DE GRUYTER Adv. Nonlinear Stud. 2020; 20(1): 213–241

Research Article

Fausto Ferrari and Antonio Vitolo*

Regularity Properties for a Class of Non-uniformly Elliptic Isaacs Operators

https://doi.org/10.1515/ans-2019-2069
Received September 11, 2019; revised October 14, 2019; accepted October 15, 2019

Abstract: We consider the elliptic differential operator defined as the sum of the minimum and the maximum eigenvalue of the Hessian matrix, which can be viewed as a degenerate elliptic Isaacs operator, in dimension larger than two. Despite of nonlinearity, degeneracy, non-concavity and non-convexity, such an operator generally enjoys the qualitative properties of the Laplace operator, as for instance maximum and comparison principles, ABP and Harnack inequalities, Liouville theorems for subsolutions or supersolutions. Existence and uniqueness for the Dirichlet problem are also proved as well as local and global Hölder estimates for viscosity solutions. All results are discussed for a more general class of weighted partial trace operators.

Keywords: Weighted Partial Trace Operators, Bellman–Isaacs Equations, Viscosity Solutions, Global Hölder Estimates

MSC 2010: 35J60, 35J70, 35J25, 35B50, 35B53, 35B65, 35D40

Communicated by: Enrico Valdinoci

1 Introduction and main results

In this paper we investigate the properties of weighted partial trace operators

$$\mathcal{M}_{\mathbf{a}} := \sum_{i=1}^{n} a_i \lambda_i(X), \tag{1.1}$$

where $\lambda_i(X)$ are the eigenvalues of $X \in \mathbb{S}^n$, the set of $n \times n$ real symmetric matrices, in increasing order, that is

$$\lambda_1(X) \leq \cdots \leq \lambda_n(X),$$

and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is an n-ple of non-negative coefficients a_i such that $a_j > 0$ for at least one $j \in \{1, \ldots, n\}$.

The class $\overline{\mathcal{A}}$ of such operators includes the partial trace operators

$$\mathcal{P}_k^-(X) = \sum_{i=1}^{k} \lambda_i(X), \quad \mathcal{P}_k^+(X) = \sum_{i=n-k+1}^{n} \lambda_i(X), \tag{1.2}$$

considered by Harvey and Lawson [49, 50] and Caffarelli, Li and Nirenberg [22, 23].

Here we introduce the subclass $\mathcal{A}$, characterized by non-negative coefficients a_i such that $a_1 > 0$ and $a_n > 0$, which in some sense complements the set of operators $\mathcal{P}_k^{\pm}(X)$ with $k < n$. In fact, the prototype of $\mathcal{A}$

***Corresponding author: Antonio Vitolo,** Dipartimento di Ingegneria Civile, Università di Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano (SA); and Istituto Nazionale di Alta Matematica, INdAM – GNAMPA, Italy, e-mail: vitolo@unisa.it
Fausto Ferrari, Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy, e-mail: fausto.ferrari@unibo.it

is the min-max operator

$$\mathcal{M}(X) := \lambda_1(X) + \lambda_n(X).$$

As we will see later, $\mathcal{M}$ can be in fact viewed as a degenerate elliptic Isaacs operator (for $n \geq 2$) whereas $\mathcal{P}_k^{\pm}(X)$ results in a degenerate elliptic Bellman operator (for $k < n$).

Of course, the case $n = 2$ is by far well known, because $\mathcal{M}$ reduces to the classical Laplace operator. However, in higher dimension, namely for $n > 2$, the operator ceases to be uniformly elliptic, it becomes a fully nonlinear non-convex degenerate elliptic operator. Nonetheless, we will see, rather surprisingly, that it retains many properties of the Laplace operator.

It also worth noticing that the operators $\mathcal{M}_{\mathbf{a}}$ of the smaller subclass $\underline{\mathcal{A}}$, characterized by weights $a_i > 0$ for all $i = 1, \dots, n$, are uniformly elliptic, as we will see in Section 2.

After introducing our main results we shall further come back to the original motivation for studying the operators of $\overline{\mathcal{A}}$, and in particular the subclass $\mathcal{A}$.

A good number of results will depend on the dimension n and on the following two quantities, namely the minimum between the coefficients of the smallest and the greatest eigenvalue, and the arithmetic mean of the coefficients, namely

$$a^* := \min(a_1, a_n), \quad \tilde{a} := \frac{a_1 + \cdots + a_n}{n},$$

in the sense that the involved constants are uniformly bounded when a positive upper bound of the first one and a finite upper bound of the latter one are avalaible.

The constants which depend only on n, a_1, a_n and $\tilde{a}$ will be also called universal constants.

The following result is a revisitation of the bilateral Alexandroff–Bakelman–Pucci (ABP) estimate for the class $\mathcal{A}$, only depending on n and a^*.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain of diameter d. Let $f \in C(\Omega)$ be bounded in Ω. If $u \in C(\bar{\Omega})$ is a viscosity solution to the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in Ω, with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$, then*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} u^+ + \frac{C_n}{a^*} d \|f\|_{L^n(\Omega)}, \tag{1.3}$$

where $C_n > 0$ is a positive constant depending only on n.

We emphasize the following difference between estimate (1.3) and the standard ABP estimates, see for instance [47, Theorem 9.1]: the denominator of the right-hand side is $a^* = \min(a_1, a_n)$ instead of the geometric mean $\mathcal{D}^* = (a_1 \cdots a_n)^{\frac{1}{n}}$, the geometric mean of the coefficients, which would be useless in the non-uniformly elliptic case, as soon as one of the coefficients a_j is zero, while a^* is positive for the class $\mathcal{A}$.

The above result is obtained as a consequence of two unilateral ABP estimates for subsolutions (4.6) and supersolutions (4.7).

The ABP estimate stated before also underlies a corresponding Harnack inequality for the equation $\mathcal{M}_{\mathbf{a}}[u] = f$, depending on n, a^* and $\tilde{a}$, instead of the elliptic constants λ and $\Lambda \geq \lambda$, which would be ineffective in the degenerate elliptic case in which $\lambda = 0$. This Harnack inequality cannot be extended to arbitrary degenerate elliptic operators of the class $\overline{\mathcal{A}}$, and in particular it fails to hold for partial trace equations $\mathcal{P}_k^{\pm}[u] = f$ when $k < n$.

Theorem 1.2 (Harnack inequality). *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$. Let u be a viscosity solution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in the unit cube Q_1 such that $u \geq 0$ in Q_1, where f is continuous and bounded. Then*

$$\sup_{Q_{1/2}} u \leq C\Big(\inf_{Q_{3/4}} u + \|f\|_{L^n(Q_1)}\Big),$$

where C is a positive constant depending only on n, a^* and $\tilde{a}$.

We prove Theorem 1.2 via two inequalities for subsolutions and non-negative supersolutions, known in literature respectively as the local maximum principle (Theorem 5.1) and the weak Harnack inequality (Theorem 5.2), suitably adapted to this framework, by comparison with Pucci extremal operators.

From the Harnack inequality, the interior C^α estimates of Theorem 5.3 in Section 5 follow with a universal exponent $\alpha \in (0, 1)$, in the same way of the uniformly elliptic case [25].

Here, in Lemma 5.4, we get boundary Hölder estimates assuming for Ω a uniform exterior sphere property, with radius $R > 0$:

(S) *for all $y \in \partial\Omega$ there is a ball B_R of radius R such that $y \in \partial B_R$ and $\overline{\Omega} \subset \overline{B}_R$.*

We obtain the following estimates for the Hölder seminorm $[u]_{\gamma,\Omega}$. See the notation (5.3) in Section 5.

Theorem 1.3 (Global Hölder estimates). *Let $u \in C(\overline{\Omega})$ be a viscosity solution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in a bounded domain Ω. We assume that with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$ and f is continuous and bounded in Ω. Let also $\alpha \in (0, 1)$ be the exponent of the interior C^α estimates.*

(i) *Suppose that Ω satisfies a uniform exterior sphere condition* (S) *with radius $R > 0$. If $u = g$ on $\partial\Omega$ with $g \in C^\beta(\partial\Omega)$ and $\beta \in (0, 1]$, then $u \in C^\gamma(\overline{\Omega})$ with $\gamma = \min(\alpha, \frac{\beta}{2})$, and*

$$[u]_{\gamma,\Omega} \le C(\|g\|_{C^\beta(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}), \tag{1.4}$$

where C is a positive constants depending only on n, a^*, $\tilde{a}_n$, R, L and β.

(ii) *Suppose in addition that Ω has a uniform Lipschitz boundary with Lipschitz constant L. If $g \in C^{1,\beta}(\partial\Omega)$ with $\beta \in [0, 1)$, then $u \in C^{(1+\beta)/2}(\overline{\Omega})$, where $\gamma = \min(\alpha, \frac{1}{2}(\beta + 1))$, and*

$$[u]_{\frac{1}{2}(1+\beta),\Omega} \le C(\|g\|_{C^{1,\beta}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}). \tag{1.5}$$

A global estimate for the Hölder norm $\|u\|_{C^{0,\gamma}(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{\gamma,\Omega}$ can be obtained combining the above estimates with the uniform estimate of Corollary 3.3.

In some cases, we can obtain an explicit interior Hölder exponent. For instance, in the case of asymmetric distributions of weights, concentrated on the smallest or the largest eigenvalue, as for the upper and lower partial trace operators $\mathcal{P}_k^\pm$, $k < n$, see Lemma 5.5. The result depends in fact on the smallness of the quotients $\frac{\hat{a}_1}{a_1}$ and $\frac{\hat{a}_n}{a_n}$ (see Section 3.3), as it can be seen in the statement below.

Theorem 1.4. *Let $u \in C(\overline{\Omega})$ be a viscosity solution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in a bounded domain Ω, where f is continuous and bounded. Suppose $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ with $a_1 \ge \hat{a}_1$, resp. with $a_n \ge \hat{a}_n$. Then the global Hölder estimates of Theorem 1.3 hold, namely* (1.4) *in case* (i) *and* (1.5) *in case* (ii), *with*

$$\alpha = \max\Big(1 - \frac{\hat{a}_1}{a_1}, 1 - \frac{\hat{a}_n}{a_n}\Big).$$

In particular, we deduce the following C^α estimates.

(i) *Suppose that Ω satisfies a uniform exterior sphere condition* (S) *with radius $R > 0$. If $u = g$ on $\partial\Omega$ with $g \in C^{2\alpha}(\partial\Omega)$ and $\alpha \in (0, \frac{1}{2}]$, then $u \in C^\alpha(\overline{\Omega})$, and*

$$[u]_{\alpha,\Omega} \le C(\|g\|_{C^{2\alpha}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}), \tag{1.6}$$

where C is a positive constant depending only on n, a_1, $\hat{a}_1$, a_n, $\hat{a}_n$, $\tilde{a}$, R and α.

(ii) *Suppose in addition that Ω has a uniform Lipschitz boundary with Lipschitz constant L. If $g \in C^{1,2\alpha-1}(\partial\Omega)$ with $\alpha \in [\frac{1}{2}, 1]$, then*

$$[u]_{\alpha,\Omega} \le C(\|g\|_{C^{1,2\alpha-1}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}), \tag{1.7}$$

where C is a positive constant also depending on L.

The optimal regularity of solutions in the case of degenerate, non-uniform ellipticity, is an open problem.

Partial answers are contained for instance in [65] for the special case $a_1 = 1$, $a_i = 0$ for $i = 2, \dots, n$, as mentioned in the sequel, [10, 52] for other kind of singular or degenerate elliptic operators and [37, 40] for non-commutative structures.

Concerning higher regularity, one could borrow the techniques of [11, 19, 24, 27, 28, 53, 54, 56, 57, 73, 75], which however do not seem at the moment directly applicable in the more general non-uniformly elliptic setting.

It is remarkable the particular case of the interior $C^{1,\alpha}$ regularity proved in [65] for the equation $\lambda_1[u] = f(x)$ with $C^{1,\alpha}$ boundary data.

Further aspects of the qualitative theory, like the strong maximum principle and Liouville theorems, will be discussed in the last sections of the paper. New results for operators $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ will be shown there, depending on the relative magnitude of a_1 and a_n, and their complements $\hat{a}_1$ and $\hat{a}_n$ with respect to $|\mathbf{a}| = a_1 + \cdots + a_n$, see Section 3.3.

Turning to the motivations about the importance of this research, we recall that the partial trace operators $\mathcal{P}_k^+(X)$ are degenerate elliptic operators, which can be represented as Bellman operators

$$\mathcal{P}_k^+(X) = \sup_{W\in\mathcal{G}_k} \operatorname{Tr}(X_W), \quad \mathcal{P}_k^-(X) = \inf_{W\in\mathcal{G}_k} \operatorname{Tr}(X_W), \tag{1.8}$$

where $\mathcal{G}_k$ is the Grassmanian of the k-dimensional subspace W of $\mathbb{R}^n$ and X_W is tha matrix of the quadratic form associated to X restricted to W, see [49].

Upper and lower partial trace operators arise in geometric problems of mean partial curvature considered by Wu [80] and Sha [71, 72]. Following the interest generated by the previous works, a number of papers has been devoted to the properties of these operators, we recall for instance [3, 26, 46, 79].

On the other hand, it is also worth noticing that Bellman equations arise in stochastic control problems, see Krylov [58], Fleming and Rishel [42], Fleming and Soner [43], Fleming and Souganidis [44] and the references therein.

As well as the partial trace operators $\mathcal{P}_k^\pm$ with $k < n$ constitute a model for degenerate elliptic Bellman operators, the min-max operator $\mathcal{M}$ provides for $n \geq 3$ a prototype of degenerate elliptic Isaacs operators by the representation

$$\mathcal{M}(X) = \sup_{|\xi|=1} \inf_{|\eta|=1} \operatorname{Tr}(X_{\xi,\eta}), \tag{1.9}$$

where $\operatorname{Tr}(X_{\xi,\eta})$ is the trace the matrix $X_{\xi,\eta}$ of the quadratic form associated to X restricted to $L(\xi,\eta)$, the subspace of $\mathbb{R}^n$ spanned by ξ and η.

The alternative representation

$$\mathcal{M}(X) = \max_{|\xi|=1} \langle X\xi, \xi\rangle + \min_{|\xi|=1} \langle X\xi, \xi\rangle$$

suggests the relationship between $\mathcal{M}(X)$ and stochastic zero-sum, two-players differential games and Isaacs equations, for which we refer for instance to [41, 43, 64] and the references therein and to [16, 17] for more recent contributions.

