded and connected open set with sufficiently smooth boundary $\partial\Omega$. The aim of this short note is to prove an optimal solvability-type result which is somehow complementary to [1]: see Remark 1.1 for a comparison between our result and the one in [1]. In particular, we are interested in the following Kirchhoff problem

$$\begin{cases} -M\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We immediately fix the standing assumptions on both the Kirchhoff function M and the nonlinearity f.

($f1$) $f : \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function.
($f2$) $f(\cdot,t) \in L^\infty(\Omega)$ for every $t \geq 0$.
($f3$) There exists a positive constant $c > 0$ such that

$$|f(x,t)| \leq c(1+t) \qquad \text{for a.e. } x \in \Omega \text{ and every } t \geq 0.$$

($f4$) For a.e. $x \in \Omega$, the function $t \mapsto \dfrac{f(x,t)}{t}$ is strictly decreasing in $(0,+\infty)$.
(M) $M : [0,+\infty) \to \mathbb{R}$ is a non-negative, non-decreasing and continuous function such that $M(s) > 0$ for every $s > 0$. For a future reference, we set

$$M(0) = m_0 \geq 0.$$

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As is customary in the literature, if $m_0 > 0$ (hence, $M(s) \geq m_0 > 0 \; \forall \; s \geq 0$), we say that M *is non-degenerate*; if, instead, $m_0 = 0$, we say that M *is degenerate*. We stress right now that our technique does not allow us to consider Kirchhoff functions which are degenerate on an interval $[0, t_0)$, see Remark 2.3.

Remark 1.1. We observe that all the assumptions $(f1)$-to-$(f4)$ are trivially satisfied in the particular case of power-type linearities $f(x,u) = u^\theta$, with $0 \leq \theta < 1$: this has been considered in [1], where the authors prove existence and uniqueness of a solution by means of sub/supersolution methods, *but with a non-increasing* M, which is usually not the standard assumption in the applications. We want to stress that the proof in [1] is heavily based on comparison principles whose validity for Kirchhoff functions M having *non-decreasing* behaviour (like ours) may fail to hold without some extra assumption, see e.g. [2, 11, 9]. For instance, it is proved in [7, Proposition 2] that the comparison principle holds, provided that M is *any* non-decreasing function (no sign assumption is required) and the map

$$t \mapsto tM(t)$$

is non-increasing, but this is clearly an assumption incompatible with (M). In this perspective, we highlight that we do not require any additional assumption on any of the maps $t^\alpha M(t)$ (with $\alpha \neq 0$). We also notice that, in working with an explicit nonlinearity (as is done in [1]), there is no need to look for optimal solvability as in the present paper. Keeping this aim in mind, it is therefore kind of natural to look for a variational approach when M is non-decreasing and/or possibly degenerate. We recall that there exists a huge literature concerning variational methods for Kirchhoff problem, see e.g. [20, 3, 15, 21, 16] and the references therein for a definitely non exhaustive list of contributions.

In what follows, we will exploit the variational nature of the problem, in order to provide necessary and sufficient conditions for solvability of (1.1) in the spirit of the celebrated paper [6] by Brezis and Oswald. To this aim, owing to assumption $(f4)$, we introduce the following functions:

$$a_0(x) := \lim_{t \downarrow 0} \frac{f(x,t)}{t} \qquad a_\infty(x) := \lim_{t \uparrow +\infty} \frac{f(x,t)}{t} \qquad (\text{for } x \in \Omega).$$

Remark 1.2. We explicitly notice that, since we do not require any bound on the map $t \mapsto f(\cdot,t)/t$ near $t = 0$, the function $a_0(\cdot)$ can be *unbounded from above*: more precisely, by assumptions $(f2)$ and $(f4)$ we have

$$+\infty \geq a_0(x) = \lim_{t \downarrow 0} \frac{f(x,t)}{t} \geq f(x,1) \geq -\|f(x,1)\|_{L^\infty(\Omega)} > -\infty. \tag{1.2}$$

On the other hand, by assumption $(f3)$ we have

$$|a_\infty(x)| = \lim_{t \to +\infty} \left| \frac{f(x,t)}{t} \right| \leq \lim_{t \to +\infty} \frac{c(1+t)}{t} = c. \tag{1.3}$$

We also point out that, again by assumption $(f4)$, one has $a_\infty(x) \leq a_0(x)$ for every $x \in \Omega$ (with the usual convention that $-\infty < z < +\infty$ for all $z \in \mathbb{R}$).

Definition 1.3. Let the above assumptions and notations be in force. We say that a function $u \in H_0^1(\Omega)$ is a *weak solution* of (1.1) if

$$\int_\Omega M\left(\|\nabla u\|^2_{L^2(\Omega)}\right) \langle \nabla u, \nabla \varphi \rangle \, dx = \int_\Omega f(x,u)\varphi \, dx \quad \forall \; \varphi \in H_0^1(\Omega), \tag{1.4}$$

and $u \geq 0$ a.e. in Ω and $|\{x \in \Omega : \, u(x) > 0\}| > 0$ (here and throughout, $|\cdot|$ denotes the n-dimensional Lebesgue measure of a measurable set, while $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$).

As commonly exploited in the literature, weak solutions to the equation in (1.1) can be obtained as critical points of the functional

$$J_{\mathcal{M}}(u) := \frac{1}{2}\mathcal{M}\left(\|\nabla u\|^2_{L^2(\Omega)}\right) - \int_\Omega F(x,u)\,dx, \quad u \in H^1_0(\Omega), \tag{1.5}$$

where $\mathcal{M}$ and F are primitives of M and f respectively, that is,

$$\mathcal{M}(s) = \int_0^s M(\zeta)\,d\zeta \qquad \text{and} \qquad F(x,t) = \int_0^t f(x,s)\,ds.$$

We explicitly stress that, in order to define $F(x,t)$ also for *negative* t (so that the functional $J_{\mathcal{M}}$ is well-posed on $H^1_0(\Omega)$), following [6] we agree to set

$$f(x,t) = f(x,0) \text{ for } t \leq 0. \tag{1.6}$$

Remark 1.4. We explicitly highlight, for a future reference, that

$$F(x,t^+) \geq F(x,t) \quad \text{for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.$$