Following the main stream of the mean value properties of solution to linear equations, as well as in the case of the ∞-Laplacian, it is also worth to be remarked that whenever u is C^2, the following expansion yields:

$$u_1^\varepsilon(x) \equiv \min_{|\xi|=1} \frac{u(x+\varepsilon\xi) + u(x-\varepsilon\xi)}{2} = u(x) + \frac{\varepsilon^2}{2}\lambda_1(x) + o(\varepsilon^2),$$
$$u_2^\varepsilon(x) \equiv \max_{|\eta|=1} \frac{u(x+\varepsilon\eta) + u(x-\varepsilon\eta)}{2} = u(x) + \frac{\varepsilon^2}{2}\lambda_n(x) + o(\varepsilon^2).$$

As a consequence, if we consider a continuous function u, the operator given by the limit

$$\lim_{\epsilon\to 0} \frac{2}{\epsilon^2}(u_{\varepsilon,1}(x) + u_{\varepsilon,2}(x) - 2u(x)),$$

whenever it exists, may be considered as the weak version of our operator $\mathcal{M}$. For an almost compete list of references from this point of view, see [59, 63] for the p-Laplace equation, as well as [38, 39] for further applications to non-commutative fields where a lack of ellipticity occurs.

The paper is organized as follows. In Section 2 we introduce the main definitions about elliptic operators and viscosity solutions. In Section 3 we discuss in detail the properties of the weighted partial trace operators, in particular $\mathcal{M}$. We show a comparison principle, an existence and uniqueness theore, and compute the radial solutions. In Section 4 we prove Theorem 4.2. In Section 5 we show the Harnack inequality, interior and boundary Hölder estimates. We also discuss, in Section 6, the strong maximum principle via both the Hopf boundary point lemma and the Harnack inequality, showing suitable counterexamples. Finally, in Section 7, we also prove Liouville theorems and an unilateral Liouville property with the Hadamard's three circles theorem.

2 General Preliminaries

This section is organized in some subsections, mainly for introducing common notation about viscosity theory of elliptic nonlinear PDEs, see Sections 2.1 and 2.5. In Sections 2.2 and 2.3 we introduce our class of operators $\mathcal{A}$ and in particular discuss the min-max operator $\mathcal{M}$ showing by counterexamples that it is nonlinear, non-convex and non-uniformly elliptic. In Section 2.4 we discuss a comparison result with the partial trace operators operator $\mathcal{P}_k^{\pm}$.

2.1 Ellipticity and Viscosity Solutions

We start recalling some ellipticity notions. Let $\mathcal{S}^n$ be the set of $n \times n$ symmetric matrices with real entries, partially ordered with the relationship $X \leq Y$ if and only if $Y - X$ is semidefinite positive.

A fully nonlinear operator, that is a mapping $\mathcal{F} : \mathcal{S}^n \to \mathbb{R}$, is said degenerate elliptic if

$$X \leq Y \implies \mathcal{F}(X) \leq \mathcal{F}(Y), \tag{2.1}$$

and uniformly elliptic if

$$X \leq Y \implies \lambda \operatorname{Tr}(Y - X) \leq \mathcal{F}(Y) - \mathcal{F}(X) \leq \Lambda \operatorname{Tr}(Y - X), \tag{2.2}$$

for positive constants λ and Λ, called ellipticity constants. Note indeed that, by the left-hand side inequality in (2.2), a uniformly elliptic operator $\mathcal{F}$ satisfies (2.1), and so it is degenerate elliptic. The uniform ellipticity also implies the continuity of the mapping $\mathcal{F} : \mathcal{S}^n \to \mathbb{R}$. In what follows we also assume that $\mathcal{F}$ is a continuous mapping even in the degenerate elliptic case.

Suppose now $X \leq Y$. It is plain that $\operatorname{Tr}(Y - X) \geq 0$. Suppose in addition $\mathcal{F}(Y) = \mathcal{F}(X)$. If $\mathcal{F}$ is uniformly elliptic, in view of the left-hand side of (2.2), we also have $\operatorname{Tr}(Y - X) \leq 0$, so that $\operatorname{Tr}(Y - X) = 0$. Then $Y = X$. In other words, $\mathcal{F}$ is strictly increasing on ordered chains of $\mathcal{S}^n$.

The class of uniformly elliptic operators with given ellipticity constants λ and Λ is bounded by two estremal operators, the maximal and minimal Pucci operator, which are in turn uniformly elliptic with the same ellipticity constants, respectively:

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda \operatorname{Tr}(X^+) - \lambda \operatorname{Tr}(X^-),$$
$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda \operatorname{Tr}(X^+) - \Lambda \operatorname{Tr}(X^-),$$

where $X = X^+ - X^-$ is the unique decomposition of $X \in \mathcal{S}^n$ as difference of semidefinite positive matrices X^+ and X^- such that $X^+X^- = 0$.

In view of this definition, the uniformly ellipticity (2.2) of $\mathcal{F}$ can equivalently be stated as

$$\mathcal{M}^-_{\lambda,\Lambda}(Y - X) \leq \mathcal{F}(Y) - \mathcal{F}(X) \leq \mathcal{M}^+_{\lambda,\Lambda}(Y - X) \quad \text{for all } X, Y \in \mathcal{S}^n.$$

From this it also follows that, if $\mathcal{F}$ is uniformly elliptic and $\mathcal{F}(0) = 0$, then

$$\mathcal{M}^-_{\lambda,\Lambda}(X) \leq \mathcal{F}(X) \leq \mathcal{M}^+_{\lambda,\Lambda}(X) \quad \text{for all } X \in \mathcal{S}^n, \tag{2.3}$$

which shows the extremality of Pucci operators.

Throughout this paper we will assume in fact

$$\mathcal{F}(0) = 0.$$

Of course, the results can be applied, in the case $\mathcal{F}(0) \neq 0$, to the operator $\mathcal{G}(X) = \mathcal{F}(X) - \mathcal{F}(0)$.

Let Ω be an open set of $\mathbb{R}^n$. A fully nonlinear operator $\mathcal{F}$ acts on $u \in C^2(\Omega)$ through the Hessian matrix D^2u setting

$$\mathcal{F}[u](x) = \mathcal{F}(D^2u(x)).$$

Let f be a function defined in Ω. A solution $u \in C^2(\Omega)$ of the equation $\mathcal{F}[u] = f$ is called a classical solution, as well as classical subsolution or supersolution of $F[u] = f$ if $\mathcal{F}(D^2u(x)) \ge f(x)$ or $\mathcal{F}(D^2u(x)) \le f(x)$ for every $x \in \Omega$, respectively. For instance, if $\mathcal{F}(X) = \mathrm{Tr}(X)$ and $f(x)$ is a continuous function, then $\mathcal{F}[u] = \Delta u$ is the Laplacian and the equation $\mathcal{F}[u] = f$ is the Poisson equation $\Delta u = f$.

Let $\mathcal{F}$ be a degenerate elliptic operator. We can solve the equation $\mathcal{F}[u] = f(x)$ in a weaker sense, namely in the viscosity sense. We are essentially concerned in this paper with pure second-order operators $\mathcal{F}[u] = \mathcal{F}(D^2u)$. We refer to [25] and [33] for general operators, also depending on $x \in \Omega$, u and the gradient Du, and to [49] for a geometric interpretation of viscosity solutions.

We briefly recall what it means to solve the equation $\mathcal{F}[u] = f$ introducing sub/superjets basic notions. See [32, 33].

Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^n$, and $u : \mathcal{O} \to \mathbb{R}$. The second-order superjet $J^{2,+}_{\mathcal{O}}u(x_0)$ and subjet $J^{2,-}_{\mathcal{O}}u(x_0)$ of u at $x_0 \in \mathcal{O}$ are respectively the sets

$$J^{2,+}_{\mathcal{O}}u(x_0) = \{(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x) \le u(x_0) + \langle \xi, x - x_0\rangle + \tfrac{1}{2}\langle X(x - x_0), (x - x_0)\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\}$$

and

$$J^{2,-}_{\mathcal{O}}u(x_0) = \{(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x) \ge u(x_0) + \langle \xi, x - x_0\rangle + \tfrac{1}{2}\langle X(x - x_0), (x - x_0)\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\}.$$

We denote by $\mathrm{usc}(\mathcal{O})$ and $\mathrm{lsc}(\mathcal{O})$ the set of upper and lower semicontinuous functions in $\mathcal{O}$, respectively. If $u \in \mathrm{usc}(\mathcal{O})$, then u is a viscosity subsolution of a fully nonlinear elliptic equation $\mathcal{F}[u] = f$ if

$$\mathcal{F}(X) \ge f(x) \quad \text{for all } x \in \mathcal{O} \text{ and all } (\xi, X) \in J^{2,+}_{\mathcal{O}}u(x).$$

If $u \in \mathrm{lsc}(\mathcal{O})$, then u is a viscosity supersolution of the same equation if

$$\mathcal{F}(X) \le f(x) \quad \text{for all } x \in \mathcal{O} \text{ and all } (\xi, X) \in J^{2,-}_{\mathcal{O}}u(x).$$

A viscosity solution of the equation $\mathcal{M}[u] = f$ is both a subsolution and a supersolution $u \in C(\mathcal{O})$.

It is worth noticing that classical solutions are viscosity solutions. Viceversa, viscosity solutions of class C^2 are in turn classical solutions. The same holds for subsolutions and supersolutions.

2.2 The Operator Class $\mathcal{A}$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis in $\mathbb{R}^n$ such that $(\mathbf{e}_i)_j = \delta_{ij}$ for $i, j = 1, \dots, n$, and let $\lambda_i(X)$, $i = 1, \dots, n$, be the eigenvalues of $X \in \mathcal{S}^n$ in non-decreasing order.

Let $\mathbf{a} = (a_1, \dots, a_n) = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. We consider the class of degenerate elliptic weighted trace operators

$$\overline{\mathcal{A}} = \Big\{\mathcal{M}_{\mathbf{a}} : \underline{a} \equiv \min_i a_i \ge 0,\ \overline{a} \equiv \max_i a_i > 0\Big\},$$

where

$$\mathcal{M}_{\mathbf{a}}(X) = a_1\lambda_1(X) + \cdots + a_n\lambda_n(X) \quad \text{(see (1.1))}.$$

We observe that $\overline{\mathcal{A}}$ contains both uniformly and non-uniformly elliptic operators. In particular, all previously considered operators belong to this class with a suitable representation:

$$\mathrm{Tr}(X) = \mathcal{M}_{\mathbf{e}_1+\dots+\mathbf{e}_n}(X), \qquad \mathcal{M}(X) = \mathcal{M}_{\mathbf{e}_1+\mathbf{e}_n}(X),$$
$$\mathcal{P}^+(X) = \mathcal{M}_{\mathbf{e}_{n-k+1}+\dots+\mathbf{e}_n}(X), \quad \mathcal{P}^-(X) = \mathcal{M}_{\mathbf{e}_1+\dots+\mathbf{e}_k}(X).$$

Very recently, recalling the pioneeristic paper [66], Blanc and Rossi [15] have shown that it is possible to define a game satisfying a dynamic programming principle (DPP) which leads to the Dirichlet problem

$$\begin{cases} \mathcal{M}_{\mathbf{a}}[u] = 0 & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega. \end{cases}$$

Moreover, an associated evolution problem is considered in [14].

We point out that $\mathcal{M} = \mathcal{M}_{\mathbf{e}_1+\mathbf{e}_n}$ is neither linear nor uniformly elliptic, neither concave nor convex, except when $n = 2$, as it follows from the representation (1.9) and it will be proved in the next section with suitable counterexamples.

Actually, $\mathcal{M}$ is a model of a larger class of degenerate, possibly non-uniformly elliptic operators

$$\mathcal{A} = \{\mathcal{M}_{\mathbf{a}} : \underline{a} \geq 0,\ a^* \equiv \min(a_1, a_n) > 0\},$$

which can be seen as $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_n$, where

$$\mathcal{A}_j = \{\mathcal{M}_{\mathbf{a}} : \underline{a} \geq 0,\ a_j > 0\}.$$

Setting in addition

$$\underline{\mathcal{A}} = \{\mathcal{M}_{\mathbf{a}} : \underline{a} > 0\},$$

we notice that

$$\underline{\mathcal{A}} \subset \mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_n \subset \overline{\mathcal{A}}.$$

We remark for instance that, while the min-max operator $\mathcal{M}$ belongs to $\mathcal{A}$, the partial trace operators $\mathcal{P}_k^- \in \mathcal{A}_1$ and $\mathcal{P}_k^+ \in \mathcal{A}_n$ do not belong to $\mathcal{A}$ for $k < n$.

On the other hand, every $\mathcal{M}_{\mathbf{a}} \in \underline{\mathcal{A}}$ is uniformly elliptic. In fact, if $X \leq Y$, then

$$\mathcal{M}_{\mathbf{a}}(Y) - \mathcal{M}_{\mathbf{a}}(X) = \sum_{i=1}^{n} a_i(\lambda_i(Y) - \lambda_i(X)) \geq \underline{a}\,\mathrm{Tr}(Y - X), \tag{2.4}$$

so that every $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ is degenerate elliptic. Since $X \leq Y$ also implies

$$\mathcal{M}_{\mathbf{a}}(Y) - \mathcal{M}_{\mathbf{a}}(X) = \sum_{i=1}^{n} a_i(\lambda_i(Y) - \lambda_i(X)) \leq \overline{a}\,\mathrm{Tr}(Y - X), \tag{2.5}$$

we conclude that $\mathcal{M}_{\mathbf{a}} \in \underline{\mathcal{A}}$ is uniformly elliptic with ellipticity constants $\lambda = \underline{a} \equiv \min_i a_i$ and $\Lambda = \overline{a} \equiv \max_i a_i$.

We also observe that the operators $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ are invariant by rotation, since $\mathcal{M}_{\mathbf{a}}(\mathcal{R}^T X \mathcal{R}) = \mathcal{M}(X)$ for all orthogonal matrices $\mathcal{R}$, and are positively homogeneous of degree one:

$$\mathcal{M}_{\mathbf{a}}(\rho X) = \sum_{i=1}^{n} a_i \lambda_i(\rho X) = \rho \sum_{i=1}^{n} a_i \lambda_i(X) = \rho \mathcal{M}_{\mathbf{a}}(X), \quad \rho \geq 0. \tag{2.6}$$

Next, we investigate more closely the peculiar properties of the min-max operator $\mathcal{M}(X) = \lambda_1(X) + \lambda_n(X)$.