To prove this fact (in the meaningful case when $t \leq 0$ and $t^+ = 0$) we first observe that, by exploiting assumptions $(f2) - (f4)$, we have

$$f(x,t) = \frac{f(x,t)}{t}\cdot t \geq f(x,1)t \geq -\|f(x,1)\|_{L^\infty(\Omega)}t \equiv -c_f t,$$
$$\text{for a.e. } x \in \Omega \text{ and } 0 \leq t \leq 1;$$

this, together with assumption $(f1)$, ensures that $f(x,0) \geq 0$. As a consequence of this last fact, and taking into account (1.6), we then obtain

$$F(x,t) = \int_0^t f(x,s)\,ds = f(x,0)t \leq 0 = F(x,t^+) \quad \forall\, t \leq 0.$$

Remark 1.5. In the local and linear case $M(s) = 1$ (and hence $\mathcal{M}(s) = s$) considered in [6], there is a *direct relation* between the variational formulation of

$$-\Delta u = f(x,u)$$

and the minimization problem

$$\inf_{u \in H^1_0(\Omega)} J_{\mathcal{M}}(u) = \inf_{u \in H^1_0(\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)\,dx\right).$$

More precisely, when we test the equation appearing in problem (1.1) with *the weak solution* $u \in H^1_0(\Omega)$ *itself* (provided it exists), we obtain

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega f(x,u)u\,dx,$$

and the *operator part*

$$\int_\Omega |\nabla u|^2\,dx$$

coincides with $\mathcal{M}(\|\nabla u\|^2_{L^2(\Omega)}) = \|\nabla u\|^2_{L^2(\Omega)}$ in the minimization problem.

This is no more true in the case when one considers different M: in fact, using the weak solution $u \in H^1_0(\Omega)$ as a test function for the equation in (1.1), we get

$$M(\|\nabla u\|^2_{L^2(\Omega)}) \int_\Omega |\nabla u|^2\,dx = \int_\Omega f(x,u)u\,dx,$$

but the operator part *does not coincide with* $\mathcal{M}(\|\nabla u\|^2_{L^2(\Omega)})$, that is,

$$M(\|\nabla u\|^2_{L^2(\Omega)}) \int_\Omega |\nabla u|^2\,dx \neq \mathcal{M}(\|\nabla u\|^2_{L^2(\Omega)}).$$

As already anticipated, our aim in this note is to obtain *optimal conditions* for the existence of a unique weak solution (in the sense of Definition 1.3) to problem (1.1). To state our main result in this direction, we need to introduce a notion which is the extension to our setting of the original definition in [6]: if $a : \Omega \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is any measurable function, we define

$$\Lambda(a) := \inf_{v \in H^1_0(\Omega)\setminus\{0\}} \left\{ \frac{\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) - \int_{\{v\neq 0\}} a(x)v^2}{\|v\|^2_{L^2(\Omega)}} \right\}.$$

Remark 1.6. Due to the key role played by the numbers $\Lambda(a_0)$ and $\Lambda(a_\infty)$ in our main result, see Theorem 1.7 below, here we list for a future reference some basic properties of $\Lambda(a)$, when $a : \Omega \to \overline{\mathbb{R}}$ is a generic measurable function.

(1) Being defined as the *infimum of a subset of* $\overline{\mathbb{R}}$, $\Lambda(a)$ always exists in $\overline{\mathbb{R}}$; moreover, we have $\Lambda(a) = +\infty$ if and only if $a(x) = -\infty$ for a.e. $x \in \Omega$.

(2) If $a, b : \Omega \to \overline{\mathbb{R}}$ are measurable functions such that $a \leq b$ a.e. on Ω, it follows from the very definition of $\Lambda(\cdot)$ that $\Lambda(b) \leq \Lambda(a)$ in $\overline{\mathbb{R}}$.

(3) Assume that there exists some number $m \in [0, +\infty)$ such that $a \geq -m$ for almost every $x \in \Omega$. Owing to assumption (M), and choosing an arbitrary 'admissible' function $v \in H^1_0(\Omega)$, $v \not\equiv 0$, we then get

$$\Lambda(a) \leq \frac{\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right)}{\|v\|^2_{L^2(\Omega)}} + m < +\infty.$$

As a consequence, *if a is bounded from below* (as is the case of a_0, see (1.2)), we derive that $\Lambda(a) \in [-\infty, +\infty)$.
Analogously, if we assume that $a(x) \leq m$ for almost every $x \in \Omega$ (i.e., a is bounded from above), again by assumption (M) we have

$$\frac{\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) - \int_{\{v\neq 0\}} a(x)v^2}{\|v\|^2_{L^2(\Omega)}} \geq -m \quad \forall\, v \in H^1_0(\Omega) \setminus \{0\},$$

so that $\Lambda(a) \geq -m > -\infty$. As a consequence, *if a is bounded from above* (as is the case of a_∞, see (1.3)), we derive that $\Lambda(a) \in (-\infty, +\infty]$.

(4) In the particular case when M *is non-degenerate*, that is, $M(0) = m_0 > 0$, it follows from the very definition of $\Lambda(a)$ that

$$\begin{aligned} \Lambda(a) &\geq \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \left\{ \frac{m_0\|\nabla v\|_{L^2(\Omega)}^2 - \int_{\{v\neq 0\}} a(x)v^2}{\|v\|_{L^2(\Omega)}^2} \right\} \\ &= \inf_{\substack{v\in H_0^1(\Omega) \\ \|v\|_{L^2(\Omega)}=1}} \left\{ m_0\|\nabla v\|_{L^2(\Omega)}^2 - \int_{\{v\neq 0\}} a(x)v^2 \right\} \\ &= \lambda_1(-m_0\Delta - a), \end{aligned} \tag{1.7}$$

where $\lambda_1(-m_0\Delta - a)$ is the first eigenvalue of $-m_0\Delta - a$ (with Dirichlet boundary condition), as defined in [6].
Analogously, if we assume that the *Kirchhoff function* M is bounded from above by some $m_1 \in (0, +\infty)$ (i.e., $M(s) \leq m_1$ for all $s \geq 0$), we get

$$\Lambda(a) \leq \lambda_1(-m_1\Delta - a).$$

We explicitly observe that, *if M is non-degenerate* (that is, $M(0) = m_0 > 0$) *and if $a \leq 0$ almost everywhere in* Ω, from (1.7) we derive that

$$\Lambda(a) \geq \lambda_1(-m_0\Delta - a) \geq m_0\lambda_1(-\Delta) > 0,$$

where $\lambda_1(-\Delta)$ denotes the first Dirichlet eigenvalue of $L = -\Delta$ in Ω.