2.3 The Min-Max Operator $\mathcal{M}$

In the previous subsection, we claimed that $\mathcal{M}$ is neither linear nor uniformly elliptic, neither concave nor convex, except for $n = 2$. This is intuitive by the representation (1.9):

$$\mathcal{M}(X) = \sup_{|\xi|=1} \inf_{|\eta|=1} \mathrm{Tr}(X_{\xi,\eta}).$$

Nonetheless, we present a few counterexamples that support the above claim.

Remark 2.1. Let us consider the matrices

$$X_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_3, \quad X_2 = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, \quad X_3 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3.$$

Then $\lambda_1(X_i) = -1$ and $\lambda_3(X_i) = 1$, so that $\mathcal{M}(X_i) = 0$ for all $i = 1, 2, 3$.

(i) The operator $\mathcal{M}$ is not linear in dimension $n \geq 3$. In fact,

$$\begin{aligned} X_1 - X_2 &= \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_3 + \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \\ &= 2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned}$$

and therefore

$$\lambda_1(X_1 - X_2) = -1, \quad \lambda_3(X_1 - X_2) = 2,$$

so that

$$\mathcal{M}(X_1) - \mathcal{M}(X_2) = 0 \neq 1 = \mathcal{M}(X_1 - X_2).$$

(ii) The operator $\mathcal{M}$ is not uniformly elliptic in dimension $n \geq 3$. In fact, we note that $X_3 \leq X_1$, and

$$\mathcal{M}(X_3) = \mathcal{M}(X_1) = 0, \quad \text{but } X_3 \neq X_1,$$

against the strictly increasing property on ordered chains observed in Section 2.1 for the uniformly elliptic case.

(iii) The operator $\mathcal{M}(X)$ is neither convex nor concave. In fact, for every $t \in [0, 1]$, it turns out that

$$\begin{aligned} tX_1 + (1-t)X_2 &= t(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_3) + (1-t)(-\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) \\ &= (2t-1)\mathbf{e}_1 \otimes \mathbf{e}_1 + (1-t)\mathbf{e}_2 \otimes \mathbf{e}_2 - t\mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{M}(tX_1 + (1-t)X_2) &= \lambda_1(tX_1 + (1-t)X_2) + \lambda_3(tX_1 + (1-t)X_2) \\ &= \min\{2t-1; -t\} + \max\{2t-1; 1-t\}, \end{aligned}$$

so that $\mathcal{M}(tX_1 + (1-t)X_2) = 1 - 2t$ for $t \in (\frac{1}{3}, \frac{2}{3})$. From this

$$\mathcal{M}(tX_1 + (1-t)X_2) \begin{cases} > 0 & \text{for } t \in (\frac{1}{3}, \frac{1}{2}), \\ < 0 & \text{for } t \in (\frac{1}{2}, \frac{2}{3}), \end{cases}$$

while it is plain that for every $t \in [0, 1]$,

$$t\mathcal{M}(X_1) + (1-t)\mathcal{M}(X_2) = 0.$$

Thus $\mathcal{M}$ is neither convex nor concave.

Since $\mathcal{M} \in \mathcal{A}$ we already know that it is homogeneous of degree one (2.6): for every $\rho \geq 0$ and for every $X \in \mathcal{S}^n$,

$$\mathcal{M}(\rho X) = \rho\mathcal{M}(X). \tag{2.7}$$

On the other hand,

$$\mathcal{M}(-X) = \lambda_1(-X) + \lambda_n(-X) = -\lambda_n(X) - \lambda_1(X) = -\mathcal{M}(X),$$

and therefore (2.7) continues to hold for $\rho < 0$.

The next remark contains a few comments on the representation (1.9).

Remark 2.2. The operator $\mathcal{M}(X)$ can be put in the form

$$\mathcal{M}(X) = \sup_{|\xi|=1} \inf_{\substack{|\eta|=1 \\ \eta \perp \xi}} \mathrm{Tr}(X_{\xi,\eta}). \tag{2.8}$$

In order to prove this, we start observing that plainly

$$\mathcal{M}(X) = \sup_{|\xi|=1} \inf_{|\eta|=1} (\langle X\xi, \xi\rangle + \langle X\eta, \eta\rangle) \leq \sup_{|\xi|=1} \inf_{\substack{|\eta|=1 \\ \eta \perp \xi}} \mathrm{Tr}(X_{\xi,\eta}).$$

To have also the reverse inequality, and so (2.8), we observe that the representative matrix $X_{\xi,\eta}$ of the quadratic form associated to X restricted to $L(\xi, \eta)$, the subspace of $\mathbb{R}^n$ spanned by directions ξ and η, has trace

$$\mathrm{Tr}(X_{\xi,\eta}) = \langle X\xi, \xi\rangle + \langle X\eta, \eta\rangle,$$

and thus

$$\sup_{|\xi|=1} \inf_{\substack{|\eta|=1 \\ \eta \perp \xi}} \mathrm{Tr}(X_{\xi,\eta}) = \sup_{|\xi|=1} \Big(\langle X\xi, \xi\rangle + \inf_{\substack{|\eta|=1 \\ \eta \perp \xi}} \langle X\eta, \eta\rangle \Big).$$

To compute the inf in the latter equation, we may assume that X is diagonal, by rotational invariance, with the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ on the diagonal from the top to the bottom. Note also that in this case $\langle X\xi, \xi\rangle = \lambda_1\xi_1^2 + \cdots + \lambda_n\xi_n^2$ and $\langle X\eta, \eta\rangle = \lambda_1\eta_1^2 + \cdots + \lambda_n\eta_n^2$, so that by symmetry we may assume $\xi_i \ge 0$ and $\eta_i \ge 0$ for all $i = 1, \dots, n$, that is

$$\sup_{|\xi|=1}\inf_{\substack{|\eta|=1\\ \eta\perp\xi}} \operatorname{Tr}(X_{\xi,\eta}) = \sup_{\substack{|\xi|=1\\ \xi\ge 0}}\Big(\langle X\xi, \xi\rangle + \inf_{\substack{|\eta|=1\\ \eta\ge 0\\ \eta\perp\xi}}\langle X\eta, \eta\rangle\Big).$$

Using the Lagrange multipliers λ and μ, the inf is obtained in correspondence of a critical point of the function

$$h(\eta, \lambda, \mu) := \langle X\eta, \eta\rangle - \lambda(\langle\eta, \eta\rangle - 1) - \mu\langle\xi, \eta\rangle,$$

which solve the system

$$\begin{cases} X\eta = \lambda\eta + \dfrac{\mu}{2}\xi, \\ \langle\eta, \eta\rangle = 1, \\ \langle\xi, \eta\rangle = 0 \end{cases}$$

or equivalently

$$\begin{cases} \lambda_1\eta_1 = \lambda\eta_1 + \dfrac{\mu}{2}\xi_1, \\ \quad\vdots \\ \lambda_n\eta_n = \lambda\eta_n + \dfrac{\mu}{2}\xi_n, \\ \eta_1^2 + \cdots + \eta_n^2 = 1, \\ \xi_1\eta_1 + \cdots + \xi_n\eta_n = 0. \end{cases}$$

We can show that $\mu = 0$. Otherwise, suppose by contradiction $\mu \ne 0$. Let $I = \{i \in \{1, \dots, n\} : \xi_i \ne 0\}$, which is non-empty because $|\xi| = 1$. Then from above $(\lambda_i - \lambda)\eta_i = \frac{\mu}{2}\xi_i \ne 0$, and so $\lambda \ne \lambda_i$ for all $i \in I$. Inserting $\eta_i = \frac{\mu}{2}\frac{\xi_i}{\lambda-\lambda_i}$ in the last row of the system, we get

$$\frac{\mu}{2}\sum_{i\in I}\frac{\xi_i^2}{\lambda - \lambda_i} = 0.$$

Since $\xi_i > 0$ and $\eta_i > 0$ for $i \in I$, all the terms of the sum have the same sign (the sign of μ), and this would imply $\mu = 0$, against the assumption. Therefore critical points are not affected by the constraint $\eta\perp\xi$, and this proves the representation (2.8).

If instead of “sup inf” as in (2.8) we consider “inf sup”, we re-obtain $\mathcal{M}$:

$$\mathcal{M}(X) = \sup_{|\xi|=1}\inf_{\substack{|\eta|=1\\ \eta\perp\xi}} \operatorname{Tr}(X_{\xi,\eta}) = \inf_{|\xi|=1}\sup_{\substack{|\eta|=1\\ \eta\perp\xi}} \operatorname{Tr}(X_{\xi,\eta}) \tag{2.9}$$

with or without the constraint $\eta\perp\xi$.

2.4 Comparison with the Partial Trace Operators Operator $\mathcal{P}_k^\pm$

Let us give a comparative look to the partial trace operators (1.2):

$$\mathcal{P}_k^-(X) = \lambda_1(X) + \cdots + \lambda_k(X), \quad \mathcal{P}_k^+(X) = \lambda_{n-k+1}(X) + \cdots + \lambda_n(X).$$

Remark 2.3. If in (2.9) we consider “sup sup” or “inf inf” instead of “sup inf” or “inf sup”, it is not difficult to recognize, from (1.8), that we obtain the above partial trace operators with $k = 2$:

$$\mathcal{P}_2^-(X) = \inf_{|\xi|=1}\inf_{\substack{|\eta|=1\\ \eta\perp\xi}} \operatorname{Tr}(X_{\xi,\eta}),$$

$$\mathcal{P}_2^+(X) = \sup_{|\xi|=1}\sup_{\substack{|\eta|=1\\ \eta\perp\xi}} \operatorname{Tr}(X_{\xi,\eta}).$$

Next, we list some properties of operators $\mathcal{P}_k^\pm$. By definition, it is plain that $\mathcal{P}_k^- \le \mathcal{P}_k^+$; in addition $\mathcal{P}_k^+$ and $\mathcal{P}_k^-$ are respectively subadditive and superadditive:

$$\mathcal{P}_k^-(X) + \mathcal{P}_k^-(Y) \le \mathcal{P}_k^-(X+Y) \le \mathcal{P}_k^+(X+Y) \le \mathcal{P}_k^+(X) + \mathcal{P}_k^+(Y).$$

Moreover, $\mathcal{P}_k^-(X) = -\mathcal{P}_k^+(-X)$, so that from the left-hand inequality

$$\mathcal{P}_k^-(X+Y) \le \mathcal{P}_k^-(X) - \mathcal{P}_k^-(-Y) = \mathcal{P}_k^-(X) + \mathcal{P}_k^+(Y)$$

and from the right-hand

$$\mathcal{P}_k^+(X+Y) \ge \mathcal{P}_k^+(X) - \mathcal{P}_k^+(-Y) = \mathcal{P}_k^+(X) + \mathcal{P}_k^-(Y).$$

In particular, since $\lambda_1(X) = \mathcal{P}_1^-(X)$ and $\lambda_n(X) = \mathcal{P}_1^+(X)$,

$$\lambda_1(X) + \lambda_1(Y) \le \lambda_1(X+Y) \le \lambda_1(X) + \lambda_n(Y)$$

and

$$\lambda_1(X) + \lambda_n(Y) \le \lambda_n(X+Y) \le \lambda_n(X) + \lambda_n(Y).$$

We recall that the inequality stated above for the partial trace operators $\mathcal{P}_k^\pm$ continues to hold for the Pucci extremal operators $\mathcal{M}_{\lambda,\Lambda}^\pm$, that can be in turn regarded as Bellman operators. In fact, setting

$$\mathcal{S}_{\lambda,\Lambda}^n = \{A \in \mathcal{S}^n : \lambda I \le A \le \Lambda I\},$$

where I is the $n \times n$ identity matrix, we have

$$\mathcal{M}_{\lambda,\Lambda}^+(X) = \sup_{A \in \mathcal{S}_{\lambda,\Lambda}^n} \operatorname{Tr}(AX),$$
$$\mathcal{M}_{\lambda,\Lambda}^-(X) = \inf_{A \in \mathcal{S}_{\lambda,\Lambda}^n} \operatorname{Tr}(AX).$$

2.5 Duality

Let $\mathcal{F}$ be a fully nonlinear degenerate elliptic operator. If $\mathcal{F}$ is linear and u is a subsolution of the equation $\mathcal{F}(D^2u) = f$, then $v = -u$ is a supersolution of the equation $\mathcal{F}(D^2v) = -f$.

If we deal with an arbitrary fully nonlinear operator and u is a subsolution to $F(D^2u) = f$, then $v = -u$ is a supersolution of an equation $\tilde{\mathcal{F}}(D^2v) = -f$ for the dual operator $\tilde{\mathcal{F}}$,

$$\tilde{\mathcal{F}}(X) = -\mathcal{F}(-X),$$

which is in general different from $\mathcal{F}$. Moreover, $\overline{\mathcal{F}}$ is degnerate (uniformly) elliptic if $\mathcal{F}$ is degenerate (uniformly) elliptic.

Computing the dual of the operators introduced above, we note that by homogeneity for the min-max operator $\mathcal{M}$ we have $\tilde{\mathcal{M}} = \mathcal{M}$ as in the case of linear operators, while the upper and lower partial trace operators are each one the dual of the other one, $\tilde{\mathcal{P}}_k^\pm = \mathcal{P}_k^\pm$, as well as the maximal and the minimal the Pucci operators, $\tilde{\mathcal{M}}_{\lambda,\Lambda}^\pm = \mathcal{M}_{\lambda,\Lambda}^\mp$. In the general case $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$, we have $\tilde{\mathcal{M}}_{\mathbf{a}} = \mathcal{M}_{\mathbf{a}'}$, where $\mathbf{a}' = (a_n, a_{n-1}, \dots, a_1)$,.

3 Auxiliary Results

In this section we apply the Perron method, well known in the literature, see for instance [33] and [48], in order to show: weak maximum and comparison principles, existence and uniqueness of solutions, see respectively Sections 3.1 and 3.2. The proofs are based on the properties of our operators, suitably exploited, and an appropriate adaptation of arguments used for the uniformly elliptic case. In Section 3.3 we obtain the radial representation of the operators $\mathcal{M}_{\mathbf{a}}$.

3.1 Weak Maximum and Comparison Principles

The following comparison principle holds between viscosity subsolutions and supersolutions of the equation $\mathcal{M}_{\mathbf{a}}[u] = f$ in a bounded domain Ω, as proved for uniformly elliptic operators in the basic paper by Crandall, Ishii and Lions [33].

Theorem 3.1 (Comparison Principle). *Let $u \in \mathrm{usc}(\overline{\Omega})$ and $v \in \mathrm{lsc}(\overline{\Omega})$ such that $\mathcal{M}_{\mathbf{a}}(D^2u) \geq f$ and $\mathcal{M}_{\mathbf{a}}(D^2v) \leq f$ in Ω are satisfied in the viscosity sense, respectively, where Ω is a bounded open set of $\mathbb{R}^n$, $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ and f is a bounded continuous function in Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Letting $v \equiv 0$ and $f \equiv 0$, we obtain the following weak maximum principle.