The main result. Taking into account all the definitions and notations introduced so far, we are able to state the main result of this note.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected open set with smooth boundary $\partial\Omega$. Moreover, assume that f satisfies assumptions $(f1)$-to-$(f4)$ and that M satisfies assumption (M). Then, the following holds.*

(1) *If there exists a weak solution to* (1.1)*, then it is* unique, bounded and strictly positive in Ω*; in addition, we necessarily have $\Lambda(a_0) < 0$.*
(2) *If one has $\Lambda(a_0) < 0 < \Lambda(a_\infty)$, then there exists a weak solution to* (1.1)*, which is therefore unique, bounded and strictly positive in Ω.*

Remark 1.8 (Sharpness of Theorem 1.7)**.** We explicitly highlight that, despite its statement, Theorem 1.7 is *almost sharp*: more precisely, due to the nature of the problem, *and differently from* [6], one can find weak solutions of (1.1) *even if*

$$\Lambda(a_\infty) = 0,$$

both in the degenerate case ($m_0 = 0$) and in the non-degenerate case ($m_0 > 0$). In order to illustrate this fact, we consider the following examples.

(1) In Euclidean space $\mathbb{R}^n$ (with $n \geq 1$), let $B_1 \subseteq \mathbb{R}^n$ denote the unit ball with centre $x_0 = 0$, and let u_0 be the function defined as follows:

$$u_0(x) = 1 - \|x\|^2 \qquad (x \in \overline{B}_1).$$

A direct computation shows that this function $u_0 \in C^2(\overline{B}_1)$ is a (classical) solution of problem (1.1), where M and f are given, respectively, by

$$f(x,t) \equiv \frac{8n|B_1|}{n+1}; \qquad M(s) = s \text{ (hence, } \mathcal{M}(s) = s^2/2).$$

It is then immediate to see that the *constant function* f satisfies assumptions (f1)-to-(f4), and the Kirchhoff function $M(s) = s$ satisfies assumption (M). On the other hand, in the present context we have

$$a_\infty(x) = \lim_{t\downarrow+\infty} \frac{f(x,t)}{t} \equiv 0,$$

and a standard homogeneity argument shows that

$$\Lambda(a_\infty) = \Lambda(0) = \inf_{\substack{v\in H_0^1(B_1)\\ v\neq 0}} \left\{ \frac{\|\nabla v\|^4_{L^2(\Omega)}}{2\|v\|^2_{L^2(\Omega)}} \right\} = 0.$$

(2) In Euclidean space $\mathbb{R}^n$ (with $n \geq 1$), let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ be a *bounded and connected open set* with smooth boundary, and let $\lambda_1 = \lambda_{1,-\Delta}(\Omega) > 0$ be the first Dirichlet eigenvalue of $L = -\Delta$ in Ω (obtained as the minimum of the Rayleigh quotient). Moreover, let

$$0 < \varepsilon < \lambda_1$$

be arbitrarily fixed. Owing the result established in [6], we know that there exists a (unique) weak solution $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\Delta u = \varepsilon(1+u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \partial\Omega. \end{cases}$$

(note that, with reference to the notation in [6], we have $\lambda_1(-\Delta - a_0) = -\infty$ and $\lambda_1(-\Delta - a_\infty) = \lambda_1 - \varepsilon > 0$); as a consequence, *for every* $\beta > 0$ we see that this function u_0 is a weak solution of problem (1.1), with

- $f(x,t) = \varepsilon(1 + \beta\|\nabla u_0\|^2_{L^2(\Omega)})(1+t)$;
- $M(s) = 1 + \beta s$ (hence, $\mathcal{M}(s) = s + \beta s^2/2$).

We explicitly stress that, even if f depends on $\|\nabla u_0\|^2_{L^2(\Omega)}$, this is a *strictly positive* real number which is by now fixed.
Now, if we choose the parameter $\beta > 0$ in such a way that

$$\varepsilon(1 + \beta\|\nabla u_0\|^2_{L^2(\Omega)}) = \lambda_1$$

(note that is is always possible, since $\varepsilon < \lambda_1$), we have

$$a_\infty = \lim_{t\downarrow+\infty} \frac{f(x,t)}{t} = \varepsilon(1 + \beta\|\nabla u_0\|^2_{L^2(\Omega)}) = \lambda_1,$$

and a simple homogeneity argument shows that, in this case,

$$\begin{aligned} \Lambda(a_\infty) &= \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \left\{ \frac{\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) - \int_{\{v\neq 0\}} a_\infty(x) v^2}{\|v\|^2_{L^2(\Omega)}} \right\} \\ &= \inf_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \left\{ \frac{\|\nabla v\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\nabla v\|^4_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \right\} - \lambda_1 = 0. \end{aligned}$$

On account of Remark 1.8, we can say that our Theorem 1.7 is *almost sharp*, in the sense that we cannot hope for the condition $\Lambda(a_0) < 0 < \Lambda(a_\infty)$ to be both *necessary and sufficient* for the existence of a weak solution of (1.1). However, if we restrict our attention to a particular class of nonlinearity f and if M is non-degenerate, we can easily obtain the following *optimal solvability* result.

Corollary 1.9. *Let the assumptions and the notation of Theorem 1.7 be in force. Assume, in addition, that M* is non-degenerate *and that $a_\infty \leq 0$ a.e. in Ω. Then, there exists a weak solution of problem* (1.1) if and only if

$$\Lambda(a_0) < 0.$$

Proof. Since M *is non-degenerate* and since $a_\infty \leq 0$ a.e. in Ω, from Remark 1.6-(4) we know that $\Lambda(a_\infty) > 0$; hence, Theorem 1.7 shows that

$$\exists \text{ a weak solution of problem (1.1)} \iff \Lambda(a_0) < 0.$$

This ends the proof. ∎

We highlight that the sign assumption $a_\infty \leq 0$ a.e. in Ω in the statement of Corollary 1.9 is actually a sign assumption on the nonlinearity f.

We further notice that Corollary 1.9 allows us to obtain *optimal solvability conditions* for the problem considered in [9] (and stated in Example 1.10 below), *without the additional assumption* that $t \mapsto tM(t)$ is invertible.

Example 1.10. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and connected open set with smooth boundary $\partial\Omega$, and let $M : [0, +\infty) \to [0, +\infty)$ satisfy assumption (M). Moreover, let $\lambda \in \mathbb{R}$ and $p \in (0, 1)$ be fixed. We assume that M *is non-degenerate*, that is,

$$M(0) = m_0 > 0,$$

and we consider the following problem

$$\text{(P)} \qquad \begin{cases} -M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = \lambda u^p & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we claim that this problem admits a solution *if and only if* $\lambda > 0$.

To prove this claim we first observe that, if $\lambda \leq 0$, problem (P) cannot have solutions by the classical Weak Maximum Principle. In fact, if we assume by contradiction that there exists a weak solution $u \in H^1_0(\Omega)$ of (P) (in the sense of Definition 1.3), by standard Elliptic Regularity (and since M is non-degenerate) it is readily seen that $u \in C^2(\overline{\Omega})$; then, recalling that $\lambda \leq 0$ and $M > 0$, we get

$$-\Delta u = \frac{\lambda u^p}{M(\|\nabla u\|^2_{L^2(\Omega)})} \leq 0 \quad \text{in } \Omega.$$

On the other hand, since $u \equiv 0$ on $\partial\Omega$, by the Weak Maximum Principle we conclude that $u \leq 0$ in Ω, and this is in contradiction with the fact that $u \gneqq 0$.

If, instead, $\lambda > 0$, it is immediate to recognize that the function

$$f(x, t) = \lambda t^p$$

satisfies *all the assumptions* $(f1)$-to-$(f4)$; moreover, we have

$$a_0(x) := \lim_{t \downarrow 0} \frac{f(x,t)}{t} = +\infty, \qquad a_\infty(x) := \lim_{t \uparrow +\infty} \frac{f(x,t)}{t} = 0.$$

We are then entitled to apply Corollary 1.9, showing that problem (P) is solvable *if and only if* $\Lambda(a_0) < 0$. On the other hand, since $a_0 \equiv +\infty$ we have

$$\Lambda(a_0) = -\infty < 0,$$

and hence there exists a solution of (P) for every $\lambda > 0$.

We stress once again that we are able to establish an optimal solvability result in the case of sublinear Kirchhoff problems with a *general nonlinearity* and with the proper monotonicity behaviour of M, Our technique is so simple that we believe it can be adapted to slightly different settings like p-Kirchhoff operators or fractional Kirchhoff operators, possibly with a few modifications for the uniqueness part.

We close this introduction with a brief recall of a few other results dealing with Brezis-Oswald-type problems. The first generalization to the p-Laplacian case with Dirichlet boundary conditions is obtained in [8]. Subsequently, the case of the p-Laplacian (in particular $p = 2$) with Robin boundary conditions has been considered in [10]. Moving to the nonlocal world, in [13] the case of the fractional p-Laplacian has been recently fully addressed, while partial results (optimal only in the linear case $p = 2$) has been obtained in [19] for the fractional p-Laplacian in presence of nonlocal Robin boundary conditions. In the previous cases, the fractional (p)-Laplacian is defined on the whole of $\mathbb{R}^n$ via the Cauchy principal value; existence results in the case of nonlocal operators defined in bounded domains are obtained in [17]. In addition, we mention [4, 5], where the mixed local–nonlocal case has been fully treated. Finally, concerning the uniqueness part, we mention [14] for a class of integro-differential operators and [18] for a non–smooth setting.

The paper is organized as follows:

- in Section 2 we collect some preliminary results, mostly well-known in the literature concerning Kirchhoff problems, which will be exploited in the proof of Theorem 1.7;
- using the results in Section 2, we prove Theorem 1.7 in Section 3.

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2. Preliminary results

In this section we collect a few immediate results which already appear in the wide literature concerning Kirchhoff problems.

To begin with, we prove the following result whose proof closely follows a pretty classical truncation method. For sake of completeness, we present all the details.

Proposition 2.1. *Let $u \in H_0^1(\Omega)$ be a weak solution of problem (1.1) with f satisfying only assumption $(f1)$-to-$(f3)$. Then, $u \in L^\infty(\Omega)$.*

Proof. To begin with, we arbitrarily fix $\delta \in (0, 1)$ and we set

$$\tilde{u} := \delta u.$$

Then, recalling that u solves (1.1) (hence, $u \not\equiv 0$ in Ω), we have

$$\kappa_u \int_\Omega \langle \nabla \tilde{u}, \nabla v\rangle \, dx = \delta \int_\Omega f(x, u) v \, dx \quad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where we have used the shorthand notation

$$\kappa_u := M\left(\|\nabla u\|^2_{L^2(\Omega)}\right) > 0.$$

We explicitly stress that the *strict inequality* $\kappa_u > 0$ follows from assumption (M), both in the degenerate and in the non-degenerate case, since $u \not\equiv 0$.

Now, for every $k \geq 0$, we define $C_k := 1 - 2^{-k}$ and

$$v_k := \tilde{u} - C_k, \quad w_k := (v_k)_+ := \max\{v_k, 0\}, \quad U_k := \|w_k\|^2_{L^2(\Omega)}.$$

We explicitly point out that, in view of these definitions, one has

(a) $\|\tilde{u}\|^2_{L^2(\Omega)} = \delta^2 \|u\|^2_{L^2(\Omega)}$;
(b) $w_0 = v_0 = \tilde{u}$ (since $C_0 = 0$ and $\tilde{u} \geq 0$ in Ω);
(c) $v_{k+1} \leq v_k$ and $w_{k+1} \leq w_k$ (since $C_k < C_{k+1}$).

We then observe that, since $u \in H^1_0(\Omega)$ and $0 \leq w_k \leq \tilde{u}$, we have $w_k \in H^1_0(\Omega)$; we are then entitled to use the function w_k as a *test function* in (2.1), obtaining

$$\kappa_u \int_\Omega |\nabla w_k|^2 \, dx = \kappa_u \int_\Omega \langle \nabla \tilde{u}, \nabla w_k \rangle \, dx = \delta \int_\Omega f(x, u) w_k \, dx.$$

From this, by exploiting assumption $(f3)$ (and since $w_k \geq 0$), we obtain

$$\kappa_u \int_\Omega |\nabla w_k|^2 \, dx \leq c\,\delta \int_\Omega (1 + u)\, w_k \, dx \leq c \int_\Omega (1 + \tilde{u}) w_k \, dx, \tag{2.2}$$

since $\delta < 1$. To proceed further we note that, for every $k \geq 1$, one has

$$\tilde{u}(x) < (2^k - 1) w_{k-1}(x) \quad \text{for} \quad x \in \{w_k > 0\}, \tag{2.3}$$

and the inclusions

$$\{w_k > 0\} = \{\tilde{u} > C_k\} \subseteq \{w_{k-1} > 2^{-k}\} \tag{2.4}$$

hold true for every $k \geq 1$. By combining (2.3)-(2.4) with (2.2), and by taking into account that $w_k \leq w_{k-1}$ a.e. in Ω, for every $k \geq 1$, we get

$$\begin{aligned}
\kappa_u \int_\Omega |\nabla w_k|^2 \, dx &\leq c \int_{\{w_k > 0\}} (1 + \tilde{u}) w_k \, dx \\
&\leq c \int_{\{w_k > 0\}} \Big[w_{k-1} + (2^k - 1) w_{k-1}^2 \Big] \, dx \\
&\leq c \int_{\{w_{k-1} > 2^{-k}\}} \Big[2^k w_{k-1}^2 + (2^k - 1) w_{k-1}^2 \Big] \, dx \\
&\leq c\, 2^{k+1} \int_{\{w_{k-1} > 2^{-k}\}} w_{k-1}^2 dx \\
&\leq c\, 2^{k+1} \int_\Omega w_{k-1}^2 dx = c\, 2^{k+1} U_{k-1}.
\end{aligned} \tag{2.5}$$