Corollary 3.2 (Weak Maximum Principle). *Let $u \in \mathrm{usc}(\overline{\Omega})$, where Ω is a bounded domain of $\mathbb{R}^n$. If one has $\mathcal{M}_{\mathbf{a}}(D^2u(x)) \geq 0$ in Ω in the viscosity sense for some $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$, then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

On the other hand, if $u \in \mathrm{lsc}(\overline{\Omega})$ is a viscosity solution of the differential inequality $\mathcal{M}_{\mathbf{a}}(D^2u(x)) \leq 0$ in Ω for some $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$, then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Proof of Theorem 3.1. The case of $f(x) \equiv 0$ is covered in [48, Theorem 6.5]. In fact, considering the Dirichlet set $F = \{X \in \mathcal{S}^n : \mathcal{M}_{\mathbf{a}}(X) \geq 0\}$ and its dual set $\tilde{F} = \{X \in \mathcal{S}^n : \mathcal{M}_{\mathbf{a}'}(X) \geq 0\}$ in the geometric setting of Harvey and Lawson [48], then by our assumptions $u, -v \in \mathrm{usc}(\overline{\Omega})$ are of type F and $\tilde{F}$ in Ω, and our comparison principle is deduced the subaffinity of $u - v$ established there.

For sake of completeness, we give an analytic proof based on the device contained in the proof of [33, Theorem 3.3] by Crandall, Ishii and Lions. See also [8].

We have to show that, under the given assumptions, the maximum of $u - v$ must be realized on $\partial\Omega$.

(i) Firstly, setting $u_\varepsilon(x) = u(x) + \frac{1}{2}\varepsilon|x|^2$, we prove that $u_\varepsilon - v$ cannot have a positive maximum in Ω, for all fixed $\varepsilon > 0$. Actually,

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(D^2u_\varepsilon) &:= \sum_{i=1}^n a_i\lambda_i(D^2u_\varepsilon) = \sum_{i=1}^n a_i\lambda_i(D^2u + \varepsilon I) \geq f(x) + |\mathbf{a}|\varepsilon, \\ \mathcal{M}_{\mathbf{a}}(D^2v) &:= \sum_{i=1}^n a_i\lambda_i(D^2v) \leq f(x), \end{aligned}$$

where $|\mathbf{a}| = a_1 + \cdots + a_n > 0$. Supposing, by contradiction, that $u_\varepsilon - v$ has a positive maximum in Ω and following the proof of [33, Theorem 3.3], for all $\alpha > 0$ there exist points $x_\alpha, y_\alpha \in \Omega$ and matrices $X_\alpha, Y_\alpha \in \mathcal{S}^n$ such that

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{3.1}$$

and

$$\sum_{i=1}^n a_i\lambda_i(X_\alpha) \geq f(x_\alpha) + |\mathbf{a}|\varepsilon, \qquad \sum_{i=1}^n a_i\lambda_i(Y_\alpha) \leq f(y_\alpha). \tag{3.2}$$

Moreover,

$$\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0. \tag{3.3}$$

Noting that (3.1) implies $X_\alpha \leq Y_\alpha$, from (3.2) we get

$$f(x_\alpha) + |\mathbf{a}|\varepsilon \leq \sum_{i=1}^n a_i\lambda_i(X_\alpha) \leq \sum_{i=1}^n a_i\lambda_i(Y_\alpha) \leq f(y_\alpha).$$

Taking the limit as $\alpha \to \infty$ and using (3.3), by the continuity of $f(x)$ we have a contradiction: $\varepsilon \leq 0$. Therefore $u_\varepsilon - v$ cannot have a positive maximum in Ω.

(ii) From (i) it follows, for all $\varepsilon > 0$, that $\max_{\overline{\Omega}}(u_\varepsilon - v) \le \max_{\partial\Omega}(u_\varepsilon - v)$. Taking into account that $u \le v$ on $\partial\Omega$, then we have

$$u(x) + \frac{1}{2}\varepsilon|x|^2 - v(x) \le \frac{1}{2}\varepsilon R^2 \quad \text{for } x \in \Omega,$$

where $R > 0$ is the radius of a ball B_R centered at the origin such that $\Omega \subset B_R$. Letting $\varepsilon \to 0^+$, we conclude that $u \le v$ in Ω, as claimed. □

From Corollary 3.2 we deduce the following uniform estimates for viscosity solutions of the equation $\mathcal{M}_{\mathbf{a}}[u] = f$ in a bounded domain Ω

Proposition 3.3 (Uniform Estimate). *Let $u \in \operatorname{usc}(\overline{\Omega})$, where Ω is a bounded domain of $\mathbb{R}^n$. If $\mathcal{M}_{\mathbf{a}}(D^2u(x)) \ge f(x)$ in Ω in the viscosity sense for some $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ and f is bounded below in Ω, then*

$$u(x) \le \max_{\partial\Omega} u^+ Cd^2 \|f^-\|_{L^\infty(\Omega)} \quad \text{for all } x \in \overline{\Omega},$$

where C is a positive constant, which can be chosen equal to $1/|\mathbf{a}|$.

On the other hand, if $u \in \operatorname{lsc}(\overline{\Omega})$ is a viscosity solution of the differential inequality $\mathcal{M}_{\mathbf{a}}(D^2u(x)) \le f(x)$ in Ω for some $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ and f bounded above in Ω, then

$$u(x) \ge \min_{\partial\Omega} u - Cd^2 \|f^+\|_{L^\infty(\Omega)} \quad \text{for all } x \in \overline{\Omega}.$$

Proof. Let us prove the first one. Setting $K^- = \|f^-\|_{L^\infty(\Omega)}$, the function $v = u + \frac{K^-}{2|\mathbf{a}|}|x|^2$ is a subsolution of the equation $\mathcal{M}_{\mathbf{a}}[v] = 0$. By Corollary 3.2 we get $v(x) \le \max_{\overline{\Omega}} v$, so that

$$u(x) \le v(x) \le \max_{\partial\Omega} u + \frac{K^-}{2|\mathbf{a}|}d^2,$$

which yields the first inequality of the estatement. □

3.2 Existence and Uniqueness

As a consequence of the above comparison principle, we can also prove an existence and uniqueness result for the Dirichlet problem in bounded domains Ω via the Perron method, assuming that Ω has a uniform exterior cone condition, see [60] and [21]: there exist $\theta_0 \in (0, \pi)$ and $r_0 > 0$ so that for every $y \in \partial\Omega$ there is a rotation $\mathcal{R} = \mathcal{R}(y)$ such that

$$\overline{\Omega} \cap B_{r_0}(y) \subset y + \mathcal{R}\Sigma_{\theta_0},$$

where

$$\Sigma_{\theta_0} = \{x \in \mathbb{R}^n : x_n \ge |x| \cos\theta_0\}.$$

Theorem 3.4. *Let Ω be a bounded domain of $\mathbb{R}^n$ endowed with a uniform exterior cone condition. Let g be a continuous function on the boundary $\partial\Omega$, and let f be a continuous and bounded function in Ω. Then for $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$ the Dirichlet problem*

$$\begin{cases} \mathcal{M}_{\mathbf{a}}(D^2u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has a unique viscosity solution $u \in C(\overline{\Omega})$.

Proof. According to the Perron method [33, Theorem 4.1], we need a comparison principle, and the existence of a subsolution and a supersolution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$. Since the comparison principle holds by Theorem 3.1, we only need to look for a viscosity subsolution $\underline{u} \in \operatorname{usc}(\overline{\Omega})$ and a viscosity supersolution $\overline{u} \in \operatorname{lsc}(\overline{\Omega})$ of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f(x)$ such that $\underline{u} = g = \overline{u}$ on $\partial\Omega$.

To do this, we will use the following inequalities, see (2.4) and (2.5):

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(X) &= a_1\lambda_1(X) + \cdots + a_n\lambda_n(X) = n\frac{a_1}{n}\lambda_1(X) + \cdots + a_n\lambda_n(X) \\ &\le \frac{a_1}{n}\lambda_1(X) + \sum_{i=2}^{n}\Big(\frac{a_1}{n} + a_i\Big)\lambda_i(X) =: \mathcal{M}_{\overline{\mathbf{a}}}(X) \end{aligned}$$

and

$$\mathcal{M}_{\mathbf{a}}(X) = a_1\lambda_1(X) + \cdots + a_n\lambda_n(X) = a_1\lambda_1(X) + \cdots + n\frac{a_n}{n}\lambda_n(X)$$
$$\geq \sum_{i=1}^{n-1}\Big(a_i + \frac{a_n}{n}\Big)\lambda_i(X) + \frac{a_n}{n}\lambda_n(X) =: \mathcal{M}_{\underline{\mathbf{a}}}(X).$$

If $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_1$, then $\mathcal{M}_{\overline{\mathbf{a}}}$ is uniformly elliptic with ellipticity constants

$$\overline{\lambda} = \frac{a_1}{n}, \quad \overline{\Lambda} = \frac{a_1}{n} + \max_{2\leq i\leq n} a_i,$$

so that

$$\mathcal{M}_{\mathbf{a}}(X) \leq \mathcal{M}_{\overline{\mathbf{a}}}(X) \leq \mathcal{M}^+_{\frac{a_1}{n},|\mathbf{a}|} \tag{3.5}$$

and, if $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_n$, then $\mathcal{M}_{\underline{\mathbf{a}}}$ is uniformly elliptic with ellipticity constants

$$\underline{\lambda} = \frac{a_n}{n}, \quad \underline{\Lambda} = \frac{a_n}{n} + \max_{1\leq i\leq n-1} a_i,$$

so that

$$\mathcal{M}_{\mathbf{a}}(X) \geq \mathcal{M}_{\underline{\mathbf{a}}}(X) \geq \mathcal{M}^-_{\frac{a_n}{n},|\mathbf{a}|}(X). \tag{3.6}$$

Therefore, if $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$, and λ^* and Λ^* are positive numbers such that

$$\lambda^* \leq \min(\underline{\lambda}, \overline{\lambda}) = \frac{a^*}{n} \equiv \frac{\min(a_1, a_n)}{n}, \quad \Lambda^* \geq \max(\underline{\Lambda}, \overline{\Lambda}) \geq |\mathbf{a}| \equiv a_1 + \cdots + a_n,$$

by the extremality properties (2.3) of Pucci operators, from (3.5) and (3.6) we have

$$\mathcal{M}^-_{\frac{a^*}{n},|\mathbf{a}|}(X) \leq \mathcal{M}_{\mathbf{a}}(X) \leq \mathcal{M}^+_{\frac{a^*}{n},|\mathbf{a}|}(X). \tag{3.7}$$

Next, setting $K = \sup_\Omega |f|$, we solve by [21, Proposition 3.2] the Dirichlet problems

$$\begin{cases} \mathcal{M}^-_{\frac{a^*}{n},|\mathbf{a}|}(D^2\underline{u}) = K & \text{in } \Omega, \\ \underline{u} = g & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathcal{M}^+_{\frac{a^*}{n},|\mathbf{a}|}(D^2\overline{u}) = -K & \text{in } \Omega, \\ \overline{u} = g & \text{on } \partial\Omega. \end{cases}$$

Since obviously $-K \leq f(x) \leq K$ for all $x \in \Omega$, from (3.7) it follows that $\underline{u}$ and $\overline{u}$ provide a subsolution and a supersolution that we were searching for, concluding the proof. □

An existence and uniqueness result is provided for all the class $\mathcal{A}$ by [48, Theorem 6.2] for smooth boundaries. A weaker condition can be obtained from [15], where the authors consider in detail the case $\mathbf{a} = \mathbf{e}_j$, namely the equation $\lambda_j[u] = 0$, and prove an existence and uniqueness theorem for the Dirichlet problem (3.4) with a sharp geometric condition on the boundary of Ω, depending on j. From there, we take a sufficient condition to solve the Dirichlet problem for any equation $\lambda_j(D^2u) = 0$, $j = 1, \dots, n$: given $y \in \partial\Omega$, for every $r > 0$ there exists $\delta > 0$ such that, for every $x \in B_\delta(y)$ and direction $v \in \mathbb{R}^n$ ($|v| = 1$),

$$(x + \mathbb{R}v) \cap B_r(y) \cap \partial\Omega \neq \emptyset. \tag{G_1}$$

This condition does not require smooth boundary, but it is nevertheless stronger than the exterior cone property.

Theorem 3.5. *Let Ω be a bounded domain of $\mathbb{R}^n$ satisfying condition (G_1). Let g be a continuous function on the boundary $\partial\Omega$, and f be a continuous and bounded function in Ω. Then for $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ the Dirichlet problem*

$$\begin{cases} \mathcal{M}_{\mathbf{a}}(D^2u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has a unique viscosity solution $u \in C(\overline{\Omega})$.

Proof. Following the same lines of the proof of Theorem 3.4, we only need to look for a viscosity subsolution $\underline{u} \in \mathrm{usc}(\overline{\Omega})$ and a viscosity supersolution $\overline{u} \in \mathrm{lsc}(\overline{\Omega})$ of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f(x)$ such that $\underline{u} = g = \overline{u}$ on $\partial\Omega$. To do this, we observe this time

$$\mathcal{M}_{\mathbf{a}}(X) = a_1\lambda_1(X) + \cdots + a_n\lambda_n(X) \le |\mathbf{a}|\lambda_n(X)$$

and

$$\mathcal{M}_{\mathbf{a}}(X) = a_1\lambda_1(X) + \cdots + a_n\lambda_n(X) \ge |\mathbf{a}|\lambda_1(X).$$

Next, setting $K = \sup_\Omega |f|$, we solve by [15, Theorem 1] the Dirichlet problems

$$\begin{cases} |\mathbf{a}|\lambda_1(D^2\underline{u}) = K & \text{in } \Omega, \\ \underline{u} = g & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} |\mathbf{a}|\lambda_n(D^2\overline{u}) = -K & \text{in } \Omega, \\ \overline{u} = g & \text{on } \partial\Omega. \end{cases}$$

As in the proof of Theorem 3.4, $\underline{u}$ and $\overline{u}$ provide a subsolution and a supersolution, concluding the proof. □

3.3 Radial Solutions

We compute $\mathcal{M}$ on radial functions $u(x) = v(|x|)$. Suppose v is C^2, we recall that for $x \ne 0$,

$$Du(x) = v'(|x|)\frac{x}{|x|},$$
$$D^2u(x) = v''(|x|)\frac{x}{|x|} \otimes \frac{x}{|x|} + \frac{v'(|x|)}{|x|}\Big(I - \frac{x}{|x|} \otimes \frac{x}{|x|}\Big),$$

where $\frac{x}{|x|} \otimes \frac{x}{|x|} \ge 0$, $I - \frac{x}{|x|} \otimes \frac{x}{|x|} \ge 0$ and

$$\Big\langle \frac{x}{|x|} \otimes \frac{x}{|x|}h, h\Big\rangle = \Big\langle \frac{x}{|x|}, h\Big\rangle^2,$$
$$\Big\langle \Big(I - \frac{x}{|x|} \otimes \frac{x}{|x|}\Big)h, h\Big\rangle = |h|^2 - \Big\langle \frac{x}{|x|}, h\Big\rangle^2.$$

As a consequence, $\frac{x}{|x|}$ is eigenvector of $\frac{x}{|x|} \otimes \frac{x}{|x|}$ with eigenvalue 1, and of $I - \frac{x}{|x|} \otimes \frac{x}{|x|}$ with eigenvalue 0. Conversely, all non-zero vectors orthogonal to $\frac{x}{|x|}$ are eigenvectors of $\frac{x}{|x|} \otimes \frac{x}{|x|}$ with eigenvalue 0 and of $I - \frac{x}{|x|} \otimes \frac{x}{|x|}$ with eigenvalue 1. It follows that

$$\lambda_1(D^2u(x)) + \lambda_n(D^2u(x)) = v''(|x|) + \frac{v'(|x|)}{|x|}.$$

From this we deduce useful properties which are collected in the following remark.