We now estimate from below the term U_{k-1} in the right-hand side of (2.5). To this end we first observe that, as a consequence of (2.4), we get

$$\begin{aligned}
U_{k-1} &= \int_\Omega w_{k-1}^2 \, dx \geq \int_{\{w_{k-1} > 2^{-k}\}} w_{k-1}^2 \, dx \\
&\geq 2^{-2k} \left| \{w_{k-1} > 2^{-k}\} \right| \geq 2^{-2k} \left| \{w_k > 0\} \right|.
\end{aligned} \tag{2.6}$$

Choosing an exponent $p \in (2, 2^*)$ (with the convention that $2^* = \infty$ if $n = 1, 2$), by Hölder's inequality (with exponents $p/2$ and $p/(p-2)$) and the Sobolev Embedding

Theorem, from (2.5)-(2.6) we obtain the following estimate:

$$
\begin{aligned}
U_k = \|w_k\|^2_{L^2(\Omega)} &\leq \left(\int_\Omega w_k^p\,dx\right)^{2/p} \big|\{w_k > 0\}\big|^{1-2/p} \\
&\leq \mathbf{c}_p \int_\Omega |\nabla w_k|^2\,dx \cdot \big|\{w_k > 0\}\big|^{1-2/p} \\
&\leq \mathbf{c}_p \Big(\frac{c}{\kappa_u}\,2^{k+1}\,U_{k-1}\Big)\,\big(2^{2k}U_{k-1}\big)^{1-2/p} \\
&= \mathbf{c}'\,\big(2^{1+2(1-2/p)}\big)^{k-1}\,U_{k-1}^{2-2/p} \qquad \Big(\text{with } \mathbf{c}' := \frac{c}{\kappa_u}\,2^{2+2(1-2/p)}\,\mathbf{c}_p\Big),
\end{aligned} \tag{2.7}
$$

for every $k \geq 1$, where $\mathbf{c}_p > 0$ is the constant in the Sobolev Embedding Theorem (which may also depends on the n-dimensional measure of Ω).

Recalling that $p > 2$, estimate (2.7) can be re-written as

$$U_k \leq \mathbf{c}' B^{k-1} U_{k-1}^{1+\alpha},$$

where

$$B := 2^{1+2(1-2/p)} > 1 \quad \text{and} \quad \alpha := 1 - 2/p > 0$$

Hence, from [12, Lemma 7.1] we get that $U_k \to 0$ as $k \to +\infty$, provided that

$$U_0 = \|\tilde{u}\|^2_{L^2(\Omega)} = \delta^2 \|u\|^2_{L^2(\Omega)} < (\mathbf{c}')^{-1/\alpha}\, B^{-1/\alpha^2}.$$

As a consequence, if $\delta > 0$ is small enough, we obtain

$$0 = \lim_{k\to+\infty} U_k = \lim_{k\to+\infty} \int_\Omega (\tilde{u} - C_k)_+^2\,dx = \int_\Omega (\tilde{u} - 1)_+^2\,dx.$$

Bearing in mind that $\tilde{u} = \delta u$ (and $u \geq 0$), we then get

$$0 \leq u \leq \frac{1}{\delta} \qquad \text{a.e. in } \Omega,$$

from which we conclude that $u \in L^\infty(\Omega)$. □

Proposition 2.2. *Let $u \in H_0^1(\Omega)$ be a weak solution of problem* (1.1) (*according to Definition 1.3*). *Then, the following assertions hold:*

- *$u \in W^{2,p}(\Omega)$ for all $p \in [1, +\infty)$ (hence, $u \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$);*
- *$u(x) > 0$ for every $x \in \Omega$;*
- *$\partial_\nu u(x) < 0$ for every $x \in \partial\Omega$.*

Proof. First of all we observe that, since $u \in H_0^1(\Omega)$ is a weak solution of (1.1), from assumption (M) we have

$$\kappa_u := M\left(\|\nabla u\|^2_{L^2(\Omega)}\right) > 0;$$

as a consequence, u is also a weak solution of

$$-\Delta u = \frac{f(x,u)}{\kappa_u}, \quad \text{in } \Omega. \tag{2.8}$$

In view of (2.8), and since *we already know* from Proposition 2.1 that $u \in L^\infty(\Omega)$, by arguing exactly as in [6] we deduce that $x \mapsto f(x, u(x)) \in L^\infty(\Omega)$, and hence

$$u \in W^{2,p}(\Omega) \quad \forall\ 1 \leq p < \infty.$$

Moreover, by [6, Lemma 1] we infer that $u > 0$ in Ω and $\partial_\nu u < 0$ on $\partial\Omega$. □

Remark 2.3. We notice that the above argument cannot be run in presence of a *more degenerate* Kirchhoff function M, e.g., vanishing on some interval of the form $[0, s_0]$ for some $s_0 > 0$.

The previous Proposition 2.2, together with the proper monotonicity of M, plays a major role in the proof of the uniqueness.