Remark 3.6. (i) The operator $\mathcal{M}$ is linear on the radial functions $u(x) = v(|x|)$.
(ii) Any function of the form

$$\varphi(x) = a + b\log|x|,$$

with a and b constant, is a solution of $\mathcal{M}[u] = 0$ in $\mathbb{R}^n \setminus \{0\}$.
(iii) Recall that the k-th Hessian operator, $k = 1, \dots, n$, for radial functions is

$$S_k(D^2u) = \binom{n-1}{k-1}\Big(\frac{v'}{|x|}\Big)^{k-1}\Big(v'' + \frac{n-k}{k}\frac{v'}{|x|}\Big).$$

In case $n = 2k$ the radial solutions of the equation

$$S_{\frac{n}{2}}(D^2u) = 0$$

are just the radial solutions of $\mathcal{M}(D^2u) = 0$.

Recalling that $|\mathbf{a}| = a_1 + \cdots + a_n$, let $\hat{a}_j = |\mathbf{a}| - a_j$, $j = 1, \ldots, n$. More generally, for $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ the non-constant radial solutions in $\mathbb{R}^n \setminus \{0\}$, up to a multiplicative constant, are

$$\varphi(x) = \begin{cases} |x|^{-\gamma_n} & \text{if } \hat{a}_n > a_n, \\ \log|x|^{-1} & \text{if } \hat{a}_n = a_n, \\ |x|^{\gamma_1} & \text{if } a_1 > \hat{a}_1, \\ \log|x| & \text{if } a_1 = \hat{a}_1, \end{cases} \tag{3.8}$$

where $\gamma_n = \frac{\hat{a}_n}{a_n} - 1$.

4 The ABP Estimate

The celebrated ABP estimate provides a uniform estimate for the solution of an elliptic equation $F[u] = f$ with the L^n-norm of f. The original inequality, for linear uniformly elliptic operators in bounded domains, goes back to Alexandroff [1, 2], but it already appears in Bakel'man [5]. A different version has been later obtained by Pucci [68].

In [18] it was also proved for the first time an ABP estimate for solutions in $W^{2,p}_{\mathrm{loc}}(\Omega)$ of the equation $F[u] = f$ with $f \in L^p$ and $p \in (\frac{n}{2}, n)$. A result of this kind is known in the framework of L^p-viscosity solutions [21] as the generalized maximum principle, which can be found in [45] and [34] in the fully nonlinear uniformly elliptic case. See also [55] for the case of L^p viscosity solutions.

It is worth noticing that an ABP estimate for degenerate elliptic equations of p-Laplacian type has been proved by Imbert [52].

An extension of this inequality to unbounded domains Ω of cylindrical type for bounded solutions in $W^{2,n}_{\mathrm{loc}}(\Omega)$ is due to Cabré [18]. By domains of cylindrical type we intend here a measure-geometric condition, which is satisfied by cylinders and goes back to a famous paper of Berestycki, Nirenberg and Vardhan [9], containing a characterization of the weak maximum principle. In subsequent papers the results of [18] have been generalized to domains of conical type [20, 77, 78] and to viscosity solutions of fully nonlinear uniformly elliptic equations [26], and then to different classes of degenerate elliptic equations [10, 29, 30].

The proof of the ABP estimates of Theorem 1.1 is based on the geometrical argument used in [47, proof of Theorem 9.1] for classical solutions.

We denote by Γ_u^+ the upper convex envelope of u, the smallest concave function greater than u in Ω, and by Γ_u^- the lower convex envelope of u, the largest convex function smaller than u in Ω.

Lemma 4.1. *Let Ω be a bounded domain with diameter d, and $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_1$. For every $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $u \le 0$ on $\partial\Omega$ we have*

$$\sup_\Omega u^+ \le \frac{1}{a_1} \frac{d}{\omega_n^{1/n}} \|\mathcal{M}_{\mathbf{a}}(D^2u(x))^-\|_{L^n(\{\Gamma_u^+ = u\})}, \tag{4.1}$$

where ω_n denotes the Lebesgue measure of the n-dimensional unit ball.

On the other hand, let us assume $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_n$. For every $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $u \ge 0$ on $\partial\Omega$ we have

$$\sup_\Omega u^- \le \frac{1}{a_n} \frac{d}{\omega_n^{1/n}} \|\mathcal{M}_{\mathbf{a}}(D^2u(x))^+\|_{L^n(\{\Gamma_u^- = u\})}. \tag{4.2}$$

Proof. Let us prove the first estimate (4.1). We argue following [47, proof of Lemma 9.2] and [25, proof of Lemma 3.], denoting by $\chi_u : \Omega \to \mathbb{R}^n$ the normal mapping

$$\chi_u(z) = \{p \in \mathbb{R}^n : u(x) \le u(y) + \langle p, x - z\rangle \text{ for all } x \in \Omega\}, \quad z \in \Omega.$$

We remark that on the upper contact set $\{\Gamma_u^+ = u\}$ the eigenvalues of D^2u are non-positive, and the Lebesgue measure of χ_u can be estimated as

$$|\chi_u(\Omega)| \le \int_{\Gamma_u^+ = u} |\det D^2u(x)|\, dx. \tag{4.3}$$

If $u \le 0$ in Ω, then inequality (4.1) is obvious. Suppose then u realizes a positive maximum at a point $y \in \Omega$, and recall that $\Omega \subset B_d(y)$.

Let κ be the function whose graph is the cone K with vertex $(y, u(y))$ and base $\partial B_d(y)$; then $\chi_\kappa(\Omega) \subset \chi_u(\Omega)$. Then $\chi_u(\Omega)$ and contains all the slopes of $B_{u(y)/d}$, so that $\omega_n(\frac{u(y)}{d})^n \le |\chi_u(\Omega)|$ and by (4.3),

$$u^+(y) \le \frac{d}{\omega_n^{1/n}}\bigg(\int_{\Gamma_u^+=u} |\det D^2 u(x)|\,dx\bigg)^{\frac{1}{n}}. \tag{4.4}$$

Since on the contact set we have $|\lambda_n| \le |\lambda_{n-1}| \le \cdots \le |\lambda_1|$, it follows that

$$\begin{aligned} |\det D^2 u| &= |\lambda_1(D^2u)| \cdots |\lambda_n(D^2u)| \le |\lambda_1(D^2u)|^n \\ &= \frac{1}{a_1^n}|a_1\lambda_1(D^2u)|^n \le \frac{1}{a_1^n}\bigg|\sum_{i=1}^n a_i\lambda_i(D^2u)\bigg|^n \\ &= \frac{1}{a_1^n}((\mathcal{M}_{\mathbf{a}}(D^2u))^-)^n. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5) we obtain the estimate from above (4.1).

For the estimate from below, we can apply (4.1) with $v = -u$ instead of u, observing that by assumption $v \le 0$ on $\partial\Omega$ and by duality

$$\mathcal{M}_{\mathbf{a}'}(D^2v)) = -\mathcal{M}_{\mathbf{a}}(D^2u)).$$

Then

$$\begin{aligned} \sup_\Omega u^- &= \sup_\Omega v^+ \\ &\le \frac{1}{a_1'}\frac{d}{\omega_n^{1/n}}\|\mathcal{M}_{\mathbf{a}'}(D^2v(x))^+\|_{L^n(\{\Gamma_v^+=v\})} \\ &= \frac{1}{a_n}\frac{d}{\omega_n^{1/n}}\|\mathcal{M}_{\mathbf{a}}(D^2u(x))^-\|_{L^n(\{\Gamma_u^-=u\})}. \end{aligned}$$

□

Theorem 1.1 is obtained combining the two unilateral ABP estimates, which hold separately for subsolutions and supersolutions, contained in the following result.

Theorem 4.2. *Let Ω be a bounded domain of diameter d. Let f be continuous and bounded in Ω. There exist an universal constant $C_n > 0$, depending only on n,*

(i) *for viscosity subsolutions $u \in \operatorname{usc}(\Omega)$ of the equation $\mathcal{M}_{\mathbf{a}}[u] = f$ in Ω with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_1$ and*

$$\sup_\Omega u^+ \le \sup_{\partial\Omega} u^+ + \frac{C_n}{a_1} d\|f\|_{L^n(\Omega)}, \tag{4.6}$$

(ii) *for viscosity supersolutions $u \in \operatorname{lsc}(\Omega)$ of the equation $\mathcal{M}_{\mathbf{a}}[u] = f$ in Ω with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_n$ and*

$$\sup_\Omega u^- \le \sup_{\partial\Omega} u^- + \frac{C_n}{a_n} d\|f\|_{L^n(\Omega)}. \tag{4.7}$$

For classical solutions the proof follows directly from Lemma 4.1.

Proof of Theorem 4.2: Classical Solutions. For subsolutions, supposing $\mathcal{M}_{\mathbf{a}}(D^2u) \ge f \ge -f^-$, we have

$$\mathcal{M}_{\mathbf{a}}(D^2u)^- \le f^-.$$

From Lemma 4.1, passing to $u - \sup_{\partial\Omega} u$ in (4.1), we get inequality (4.6). For supersolutions, supposing $\mathcal{M}_{\mathbf{a}}(D^2u) \le f \le -f^+$, we have

$$\mathcal{M}_{\mathbf{a}}(D^2u)^+ \le f^+.$$

From Lemma 4.1, passing to $u - \inf_{\partial\Omega} u$ in (4.2), we get inequality (4.7). □

To consider viscosity subsolutions, we extend $u^+ = \max(u, 0)$ and f^- to zero outside Ω, keeping the respective notations, and observing that in the viscosity setting $\mathcal{M}_{\mathbf{a}}(D^2u^+) \ge -f^-$ in $\mathbb{R}^n$. For viscosity supersolutions we extend $u^- = \max(u, 0)$ and f^+ to zero outside Ω so that $\mathcal{M}_{\mathbf{a}}(D^2u^-) \le f^+$ in $\mathbb{R}^n$.

In what follows we will refer to Γ_u^+ and Γ_u^- as to the upper and the lower convex envelope of u^+ and $-u^-$, respectively, relative to the ball B_{2d} concentric with a ball B_d of radius d containing Ω.

The key tool is the following lemma, which allows to apply the classical ABP estimates obtained before to viscosity subsolutions and supersolutions and is the counterpart of [25, Lemma 3.3].

Lemma 4.3. *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$. Let $u \in \operatorname{lsc}(\overline{B}_\delta)$, where $B_\delta = \{|x - x_0| < \delta\}$ such that*

$$\mathcal{M}_{\mathbf{a}}(D^2 u) \le f \quad \text{in } B_\delta$$

in the viscosity sense, and w be a convex function such that

$$w(x_0) = u(x_0), \quad w(x) \le u(x) \quad \text{in } B_\delta.$$

For sufficiently small $\varepsilon \in (0, \varepsilon_0)$ and any function f, bounded above, we have

$$\ell(x) \le w(x) \le \ell(x) + \frac{1}{2} C_\varepsilon \Big(\sup_{B_\delta} f^+ \Big) |x - x_0|^2 \quad \text{in } B_{\varepsilon\delta}, \tag{4.8}$$

where $\ell(x)$ is the supporting hyperplane for w at x_0. In particular, there exists a convex paraboloid of opening $\frac{C_\varepsilon}{a_n}$ touching the graph of w from above.

Here $\varepsilon_0 > 0$ depends on (a positive lower bound of) a_n and (an upper bound of) $\hat{a}_n$ defined in Section 3.3; moreover $C_\varepsilon \to \frac{1}{a_n}$ as $\varepsilon \to 0$. Therefore, when u is second-order differentiable and f is continuous at $x = x_0$, we get

$$\lambda_n^+(D^2 w(x_0)) \le \frac{1}{a_n} f^+(x_0).$$

Proof. The first one inequality in (4.8) depends on the fact that $\ell(x)$ is the supporting hyperplane of w at x_0.

Concerning the second one, we may proceed assuming $x_0 = 0$ and $\delta = 1$.