Proposition 2.4. *The problem* (1.1) *has a unique weak solution.*

Proof. Let $u_1, u_2 \in H_0^1(\Omega)$ be two weak solutions of problem (1.1). As in [6], thanks to Proposition 2.2, we can consider

$$\varphi_1 := \frac{u_2^2}{u_1} - u_1$$

as test function in the equation solved by u_1, and

$$\varphi_2 := \frac{u_1^2}{u_2} - u_2$$

as test function in the equation solved by u_2. Recalling that

$$\nabla\varphi_1 = 2\frac{u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1 - \nabla u_1, \quad \text{and} \quad \nabla\varphi_2 = 2\frac{u_1}{u_2}\nabla u_1 - \frac{u_1^2}{u_2^2}\nabla u_2 - \nabla u_2,$$

we then sum the equations up, finding

$$\begin{aligned}
&\int_\Omega (f(x,u_1)\varphi_1 + f(x,u_2)\varphi_2)\, dx \\
&\quad = \int_\Omega M\left(\|\nabla u_1\|_{L^2(\Omega)}^2\right)\langle \nabla u_1, \nabla\varphi_1\rangle\, dx + \int_\Omega M\left(\|\nabla u_2\|_{L^2(\Omega)}^2\right)\langle \nabla u_2, \nabla\varphi_2\rangle\, dx \\
&\quad = -M\left(\|\nabla u_1\|_{L^2(\Omega)}^2\right)\int_\Omega \left|\nabla u_1 - \frac{u_2}{u_1}\nabla u_1\right|^2 dx \\
&\qquad - M\left(\|\nabla u_2\|_{L^2(\Omega)}^2\right)\int_\Omega \left|\nabla u_2 - \frac{u_1}{u_2}\nabla u_2\right|^2 dx \\
&\qquad + \left(M\left(\|\nabla u_1\|_{L^2(\Omega)}^2\right) - M\left(\|\nabla u_2\|_{L^2(\Omega)}^2\right)\right)\int_\Omega \left(|\nabla u_2|^2 - |\nabla u_1|^2\right) dx \\
&\quad \le \left(M\left(\|\nabla u_1\|_{L^2(\Omega)}^2\right) - M\left(\|\nabla u_2\|_{L^2(\Omega)}^2\right)\right)\int_\Omega \left(|\nabla u_2|^2 - |\nabla u_1|^2\right) dx \le 0,
\end{aligned}$$

because M is non-decreasing. Therefore, as in [6],

$$\int_\Omega \left(\frac{f(x,u_1)}{u_1} - \frac{f(x,u_2)}{u_2}\right)(u_2^2 - u_1^2)\, dx \le 0,$$

which, together with $(f4)$ shows that $u_1 = u_2$, for a.e. $x \in \Omega$. □

Remark 2.5. We stress that the previous *variational proof* seems to work only with M non-decreasing.

3. Proof of Theorem 1.7

In this section we will prove Theorem 1.7. We have already showed that if there exists a weak solution of problem (1.1), it must be unique.

Proposition 3.1. *Assume that there exists a weak solution $u \in H_0^1(\Omega)$ of problem* (1.1)*. Then, we have $\Lambda(a_0) < 0$.*

Proof. By definition of $\Lambda(a_0)$, we have

$$\Lambda(a_0) \leq \frac{\mathcal{M}\left(\|\nabla u\|^2_{L^2(\Omega)}\right) - \int_\Omega a_0(x)u^2}{\|u\|^2_{L^2(\Omega)}}; \tag{3.1}$$

moreover, since $u \gneqq 0$ in Ω (as u is a weak solution of problem (1.1), see Definition 1.3), by definition of a_0 and assumption $(f4)$ we also have

$$\int_\Omega f(x,u)u\,dx < \int_\Omega a_0(x)u^2\,dx. \tag{3.2}$$

On the other hand,

$$\mathcal{M}\left(\|\nabla u\|^2_{L^2(\Omega)}\right) = \int_0^{\|\nabla u\|^2_{L^2(\Omega)}} M(s)\,ds \leq M\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\|\nabla u\|^2_{L^2(\Omega)}, \tag{3.3}$$

and hence, combining (3.1) with (3.2) and (3.3), we get

$$\Lambda(a_0) < \frac{M\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\int_\Omega \langle \nabla u, \nabla u\rangle\,dx - \int_\Omega f(x,u)u\,dx}{\|u\|^2_{L^2(\Omega)}} = 0,$$

where the last inequality follows by choosing $v = u$ in (1.4) (recall that, by assumption, u is a weak solution of problem (1.1)). This closes the proof. $\square$

Remark 3.2. We notice once again that the non-decreasing assumption made on M plays a key role in (3.3), allowing to relate (towards the proper direction) the variational definition of $\Lambda(a_0)$ and the weak formulation of (1.1).

To prove the existence of a weak solution of problem (1.1), we now follow the scheme of the proof in [6].

Proposition 3.3. *The functional $J_\mathcal{M}$ defined in* (1.5) *is sequentially weakly lower semicontinuous with respect to the H^1_0-topology.*

Proof. Assume that $u_n \to u$ weakly in $H^1_0(\Omega)$. As in [6], thanks to $(f3)$, by Fatou Lemma we have that

$$\limsup_{n\to+\infty} \int_\Omega F(x,u_n)\,dx \leq \int_\Omega F(x,u)\,dx. \tag{3.4}$$

Now, the L^2-norm is trivially w.l.s.c., and therefore we use that M is non-decreasing and continuous to get

$$\begin{aligned} \mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) &\leq \mathcal{M}\left(\liminf_{n\to+\infty}\|\nabla v_n\|^2_{L^2(\Omega)}\right) = \lim_{n\to+\infty}\mathcal{M}\left(\inf_{k\geq n}\|\nabla v_k\|^2_{L^2(\Omega)}\right) \\ &\leq \liminf_{n\to+\infty}\mathcal{M}\left(\|\nabla v_n\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{3.5}$$

where in the last step we used that

$$\inf_{k\geq n}\|\nabla v_k\|^2_{L^2(\Omega)} \leq \|\nabla v_k\|^2_{L^2(\Omega)},$$

for every $k \geq n$. Combining (3.4) and (3.5) we get the desired conclusion. $\square$

Proposition 3.4. *Assume that $\Lambda(a_\infty) > 0$. Then, the functional $J_\mathcal{M}$ defined in* (1.5) *is coercive on $H^1_0(\Omega)$.*

Proof. Let us assume by contradiction that there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$\|u_n\|_{H_0^1(\Omega)} \to +\infty, \quad \text{as } n \to +\infty$$

and $J_{\mathcal{M}}(u_n) \le C$, for some $C > 0$. Exploiting the growth assumption $(f3)$, we get

$$\frac{1}{2}\mathcal{M}\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right) \le C + \int_\Omega F(x, u_n)\, dx \le C + 2c\int_\Omega (1 + u_n^2)\, dx. \tag{3.6}$$

On the other hand, by definition of $\mathcal{M}$ and using the fact that M is non-decreasing, we have that

$$\begin{aligned} \mathcal{M}\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right) &= \int_0^{\|\nabla u_n\|^2_{L^2(\Omega)}} M(s)\, ds \ge \int_1^{\|\nabla u_n\|^2_{L^2(\Omega)}} M(s)\, ds \\ &\ge M(1)\left(\|\nabla u_n\|^2_{L^2(\Omega)} - 1\right) \to +\infty, \end{aligned} \tag{3.7}$$

as $n \to +\infty$. Combining (3.6) and (3.7), we can then define $t_n := \|u_n\|_{L^2(\Omega)}$ which is such that

$$t_n \to +\infty, \quad \text{as } n \to +\infty.$$

We further define the sequence $v_n := \frac{u_n}{t_n}$ for which the following holds:

$$\|v_n\|_{L^2(\Omega)} = 1, \quad \|u_n\|_{H_0^1(\Omega)} \le C;$$

as a consequence, since $H_0^1(\Omega)$ is a Hilbert space, we can infer the existence of a function $v \in H_0^1(\Omega)$ such that v_n converges to v weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere in Ω. Moreover, $\|v\|_{L^2(\Omega)} = 1$.

We now turn to estimate (3.6), where we now write $u_n = t_n v_n$ and we further divide each side by t_n^2, finding

$$\frac{1}{2t_n^2}\int_0^{t_n^2\|\nabla v_n\|^2_{L^2(\Omega)}} M(s)\, ds \le \frac{1}{t_n^2}\int_\Omega F(x, t_n v_n)\, dx + \frac{C}{t_n^2}.$$

With a change of variable ($s = t_n^2\sigma$), the left hand side becomes

$$\frac{1}{2t_n^2}\int_0^{t_n^2\|\nabla v_n\|^2_{L^2(\Omega)}} M(s)\, ds = \frac{1}{2}\int_0^{\|\nabla v_n\|^2_{L^2(\Omega)}} M(t_n^2\sigma)\, d\sigma.$$

Since $t_n \to +\infty$ as $n \to +\infty$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\int_0^{\|\nabla v_n\|^2_{L^2(\Omega)}} M(t_n^2\sigma)\, d\sigma \ge \frac{1}{2}\int_0^{\|\nabla v_n\|^2_{L^2(\Omega)}} M(\sigma)\, d\sigma = \frac{1}{2}\mathcal{M}\left(\|\nabla v_n\|^2_{L^2(\Omega)}\right)$$

for every $n \ge n_0$. This implies that

$$\frac{1}{2}\mathcal{M}\left(\|\nabla v_n\|^2_{L^2(\Omega)}\right) \le \int_\Omega \frac{F(x, t_n v_n)}{t_n^2}\, dx + \frac{C}{t_n^2}, \quad \text{for every } n \ge n_0. \tag{3.8}$$

Now, by arguing exactly as in [6] (proof of (24)) (which is legitimate, since assertion (24) in [6] only concerns F), we also have that

$$\limsup_{n\to+\infty} \int_{\{v>0\}} \frac{F(x, t_n v_n^+)}{t_n^2}\, dx \le \frac{1}{2}\int_{\{v>0\}} a_\infty(x) v^2\, dx.$$

Thus, taking the lim inf in (3.8), and recalling (3.5), we get

$$
\begin{aligned}
0 &\geq \liminf_{n\to+\infty}\left(\frac{1}{2}\mathcal{M}\left(\|\nabla v_n\|^2_{L^2(\Omega)}\right) - \int_\Omega \frac{F(x,t_nv_n)}{t_n^2}\,dx\right) \\
&\geq \frac{1}{2}\left(\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) - \int_{\{v>0\}} a_\infty(x)v^2\,dx\right) \\
&\geq \frac{1}{2}\left(\mathcal{M}\left(\|\nabla v^+\|^2_{L^2(\Omega)}\right) - \int_{\{v>0\}} a_\infty(x)(v^+)^2\,dx\right) \\
&\geq \frac{\Lambda(a_\infty)}{2}\|v^+\|^2_{L^2(\Omega)} \geq 0,
\end{aligned}
\tag{3.9}
$$

where the latter is consequence of the assumption made on $\Lambda(a_\infty)$, while the third inequality is due to the non-decreasing behaviour of M.

With (3.9) at hand, we can easily complete the proof of the proposition: indeed, since we are assuming $\Lambda(a_\infty) > 0$, from (3.9) we derive that $v^+ \equiv 0$ a.e. in Ω; from this, since all the inequalities in (3.9) are actually *equalities*, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(\mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) - \int_{\{v>0\}} a_\infty(x)v^2\,dx\right) = 0 \\
&\qquad\Longleftrightarrow\ \mathcal{M}\left(\|\nabla v\|^2_{L^2(\Omega)}\right) = 0 \ \Longleftrightarrow\ \|\nabla v\|^2_{L^2(\Omega)} = 0,
\end{aligned}
$$

where we have also used the fact that $M(s) > 0$ for all $s > 0$. Summing up, we conclude that $v = 0$ a.e. in Ω, which is in contradiction with the fact that

$$\|v\|_{L^2(\Omega)} = 1.$$

This ends the proof. □

Proposition 3.5. *Assume that* $\Lambda(a_0) < 0$. *Then, there exists* $\varphi \in H^1_0(\Omega)$ *such that*

$$J_{\mathcal{M}}(\varphi) < 0.$$

Proof. By definition of $\Lambda(a_0)$, there exists $\varphi \in H^1_0(\Omega)$, $\varphi \not\equiv 0$ such that

$$\mathcal{M}\left(\|\nabla\varphi\|^2_{L^2(\Omega)}\right) - \int_{\{\varphi\neq 0\}} a_0(x)\varphi^2\,dx < 0. \tag{3.10}$$

We then claim that we can assume $\varphi \in L^\infty(\Omega)$ and $\varphi \geq 0$ a.e. in Ω. In fact, taking into account that $|\varphi| \in H^1_0(\Omega)\setminus\{0\}$ and $\nabla|\varphi| = \nabla\varphi$ a.e. in Ω (recall that $\nabla\varphi = 0$ a.e. on the set $\{\varphi = 0\}$), from (3.10) we find

$$\mathcal{M}\left(\|\nabla|\varphi|\|^2_{L^2(\Omega)}\right) - \int_{\{\varphi\neq 0\}} a_0(x)\varphi^2\,dx < 0,$$

and thus we can assume $\varphi \geq 0$ (by possibly replacing φ with $|\varphi|$). As regards the assumption $\varphi \in L^\infty(\mathbb{R}^n)$, for every $k \in \mathbb{N}$ we define

$$\varphi_k = \min\{\varphi, k\}.$$

Since $0 \leq \varphi_k \leq \varphi$, we have $\varphi_k \in H^1_0(\Omega)$; moreover, since

$$\|\nabla\varphi_k\|_{L^2(\Omega)} \leq \|\nabla\varphi\|_{L^2(\Omega)}$$

(which is a trivial consequence of the identity $\nabla\varphi_k = \nabla\varphi \cdot \mathbf{1}_{\{\varphi<k\}}$ a.e. in Ω) and since the function M is non-decreasing, again from (3.10) we get

$$\mathcal{M}\left(\|\nabla\varphi_k\|^2_{L^2(\Omega)}\right) \leq \mathcal{M}\left(\|\nabla\varphi\|^2_{L^2(\Omega)}\right) < \int_{\{\varphi\neq 0\}} a_0(x)\varphi^2\,dx.$$