(i) Subtracting $\ell(x)$, we consider the functions $v(x) = u(x) - \ell(x)$ and $\varphi(x) = w(x) - \ell(x)$, which satisfy in turn the assumptions on $u(x)$ and $w(x)$, respectively. This simplifies the computations, since $\varphi(0) = 0$ and the supporting hyperplane for $v(x)$ at $x = 0$ is now horizontal, so that $\varphi(x) \ge 0$ in B_1. In this way, we are reduced to show

$$\varphi(x) \le \frac{1}{2} C_\varepsilon K |x|^2 \quad \text{in } B_\varepsilon,$$

with $K = \sup_{B_1} f^+$, under the assumptions

$$\varphi(0) = v(0) = 0, \quad \varphi(x) \le v(x) \quad \text{in } B_1, \tag{4.9}$$

and

$$a_1 \lambda_1(D^2 v) + \cdots + a_n \lambda_n(D^2 v) \le f^+(x) \quad \text{in } B_1, \tag{4.10}$$

which implies

$$\hat{a}_n \lambda_1(D^2 v) + a_n \lambda_n(D^2 v) \le f^+(x) \quad \text{in } B_1.$$

(ii) Let $0 < \rho < \varepsilon$ and let M_ρ be the maximum of φ on $\overline{B}_\rho$. We may suppose that a maximum point is $x_\rho = (0, \dots, 0, \rho) \in \partial B_\rho$. Since the supporting hyperplane for $\varphi(x)$ at x_ρ is constant on the tangent line to B_ρ through x_ρ, we have

$$\varphi(x) \ge M_\rho \quad \text{for } x = (x', \rho),$$

where $x' = (x_1, \dots, x_{n-1})$. Let us consider now the cylindrical box

$$R = \Big\{ x = (x', x_n) : |x'| < \sqrt{1 - \rho^2},\ -\varepsilon\rho < x_n < \rho \Big\} \subset B_1,$$

and the paraboloid

$$P(x) = \frac{1}{2}(x_n + \varepsilon\rho)^2 - \frac{1}{2} \frac{(1+\varepsilon)^2}{1 - \rho^2} \rho^2 |x'|^2.$$

Evaluating $P(x)$ on ∂R, when $x_n = -\varepsilon\rho$ or $|x'| = \sqrt{1 - \rho^2}$, we have $P(x) \le 0$. On the remaining part of ∂R, $x_n = \rho$, we have $P(x) \le \frac{1}{2}(1+\varepsilon)^2 \rho^2$, from which

$$\frac{M_\rho}{\frac{1}{2}(1+\varepsilon)^2 \rho^2} P(x) \le \varphi(x) \quad \text{on } \partial R. \tag{4.11}$$

(iii) Since $\rho < \varepsilon$, it follows that $P(x)$ is solution of the differential inequality

$$\begin{aligned}\hat{a}_n\lambda_1(D^2P) + a_n\lambda_n(D^2P) &\geq a_n - \frac{(1+\varepsilon)^2}{1-\rho^2}\hat{a}_n\rho^2\\ &\geq a_n - \frac{1+\varepsilon}{1-\varepsilon}\hat{a}_n\varepsilon^2\\ &\equiv a_n - \hat{a}_n c_\varepsilon,\end{aligned}$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0^+$, so that $a_n - \hat{a}_n c_\varepsilon > 0$ for $\varepsilon < \varepsilon_0$ small enough, and the function $Q(x) \equiv \frac{K}{a_n - \hat{a}_n c_\varepsilon}P(x)$ satisfies the differential inequality

$$\hat{a}_n\lambda_1(D^2Q) + a_n\lambda_n(D^2Q) \geq K \geq f^+ \quad \text{in } B_1. \tag{4.12}$$

(iv) We claim that

$$M_\rho = \max_{\overline{B}_\rho}\varphi(x) \leq \frac{1}{2}\frac{(1+\varepsilon)^2}{a_n - \hat{a}_n c_\varepsilon}K\rho^2. \tag{4.13}$$

In fact, arguing by contradiction, suppose that $M_\rho > \frac{1}{2}\frac{(1+\varepsilon)^2}{a_n-\hat{a}_n c_\varepsilon}K\rho^2$. Then using (4.11) and (4.9),

$$Q(x) = \frac{K}{a_n - \hat{a}_n c_\varepsilon}P(x) < \frac{M_\rho}{\frac{1}{2}(1+\varepsilon)^2\rho^2}P(x) \leq \varphi(x) \leq v(x) \quad \text{on } \partial R.$$

By (4.10) and (4.12), the comparison principle would imply $Q(x) \leq v(x)$ in R, and this is a contradiction with $v(0) = 0 < Q(0)$, which proves the claim.

Setting $\rho = |x|$ in (4.13), as in the proof of [25, Theorem 3.2], we conclude that the statement of the theorem holds with $C_\varepsilon = \frac{(1+\varepsilon)^2}{a_n - \hat{a}_n c_\varepsilon}$, where $c_\varepsilon \to 0$ as $\varepsilon \to 0^+$. □

Proof of Theorem 4.2: Viscosity Solutions. We follow the lines of the proof of [25, Theorem 3.6], considering subsolutions $u \in \operatorname{usc}(\overline{\Omega})$. The case of supersolutions $u \in \operatorname{lsc}(\overline{\Omega})$ with estimate (4.7) from below can be obtained by duality, passing to $-u$.

Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$. From Lemma 4.3 and duality we deduce a similar conclusion for subsolutions $u \in \operatorname{lsc}(\overline{B}_\delta)$, where $B_\delta = \{|x - x_0| < \delta\}$ such that

$$\mathcal{M}_{\mathbf{a}}(D^2u) \geq f(x) \quad \text{in } B_\delta$$

in the viscosity sense. Let w be a concave function such that

$$w(x_0) = u(x_0), \quad w(x) \leq u(x) \quad \text{in } B_\delta.$$

For sufficiently small $\varepsilon \in (0, \varepsilon_0)$ and any function f, bounded above, we have

$$\ell(x) - \frac{1}{2}C_\varepsilon\Big(\sup_{B_\delta} f^+\Big)|x - x_0|^2 \leq w(x) \leq \ell(x) \quad \text{in } B_{\varepsilon\delta},$$

where $\ell(x)$ is the supporting hyperplane for w at x_0. In particular, there exists a concave paraboloid of opening C_ε touching the graph of w from below.

Here $\varepsilon_0 > 0$ depends on a lower bound for a_1 and an upper bound for $\hat{a}_1$ defined in Section 3.3, and $C_\varepsilon \to \frac{1}{a_1}$ as $\varepsilon \to 0$. Therefore, when u is second-order differentiable and f is continuous at $x = x_0$, we get

$$\lambda_1^-(D^2w(x_0)) \leq \frac{1}{a_1}f^-(x_0). \tag{4.14}$$

Using [25, Lemma 3.5], we deduce from the above that $\Gamma_u \in C^{1,1}(\overline{B}_d)$. Hence Γ_u is second-order differentiable a.e. in $\overline{B}_d$ and (4.4) holds for Γ_u^+ in B_d.

If $u \leq 0$ on $\partial\Omega$, then we have

$$\sup_{B_d} u^+ \leq C_n d\bigg(\int_{\Gamma_u^+ = u} |\det D^2\Gamma_u^+(x)|\,dx\bigg)^{\frac{1}{n}}. \tag{4.15}$$

Reasoning as in the proof of [25, Theorem 3.6], that is observing that the upper contact points are in Ω and Γ_u^+ is second-order differentiable a.e. on $\{\Gamma_u^+ = u\}$, where f is continuous and therefore, by (4.14)

$$|\det D^2(\Gamma_u^+(x))| \le (\lambda_1^-(D^2\Gamma_u^+(x)))^n \le \frac{1}{a_1^n}(f^-(x))^n. \tag{4.16}$$

Estimating (4.15) with (4.16), we get the ABP estimate (4.6) for $u \le 0$ on $\partial\Omega$. Passing to $u - \sup_{\partial\Omega} u$, which is ≤ 0 on $\partial\Omega$, we conclude that (4.6) holds. □

5 Harnack Inequality and C^α Estimates

The Harnack inequality, classically related to the mean properties of the Laplace operator, is a powerful nonlinear technique for regularity in the framework of fully nonlinear equations. We refer to [47] for solutions of linear uniformly elliptic equations in Sobolev spaces, to [74] for quasi-linear uniformly elliptic equations and to [24, 25] for viscosity solutions of fully nonlinear equations. See also [4, 52, 61] for further contributions.

In order to prove the Harnack inequality for non-negative solutions and the related local estimates for subsolutions and non-negative supersolutions, respectively known in literature (see for instance [47]) as the local maximum principle and the weak Harnack inequality, we could employ the same strategy of [25, Chapter 4].

A quicker way, sufficient for the applications, is based on inequalities (3.7) obtained in Section 3. The results are given in cubes, and here $\mathcal{Q}_\ell$ is a cube of $\mathbb{R}^n$ of edge ℓ centered at the origin, i.e.

$$Q_\ell = \left\{|x_i| < \frac{\ell}{2} : i = 1, \dots, n\right\},$$

but they could be equivalently stated in balls.

Theorem 5.1 (Local Maximum Principle). *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_1$. Let u be a viscosity subsolution of the equation*

$$\mathcal{M}_{\mathbf{a}}(D^2u) = f$$

in Q_1, where f is continuous and bounded. Then

$$\sup_{Q_{1/2}} u \le C_p(\|u^+\|_{L^p(Q_{2/3})} + \|f^-\|_{L^n(Q_1)}), \tag{5.1}$$

where C_p is a constant depending only on n, p, a_1 and $\tilde{a}$.

Proof. In view of inequalities (3.7), we have $\mathcal{M}^+_{a_1/n,\tilde{a}n}(D^2u) \ge \mathcal{M}_{\mathbf{a}}(D^2u) \ge f(x) \ge -f^-(x)$, and therefore we can apply [25, Theorem 4.8 (2)] to obtain (5.1). □

Theorem 5.2 (Weak Harnack Inequality). *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_n$. Let $u \ge 0$ be a viscosity supersolution of the equation*

$$\mathcal{M}_{\mathbf{a}}(D^2u) = f$$

in Q_1, where f is continuous and bounded. Then

$$\|u\|_{L^{p_0}(Q_{2/3})} \le C_0\Big(\inf_{Q_{3/4}} u + \|f^+\|_{L^n(Q_1)}\Big), \tag{5.2}$$

where $p_0 > 0$ and C_0 are universal constants, depending only on n, p, a_n and $\tilde{a}$.

Proof. In view of inequalities (3.7), we have $\mathcal{M}^-_{a_n/n,\tilde{a}n}(D^2u) \le \mathcal{M}_{\mathbf{a}}(D^2u) \le f(x) \le f^+(x)$, and therefore we can apply [25, Theorem 4.8 (1)] to obtain (5.2). □

The proof of Theorem 1.2 (Harnack inequality) follows at once.

Proof of Theorem 1.2. Let $p_0 > 0$ be the exponent of Theorem 5.2. From (5.2) and (5.1) it follows that

$$\sup_{Q_{1/2}} u \le C_{p_0}(\|u\|_{L^{p_0}(Q_{2/3})} + \|f^-\|_{L^n(Q_1)}) \le C_{p_0}\Big(C_0\Big(\inf_{Q_{3/4}} u + \|f^+\|_{L^n(Q_1)}\Big) + \|f^-\|_{L^n(Q_1)}\Big),$$

which yields the result. □

From the Harnack inequality, in a standard way, using the technique for the proof of [25, Proposition 4.10] and [47, Lemma 8.23], the following Hölder regularity results and C^α interior estimates can be obtained. We give the result with concentric balls B_1 and $B_{1/2}$ of radius 1 and $\frac{1}{2}$, respectively.

Theorem 5.3 (Interior Hölder Continuity). *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$. Let u be a viscosity solution of the equation*

$$\mathcal{M}_{\mathbf{a}}(D^2u) = f$$

in B_1, where f is continuous and bounded. Then $u \in C^\alpha(\overline{B}_{1/2})$ and

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)}),$$

where C is a positive constant depending only on n, $a^* = \min(a_1, a_n)$ and $\tilde{a} = \frac{1}{n}(a_1 + \cdots + a_n)$.

Global Hölder estimates can be proved for domains with the uniform exterior sphere condition (S), see Section 1, via the boundary Hölder estimates of the lemma below. We adopt the following notations, for the Hölder seminorm ($0 < \gamma < 1$)) of a function $h : D \to \mathbb{R}$ in a subset D of $\mathbb{R}^n$:

$$[h]_{\beta,D} = \sup_{\substack{x,y\in D\\ x\neq y}} \frac{|h(x) - h(y)|}{|x - y|^\beta}. \tag{5.3}$$

Lemma 5.4. *Let $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ and let u be a viscosity solution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in a bounded domain Ω, where f is continuous and bounded. We assume that Ω satisfies a uniform exterior sphere condition* (S) *with radius $R > 0$, and $u = g$ on $\partial\Omega$.*

(i) *If $g \in C^\beta(\partial\Omega)$ with $\beta \in (0, 1]$, then*

$$\sup_{\substack{x\in\Omega\\ y\in\partial\Omega}} \frac{u(x) - u(y)}{|x - y|^{\frac{\beta}{2}}} \le C([g]_{\beta,\partial\Omega} + \|f^-\|_{L^\infty(\Omega)})$$

with $C > 0$ depending only on n, $\tilde{a}$, R and β.

(ii) *Assume in addition that Ω has a uniform Lipschitz boundary with Lipschitz constant L. If $g \in C^{1,\beta}(\partial\Omega)$ with $\beta \in [0, 1)$, then*

$$\sup_{\substack{x\in\Omega\\ y\in\partial\Omega}} \frac{u(x) - u(y)}{|x - y|^{\frac{1}{2}(1+\beta)}} \le C([g]_{1,\partial\Omega} + [Dg]_{\beta,\partial\Omega} + \|f^-\|_{L^\infty(\Omega)}) \tag{5.4}$$

with $C > 0$ depending only on n, $\tilde{a}$, R, L and β.

If u is a viscosity supersolution, then (i) *and* (ii) *hold with $u(y) - u(x)$ and $\|f^+\|_{L^\infty(\Omega)}$ instead of $u(x) - u(y)$ and $\|f^-\|_{L^\infty(\Omega)}$, respectively.*

Proof. We treat in detail the case of subsolutions. The result for subsolutions will follow by duality. Therefore, suppose that $u \in \mathrm{usc}(\overline{\Omega})$ is a subsolution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = f$ in Ω such that $u = g$ on $\partial\Omega$. Let $y \in \partial\Omega$ and B_R a ball of radius R, centered at $x_0 \in \mathbb{R}^n$, such that $y \in \partial B_R$ and $\overline{\Omega} \subset \overline{B}_R$, according to (S). Supposing, as we may, $y = (0, \dots, 0, 0)$ and $x_0 = (0, \dots, 0, R)$; then $\overline{B}_R$ is described by the inequality $x_1^2 + \cdots + x_{n-1}^2 + (x_n - R)^2 \le R^2$. It follows that

$$|x|^2 \le 2Rx_n, \quad x \in \overline{\Omega}. \tag{5.5}$$

Case (i). By assumption on g and (5.5), we have

$$|g(x)| \le [g]_{\beta,\Omega}|x|^\beta, \quad x \in \overline{\Omega}.$$

To simplify, we may suppose $g(0) = 0$, so that in particular

$$g(x) \le [g]_{\beta,\Omega}|x|^\beta \le (2R)^{\frac{\beta}{2}}[g]_{\beta,\Omega}x_n^{\frac{\beta}{2}}, \quad x \in \overline{\Omega}. \tag{5.6}$$

Next, we define

$$\varphi(x) = C_1([g]_{\beta,\Omega} + \varepsilon)x_n^{\frac{\beta}{2}} - \frac{1}{2|\mathbf{a}|}\|f^-\|_{L^\infty(\Omega)}|x|^2,$$

where ε is any positive number and

$$C_1 \geq (2R)^{\frac{\beta}{2}} + \frac{(2R)^{2-\frac{\beta}{2}}}{2|\mathbf{a}|} \frac{\|f^-\|}{[g]_{\beta,\Omega} + \varepsilon}.$$

Thus from (5.6)

$$u(x) = g(x) \leq \varphi(x) \quad \text{on } \partial\Omega.$$

Moreover, φ is a supersolution in Ω:

$$\mathcal{M}_{\mathbf{a}}(D^2\varphi) = a_1 C_1([g]_{\beta,\Omega} + \varepsilon)\frac{\beta}{2}\Big(\frac{\beta}{2} - 1\Big)x_n^{\frac{\beta}{2}-2} - \|f^-\|_{L^\infty(\Omega)} \leq -f^-(x) \quad \text{in } \Omega.$$

By the comparison principle $u(x) \leq \varphi(x)$ for all $x \in \Omega$, from which

$$\frac{u(x)}{|x|^{\frac{\beta}{2}}} \leq C_2([g]_{\beta,\Omega} + \|f^-\|_{L^\infty(\Omega)}).$$

Then for an arbitrary $y \in \partial\Omega$ we have

$$\frac{u(x) - u(y)}{|x-y|^{\frac{\beta}{2}}} \leq C([g]_{\beta,\Omega} + \|f^-\|_{L^\infty(\Omega)}),$$

from which (5.4) follows.

Case (ii). By assumption on g and on $\partial\Omega$, we have

$$|g(x) - g(y) - \langle Dg(y), x-y\rangle| \leq C_1[Dg]_{\beta,\Omega}|x-y|^{1+\beta}, \quad x \in \overline{\Omega},$$

where C_1 is a positive constant depending on the Lipschitz constant L for $\partial\Omega$. We adopt the above simplifications: $y = (0, \dots, 0, 0)$, $x_0 = (0, \dots, 0, R)$, $g(y) = 0$, so that in particular

$$g(x) \leq \langle Dg(0), x\rangle + C_1[Dg]_{\beta,\Omega}|x|^{1+\beta}, \quad x \in \overline{\Omega}.$$

Therefore

$$g(x) \leq \langle Dg(0), x\rangle + C_2[Dg]_{\beta,\Omega} x_n^{\frac{1}{2}(1+\beta)}, \quad x \in \overline{\Omega}, \tag{5.7}$$

where C_2 is a positive constant depending on L, R and β. Next, we define

$$\varphi(x) = \langle Dg(0), x\rangle + C_3([Dg]_{\beta,\Omega} + \varepsilon)x_n^{\frac{1}{2}(1+\beta)} - \frac{1}{2|\mathbf{a}|}\|f^-\|_{L^\infty(\Omega)}, \tag{5.8}$$

where ε is any positive number and

$$C_3 \geq C_2 + \frac{(2R)^{1-\frac{\beta}{2}}}{2|\mathbf{a}|} \frac{\|f^-\|}{[Dg]_{\beta,\Omega} + \varepsilon}.$$

Therefore by (5.7):

$$u(x) = g(x) \leq \varphi(x) \quad \text{on } \partial\Omega.$$

Moreover, φ is a supersolution in Ω:

$$\mathcal{M}_{\mathbf{a}}(D^2\varphi) = a_1 C_3([Dg]_{\beta,\Omega} + \varepsilon)\frac{\beta+1}{2}\,\frac{\beta-1}{2}\,x_n^{\frac{\beta+1}{2}-2} - \|f^-\|_{L^\infty(\Omega)} \leq -f^-(x) \quad \text{in } \Omega.$$

By the comparison principle, we get $u \leq \varphi$ in Ω, and therefore by (5.8):

$$\begin{aligned}\frac{u(x)}{|x|^{\frac{1}{2}(1+\beta)}} &\leq |Dg(0)||x|^{\frac{1}{2}(1-\beta)} + C_4([Dg]_{\beta,\Omega} + \|f^-\|_{L^\infty(\Omega)}) \\ &\leq (4R)^{\frac{1}{2}(1-\beta)}[Dg]_{0,\partial\Omega} + C_4([Dg]_{\beta,\Omega} + \|f^-\|_{L^\infty(\Omega)}).\end{aligned}$$

Then for arbitrary $y \in \partial\Omega$ we have

$$\frac{u(x) - u(y)}{|x-y|^{\frac{1}{2}(1+\beta)}} \leq (4R)^{\frac{1}{2}(1-\beta)}[Dg]_{0,\partial\Omega} + C_4([Dg]_{\beta,\Omega} + \|f^-\|_{L^\infty(\Omega)})$$

for all $x \in \Omega$, from which (5.4) follows. □

We are ready to show the global Hölder estimates of Theorem 1.3.

Proof of Theorem 1.3. Let $\alpha \in (0,1)$ be the Hölder exponent of Theorem 5.3. From the boundary Hölder estimates of Lemma 5.4 we deduce an estimate of type

$$|u(x) - u(y)| \leq C(g, f)|x - y|^{\gamma_b}, \quad x \in \Omega,\ y \in \partial\Omega, \tag{5.9}$$

where $\gamma_b = \frac{\beta}{2}$ in case (i) and $\gamma_b = \frac{1}{2}(1+\beta)$ in case (ii). We want to show a global Hölder estimate with exponent $\gamma = \min(\alpha, \gamma_b)$. For proving the result we follow the same lines of [25, Proposition 4.13]. Thus, for $x, y \in \Omega$ we set $d_x = \operatorname{dist}(x, \partial\Omega) = |x - x_0|$, $d_y = \operatorname{dist}(y, \partial\Omega) = |y - y_0|$ for $x_0, y_0 \in \partial\Omega$, and suppose $d_y \leq d_x$. Here the constants C_i will depend at most on n, a^*, $\tilde{a}$, R, L and β.

(i) Suppose $|x - y| \leq \frac{d_x}{2}$. Since $y \in \overline{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset \Omega$, we can apply Theorem 5.3 properly scaled to the function $u(x) - u(x_0)$, and then the Hölder boundary estimate (5.9) obtaining

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} d_x^\alpha \leq C_1 \|u - u(x_0)\|_{L^\infty(B_{d_x}(x))} \leq C_2 K d_x^{\gamma_b}.$$

Recall that $\gamma \leq \alpha$. Since $\frac{d_x}{|x-y|} \geq 2$, from this we get

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} d_x^\gamma \leq \frac{u(x) - u(y)}{|x - y|^\alpha} d_x^\alpha \leq C_2 K d_x^{\gamma_b}.$$

Since also $\gamma \leq \gamma_b$,

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq C_2 K d_x^{\gamma_b - \gamma} \leq C_2 K d^{\gamma_b - \gamma} \equiv C_3 K. \tag{5.10}$$

(ii) Suppose now $|x - y| \geq \frac{d_x}{2}$. Since $d_y \leq d_x \leq 2|x - y|$ and $|x_0 - y_0| \leq d_x + |x - y| + d_y$, it follows from (5.9) that

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\leq C_4 K(d_x^{\gamma_b} + |x_0 - y_0|^{\gamma_b} + d_y^{\gamma_b}) \leq C_5 K |x - y|^\gamma. \end{aligned} \tag{5.11}$$

From (5.10) and (5.11), letting $\overline{C} = \max(C_3, C_5)$, we deduce the desired estimate $[u]_{\gamma,\Omega} \leq \overline{C} K$. ☐

In some cases, when the weights a_i are concentrated near the one of the extremal eigenvalues, we obtain an explicit interior Hölder exponent.

Lemma 5.5. *Let $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ be such that $a_1 \geq \hat{a}_1$ (resp. $a_n \geq \hat{a}_n$). Suppose that $u \in \operatorname{usc}(B_1)$ (resp. $u \in \operatorname{lsc}(B_1)$) is a viscosity subsolution (resp. supersolution) of the equation $\mathcal{M}_{\mathbf{a}}(D^2 u) = f$ in B_1, a ball of radius 1, and f is continuous and bounded above (resp. below) in B_1. Then $u \in C^\alpha(B_1)$ and the following interior C^α estimate holds:*

$$[u]_{\alpha, B_{1/2}} \leq C(\|u\|_{L^\infty(B_1)} + \|f^-\|_{L^\infty(B_1)}) \tag{5.12}$$

resp.

$$[u]_{\alpha, B_{1/2}} \leq C(\|u\|_{L^\infty(B_1)} + \|f^+\|_{L^\infty(B_1)}),$$

where $B_{1/2}$ is a ball of radius $\frac{1}{2}$ concentric with B_1, $\alpha = 1 - \frac{\hat{a}_1}{a_1}$ (resp. $\alpha = 1 - \frac{\hat{a}_n}{a_n}$), and C a positive constant depending on n, a_1 and $\hat{a}_1$ (resp. a_n and $\hat{a}_n$).

Proof. We only treat the case of subsolution, when $a_1 \geq \hat{a}_1$. The case of supersolutions, when $a_n \geq \hat{a}_n$, will follow by duality. We assume that the balls B_1 and $B_{1/2}$ are centered at 0. Then we take $x', x'' \in B_{1/2}$, and consider the ball $B_{1/2}(x')$. We note that on $\partial B_{1/2}(x')$:

$$u(x) - u(x') \leq 2\|u\|_{L^\infty(B_1)} \leq 2^{1+\alpha}\|u\|_{L^\infty(B_1)}|x - x'|^\alpha.$$

Next, we define

$$\varphi(x) = C_1 \|u\|_{L^\infty(B_1)}|x|^\alpha - \frac{1}{2|\mathbf{a}|}\|f^-\|_{L^\infty(B_1)}|x|^2,$$

where $C_1 = 2^{1+\alpha} + \frac{1}{2|\mathbf{a}|}\frac{\|f^-\|}{\|u\|}$ (in the nontrivial case $u \not\equiv 0$). Thus on $\partial B_{1/2}(x')$:

$$u(x) - u(x') \leq 2^{1-\alpha}\|u\|_{L^\infty(B_1)}|x - x'|^\alpha \leq \varphi(x - x'). \tag{5.13}$$

On the other hand,

$$\mathcal{M}_{\mathbf{a}}(D^2\varphi(x-x')) = C_1(a_1(\alpha-1)+\hat{a}_1)|x-x'|^{\alpha-2} - \|f^-\|_{L^\infty(\Omega)} \le -f^-(x). \tag{5.14}$$

By (5.14) and (5.13), using the comparison principle, we get $u(x) - u(x') \le \varphi(x-x')$ in $B_{1/2}(x')$, from which in particular

$$u(x'') - u(x') \le C(\|u\|_{L^\infty(B_1)} + \|f^-\|_{L^\infty(B_1)})|x''-x'|^\alpha.$$

Interchanging the role of x' and x'', we get (5.12). □

Combining Lemma 5.4 with Lemma 5.5, we obtain the global estimates of Theorem 1.4.

Proof of Theorem 1.4. To obtain (1.4) and (1.5), it is sufficient to follow the proof of Theorem 1.3. The above estimates (1.6) and (1.7) are in particular obtained from this proof taking $\gamma = \alpha$. We use once more the boundary Hölder estimates of Lemma 5.4, with $\beta = 2\alpha$ for (1.6) and $\beta = 2\alpha - 1$ for (1.7), as there. But we use here the interior C^α estimates of Lemma 5.5, instead of Theorem 5.3. □

Remark 5.6. Note that Theorem 1.4 provides Lipschitz estimates only in the case $\hat{a}_1 = 0$ and $\hat{a}_n = 0$, corresponding to the operators $\mathcal{M}_{\mathbf{e}_1}[u] = \lambda_1[u]$ and $\mathcal{M}_{\mathbf{e}_n}[u] = \lambda_n[u]$. See for instance [12].

Remark 5.7. Asking for higher regularity of viscosity solutions, we cannot expect viscosity solutions more regular than C^2. Indeed, we may consider the function $u := x_1^2 + \omega(x_2) - x_3^2$, in $\mathbb{R}^3$, where ω is a C^2 function but no more regular. The same regularity holds for u. Assuming in addition $|\omega''(x_2)| < 2$, by a straightforward computation we get

$$D^2u(x) = \mathrm{Diag}[(2, \omega''(x_2), -2)],$$

so that $\mathcal{M}(D^2u(x) = 0$. So we have found a solution $u \in C^2$, which does not belong to any $C^{2,\beta}$ space, $\beta \in (0,1)$.

6 The Strong Maximum Principle

The strong maximum principle for an elliptic operator F, such that $F(0) = 0$, means that a subsolution of the equation $F[u] = 0$ in an open set Ω cannot have a maximum at a point of Ω unless to be constant. Analogously, the strong minimum principle means that a supersolution u cannot have a minimum at a point of Ω unless u is constant.

One of the most elegant proof of the strong maximum principle, also known for this reason as the celebrated Hopf maximum principle [51], is based on boundary point lemma, which we establishes here below for the class of weigthed partial trace operators $\mathcal{M}_{\mathbf{a}}$. To obtain a strong maximum principle, it is sufficient to state this lemma just for a ball.