On the other hand, since a_0 is bounded from below (see (1.2)), we are entitled to apply the Fatou Lemma, thus obtaining

$$\int_{\{\varphi\neq 0\}} a_0(x)\varphi^2 \leq \liminf_{k\to+\infty}\int_{\{\varphi\neq 0\}} a_0(x)\varphi_k^2;$$

thus, we can find $k>0$ large enough so that

$$\mathcal{M}\left(\|\nabla\varphi_k\|^2_{L^2(\Omega)}\right) < \int_{\{\varphi\neq 0\}} a_0(x)\varphi_k^2 = \int_{\{\varphi_k\neq 0\}} a_0(x)\varphi_k^2.$$

Summing up, by replacing φ with φ_k, we can also assume $\varphi\in L^\infty(\Omega)$.

Now we have established the claim, we can proceed with the proof of the proposition. To this end we observe that, arguing as in [6], once again because it involves only the function F, we have the following estimate

$$\liminf_{\varepsilon\to 0^+}\int_\Omega \frac{F(x,\varepsilon\varphi)}{\varepsilon^2}\,dx \geq \frac{1}{2}\int_{\{\varphi\neq 0\}} a_0(x)\varphi^2\,dx.$$

Therefore, there exist $0<\varepsilon_0<<1$ such that

$$\mathcal{M}\left(\|\nabla\varphi\|^2_{L^2(\Omega)}\right) - \int_\Omega \frac{F(x,\varepsilon\varphi)}{\varepsilon^2}\,dx < 0,$$

for every $\varepsilon\in(0,\varepsilon_0)$. On the other hand,

$$\begin{aligned}\mathcal{M}\left(\|\nabla\varphi\|^2_{L^2(\Omega)}\right) &= \int_0^{\|\nabla\varphi\|^2_{L^2(\Omega)}} M(s)\,ds \overset{(\sigma=\varepsilon^2 s)}{=} \int_0^{\|\nabla(\varepsilon\varphi)\|^2_{L^2(\Omega)}} M\left(\frac{\sigma}{\varepsilon^2}\right)\frac{d\sigma}{\varepsilon^2} \\ &\geq \frac{1}{\varepsilon^2}\mathcal{M}\left(\|\nabla(\varepsilon\varphi)\|^2_{L^2(\Omega)}\right),\end{aligned} \tag{3.11}$$

where in the last step we used the monotonicity of M, together with the fact that $\frac{1}{\varepsilon^2}>1$ (being $\varepsilon<\varepsilon_0<1$). In particular, this implies that

$$\frac{J_{\mathcal{M}}(\varepsilon\varphi)}{\varepsilon^2} < 0, \qquad \text{for every } \varepsilon\in(0,\varepsilon_0).$$

This closes the proof. □

By combining all the results established so far, we can prove Theorem 1.7.

Proof (of Theorem 1.7). We prove assertions (1)-(2) separately.
(1) If there exists a weak solution $u\in H^1_0(\Omega)\setminus\{0\}$ of problem (1.1), we know from Proposition 2.4 that this u is unique; moreover, from Propositions 2.1-2.2 we infer that $u\in L^\infty(\Omega)$ and $u>0$ (a.e.) in Ω. Finally, by Proposition 3.1 we get

$$\Lambda(a_0)<0,$$

and this ends the proof of assertion (1).
(2) If $\Lambda(a_0)<0<\Lambda(a_\infty)$, we derive from Propositions 3.3-3.4 that J_M possesses a minimum point, say $v\in H^1_0(\Omega)$; on the other hand, since $J_M(0)=0$, it follows

from Proposition 3.5 that $v \not\equiv 0$ in Ω. Setting $u = v^+ \gneqq 0$, we then have

$$\begin{aligned} J_M(u) = J_M(v^+) &= \frac{1}{2}\mathcal{M}\big(\|\nabla v^+\|_{L^2(\Omega)}\big) - \int_\Omega F(x, v^+)\,dx \\ &\leq \frac{1}{2}\mathcal{M}\big(\|\nabla v\|_{L^2(\Omega)}\big) - \int_\Omega F(x, v^+)\,dx \\ &\text{(by Remark 1.4)} \\ &\leq J_M(v) = \min_{\varphi \in H_0^1(\Omega)} J_M(\varphi), \end{aligned}$$

and this shows that u is also a minimum point for J_M. Recalling that any minimum point of J_M is a weak solution of the PDE driving (1.1), we can then conclude that u is a weak solution of problem (1.1) (as $u \geqq 0$). This ends the proof. □

Remark 3.6. The assumption that M is non-decreasing has been crucially exploited in several steps. We briefly summarize where and what is the difference with respect to [6], where $M(s) = 1$ and $\mathcal{M}(s) = s$.

- Proposition 2.4 to get uniqueness;
- (3.5) to get the weakly lower semicontinuity of $J_{\mathcal{M}}$. In [6] this follows because of w.l.s.c. of the norm;
- (3.3) to prove that $\Lambda(a_0) < 0$, (3.7) to get the coercivity of $J_{\mathcal{M}}$ and (3.11). In [6], these steps naturally follows because of the linear nature of the operator, see Remark 1.5.

On the other hand, the *two-sided bound* in assumption $(f3)$ is needed *only* to ensure that the functional J_M is differentiable, so that its minimum points are actually weak solutions of the PDE driving (1.1); in particular, *all the results in this paper* not exploiting this fact hold by requiring the *one-side bound*

$$(\star) \qquad\qquad f(x,t) \leq c(1+t),$$

as in the paper by Brezis-Oswald [6]. The main issue in establishing Theorem 1.7 under the weak assumption $(\star)$ in the present context is to show that any minimum point u of J_M can be assumed to be *globally bounded*, thus ensuring that u is a weak solution of the PDE driving (1.1); this is done in [6] via a suitable truncation argument, which seems not applicable when $M(s) \not\equiv 1$.

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