Lemma 6.1 (Hopf Boundary Point Lemma). *Let $u \in \mathrm{usc}(\overline{B})$ be a viscosity subsolution of the equation $\mathcal{M}_{\mathbf{a}}[u] = 0$ in a ball B, with $M_{\mathbf{a}} \in \mathcal{A}_1$. Let $x_0 \in \partial B$. If $u(x_0) > u(x)$ for all $x \in B$, then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial \nu}(x_0) > 0. \tag{6.1}$$

On the other hand, let $u \in \mathrm{lsc}(\overline{B})$ be a viscosity supersolution of the equation $\mathcal{M}_{\mathbf{a}}[u] = 0$ in a ball B, with $M_{\mathbf{a}} \in \mathcal{A}_n$. If $u(x_0) < u(x)$ for all $x \in B$, then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial \nu}(x_0) < 0. \tag{6.2}$$

Proof. We just prove the theorem for subsolutions. We may suppose that B is centered at the origin, i.e. $B = \{|x| < R\}$ for $R > 0$. Arguing as in [47, Section 3.2], and considering $0 < \rho < R$, we introduce the radial test function $v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}$, with $r = |x|$. By direct computation, see Remark 3.6 in Section 3, we get

$$\mathcal{M}_{\mathbf{a}'}(D^2v) \ge 2\alpha\left(a_1(2\alpha\rho^2 - 1) - \sum_{i=2}^n a_i\right)e^{-\alpha r^2}$$

for $r \geq \rho$ and $\mathcal{M}[v] \geq 0$ for $\alpha > 0$ large enough. Since $u(x_0) - u(x) > 0$ on $|x| = \rho$, there is a constant $\varepsilon > 0$ such that $u(x_0) - u(x) - \varepsilon v(x) \geq 0$ on $|x| = \rho$, as well as on $|x| = R$. Therefore $\varepsilon v(x) \leq u(x_0) - u(x)$ on the boundary of the annulus $A_{\rho,R} = \{\rho < |x| < R\}$. By the comparison principle, the same inequality holds in $A_{\rho,R}$. In fact, $\mathcal{M}_{\mathbf{a}'}[\varepsilon v] = \varepsilon \mathcal{M}_{\mathbf{a}'}[v] \geq 0$, by positive homogeneity, and by duality $\mathcal{M}_{\mathbf{a}'}[u(x_0) - u] \leq 0$, so that εv and $u(x_0) - u$ are respectively a subsolution and a supersolution in A, and we can apply Theorem 3.1 to deduce that

$$u(x_0) - u(x) \geq \varepsilon v(x) \quad \text{for all } x \in A.$$

Taking $x = x_0 - t\frac{x_0}{R}$ in the latter inequality, dividing by $t > 0$ and letting $t \to 0^+$, we get

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{d}{dr}(e^{-\alpha r^2})\Big|_{r=R} = 2\varepsilon\alpha R e^{-\alpha R^2},$$

which proves (6.1). □

Following [47], we remark that, whether or not the normal derivative exists, we have instead of (6.1) and (6.2), respectively, the inequalities

$$\liminf_{\substack{x \to x_0 \\ x \in \Sigma}} \frac{u(x_0) - u(x)}{|x - x_0|} > 0$$

and

$$\liminf_{\substack{x \to x_0 \\ x \in \Sigma}} \frac{u(x_0) - u(x)}{|x - x_0|} < 0,$$

where Σ is any circular cone of vertex x_0 and opening less than π with axis along the normal direction at the boundary point x_0.

The Hopf boundary point lemma can be used to prove the strong maximum principle for classical subsolutions or viscosity subsolutions which are differentiable. A strong maximum principle, valid also for nonsmooth viscosity solutions, can be obtained through the weak Harnack inequality of Lemma 5.2.

For a detailed discussion on the strong maximum principle, we refer to the paper [70] and the papers quoted therein. In the case of fully nonlinear elliptic operators, see for instance [6, 7]. For further references see [36, 62, 69, 76].

Theorem 6.2 (Strong Maximum Principle). *Let u be a non-negative continuous viscosity supersolution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = 0$, with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_n$, in a domain Ω of $\mathbb{R}^n$. If u has a minimum m at some point $x_0 \in \Omega$, then $u \equiv m$ in Ω. Similarly, let u be a continuous viscosity subsolution of the equation $\mathcal{M}_{\mathbf{a}}(D^2u) = 0$, with $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}_1$. If u has a maximum M at $x_0 \in \Omega$, then $u \equiv M$ in Ω.*

Proof. For the proof in the case of differentiable solutions u, based on the Hopf lemma, we refer to the proof of [47, Theorem 3.5].

Concerning viscosity supersolutions (strong minimum principle), let $A = \{x \in \Omega : u(x) = m\}$ and $B = \Omega \setminus A$, so that $A \cup B = \Omega$, $A \cap B = \emptyset$ with $A \neq \emptyset$ and B is open. Moreover, we claim that A is also open. Recalling that Ω is a open connected set, then $B = \emptyset$, otherwise we would have a contradiction. Then $\Omega = A$, and the first part of the theorem is proved.

We are left with proving that A is open. Let $x_0 \in A$, that is $u(x_0) = m$, and suppose that the cube Q_ℓ of side ℓ centered at x_0 is contained in Ω. By the weak Harnack inequality (5.2), properly scaled and applied to $u - m \geq 0$, we have

$$\|u - m\|_{L^{p_0}(Q_{2\ell/3})} \leq C_0 \inf_{Q_{3\ell/4}} (u - m) = u(x_0) - m = 0.$$

The function $u(x) - m$ is constant in $Q_{2/3}$ and by continuity $u(x) - m = u(x_0) - m = 0$ for all $x \in Q_{2/3}$, so that $Q_{2/3} \subset A$. This shows that A is open, thereby proving the claim and concluding the proof of the first part.

In the case of viscosity subsolutions (strong maximum principle), we argue in a similar manner, considering the set $A = \{x \in \Omega : u(x) = M\}$. By duality $v = M - u$ is a non-negative supersolution of the equation $\mathcal{M}_{\mathbf{a}'}(D^2v) = 0$ such that $v(x_0) = 0$. Then by the case of supersolutions v is constant and therefore $u(x) = M$ for all $x \in \Omega$. □

It follows that for elliptic operators $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$ both the strong maximum and minimum principle are satisfied.

It is plain that the strong maximum principle implies the weak maximum principle (see Section 2) in bounded domains. This is no more true in unbounded domains, where the strong maximum principle may hold while the weak maximum principle fails to hold. An elementary example of this fact is given by the function $u(x_1, x_2) = x_1 x_2$, which is harmonic in the whole plane, and therefore satisfies the strong maximum principle in all domains of $\mathbb{R}^2$, but is positive in the quarter plane $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$ and zero on $\partial\Omega$, so that the weak maximum principle does not hold in Ω.

Turning to bounded domains, as observed in Section 2, it is sufficient that $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ to have both the weak maximum and minimum principle. Theorem 6.2 requires instead $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$ to have the strong maximum and minimum principle hold together.

Actually, the strong maximum and minimum principle may fail when $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$, but $\mathcal{M}_{\mathbf{a}} \notin \mathcal{A}$. In fact, let us consider the partial trace operator $\mathcal{P}_k^+$ defined above for $1 \le k \le n-1$: the non-constant function $u(x) = 1 + \sin x_1$ has a maximum $M = 2$ inside the cube $]0,\pi[^n$, even though $\mathcal{P}_k^+(D^2u) = 0$ in $]0,\pi[^n$. Similarly, $u(x)$ is non-negative in the cube $]-\pi, 0[^n$ and has a zero inside, even though $\mathcal{P}_k^-(D^2u) = 0$ in the cube $]-\pi, 0[^n$ for $1 \le k \le n-1$.

From the proof of Theorem 6.2, the weak Harnack inequality, which would imply the strong minimum principle, fails to hold in general for the partial trace operator $\mathcal{P}_k^-$ as soon as $k < n$. Analogously, the Harnack inequality, which would imply both the strong maximum and minimum principle, fails to hold in general for the partial trace operators $\mathcal{P}_k^\pm$ as soon as $k < n$.

7 Liouville Theorems

A direct application of the Harnack inequality yields in a standard fashion the following Liouville result for entire solutions, defined in the whole $\mathbb{R}^n$. See for instance [4].

Theorem 7.1 (Liouville Theorem). *Let $\mathcal{M}_{\mathbf{a}} \in \mathcal{A}$. If u is an entire viscosity solution of the equation $\mathcal{M}(D^2u) = 0$ which is bounded above or below, then u is constant.*

It is well known that the above Liouville theorem holds in a stronger unilateral version for the Laplace operator in dimension $n = 2$, where instead of solutions, bounded above or below, we may consider subsolutions bounded above and supersolutions bounded below. This is due to the fact that the fundamental solutions are of logarithmic type. See [67, Theorem 29].

On the other hand, this is no longer true in higher dimension. For instance, the function

$$u(x) = \begin{cases} -\frac{1}{8}(15 - 10|x|^2 + 3|x|^4) & \text{for } |x| \le 1, \\ -\frac{1}{|x|} & \text{for } |x| > 1, \end{cases} \tag{7.1}$$

is a non-constant subharmonic, bounded function in $\mathbb{R}^3$. We refer to [67, Chapter 2, Section 12].

As well, the unilateral Liouville theorem does not hold for general elliptic operators even in dimension $n = 2$. Actually, as soon as $\lambda < \Lambda$ we can find subsolutions u, bounded above, of the equation $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = 0$ in $\mathbb{R}^2$. For instance, the function (7.1), regarded as a function of $x \in \mathbb{R}^2$, is a subsolution of the equation $\mathcal{M}^+_{\lambda,2\lambda}(D^2u) = 0$ in $\mathbb{R}^2$.

Therefore, the uniform ellipticity is not sufficient by itself to guarantee such an unilateral Liouville property, even in dimension 2.

However, for particular uniformly elliptic operators as the minimal Pucci operators $\mathcal{M}^-_{\lambda,\Lambda}$, which are suitably smaller than the Laplace operator, precisely when $n \le 1 + \frac{\Lambda}{\lambda}$, the Liouville property still holds for subsolutions, bounded above (see [35]). We thank Dr. Goffi for drawing our attention to the latter issue during a workshop where the results of this paper have been announced for the first time.[1]

We notice here that the same is true for the min-max operator $\mathcal{M}(X) = \lambda_1(X) + \lambda_n(X)$, and more generally for the operators $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ such that $a_1 = \hat{a}_1$. See also [13].

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Theorem 7.2 (Hadamard Three Circles Theorem). *Let* $u \in \mathrm{usc}(\mathbb{R}^n \setminus \{0\})$ *be a subsolution of the equation*

$$\mathcal{M}_{\mathbf{a}}(D^2 u) = 0$$

with $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ *such that* $a_1 = \hat{a}_1$. *Setting* $M(r) = \max_{B_r} u$ *for* $r > 0$, *we have that* $M(r)$ *is a convex function of* $\log r$, *namely for* $0 < r_1 < r_2$,

$$M(r) \leq \frac{M(r_1)\log(\frac{r_2}{r}) + M(r_2)\log(\frac{r}{r_1})}{\log(\frac{r_2}{r_1})}, \quad r_1 \leq r \leq r_2. \tag{7.2}$$

Proof. Actually, by Remark 3.6 the function

$$\varphi(x) = \frac{M(r_1)\log(\frac{r_2}{|x|}) + M(r_2)\log(\frac{|x|}{r_1})}{\log(\frac{r_2}{r_1})}$$

satisfies the equation $\mathcal{M}_{\mathbf{a}}(D^2\varphi) = 0$ as linear combination of a constant and $\log|x|$ with non-negative coefficients, by positive homogeneity. Moreover, $u(x) \leq \varphi(x)$ on the boundary of the annulus $A_{r_1,r_2} = \{r_1 < |x| < r_2\}$. From the comparison principle (Theorem 3.1) in A_{r_1,r_2} then we obtain (7.2). □

Note that $\mathcal{M} = \mathcal{M}_{\mathbf{e}_1+\mathbf{e}_n}$ satisfies the condition $a_1 = \hat{a}_1$ and therefore the Hadamard Three Circles Theorem holds for min-max operator $\mathcal{M}$.

From Theorem 7.2 it follows that such operators satisfy the same Liouville unilateral property which holds for the Laplace operator in dimension $n = 2$: if $a_1 = \hat{a}_1$, the constant functions are the only viscosity subsolutions, bounded above, of the equation $\mathcal{M}_{\mathbf{a}} = 0$ in $\mathbb{R}^n$.

Theorem 7.3 (Unilateral Liouville Property). *Let* $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ *be such that* $a_1 = \hat{a}_1$. *Let u be a viscosity subsolution of the equation* $\mathcal{M}(D^2 u) = 0$ *in* $\mathbb{R}^n \setminus \{0\}$, *which is bounded above. Then u is constant. If u is a subsolution in the whole* $\mathbb{R}^n$ *bounded above, then the same conclusion holds if* $a_1 > \hat{a}_1$.

On the other hand, suppose $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ *such that* $a_n = \hat{a}_n$. *Let v be a viscosity supersolution of the equation* $\mathcal{M}(D^2 v) = 0$ *in* $\mathbb{R}^n \setminus \{0\}$ *which is bounded below. Then v is constant. If u is a supersolution in the whole* $\mathbb{R}^n$, *bounded below, then the same conclusion holds if* $a_n > \hat{a}_n$.

Proof. Let $\mathcal{M}_{\mathbf{a}} \in \overline{\mathcal{A}}$ with $a_1 = \hat{a}_1$. Reasoning as in [67, Section 12], we take alternatively the limits as $r_1 \to 0^+$ and $r_2 \to \infty$ in (7.2). So we get

$$M(r) \leq M(r_2) \quad \text{for } r \leq r_2, \qquad M(r) \leq M(r_1) \quad \text{for } r \geq r_1$$

concluding that $M(r_1) = M(r_2)$ for arbitrary pairs of positive numbers r_1, r_2. Then $M(r)$ is constant, and by the strong maximum principle u is in turn a constant function.

Supposing $a_1 > \hat{a}_1$, for any arbitrary $x_0 \in \mathbb{R}^n$ we set

$$v(x) = u(x_1) + C|x - x_1|^{\gamma_2},$$

with $\gamma_2 = 1 - \frac{\hat{a}_1}{a}{}_1$ and $C \geq 0$ to be determined, recalling that by (3.8) we have $\mathcal{M}_{\mathbf{a}}(D^2 v) = 0$ in $\mathbb{R}^n \setminus \{0\}$. We will compare the entire subsolution bounded above, say $u(x) \leq M$, in every punctured ball $B_R(x_1) \setminus \{x_1\}$, noting that $u(x_1) = v(x_1)$ and on $\partial B_R(x_1)$ we have

$$u(x) \leq M \leq u(x_1) + C|x - x_1|^{\gamma_2}$$

choosing $C = (M - u(x_1)R^{-\gamma_2}$. Using the comparison principle (Theorem 3.1), we infer that this inequality holds in $B_R(x_1)$. Letting $R \to \infty$, we will have $u(x) \leq u(x_1)$ for all $x \in \mathbb{R}^n$. The same holds true for any other $x_2 \in \mathbb{R}^n$ so that $u(x_1) = u(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$.

Concerning supersolutions v, bounded below, of the same equation in $\mathbb{R}^n$, it is sufficient to note that by duality the function $u = -v$ is a subsolution, bounded above, of the equation $\mathcal{M}_{\mathbf{a}'}[u] = 0$, where $\mathbf{a}' = (a_n, a_{n-1}, \ldots, a_1)$, in $\mathbb{R}^n$, and then to use the result proved before for subsolutions. □

Funding: Fausto Ferrari is partially funded by INDAM-GNAMPA project 2018: *Costanti critiche e problemi asintotici per equazioni completamente non lineari* and INDAM-GNAMPA project 2019: *Proprietà di regolarità delle soluzioni viscose con applicazioni a problemi di frontiera libera*.

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